Sharp decay characterization for the compressible Navier-Stokes equations

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Abstract

The low-frequency L^1 assumption has been extensively applied to the large-time asymptotics of solutions to the compressible Navier-Stokes equations and incompressible Navier-Stokes equations since the classical efforts due to Kawashima, Matsumura, Nishida, Ponce, Schonbek and Wiegner. In this paper, we establish a sharp decay characterization for the compressible Navier-Stokes equations in the critical L^p framework. Precisely, it is proved that the Besov space $\dot{B}^{\sigma_1}_{2,\infty}$-boundedness condition (with $\frac{d}{2}-\frac{2d}{p}\leq\sigma_1<\frac{d}{2}-1$) of the low-frequency part of initial perturbation is *not only sufficient, but also necessary* to achieve those upper bounds of time-decay estimates. Furthermore, we show that the upper and lower bounds of time-decay estimates hold *if and only if* the low-frequency part of the initial perturbation belongs to *a nontrivial subset* of $\dot{B}^{\sigma_1}_{2,\infty}$. To the best of our knowledge, our work is the first one addressing the inverse problem for the large-time asymptotics of compressible viscous fluids.

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1 Introduction and main results

In 1933, J. Leray in his pioneering work [32] introduced the concept of weak (turbulent) solutions to the incompressible Navier-Stokes equations and established the global in time existence of solutions with energy bounded initial data. Also, he addressed the question whether or not the energy of weak solutions uniformly decays in $L^2(\mathbb{R}^3)$ as the time t goes to infinity. Schonbek [41–43] introduced the Fourier splitting method and deduced uniform decay for solutions in the L^2-energy space, provided the L^1-assumption on the initial data was additionally imposed. Wiegner [49] addressed the optimal decay by a careful analysis of the relationship between the heat kernel and incompressible Navier-Stokes equations. See for instance the recent survey by the first author and Schonbek [4].

In this paper, we are concerned with the following compressible Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \mathcal{A}u, \end{cases} \tag{1.1}$$

which govern the motion of a general barotropic compressible fluid in whole space $\mathbb{R}^d$ $(d \geq 2)$. Here $u = u(t,x) \in \mathbb{R}^d$, with $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\rho = \rho(t,x) \in \mathbb{R}_+$ denote the velocity and density of the fluid, respectively. The pressure function $P(\rho)$ depends only upon the density and is assumed to be suitably smooth. The Lamé operator $\mathcal{A}$ takes the form

$$\mathcal{A}u \triangleq \mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u,$$

where the shear viscosity μ and the bulk viscosity λ are assumed to be constants for simplicity and to satisfy

$$\mu > 0, \qquad \nu \triangleq 2\mu + \lambda > 0.$$

System (1.1) is supplemented with the initial data

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad x \in \mathbb{R}^d. \tag{1.2}$$

We investigate the solution (ρ, u) to the Cauchy problem (1.1)-(1.2) fulfilling the constant far-field behavior

$$(\rho, u) \to (\bar{\rho}, 0), \qquad |x| \to \infty,$$

where $\bar{\rho} > 0$ is a given constant.

The local existence and uniqueness of smooth solutions for System (1.1) were proved by Serrin [45] and Nash [37]. The local existence of strong solutions with Sobolev regularity was obtained by Solonnikov [47], Valli [48] and Fiszdon-Zajaczkowski [19]. The global smooth solutions to the Cauchy problem of compressible and heat-conductive Navier-Stokes equations were first established by Matsumura and Nishida [35, 36], in the case that initial data are small perturbations of a linearly stable constant state in three dimensions. With the additional $L^1(\mathbb{R}^3)$ assumption of initial data, they deduced the following decay rate of smooth solutions:

$$\|(\rho - \bar{\rho}, u)(t)\|_{L^2(\mathbb{R}^3)} \lesssim \langle t \rangle^{-\frac{3}{4}} \quad \text{with} \quad \langle t \rangle \triangleq \sqrt{1 + t^2}, \tag{1.3}$$

which coincides with that of heat kernel. Furthermore, Kawashima, Matsumura and Nishida [28] proved that the solutions to the Boltzmann equation and the incompressible Navier-Stokes equations for small initial data were asymptotically equivalent to that of the compressible and heat-conductive Navier-Stokes equations at the rate with $\langle t \rangle^{-5/4}$, as $t \to \infty$. Ponce [40] obtained decay estimates in the general L^r norm:

$$\|\nabla^k(\rho - \bar{\rho}, u)(t)\|_{L^r(\mathbb{R}^d)} \lesssim \langle t \rangle^{-\frac{d}{2}(1-\frac{1}{r})-\frac{k}{2}}, \quad 2 \le r \le \infty, \quad 0 \le k \le 2, \quad d = 2, 3. \tag{1.4}$$

