A New Formulation of Immiscible Compressible Two-Phase Flow in Porous Media Via the Concept of Global Pressure

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Outline

Two-phase immiscible, compressible flow equations

Fractional flow formulation

Simplified global pressure formulation

New global pressure formulation

Numerical comparison of coefficients

Conclusion

Compressible two-phase flow

We consider two-phase compressible, immiscible flow through heterogeneous porous medium. For example, water and gas.

Assumptions:

- ▶ An incompresible fluid phase: water
- ▶ A compressible fluid phase: gas;
- ▶ No mass exchange between the phases;
- ▶ The temperature is constant;
- ▶ Porous medium my be heterogeneous but with a single rock-type.

Flow equations

Mass conservation: for $\alpha \in \{w, g\}$,

$$\Phi \frac{\partial}{\partial t}(\rho_\alpha S_\alpha) + \text{div}(\rho_\alpha \mathbf{q}_\alpha) = \mathcal{F}_\alpha,$$

The Darcy-Muscat law: for $\alpha \in \{w, g\}$,

$$\mathbf{q}_\alpha = -\mathbb{K}\frac{kr_\alpha(S_\alpha)}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha \mathbf{g}),$$

Capillary law:

$$p_c(S_w) = p_g - p_w,$$

No void space.

$$S_w + S_g = 1.$$

Flow equations

Compressibility: Phase density depends on phase pressure:

$$\rho_w = \rho_w(p_w), \quad \rho_g = \rho_g(p_g).$$

Simplifying assumptions:

- ▶ Viscosities μ_w and μ_g are constant.
- ▶ Mass sources neglected: $\mathcal{F}_w = \mathcal{F}_g = 0$.
- ▶ Water incompressible: $\rho_w = $ const.
- ▶ Gas follows ideal gas law: $\rho_g = c_g p_g$.

Note that the assumptions on the form of mass densities are not essential.

Independent variables: water saturation S_w and gas pressure p_g
($p_w = p_g - p_c(S_w)$, $S_g = 1 - S_w$).

Fractional flow formulation

▶ Reformulate flow equations in a form giving less tight coupling between the two differential equations, alowing a sort of IMPES (implicite in pressure and explicite in saturation) numerical treatment.

There are two approaches:

1. Introduce total velocity: $\mathbf{Q}_t = \mathbf{q}_w + \mathbf{q}_g$: leads to non-conservative equations

2. Introduce total flow: $\mathbf{Q}_t = \rho_w \mathbf{q}_w + \rho_g \mathbf{q}_g$: leads to conservative equations.

We work with the total flow formulation.

Fractional flow formulation: equations

Total flow:

$$\mathbf{Q}_t = -\lambda(S_w, p_g)\mathbb{K}\left(\nabla p_g - f_w(S_w, p_g)\nabla p_c(S_w) - \bar{\rho}(S_w, p_g)\mathbf{g}\right),$$

Total mass conservation:

$$\Phi\frac{\partial}{\partial t}(S_w\rho_w + (1 - S_w)\rho_g(p_g)) + \operatorname{div}(\mathbf{Q}_t) = 0,$$

Water mass conservation:

$$\Phi\rho_w\frac{\partial S_w}{\partial t} + \operatorname{div}(f_w(S_w, p_g)\mathbf{Q}_t + \mathbb{K}\mathbf{g}b_g(S_w, p_g)) = \operatorname{div}(\mathbb{K}a(S_w, p_g)\nabla S_w).$$

Fractional flow formulation: coefficients

phase mobilities $\qquad \lambda_w(S_w) = \dfrac{kr_w(S_w)}{\mu_w}, \quad \lambda_g(S_w) = \dfrac{kr_g(S_w)}{\mu_g},$

total mobility $\qquad \lambda(S_w, p_g) = \rho_w \lambda_w(S_w) + \rho_g(p_g)\lambda_g(S_w),$

water fractional flow $\quad f_w(S_w, p_g) = \dfrac{\rho_w \lambda_w(S_w)}{\lambda(S_w, p_g)},$

mean density $\qquad \bar{\rho}(S_w, p_g) = \dfrac{\lambda_w(S_w)\rho_w^2 + \lambda_g(S_w)\rho_g(p_g)^2}{\lambda(S_w, p_g)},$

