Convergence of a Finite Volume Scheme and Numerical Simulations for Water-Gas Flow in Porous Media

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PLAN

Mathematical model

We consider the flow of two immiscible compressible fluids (w=water and g=gas) in a porous medium $\mathrm{I} =]a, b[= \bigcup_{m=1}^{2} \mathrm{I}^m$.

Gas pressure - water saturation formulation

$$\begin{cases} 0 \leq S(x,t) \leq 1 & \text{in } \mathrm{I} \times]0, T[, \\ \Phi \partial_t S + \partial_x \big(f_w(S) q \big) - \partial_x \big(\mathbf{K} \partial_x \alpha(S) \big) = \frac{Q_w}{\rho} & \text{in } \mathrm{I} \times]0, T[, \\ \Phi \partial_t \big(P(1-S) \big) - \partial_x \big(P \lambda_g(S) \mathbf{K} \partial_x P \big) = \frac{Q_g}{\sigma_g} & \text{in } \mathrm{I} \times]0, T[, \\ q = -\lambda(S) \mathbf{K} \partial_x \big(P + \beta(S) \big) & \text{in } \mathrm{I} \times]0, T[, \end{cases} \qquad (1)$$

- $\Phi(x)$ is the porosity,
- $\mathbf{K}(x)$ absolute permeability,
- Q_ν the source term of phase $\nu = w, g$,
- $kr_\nu(S)$ the ν-phase relative permeability,
- $\lambda_\nu(S) = \frac{kr_\nu(S)}{\mu_\nu}$ where μ_ν the ν-phase viscosity,
- $\lambda(S) = \lambda_w(S) + \lambda_g(S)$ is the total mobility.

Finite Volume Scheme

- Let $\{t_0, \ldots, t_N\}$ be a partition of $\mathrm{J} = [0, T]$; $\Delta t^n = t^{n+1} - t^n$;
- Let $(x_i)_{i=0}^{N_x}$ be a partition of I and $x_{i+\frac{1}{2}} = (x_{i+1} + x_i)/2$;
- Vertex-Centred control volumes $\mathrm{I}_i := [x_{i-\frac{1}{2}}; x_{i+\frac{1}{2}}]$; $h_i = |\mathrm{I}_i|$

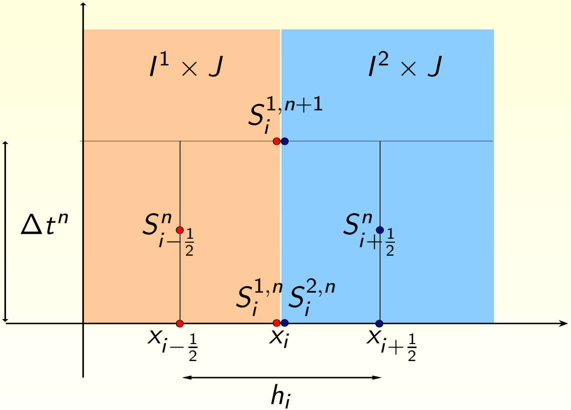

Figure: The control volume $I_i \times J_n$ at the interface for $i = i_0$.

Finite Volume Scheme

Saturation equation

$$S_i^{n+1} + \frac{\Delta t^n}{\Phi_i h_i} \sum_{j=\pm 1/2} \left(\alpha^m(S_i^{n+1}) - \alpha^m(S_{i+2j}^{n+1})\right) \frac{\mathbf{K}_{i+j}}{\Delta x_{i+j-\frac{1}{2}}} \tag{2}$$

$$= S_i^n - \frac{\Delta t^n}{\Phi_i h_i} \sum_{j=\pm 1/2} \left[f_w^m(S_i^n)\left(2jq_{i+j}^n\right)^+ - f_w^m(S_{i+2j}^n)\left(-2jq_{i+j}^n\right)^+ \right] + \frac{\Delta t^n}{\Phi_i \rho} Q_{w,i}^n$$

Pressure equation

$$P_i^{n+1}(1 - S_i^{n+1}) + \frac{\Delta t^n}{\Phi_i h_i} \sum_{j=\pm 1/2} \left((P_i^{n+1})^2 - (P_{i+2j}^{n+1})^2\right) \frac{\lambda_g^m(S^{n+1})_{i+j}\mathbf{K}_{i+j}}{2\Delta x_{i+j-\frac{1}{2}}}$$

$$= P_i^n(1 - S_i^n) + \frac{\Delta t^n}{\Phi_i \sigma_g} Q_{g,i}^n \tag{3}$$

With **continuity of the capillary pressure at the interface**

$$S_{i_0}^{m=2} = \left(P_c^2\right)^{-1}\left(P_c^1\left(S_{i_0}^{m=1}\right)\right)$$