Later, Matsumura-Nishida's results were extended to more physical situations, where the fluid domain is not the whole $\mathbb{R}^d$. For example, the exterior domain was investigated by Kobayashi [29] and Kobayashi-Shibata [30], the half-space by Kagei & Kobayashi [26, 27]. For more general data, Xin [50] found that any smooth solution to the Cauchy problem of the full compressible Navier-Stokes system without heat conduction (including the baratropic case) would blow up in finite time if the initial density contains vacuum. Huang, Li and Xin [22] constructed the global existence of classical solutions that have large highly oscillations and can contain vacuum states. For the theory of weak solutions, a breakthrough is due to P.-L. Lions [34], who obtained the global existence of weak solutions with finite energy initial data. Later further developments were achieved by Feireisl, Novotny and Petzeltová [18] and Jiang & Zhang [25] and since then this remained a very active research field.

As shown by earlier works [35, 36, 40–43, 49], the additional L^1 assumption for the data usually plays a key role in the derivation of large-time decay rates for the solutions. Notice that the following Sobolev embeddings

$$L^1(\mathbb{R}^d) \hookrightarrow \dot{B}^0_{1,\infty} \hookrightarrow \dot{B}^{-\frac{d}{2}}_{2,\infty}. \tag{1.5}$$

Although the latter space does not embed into $\dot{B}^0_{2,2}$, any function belonging to this space and concentrated in low frequencies does also belong to $\dot{B}^0_{2,2} \sim L^2(\mathbb{R}^d)$. This actually indicates the L^1 regularity is stronger than the L^2 regularity at low frequencies. Inspired by this simple observation, it will be natural to investigate the decay properties of the solutions, not under the stringent L^1-condition, but rather under a more general low-frequency assumption for viscous compressible fluids in the Besov framework with critical (minimal) regularity, in which the uniqueness of solutions holds.

As for many evolutionary equations coming from mathematical physics, *scaling invariance* plays a fundamental role and suitable critical quantities (scaling invariance norms) may control the possible blow-up of solutions. This approach is now classic. Recall the global existence results for the incompressible Navier-Stokes equations which go back to the pioneering work [20] by Fujita-Kato (see also results by Kozono-Yamazaki [31], Cannone [6], Cannone-Planchon [7], Chemin [9] for a small sample of the vast literature). Observe that the compressible Navier-Stokes system (1.1) is invariant by the transform

$$\rho(x,t) \rightsquigarrow \rho(lx, l^2 t), \qquad u(x,t) \rightsquigarrow lu(lx, l^2 t), \qquad l > 0,$$

up to a change of the pressure term P into $l^2 P$. Danchin [12] solved (1.1)-(1.2) globally in the critical homogeneous Besov space $(\dot{B}^{\frac{d}{2}-1}_{2,1} \cap \dot{B}^{\frac{d}{2}}_{2,1}) \times \dot{B}^{\frac{d}{2}-1}_{2,1}$. Subsequently, the result of [12] has been extended to the general Besov spaces modelled on L^p-norms by Charve-Danchin [8] and Chen-Miao-Zhang [11] independently. Inspired by Hoff's viscous effective flux in [23], Haspot [21] developed the L^p energy argument and achieved essentially the same result. The readers are also referred to [13, 14] on the local well-posedness subject to general initial data with critical regularity.

For convenience of the readers, we would like to recall a result about the global existence and uniqueness of solutions to the Cauchy problem (1.1)-(1.2) in the critical L^p framework. Denote by X_p and $\mathcal{X}_p$ the functional space and the corresponding energy norm:

$$\begin{aligned} X_p \triangleq \big\{(a,u) |\ (a,u)^\ell \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}+1}_{2,1}), \\ a^h \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{p}}_{p,1}), \quad u^h \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{p}-1}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{p}+1}_{p,1})\big\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{X}_p \triangleq \|(a,u)\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|a\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \\ + \|(a,u)\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|a\|^h_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|u\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}. \end{aligned} \tag{1.6}$$

The definition of Besov spaces and the ℓ/h notation of low/high-frequency are referred to Section 2 below. Moreover, $\widetilde{L}^\infty_t(\dot{B}^s_{p,q}) = \widetilde{L}^\infty(\mathbb{R}_+, \dot{B}^s_{p,q})$ denotes a class of mixed space-time spaces, which are first introduced by Chemin and Lerner [10] and can be regarded as the refinement of the usual spaces $L^\infty_t(\dot{B}^s_{p,q})$. Some assumptions are labeled as follows.