"gravity" coeff. $\qquad b_g(S_w, p_g) = \rho_w \rho_g(p_g)\dfrac{\lambda_w(S_w)\lambda_g(S_w)}{\lambda(S_w, p_g)}(\rho_w - \rho_g(p_g)),$

"diffusivity" coeff. $\qquad a(S_w, p_g) = -\rho_w \rho_g(p_g)\dfrac{\lambda_w(S_w)\lambda_g(S_w)}{\lambda(S_w, p_g)}p_c'(S_w).$

Fractional flow formulation:

Water mass conservation equation is of convection-diffusion type for saturaton S_w:

$$\Phi \rho_w \frac{\partial S_w}{\partial t} + \mathrm{div}(f_w(S_w, p_g)\mathbf{Q}_t + \mathbb{K}\mathbf{g}b_g(S_w, p_g)) = \mathrm{div}(\mathbb{K}a(S_w, p_g)\nabla S_w).$$

In the total flow equation we want to eliminate the saturation gradient:

$$\mathbf{Q}_t = -\lambda(S_w, p_g)\mathbb{K}\left(\nabla p_g - f_w(S_w, p_g)p_c'(S_w)\nabla S_w - \bar{\rho}(S_w, p_g)\mathbf{g}\right),$$

▶ Idea: introduce a new pressure like variable that will eliminate ∇S_w term (Antontsev et al., Chavent-Jaffré, Chen et al., ...)

Simplified global pressure formulation

Find a *mean pressure p* and a function $\omega(S_w, p)$ such that:

$$\nabla p_g - f_w(S_w, p_g) p_c'(S_w) \nabla S_w = \omega(S_w, p) \nabla p \tag{1}$$

Simplification A:

▶ Fractional flow function $f_w(S_w, p_g)$ can be replaced by $f_w(S_w, p)$ without introducing a significant error.

Therefore, (1) reduces to:

$$\nabla p_g = \omega(S_w, p) \nabla p + f_w(S_w, p) p_c'(S_w) \nabla S_w.$$

Simplified global pressure formulation

Now we can write the solution:

$$p = p_g - \gamma(S_w, p), \quad \gamma(S_w, p) = \int_1^{S_w} f_w(s, p) p_c'(s) \, ds, \qquad (2)$$

[here for simplicity $S_{girr} = 0$ and $p_c(1) = 0$.]

$$\omega(S_w, p) = 1 + \frac{\partial}{\partial p} \gamma(S_w, p). \qquad (3)$$

Note that (2) is a nonlinear equation to be solved.
The total flow is now given in "*generalized Darcy form*":

$$\mathbf{Q}_t = -\lambda(S_w, p) \mathbb{K}(\omega(S_w, p) \nabla p - \bar{\rho}(S_w, p) \mathbf{g}). \qquad (4)$$

Simplified global pressure formulation

Simplification B:

- ▶ Gas density $\rho(p_g)$ can be replaced by $\rho(p)$ without introducing a significant error (see, e.g., Chavent-Jaffré, Galusinski-Saad).