Finite Volume Scheme

$$q^n_{i+\frac{1}{2}} := \frac{K_{i+\frac{1}{2}}}{\Delta x_i}\lambda^m(S^n)_{i+\frac{1}{2}}\left[\left(P^n_i - P^n_{i+1}\right) + \left(\beta^m(S^n_i) - \beta^m(S^n_{i+1})\right)\right]$$

Matrix form

$$\begin{aligned}\left[\mathbb{A}^n\left(\mathbb{S}^{n+1}\right)\right]\mathbb{S}^{n+1} &= \mathbb{F}^n \\ \left[\mathbb{B}^n\left(\mathbb{S}^{n+1},\mathbb{V}^{n+1}\right)\right]\mathbb{V}^{n+1} &= \mathbb{G}^n\end{aligned}$$

where for all $n = 0, ..., N-1$

$$\begin{aligned}\mathbb{S}^n := (S^n_i)^{N_x}_{i=0} \quad &\text{and} \quad \mathbb{V}^n := (v^n_i := P^n_i(1-S^n_i))^{N_x}_{i=0} \\ \left[\mathbb{A}^n\left(\mathbb{S}^{n+1}\right)\right] := \left(A^n_{ij}\right)^{N_x}_{i,j=0} \quad &\text{and} \quad \left[\mathbb{B}^n\left(\mathbb{S}^{n+1},\mathbb{V}^{n+1}\right)\right] := \left(B^n_{ij}\right)^{N_x}_{i,j=0} \\ \mathbb{F}^n := (F^n_i)^{N_x}_{i=0} \quad &\text{and} \quad \mathbb{G}^n := (G^n_i)^{N_x}_{i=0}.\end{aligned}$$

Finite Volume Scheme

$\mathbb{A}^n$ and $\mathbb{B}^n$ are the sparse matrix with non nulls entries:

$$
\begin{aligned}
A^n_{ii}: &= 1 + \frac{\Delta t^n}{\Phi_i h_i} \sum_{j=\pm 1/2} \frac{\mathbf{K}_{i+j}}{\Delta x_{i+j-\frac{1}{2}}} \alpha'^m\left(S^{n+1}\right)_{i+j}, \\
A^n_{i,i+2j}: &= -\frac{\Delta t^n}{\Phi_i h_i} \frac{\mathbf{K}_{i+j}}{\Delta x_{i+j-\frac{1}{2}}} \alpha'^m\left(S^{n+1}\right)_{i+j}, \quad j = \pm 1/2 \\
B^n_{ii}: &= 1 + \frac{\Delta t^n}{\Phi_i h_i} \sum_{j=\pm 1/2} \left(v^{n+1} \widetilde{\lambda}^m_g\left(S^{n+1}\right)\right)_{i+j} \frac{\mathbf{K}_{i+j}}{\Delta x_{i+j-\frac{1}{2}}}, \\
B^n_{i,i+2j}: &= -\frac{\Delta t^n}{\Phi_i h_i} \left(v^{n+1} \widetilde{\lambda}^m_g\left(S^{n+1}\right)\right)_{i+j} \frac{\mathbf{K}_{i+j}}{\Delta x_{i+j-\frac{1}{2}}}, \quad j = \pm 1/2,
\end{aligned}
$$

where for $\gamma^m(S) := -\int_S^1 \breve{\lambda}^m_g(s)ds$,

$$
\widetilde{\lambda}^m_g(S)_{i+j} := \begin{cases} \breve{\lambda}^m_g(S_i) & \text{if } S_i = S_{i+2j} \\ \dfrac{\gamma^m(S_{i+2j}) - \gamma^m(S_i)}{S_{i+2j} - S_i} & \text{otherwise.} \end{cases}
$$

L^∞ and weak BV estimates

(A0) $\rho, \sigma_g, \mu_\nu(\nu = w, g)$, are positive constants.

(A1) $\Phi \in L^\infty(I)$ such that, $0 < \Phi_- \leq \Phi(x) \leq \Phi^+ \leq 1$ a.e. in I.

(A2) $0 < K_- \leq \mathbf{K}(x) \leq K^+ < +\infty$ a.e. in I.

(A3) $S^0, \mathbf{K}\partial_x \alpha(S^0) \in L^\infty(I) \cap \overline{BV}(I)$, $0 \leq S^0(x) \leq 1 - \varepsilon$.

(A4) $P^0, \mathbf{K}\partial_x P^0 \in L^\infty(I) \cap \overline{BV}(I)$, $0 < P^0_- \leq P^0 \leq P^0_+ < +\infty$.

(A5) $\lambda_\nu, \widetilde{\lambda}_g \in C^1([0,1]; \mathbb{R}^+)$ such that $\forall s \in]0,1[$, $\lambda_\nu(s) > 0$ and $\widetilde{\lambda}_g(s) \geq \widetilde{\lambda}_g^- > 0$.

