Discontinuous Galerkin finite element method for two-phase two-component flow in heterogeneous porous media with discontinuous capillary pressure

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Outline

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  - Model
  - Sequential scheme in time
  - dG method in space
  - Numerical results
- Heterogeneous setting
  - Interface conditions
  - dG setting
  - Numerical results

- Let Ω be a bounded, open, polyhedral domain in $\mathbb{R}^d$, $d \geq 1$, with boundary $\partial\Omega$ (Ω represents an indeformable, isothermal porous medium)
- Consider 2-phase (liquid (wetting) and gas (non-wetting)) 2-component (hydrogen and water) partially miscible (no water vaporisation) and partially compressible (water incompressibility) flow in Ω
- The two phases are in thermodynamical equilibrium

Homogeneous setting (continued)

- The flow is described by
  - Darcy-Muskat velocity for each phase (no gravity) :

$$\mathbf{u}_\alpha = -\mathbb{K}\lambda_\alpha \nabla p_\alpha, \quad \alpha \in \{l, g\}$$

  - saturation relation : $s_l + s_g = 1$
  - capillary pressure law : $\pi(s_g) = p_g - p_l$
  - mass conservation equation for each component :

$$\partial_t(m_\beta) + \nabla \cdot \mathbf{F}^\beta = Q^\beta, \quad \beta \in \{w, h\}$$

  - Henry model is used to close the system

Including phase appearance /disappearance

- We treat gas phase appearance and disappearance
- To avoid the change of variables and equations in saturated / unsaturated regions we introduce as new unknown the normalized total hydrogen mass density

$$\chi = \Upsilon(p_l, s_g) := (1 - s_g)R_s + C_v p_g s_g,$$

where $R_s = \chi_l^h / \chi_g^{std}$ is the solution gas/liquid ratio and $C_v = M^h / (RT\rho_g^{std})$ (see Bourgeat, Jurak & Smaï (09) for more details)

- Use new unknown χ and $p = p_l$ (liquid-phase pressure) to write the model as follows in $\Omega \times [0, T]$

$$\partial_t b(p,\chi) + \operatorname{div}(\mathbf{u}_{tot}) = Q_1$$
$$\phi \partial_t \chi + \operatorname{div}(\mathbf{u}_h) = Q_2$$

where

$$\mathbf{u}_{tot} = -A_{1,1}(p,\chi)\nabla p - A_{1,2}(p,\chi)\nabla\chi,$$
$$\mathbf{u}_h = -A_{2,1}(p,\chi)\nabla p - A_{2,2}(p,\chi)\nabla\chi$$

- Initial conditions for p and χ
- Dirichlet boundary conditions on the sets $\partial\Omega_p^{\mathrm{D}},\ \partial\Omega_\chi^{\mathrm{D}} \subset \partial\Omega$ for p and χ respectively
- Neumann boundary conditions that prescribe the normal component of the fluxes $\mathbf{u}_{tot}$ and $\mathbf{u}_{tot}$ on the rest of $\partial\Omega$

- For $m = 0, 1, \ldots, N$:
  - Solve quase-linear elliptic equation (pressure equation)

$$\tau_m^{-1} b(p^{m+1}, \chi^m) - \nabla \cdot (A_{1,1}(p^m, \chi^m) \nabla p^{m+1}) = \nabla \cdot (A_{1,2}(p^m, \chi^m) \nabla \chi^m) + F_1 + \tau_m^{-1} b(p^m, \chi^m)$$

    with respective boundary conditions
  - Calculate $U(p^{m+1}, \chi^m)) = -A_{1,1}(p^{m+1}, \chi^m) \nabla p^{m+1}$
  - Solve quase-linear reaction-advection-diffusion equation (mass transport equation)

$$\phi \tau_m^{-1} \chi^{m+1} + \nabla \cdot (f(p_l^{m+1}, \chi^{m+1}) U(p^{m+1}, \chi^m)) - \nabla \cdot (A_{2,2}(p^{m+1}, \chi^m) \nabla \chi^{m+1}) = F_2 + \phi \tau_m^{-1} \chi^m,$$

    with respective boundary conditions, here

$$f(p, \chi) = A_{2,1}(p, \chi) / A_{1,1}(p, \chi)$$

  - p^0 and χ^0 are given from the initial conditions

Sequential scheme in time : advantages

- decompose the system in an (non-linear) elliptic-parabolic equation and a (non-linear) reaction-advection-diffusion equation, weakly coupled by Darcy velocity of liquid phase
- in the absence of gas-phase, the equations are coupled via coefficients only
- reduce computational cost with respect to fully coupled approach (one step of fixed point iteration)