The system written in unknowns p and S_w. Total flow:

$$\mathbf{Q}_t = -\lambda(S_w, p)\mathbb{K}(\omega(S_w, p)\nabla p - \bar{\rho}(S_w, p)\mathbf{g}),$$

Total mass conservation:

$$\Phi\frac{\partial}{\partial t}(S_w\rho_w + (1 - S_w)\rho_g(p)) + \text{div}\mathbf{Q}_t = 0,$$

Water mass conservation:

$$\Phi\rho_w\frac{\partial S_w}{\partial t} + \text{div}(f_w(S_w, p)\mathbf{Q}_t + \mathbb{K}\mathbf{g}b_g(S_w, p)) = \text{div}(\mathbb{K}a(S_w, p)\nabla S_w).$$

Simplified global pressure formulation: Applicability

- Global pressure is well defined for all $p_g \geq 0$ and $S_w \in (0, 1]$ such that $p_w = p_g - p_c(S_w) \geq 0$;
- $p_w \leq p \leq p_g$;
- $\omega(S_w, p) > 0$ holds only if certain additional condition is fullfield. For example: $p \geq p_{min} > 0$ and

$$\forall S_w \in (0, 1], \quad \int_{S_w}^1 \frac{c_g \rho_w \lambda_w(s) \lambda_g(s)}{(\rho_w \lambda_w(s) + c_g \lambda_g(s) p_{min})^2} |p_c'(s)| \, ds < 1.$$

New global pressure formulation

We want to solve

$$\nabla p_g - f_w(S_w, p_g) p'_c(S_w) \nabla S_w = \omega(S_w, p) \nabla p \qquad (5)$$

without any simplifying assumptions.

Introduce an unknown function Π such that

$$p_g = \Pi(S_w, p),$$

where p is the *global pressure* to be defined. Then (5) reads

$$\nabla p_g = \omega(S_w, p) \nabla p + f_w(S_w, \Pi(S_w, p)) p'_c(S_w) \nabla S_w,$$

or,

$$\frac{\partial \Pi}{\partial S_w}(S_w, p) \nabla S_w + \frac{\partial \Pi}{\partial p}(S_w, p) \nabla p$$
$$= \omega(S_w, p) \nabla p + f_w(S_w, \Pi(S_w, p)) p'_c(S_w) \nabla S_w.$$

New global pressure formulation

Since p and S_w are independent variables we must have

$$\frac{\partial \Pi}{\partial S_w}(S_w, p) = f_w(S_w, \Pi(S_w, p))p_c'(S_w) \qquad (6)$$

$$\frac{\partial \Pi}{\partial p}(S_w, p) = \omega(S_w, p). \qquad (7)$$

Conclusion:

1. To calculate $\Pi(S_w, p)$ we have to solve the Cauchy problem:

$$\begin{cases} \dfrac{d\Pi(S, p)}{dS} = \dfrac{\rho_w \lambda_w(S) p_c'(S)}{\rho_w \lambda_w(S) + c_g \lambda_g(S)\Pi(S, p)}, & 0 < S < 1 \\ \Pi(1, p) = p. \end{cases}$$

2. Get $\omega(S_w, p)$ from (7).

New global pressure formulation: Remarks

1. It is more natural to replace saturation S_w with capillary pressure $u = p_c(S_w)$ as an independent variable. Then:

$$\begin{cases} \dfrac{d\hat{\Pi}(u, p)}{du} = \dfrac{\rho_w \hat{\lambda}_w(u)}{\rho_w \hat{\lambda}_w(u) + c_g \hat{\lambda}_g(u) \hat{\Pi}(u, p)}, & u > 0 \\ \hat{\Pi}(0, p) = p. \end{cases}$$

   and $\Pi(S_w, p) = \hat{\Pi}(p_c(S_w), p)$ [Hat denotes the change of variables.]

2. ω is now always strictly positive:

$$\omega(S_w, p) = \exp\left(- \int_0^{p_c(S_w)} \frac{c_g \rho_w \hat{\lambda}_w(u) \hat{\lambda}_g(u)}{(\rho_w \hat{\lambda}_w(u) + c_g \hat{\lambda}_g(u) \hat{\Pi}(u, p))^2} \, du \right),$$

3. $p \leq \Pi(S_w, p) \leq p + p_c(S_w)$ and therefore $p_w \leq p \leq p_g$.