(A6) $\lambda \in C^1([0,1]; \mathbb{R}^+)$ such that $\forall s \in [0,1]$, $\lambda(s) \geq \lambda_- > 0$.

(A7) $f_w, \check{f}_w \in C^1([0,1]; \mathbb{R}^+)$ such that $f_w(0) = 0$ and $\forall s \in]0,1[$, $f'_w(s) > 0$.

(A8) $\alpha, \beta \in C^1([0,1]; \mathbb{R}^+)$ such that $\alpha', \beta'(0) = 0$ and $\forall s \in]0,1[$, $\alpha', \beta'(s) > 0$.

(A9) $Q_\nu \in L^\infty(I \times J) \cap \overline{BV}(I \times J)$, $\partial_t Q_\nu \in L^\infty(I \times J)$ and $Q_\nu(x,t) \geq 0$ a.e. in $I \times J$.

L^∞ and weak BV estimates

CFL condition

$$2\frac{\Delta t^n}{h\phi_-}\left[\sup_s f'_w(s) + \sup_s \breve{f}_w(s)\right]\|q^n\|_\infty \leq 1, \qquad (4)$$

and for $i_n \in \{0, ..., N_x\}$ / $S_{i_n}^{n+1} = \max_i S_i^{n+1}$, we assume that

$$Q_{w,i_n}^n = 0 \text{ and } \partial_{i_n}(q^n) \geq 0. \qquad (5)$$

where : $\partial_i f := \frac{1}{h_i}\left(f^m_{i+\frac{1}{2}} - f^m_{i-\frac{1}{2}}\right)$ and $\breve{f}_w(S) := \begin{cases} \frac{f_w(S)}{S} & \text{if } S \neq 0 \\ f'_w(S) & \text{otherwise.} \end{cases}$

Proposition 1.

Under the assumptions **(A0)–(A9)**, the CFL condition (4) and (5), the scheme (2)–(3) is L^∞ stable. Furthermore the following discrete maximum principle holds: for all $i = 1, ..., N_x$ we have $0 \leq S_i^{n+1} \leq 1$.

L^∞ and weak BV estimates

Proposition 2.

Under the assumptions **(A0)–(A9)**, the CFL condition (4) and (5), the scheme (2)–(3) is BV stable in space for all $n = 1, ..., N$, furthermore we have the L^1 continuity in time.

Theorem 1.

Under the assumptions **(A0)–(A9)**, the CFL condition (4) and (5), the approximate solution (S_h, P_h) given by the scheme (2)–(3) converge in $L^1(\Omega_T)$ to (S, P) a weak solution of (1) as H and Δt go to zero.

Numerical results

Test case BOBG (C. Chavant, 2008)

BO	BG
$S = 0.77$	$S = 1.0$
$P_g = 10^5 \ Pa$ $P_w = P_g - P_c(S)$	$P_g = P_w = 10^5 \ Pa$

- **Capillary pressure:** $S^m(P_c) := \left(1 + \left(\frac{P_c}{A^m}\right)^{\frac{1}{1-B^m}}\right)^{-B^m}$
- **Relative permeability:** $kr_g(S) := (1 - S^2)(1 - S^{\frac{5}{3}})$ and $kr_w^m(S) := \left(1 + \frac{(S^{-C_m} - 1)^{D_m}}{F_m}\right)^{-E_m}$ for $m = 1$ *or* 2.

m	Φ_m	$\mathbf{K}_m \ (m^2)$	$A_m \ (Pa)$	B_m	C_m	D_m	E_m	F_m
1	0.30	1.E-20	1.5E+6	0.060	16.67	1.880	0.5	4.0
2	0.05	1.E-19	1.0E+7	0.412	2.429	1.176	1.0	1.0

Numerical results

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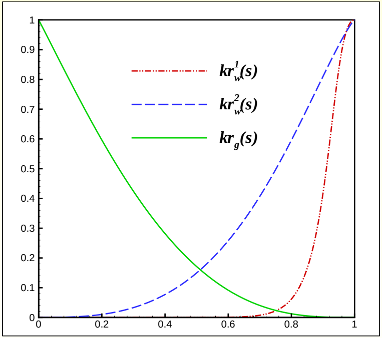
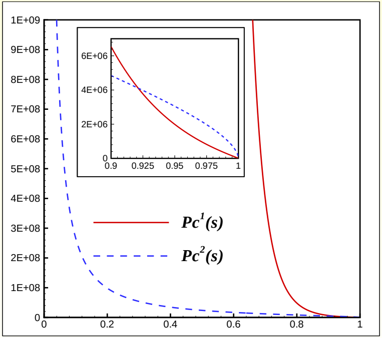
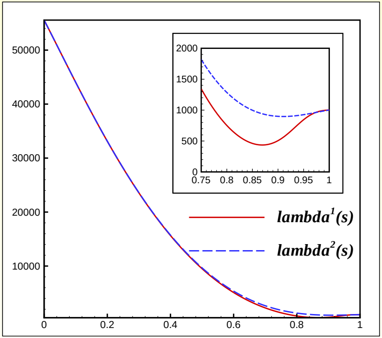