(H_1): $P'(\bar{\rho}) > 0$;

(H_2): $a_0 \triangleq \rho_0 - \bar{\rho} \in \dot{B}^{\frac{d}{p}}_{p,1}$ and $u_0 \in \dot{B}^{\frac{d}{p}-1}_{p,1}$, besides, $(a_0^\ell, u_0^\ell) \in \dot{B}^{\frac{d}{2}-1}_{2,1}$ such that

$$\mathcal{X}_{p,0} \triangleq \|(a_0,u_0)\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a_0\|^h_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|u_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \ll 1. \tag{1.7}$$

The global existence and uniqueness of solutions to (1.1)-(1.2) in the critical L^p-framework are stated as follows. See [8, 11, 12, 21].

Theorem 1.1. *Let $d \geq 2$ and p satisfy*

$$2 \leq p \leq \min(4, d^*) \quad \text{and, additionally,} \quad p \neq 4 \quad \text{if} \quad d = 2, \tag{1.8}$$

where $d^* \triangleq 2d/(d-2)$. If assumptions (H_1)-(H_2) are fulfilled, then the Cauchy problem (1.1)-(1.2) admits a unique global-in-time solution (ρ, u) with $a \triangleq \rho - \bar{\rho}$ and (a, u) in the space X_p. Furthermore, there exists some constant $C = C(p, d, \lambda, \mu, P, \bar{\rho}) > 0$ such that $\mathcal{X}_p \leq C\mathcal{X}_{p,0}$.

A natural problem is how to exhibit the large-time asymptotic behavior of the solution constructed in Theorem 1.1. Although providing an accurate long-time asymptotics picture is still open, there are a number of works concerning time-decay rates of L^r-type as in (1.3)-(1.4). Okita [39] established the decay estimates of solutions to (1.1)-(1.2) in the L^2 critical framework, by using a slight modification of the method in [12]. The low-frequency assumption with respect to $\dot{B}^0_{1,\infty}$ was additionally imposed. However, the $2D$ case could not be covered. In the survey [15], Danchin proposed another description of the time decay, which allows to handle any space dimensions $d \geq 2$. Subsequently, Danchin and the third author [17] further established the decay rates in the L^p critical spaces under the additional condition that the low-frequency part of initial perturbation is suitably small in some Besov space $\dot{B}^{\sigma_0}_{2,\infty}$ ($\sigma_0 \triangleq \frac{d}{2} - \frac{2d}{p}$) which is exactly linked with the critical embedding $L^{p/2} \hookrightarrow \dot{B}^{\sigma_0}_{2,\infty}(2 \leq p \leq \min\{4, d^*\})$. The third author [52] claimed a general low-frequency assumption in terms of $\dot{B}^{\sigma_1}_{2,\infty}$ for the upper bound of decay estimates, where the regularity exponent fulfills $\sigma_0 \leq \sigma_1 < \frac{d}{2} - 1$ (implies that $\|\cdot\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \lesssim \|\cdot\|^\ell_{\dot{B}^{\sigma_1}_{2,\infty}}$). In other words, the optimal decay rates of strong solutions in Theorem 1.1 can be obtained, provided that the low-frequency assumption is reasonably strengthened. These results all depend on the time-weighted energy approach in the Fourier semi-group framework and the smallness of low frequencies of initial data is usually needed. Later, Xin and the third author [51] developed a Lyapunov-type energy method in the L^p critical spaces to obtain the time-decay rates. Their approach still requires the $\dot{B}^{\sigma_1}_{2,\infty}$ condition on the low-frequency part of initial data but not necessarily small.

To the best of our knowledge, *whether the low-frequency assumption $\dot{B}^{\sigma_1}_{2,\infty}$ is sharp or not for the large-time behavior of strong solutions to the compressible Navier-Stokes equations in critical spaces* remains an open question. In the present paper, we shall give a positive answer to that issue and provide a necessary and sufficient condition for the sharp time-decay rates of solutions to the Cauchy problem (1.1)-(1.2). More precisely, we establish that both upper and lower bounds of time-decay estimates of solutions to the Cauchy problem for (1.1)-(1.2) in the L^p critical spaces hold if and only if the low-frequency part of initial data is bounded in a non-trivial subset of $\dot{B}^{\sigma_1}_{2,\infty}$.