Discontinuous Galerkin (dG) methods

- dG methods can be viewed as
  - FE-based methods using piecewise polynomials discontinuous across mesh elements
  - FV-based high-order methods using numerical fluxes
- Attractive features include
  - weakly imposed inter-element continuity
  - local conservation properties
  - flexibility (non-matching grids, variable polynomial degree)
  - ability to capture shocks sharply

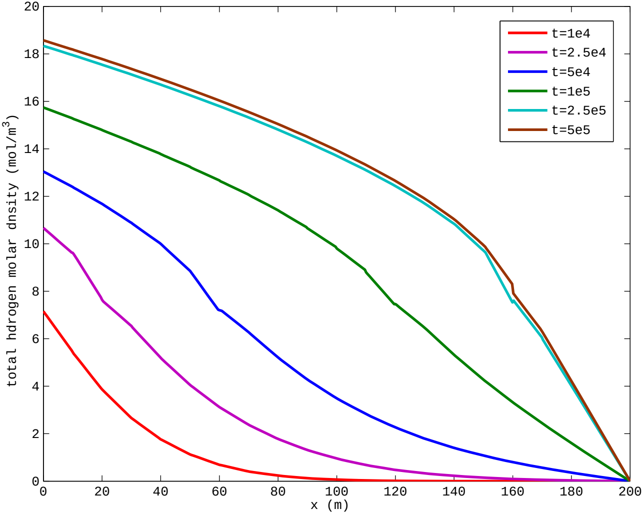
- Key ingredients (Ern, Mozolevski & Schuh (09), (10)) :
  - Sequential dG method for decoupling of the system describing two-phase two-component flows
  - Accurate (total) velocity reconstruction from pressure gradient using Raviart-Thomas FE
  - Weighted averages in the consistency terms and harmonic averages in the penalties

- Consider MOMAS benchmark Problem 1 :
  http://sources.univ-lyon1.fr/cas_ test.html
- 1D geometry, $\Omega = (0, 200)$
- The porous medias characteristics and the fluids properties are from http://sources.univ-lyon1.fr/cas_test/multi-mat.pdf, in particular $\phi = 0.15, K = 5 \cdot 10^{-20}$, $n = 1.49$ $Pr = 2 \cdot 10^{6}$, $Sgr = 0, Slr = 0.4$
- Boundary and initial conditions

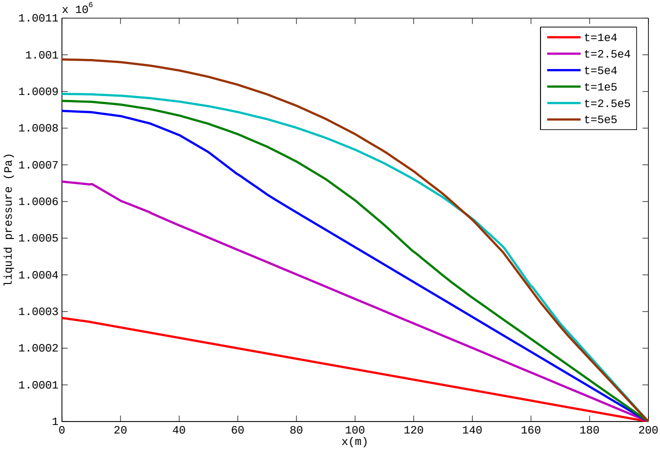
$$\begin{cases} \mathbf{u}_w|_{x=0} = 0,\ p|_{x=200} = 10^6 \text{ Pa}, \\ \mathbf{u}_h|_{x=0} = 7.5 \cdot 10^{-5} m/years,\ \chi|_{x=200} = 0, \\ p|_{t=0} = 10^6,\ \chi|_{t=0} = 0 \end{cases}$$

- Meshes : uniform in space, $nEl = 20$, adaptative in time starting from $\tau = 100$ years, $T = 7 \cdot 10^5$ years