New global pressure formulation: Flow equations

Total flow:

$$\mathbf{Q}_t = -\lambda^n(S_w, p)\mathbb{K}(\omega(S_w, p)\nabla p - \bar{\rho}^n(S_w, p)\mathbf{g}).$$

where the superscript *n* stands for **new**.

Total mass conservation:

$$\Phi \frac{\partial}{\partial t}(S_w \rho_w + c_g(1 - S_w)\Pi(S_w, p)) + \mathrm{div}(\mathbf{Q}_t) = 0.$$

Water mass conservation:

$$\Phi \rho_w \frac{\partial S_w}{\partial t} + \mathrm{div}(f_w^n(S_w, p)\mathbf{Q}_t + \mathbb{K}\mathbf{g}b_g^n(S_w, p)) - \mathrm{div}(\mathbb{K}a^n(S_w, p)\nabla S_w) = 0.$$

New global pressure formulation: Coefficients

New coefficients are obtained from the old ones by replacing p by $\Pi(S_w, p)$:

total mobility	$\lambda^n(S_w, p) = \lambda(S_w, \Pi(S_w, p))$,
water fractional flow	$f_w^n(S_w, p) = f_w(S_w, \Pi(S_w, p))$,
mean density	$\bar{\rho}^n(S_w, p) = \bar{\rho}(S_w, \Pi(S_w, p))$,
"gravity" coeff.	$b_g^n(S_w, p) = b_g(S_w, \Pi(S_w, p))$,
"diffusivity" coeff.	$a^n(S_w, p) = a(S_w, \Pi(S_w, p))$.

Only $\omega(S_w, p)$ is not simply related to the old $\omega(S_w, p)$ coefficient.

New global pressure formulation: Comparison

Comparison of gas pressure functions in the two models:

▶ Simplified global pressure formulation:

$$p_g = \Pi(S_w, p) = p + \int_1^{S_w} f_w(s, p) p_c'(s) \, ds$$

▶ New global pressure formulation:

$$p_g = \Pi(S_w, p) = p + \int_1^{S_w} f_w(s, \Pi(s, p)) p_c'(s) \, ds$$

New global pressure formulation: Comparison

Comparison of ω functions in the two models:

▶ Simplified global pressure formulation:

$$\omega(S_w, p) = 1 - \int_0^{p_c(S_w)} \frac{c_g \rho_w \hat{\lambda}_w(u) \hat{\lambda}_g(u)}{(\rho_w \hat{\lambda}_w(u) + c_g \hat{\lambda}_g(u) p)^2} \, du$$

▶ New global pressure formulation:

$$\omega(S_w, p) = \exp\left(- \int_0^{p_c(S_w)} \frac{c_g \rho_w \hat{\lambda}_w(u) \hat{\lambda}_g(u)}{(\rho_w \hat{\lambda}_w(u) + c_g \hat{\lambda}_g(u) \hat{\Pi}(u, p))^2} \, du \right)$$

Numerical comparison: Van Genuchten functions

In comparison of the coefficients we use van Genuchten's capillary pressure and relative permeability functions as in the Couplex-Gas Benchmark.
$u \geq 0$ is the capillary pressure:

$$S_{we} = \left(1 + \left(\frac{u}{P_r}\right)^n\right)^{-m}, \quad \text{(reduced water saturation)},$$

$$kr_w(u) = \left(\frac{1}{1+v^n}\right)^{m/2} \left[1 - \left(\frac{v^n}{1+v^n}\right)^m\right]^2, \quad v = \frac{u}{P_r},$$

$$kr_g(u) = \left(1 - \frac{1}{(1+v^n)^m}\right)^{1/2} \left(\frac{v^n}{1+v^n}\right)^{2m}, \quad v = \frac{u}{P_r}.$$

Parameters: $n > 1$, $m = 1 - 1/n$, $P_r > 0$.