Without loss of generality, we set $\bar{\rho} = 1$. We denote by $a = \rho - 1$ and reformulate the Cauchy problem

(1.1)-(1.2) as

$$\begin{cases} \partial_t a + \operatorname{div} u = -\operatorname{div}(au), \\ \partial_t u + \nabla a - \mathcal{A}u = g, \\ (a,u)(x,0) = (a_0,u_0)(x) \end{cases} \tag{1.9}$$

with the nonlinear term

$$g \triangleq -u\cdot\nabla u - k(a)\nabla a - I(a)\mathcal{A}u, \tag{1.10}$$

where $k(a) \triangleq \frac{P'(1+a)}{1+a} - 1$ and $I(a) \triangleq \frac{a}{a+1}$.

To study the decay characterization of solutions to (1.9), we introduce a subset of the Besov space $\dot{B}^{\sigma_1}_{2,\infty}(\sigma_1 \in \mathbb{R})$:

$$\begin{aligned} \dot{\mathcal{B}}^{\sigma_1}_{2,\infty} \triangleq \big\{ f \in \dot{B}^{\sigma_1}_{2,\infty} \mid \exists \text{ two constants } c_0, M > 0 \text{ and a sequence of integers } \{j_k\}_{k=1,2,...} \\ \text{such that } \lim_{k\to\infty} j_k = -\infty,\ |j_k - j_{k+1}| \leq M \text{ and } 2^{\sigma_1 j_k}\|\dot{\Delta}_{j_k} f\|_{L^2} \geq c_0. \big\}. \end{aligned} \tag{1.11}$$

Note that $\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ (with $\sigma_1 \in \mathbb{R}$) has a nontrivial intersection with $\dot{B}^{\sigma}_{2,1}$ when $\sigma > \sigma_1$, which will be characterized in Section 2.

Our main result is stated as follows.

Theorem 1.2. *Let (a,u) be the global solution to the Cauchy problem (1.9) constructed in Theorem 1.1. Let the real numbers σ_0, σ_1 satisfy $\sigma_0 \triangleq \frac{d}{2} - \frac{2d}{p}$ and $\sigma_0 \leq \sigma_1 < \frac{d}{2} - 1$. Then*

- (*Upper bounds*): *For any time $t_0 > 0$, the solution (a,u) fulfills*

$$\|(a,u)^{\ell}(t)\|_{\dot{B}^{\sigma_1}_{2,\infty}} \leq C, \quad t > 0, \tag{1.12}$$

$$\|(a,u)(t)\|_{\dot{\mathbb{B}}^{\sigma}_{2,p}} \leq C\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}, \quad t > t_0, \quad \sigma_1 < \sigma \leq \frac{d}{2}, \tag{1.13}$$

  if and only if $(a_0,u_0)^{\ell} \in \dot{B}^{\sigma_1}_{2,\infty}$.

- (*Upper and lower bounds*): *There exists a time $t_1 > 0$ such that the solution (a,u) fulfills (1.13) and*

$$c\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)} \leq \|(a,u)(t)\|_{\dot{\mathbb{B}}^{\sigma}_{2,p}} \leq C\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}, \quad t > t_1, \quad \sigma_1 < \sigma \leq \frac{d}{2}, \tag{1.14}$$

  if and only if $(a_0,u_0)^{\ell} \in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$.

Here the hybrid norm $\|(a,u)(\cdot,t)\|_{\dot{\mathbb{B}}^{\sigma}_{2,p}}$ is defined by

$$\|(a,u)(\cdot,t)\|_{\dot{\mathbb{B}}^{\sigma}_{2,p}} \triangleq \|(a,u)^{\ell}(\cdot,t)\|_{\dot{B}^{\sigma}_{2,1}} + \|(a,u)(\cdot,t)\|^{h}_{\dot{B}^{\frac{d}{p}}_{p,1}}.$$

The proof of Theorem 1.2 is motivated by Wiegner's argument regarding the energy decay of Leray solutions to the incompressible Navier-Stokes equations in the seminal work [49] and inverse Wiegner's argument in [46]. The inequality (1.12) can be interpreted as the nonlinear evolution of initial regularity. In the "if" part, it plays a key role in the derivation of the two-sided time-decay estimates (1.13)-(1.14). In fact, it is also indispensable in the "only if" part, see Proposition 5.1. As a direct consequence, one can also get the sharp characterization of two-sided decay estimates in the L^2 framework.