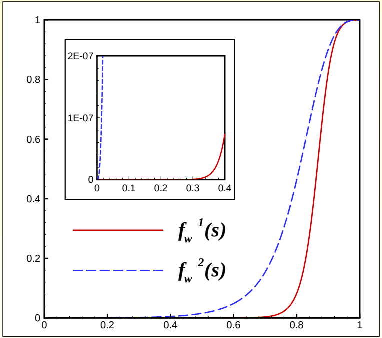


Relative permeability (left) and Capillary pressure (right)

Numerical results

- **Total mobility:** $\lambda^m(S) := \frac{kr_w^m(S)}{\mu_w} + \frac{kr_g(S)}{\mu_w}$
- **fractional flow:** $f_w^m(S) := \frac{kr_w^m(S)}{\mu_w \lambda^m(S)}$, for $m = 1$ *or* 2





Total mobility $\lambda(S)$ (left) and fractional flow f_w (right)

Numerical results

$S = 0.77$ **BO**	**BG** $S = 1.0$
$P_g = 10^5 \ Pa$ $P_w = P_g - P_c(S)$	$P_g = P_w = 10^5 \ Pa$

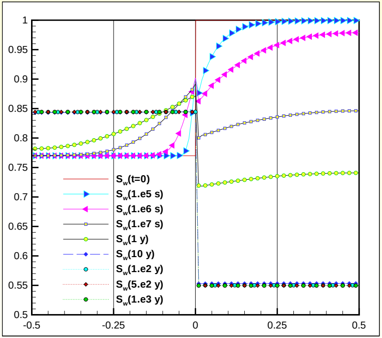


Figure: **Water saturation** S

Numerical results

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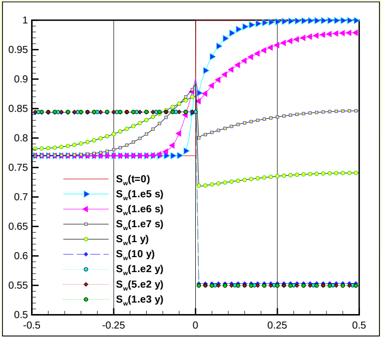

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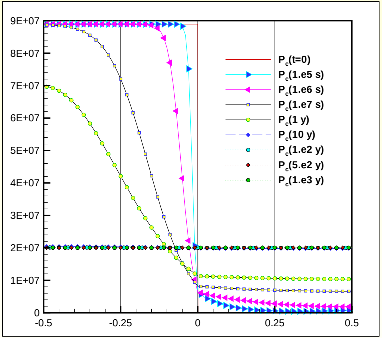


Figure: **Capillary pressure** P_c

Numerical results

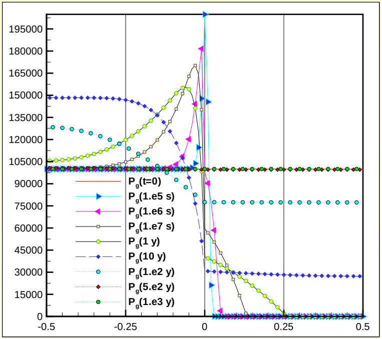
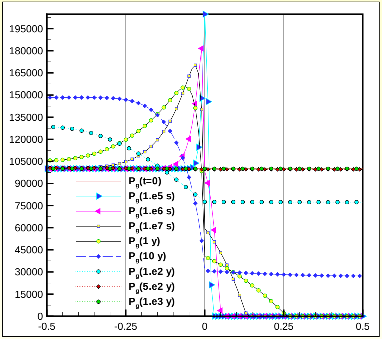


Figure: **Gas pressure** P

Numerical results



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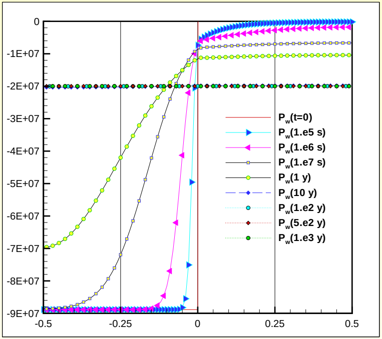

Figure: **Water pressure** P_w

References

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- O. Angelini, C. Chavant, E. Chénier, R. Eymard, S. Granet, *Finite volume approximation of a diffusion-dissolution model and application to nuclear waste storage*. Preprint submitted to Mathematics and Computers in Simulation, (2010).