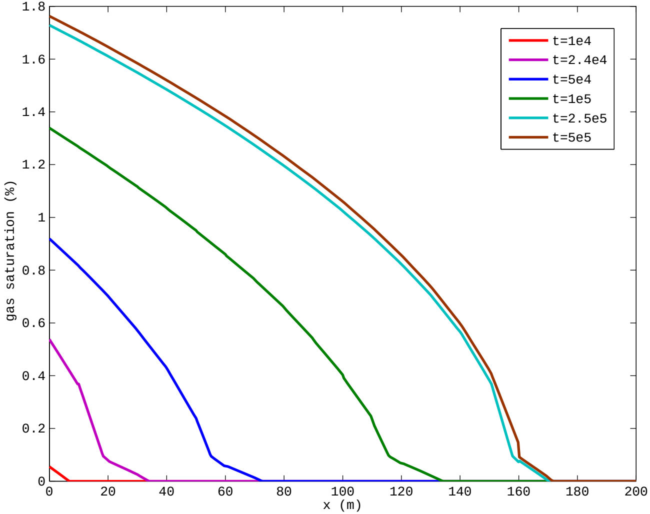
Total hydrogen molar density at several times (in years)



Liquid pressure at several times (in years)



Hydrogen saturation at several times (in years)



- Capillary pressure discontinuities lead to nonlinear interface conditions, see Bear (72); Chavent and Jaffre (78).
- Theoretical analysis of nonlinear interface problem for saturation equation, see Duijn, Molenaar, Neef, (95); Bertsch, Passo, Duijn, (03)
- Existence and uniqueness of the weak solution to the interface problem for coupled system of pressure - saturation equations, see Amaziane, Bourgeat & El Amri (96)
- FV methods for heterogeneous two-phase flows with capillary pressure discontinuities, see Enchèry, Eymard & Michel (06), Cancès (09), Cancès, Gallouët & Porretta (09)
- dG methods fo two-phase flows with capillary pressure discontinuities, see Ern, Mozolevski, Schuh (10)

- Ω is decomposed in $\Omega^{(r)},\ r \in \{1,2\}$ by an interface Γ
- The characteristics of porous media (in particular capillary pressure) could be different in each $\Omega^{(r)}$;
- Physical hypothesis : capillary pressures vanish at zero (no entry pressure), e.g. van Genuchten model

$$\pi(s_g) = p_r \left((1 - s_{ge})^{-\frac{1}{m_G}} \right)^{\frac{1}{n_G}}, \qquad (1)$$

where

$$s_{ge} = \frac{s_g - s_{gr}}{1 - s_{gr} - s_{lr}} \qquad (2)$$

is the effective saturation.

- Interface conditions :
  - Since the liquid phase is always present in both subdomains, the liquid pressure and the respective flux should be continuous
  - Owing to mass conservation the, hydrogen flux should be continuous
  - When gas phase is absent at least in one of the subdomains hydrogen mass density should be continuous at interface
  - If gas phase is present in the subdomains, normalized total hydrogen mass density can be discontinuous to ensure continuity of the capillary pressure
  - Note that dissolved hydrogen density and respective flux remain continuous

- $\forall u \in L^2(\Omega)$ let us denote by $u^{(r)}$ the restriction of u to $\Omega^{(r)}$
- Interface conditions for p :

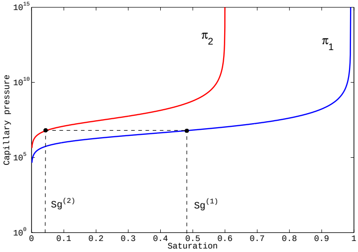
$$\mathbf{n}_\Gamma \cdot (-A_{1,1}^{(1)}(p^{(1)},\chi^{(1)})\nabla p^{(1)}) = \mathbf{n}_\Gamma \cdot (-A_{1,1}^{(2)}(p^{(2)},\chi^{(2)})\nabla p^{(2)})$$
$$p^{(1)} = p^{(2)}$$

- Interface conditions for χ :

$$\mathbf{n}_\Gamma \cdot u_h^{(1)} = \mathbf{n}_\Gamma \cdot u_h^{(2)}$$

- denote $s_g^{(r)} = S_g^{(r)}(p^{(r)},\chi^{(r)}), \quad r \in 1, 2;$
- if $s_g^{(1)} \cdot s_g^{(2)} = 0$
  - $\chi^{(1)} = \chi^{(2)}$
- else
  - $\pi^{(1)}(s_g^{(1)}) = \pi^{(2)}(s_g^{(2)})$

Capillary pressure continuity condition



- Define

$$J(p^{(1)},\chi^{(1)};p^{(2)},\chi^{(2)}) =$$

$$\begin{cases} 0,\ \text{if } s_g^{(1)} \cdot s_g^{(2)} = 0, \\ \chi^{(1)} - \Upsilon(p^{(2)}, \pi_2^{-1}(\pi_1(S_g^{(1)}(p^{(1)},\chi^{(1)}))) \\ \text{otherwise} \end{cases}$$