Corollary 1.1. *There exists a time $t_1 > 0$ such that for $\sigma_1 < \sigma \leq \frac{d}{2}$, the global-in-time solution (a, u) in Theorem 1.1 fulfills (1.12) and*

$$c\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)} \leq \|\Lambda^\sigma(a,u)(t)\|_{\dot{B}^0_{2,1}} \leq C\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}, \quad t > t_1,$$

if and only if $(a_0, u_0)^\ell \in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ with $-\frac{d}{2} \leq \sigma_1 < \frac{d}{2} - 1$.

We comment on a few points of immediate relevance:

- The low-frequency assumption in terms of $\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}(\sigma_0 \leq \sigma_1 < \frac{d}{2} - 1)$ is firstly introduced to give the sharp decay characterization for the compressible Navier-Stokes system (1.1) in critical spaces. To our knowledge, "only if" part is completely new, and this question has not been addressed for compressible fluid flows in the existing literature. In addition, Theorem 1.2 actually indicates that the upper bounds of algebraic time-decay rates obtained in [17, 51, 52] are optimal.

- It follows from Proposition 3.2 (see Section 2) that the low-frequency assumption $(a_0, u_0)^\ell \in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ is equivalent to that

$$\begin{cases} P_{\sigma_1}(a_0,u_0)_+ \triangleq \limsup\limits_{r\to 0^+} r^{-2(\sigma-\sigma_1)} \displaystyle\int_{\{|\xi|\leq r\}} |\xi|^{2\sigma}(|\widehat{a}_0(\xi)|^2 + |\widehat{u}_0(\xi)|^2)d\xi < \infty, \\ P_{\sigma_1}(a_0,u_0)_- \triangleq \liminf\limits_{r\to 0^+} r^{-2(\sigma-\sigma_1)} \displaystyle\int_{\{|\xi|\leq r\}} |\xi|^{2\sigma}(|\widehat{a}_0(\xi)|^2 + |\widehat{u}_0(\xi)|^2)d\xi > 0 \end{cases} \tag{1.15}$$

  for any $\sigma > \sigma_1$, which is closely linked with the theory of decay characters for incompressible Navier-Stokes equations and related dissipative equations (see for example, [2, 3, 38]).

- Corollary 1.1 recovers the classical L^2 decay rates of solutions if choosing $\sigma_1 = -\frac{d}{2}$. The initial data $(a_0, u_0)^\ell \in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ are sharp in comparison with previous works [26–28, 33, 36, 40], which is not only sufficient, but also necessary to achieve two-sided limits of decay estimates. For instance, Kagei and Kobayashi [26, 27] investigated the special case that initial data satisfy that $(a_0, u_0) \in L^1(\mathbb{R}^3)$ and $\widehat{a}_0(0) = \int_{\mathbb{R}^3} a_0(0)dx \neq 0$. Indeed, by the continuity of $\widehat{a}_0(\xi)$ near $\xi = 0$, there exists a small constant $r_1 > 0$ such that for $0 < r \leq r_1$, $|\widehat{a}_0(\xi)| > 0$ for $|\xi| \leq r$. Thus, it is not difficult to deduce that ($\sigma_1 = -\frac{3}{2}, \sigma = 0$ and $d = 3$)

$$r^{-3}\int_{\{|\xi|\leq r\}} (|\widehat{a}_0(\xi)|^2 + |\widehat{u}_0(\xi)|^2)d\xi \geq \frac{4}{3}\pi \inf_{|\xi|\leq r} |\widehat{a}_0(\xi)|^2 > 0, \quad 0 < r \leq r_1$$

and

$$r^{-3}\int_{\{|\xi|\leq r\}}(|\widehat{a}_0(\xi)|^2+|\widehat{u}_0(\xi)|^2)d\xi\lesssim\|(a_0,u_0)\|^2_{\dot{B}^{-\frac{3}{2}}_{2,\infty}}\lesssim\|(a_0,u_0)\|^2_{L^1},\quad r>0.$$

Hence, it follows from Proposition 3.2 that $(a_0,u_0)^\ell\in\dot{\mathcal{B}}^{-\frac{3}{2}}_{2,\infty}$. Li and Zhang [33] studied some special initial data in $\dot{B}^0_{1,\infty}$ satisfying $|\widehat{a}_0(\xi)|\gtrsim 1$ and $|\widehat{u}_0(\xi)|=0$ for $|\xi|\ll 1$, which also implies that $(a_0,u_0)^\ell\in\dot{\mathcal{B}}^{-\frac{3}{2}}_{2,\infty}$ due to (1.5).

