Analysis on systems of diophantine equations

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Not-risky-at-all observation

$N(X) \sim C \cdot X^{\text{something}}$

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If s is sufficiently large in terms of k, the number $N_{s,k}(X)$ of solutions satisfies $N_{s,k}(X) = C \cdot X^{2s-\frac{k(k+1)}{2}} + \textit{error term}$, where $C \geq 0$ does not depend on X.

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Second issue : What if $C = 0$?

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Theorem (Wooley, 2014)

If $s \geq k^2 - k + 1$, then there exists $C > 0$ such that

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$N_{s,3}(X) \sim C \cdot X^{2s-6}$ for every $s \geq 7$

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$$\prod_{j=1}^{k} \int_0^1 e((x_1^j + \cdots + x_s^j - y_1^j - \cdots - y_s^j)t)\mathrm{d}t$$

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A calculation shows that

$$N_{s,k}(X) = \int_{[0,1]^k} \left| \sum_{1 \le x \le X} e(\alpha_1 x + \cdots + \alpha_k x^k) \right|^{2s} \mathrm{d}\boldsymbol{\alpha}$$

By writing $f(\boldsymbol{\alpha}) = \sum_{1 \le x \le X} e(\alpha_1 x + \cdots + \alpha_k x^k)$, we have

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$$\int_{\mathfrak{M}} |f|^{2s} \sim C \cdot X^{2s - \frac{k(k+1)}{2}}$$

$$\int_{\mathfrak{m}} |f|^{2s} = \mathit{error\ \ term}$$

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$$\begin{cases} a_{1,1}x_1^{d_1} + \cdots + a_{1,s}x_s^{d_1} = 0 \\ \ldots \\ a_{k,1}x_1^{d_k} + \cdots + a_{k,s}x_s^{d_k} = 0 \end{cases}$$

with $a_{i,j}$ nonzero integers and d_i positive and strictly increasing integers.

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Theorem

If $s \geq 2d_k^2 - 2d_k + 1$ and if there exists one nonsingular real solution and one nonsingular p-adic solution (for every p), then there exists $C > 0$ such that

$$\mathcal{J}_{s,k}(X) \sim C \cdot X^{s-(d_1+\cdots+d_k)}$$

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Second issue It becomes incredibly difficult if we allow too many $a_{i,j}$ to be zero.

Thanks for your attention