Numerical comparison

1. We compare gas pressure functions in new and simplified models, for fixed global pressure, as functions of capillary pressure:

$$u \mapsto \Pi(u, p), \text{ new model}; \quad u \mapsto \Pi(u, p) \text{ simplified model}$$

2. We compare all the other coefficients for a fixed global pressure as functions of water saturation.

Notation: new = from new model, simpl = from simplified model.
Van Genuchten parameters:

$$n = 1.54, \ Pr = 2 \text{ MPa}, \ \mu_w = 7.98 \cdot 10^{-3} \text{ Pas}, \ \mu_g = 9 \cdot 10^{-6} \text{ Pas},$$
$$\rho_w = 10^3 \text{ kg/m3 and } c_g = 0.808.$$

Comparison of gas pressure functions
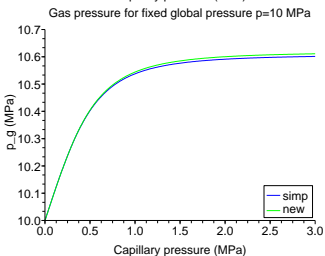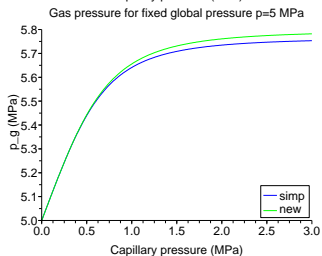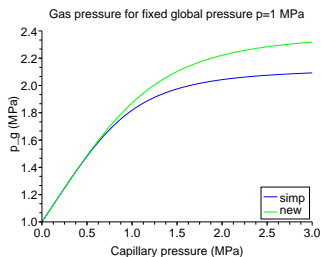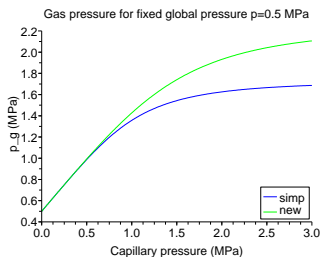
Coefficients at global pressure of 0.5 MPa
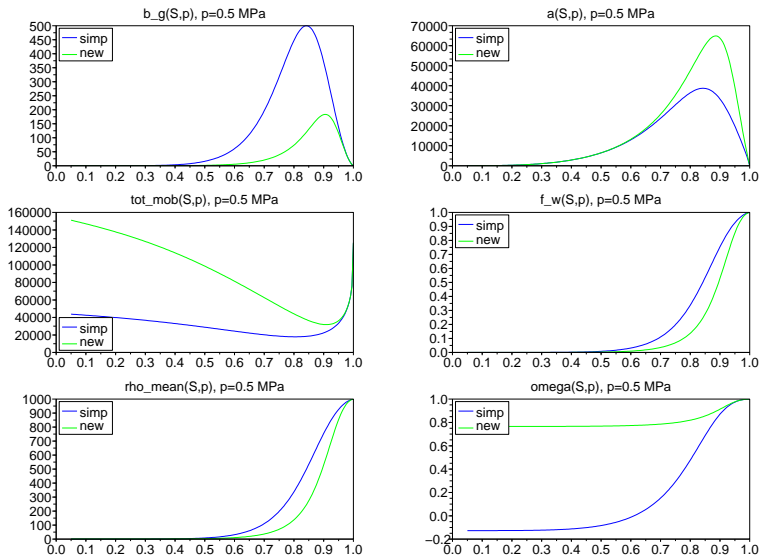
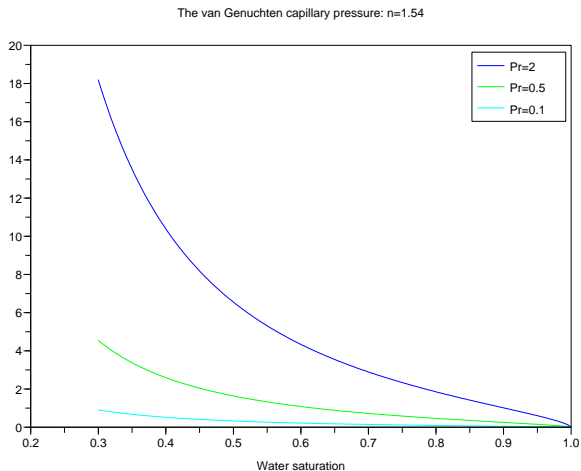# Coefficients at global pressure of 1 MPa
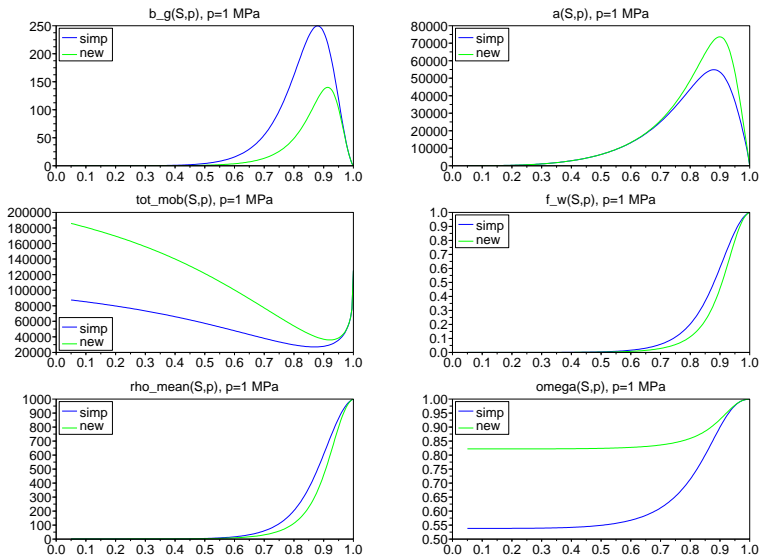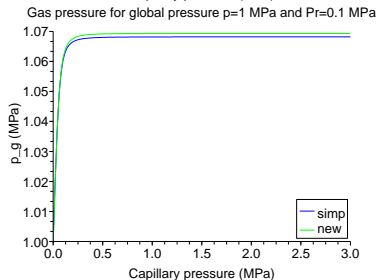
Coefficients at global pressure of 5 MPa
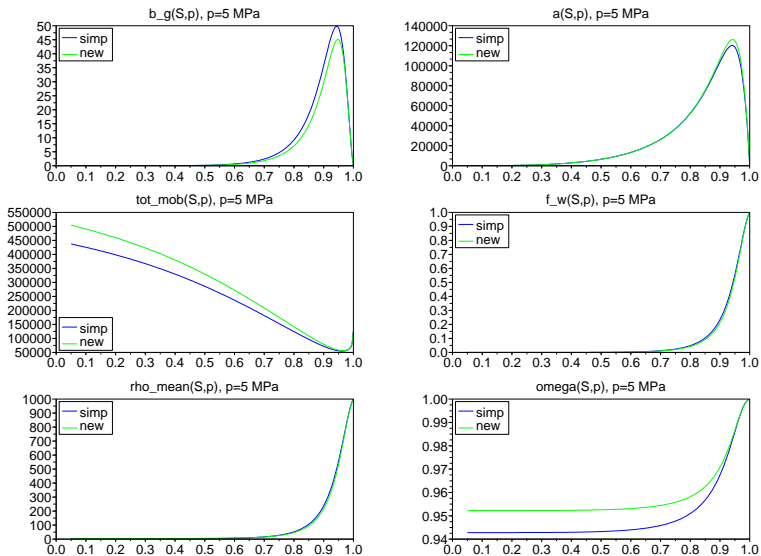
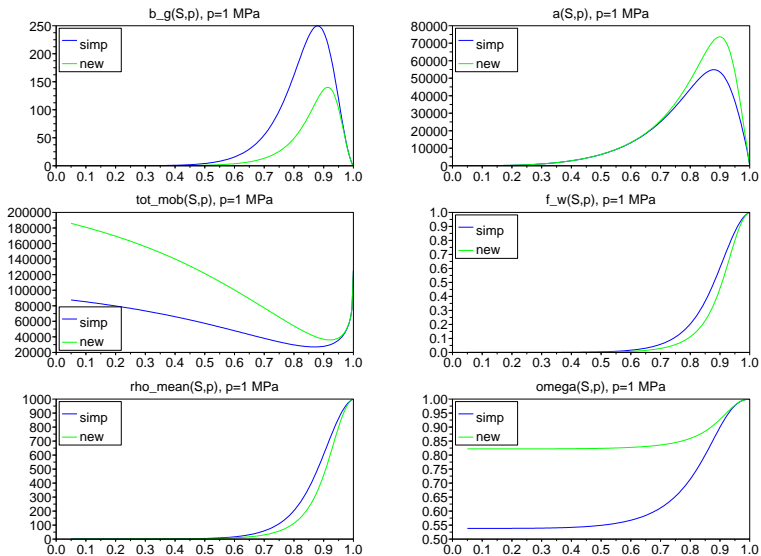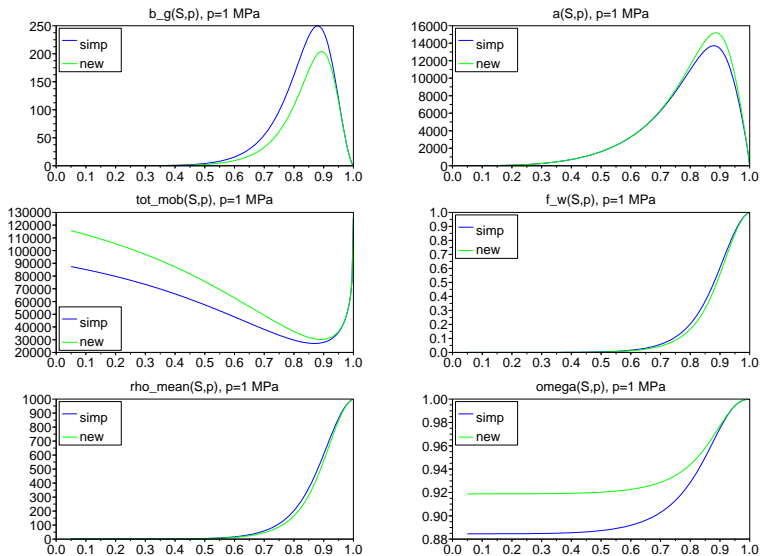# Coefficients at global pressure of 10 MPa
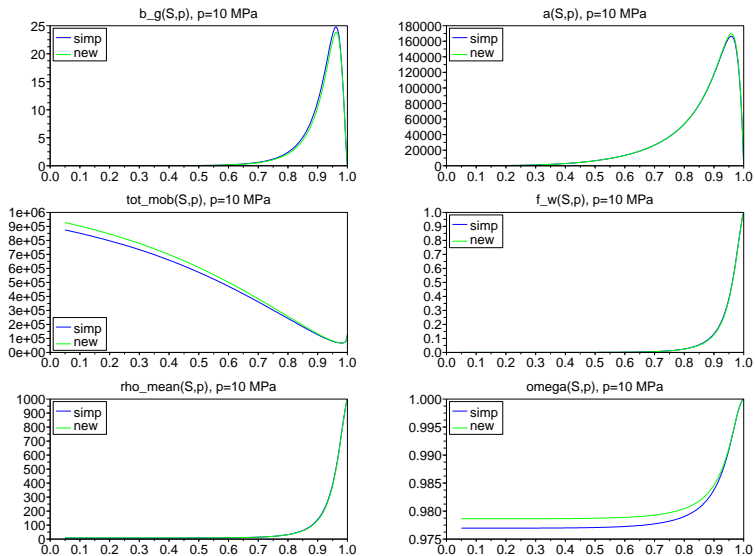
Different capillary pressures curves