- Then the above interface condition for χ is equivalent to

$$\chi^{(1)} - \chi^{(2)} = J(p^{(1)},\chi^{(1)};p^{(2)},\chi^{(2)})$$

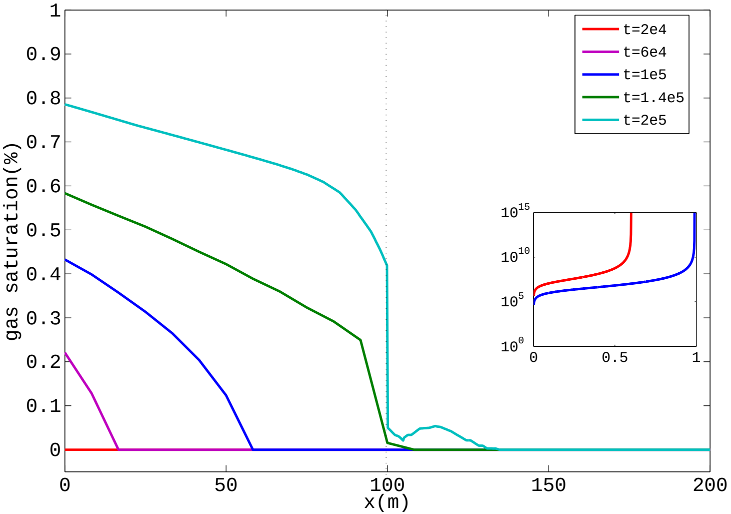
- Family of shape-regular meshes $\{\mathcal{T}_h\}_{h>0}$ exactly fitted to the partition $\Omega = \Omega^{(1)} \cup \Omega^{(2)}$
- Key ingredients, the same as in homogeneous case
  - Sequential dG method for decoupling of the system describing two-phase two-component flows
  - Accurate (total) velocity reconstruction from pressure gradient using Raviart-Thomas FE
  - Weighted averages in the consistency terms and harmonic averages in the penalties
- New : weak implementation of non-linear interface condition

- Consider MOMAS heterogeneous benchmark Problem 2 : http://sources.univ-lyon1.fr/cas_ test.html
- 1D geometry, $\Omega = (0, 200)$ with an interface at $x = 100$
- The porous medias characteristics and the fluids properties are from http://sources.univ-lyon1.fr/cas_test/multi-mat.pdf, in particular $\phi = [0.3; 0.15], K = [10^{-18}; 5 \cdot 10^{-20}]$, $n = [1.54; 1.49]$ $Pr = [2 \cdot 10^6; 15 \cdot 10^6]$, $Sgr = [0; 0], Slr = [0.01; 0.4]$
- Boundary and initial conditions

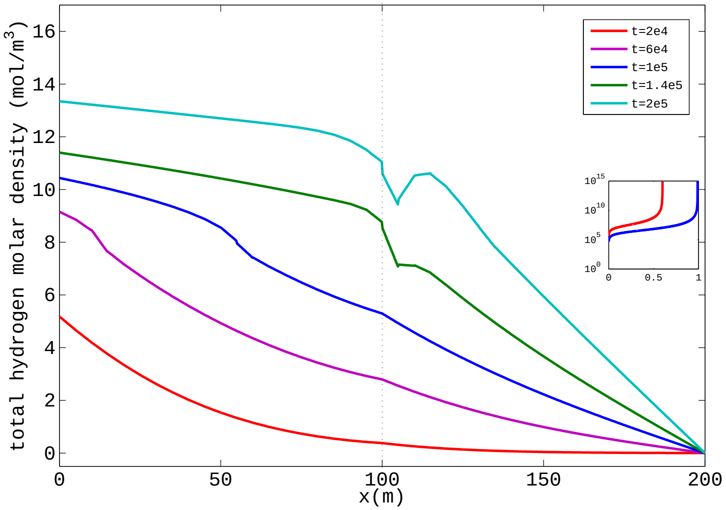
$$\begin{cases} \mathbf{u}_w|_{x=0} = 0, \ p|_{x=200} = 10^6 \text{ Pa}, \\ \mathbf{u}_h|_{x=0} = 7.5 \cdot 10^{-5} m/years, \ \chi|_{x=200} = 0, \\ p|_{t=0} = 10^6, \ \chi|_{t=0} = 0 \end{cases}$$

- Meshes : uniform in space, $nEl = 20$ in each subdomain, adaptative in time starting from $\tau = 100$ years, $T = 2.7 \cdot 10^5$ years

Hydrogen saturation at several times (in years)



Total hydrogen molar density at several times (in years)



Liquid pressure at several times (in years)