We are holding global pressure $p = 1$ MPa and we vary capillary pressure curve by varying P_r parameter (the entry pressure):
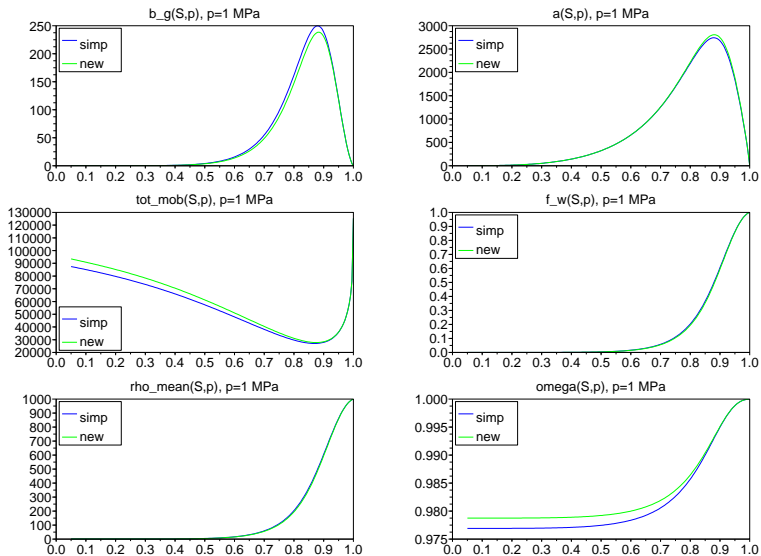

The van Genuchten capillary pressure: n=1.54

Gas pressure functions for different P_r's ($p = 1$ MPa)

Gas pressure for global pressure p=1 MPa and Pr=2 MPa

Gas pressure for global pressure p=1 MPa and Pr=0.5 MPa

Gas pressure for global pressure p=1 MPa and Pr=0.1 MPa
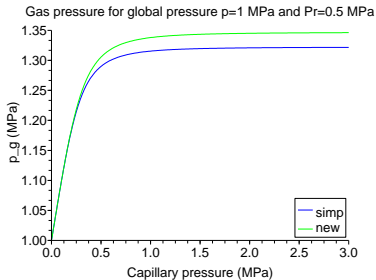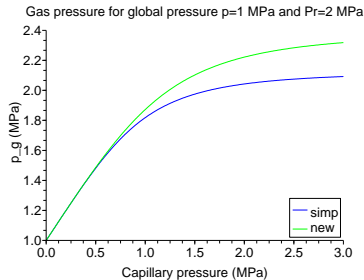
Coefficients for $P_r = 2$ MPa ($p = 1$ MPa)

Coefficients for $P_r = 0.5$ MPa ($p = 1$ MPa)

Coefficients for $P_r = 0.1$ MPa ($p = 1$ MPa)

Conclusion

- ▶ For *small capillary pressures* all the coefficients in new and simplified systems are very close.
- ▶ In the regimes with large global pressures the difference in the coefficients is small
- ▶ For small global pressures and relatively large capillary pressure the difference between new and simplified coefficients becomes significant.
- ▶ Replacing

$$\frac{\partial}{\partial t}(c_g(1 - S_w)\Pi(S_w, p)) \quad \text{by} \quad \frac{\partial}{\partial t}(c_g(1 - S_w)p)$$

can have large influence on mass conservation even in for small capillary pressure and elevated global pressure.

Work in progress

- ▶ An extension to multiphase, multicomponent models is straightforward and it is at present in the course of study.
- ▶ The new formulation is well adapted for the mathematical analysis of the model. At present we study existence, uniqueness etc.
- ▶ The discretization of the model by a vertex-centered finite volume scheme is currently studied.

Reference: B. Amaziane, M. Jurak: *A new formulation of immiscible compressible two-phase flow in porous media*, C. R. A. S. Mécanique 336 (2008) 600-605.