- We can construct the initial function (a_0,u_0) fulfilling (1.7) and $(a_0,u_0)^\ell\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$, for example,

$$a_0(x)=\varepsilon\mathcal{F}^{-1}\Big(|\xi|^{\sigma_1-\frac{d}{2}}\phi(\frac{\xi}{\varepsilon})\Big)(x),\quad u_0(x)=\mathcal{F}^{-1}\Big(\mathcal{F}\Big(\sin(\frac{x\cdot\omega_1}{\varepsilon})\omega_2\Big)(1-\phi(\frac{\xi}{\varepsilon}))\Big)(x),\tag{1.16}$$

  where $\varepsilon>0$ is a suitably small constant, $\phi(\xi)$ is a smooth cut-off function such that $\phi(\xi)=0$ for $|\xi|\geq 1$, and ω_1,ω_2 stand for any unit vectors of $\mathbb{R}^d$. Clearly, the initial data u_0 presented by (1.16) is large highly oscillating if $p>d$ in physical dimensions $d=2,3$. See [8,11] for more details.

- Last but not least, we would like to mention that the sharp decay characterization in critical spaces is of independent interest, which gives a new attempt in the Fourier semi-group framework. Indeed, our approach is to develop the theory of decay characters for linear compressible Navier-Stokes equations with respect to $\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$. On the other hand, Inspired by Hoff-Zumbrun's spectral analysis ([24]), Wiegner's argument and inverse Wiegner's argument (bounding the discrepancy between the nonlinear solution and the linear solution) are first employed in the critical framework, which allow us to remove the smallness of low frequencies of initial data in contrast to prior works [17,52]. The suitable modification of approach is likely to be effective for other incompressible/compressible fluid equations.

In what follows, let us introduce the theory of decay characters, first developed for a large class of dissipative system

$$\begin{cases}\partial_t U=\mathcal{L}U, & x\in\mathbb{R}^d,\quad t>0,\\ U(x,0)=U_0(x),\end{cases}\tag{1.17}$$

where $\mathcal{L}$ is a pseudo-differential operator with symbol

$$\mathcal{M}(\xi)\triangleq P(\xi)^{-1}D(\xi)P(\xi),\quad \text{a.e. }\xi\in\mathbb{R}^d.$$

$D(\xi)$ and $P(\xi)$ are, respectively, diagonal and orthogonal matrices of order n, with $D(\xi)_{ij}=-c_i|\xi|^{2\alpha}\delta_{ij}$ and $c_i\geq c>0$ for all $i=1,...,n$ and $\alpha>0$. $P(\xi)_{i,j}$ are homogeneous smooth functions outside $\xi=0$. Basic examples include the heat equation (in this case $\mathcal{L}=\Delta$ with $P(\xi)=\mathbf{I}_n$ and $D(\xi)=-|\xi|^2\mathbf{I}_n$) or the fractional diffusion equation ($P(\xi)=\mathbf{I}_n$ and $D(\xi)=-|\xi|^{2\alpha}\mathbf{I}_n$). Bjorland-Schonbek [2] and Niche-Schonbek [38] proved that any solution has a two-sided time decay estimate $(1+t)^{-\sigma/2\alpha}\lesssim\|e^{t\mathcal{L}}U_0\|_{L^2}\lesssim$

$(1+t)^{-\sigma/2\alpha}$ if the initial data satisfy

$$0 < \lim_{r\to 0+} r^{-2\sigma}\int_{|\xi|\le r}|\widehat{U}_0(\xi)|^2 d\xi < \infty \tag{1.18}$$

for $\sigma > 0$. The condition (1.18) is closely linked with the decay character (see [2, 38]), however, it is somehow too stringent as such a limit might not exist. In order to overcome this restriction, the first author [3] improved the original definition of decay character and proved that a slight modification of (1.18) is not only sufficient but also necessary condition for the two-sided decay estimates of solutions to (1.17). More precisely,

$$\begin{cases} \liminf\limits_{r\to 0+} r^{-2\sigma}\displaystyle\int_{|\xi|\le r}|\widehat{U}_0(\xi)|^2 d\xi > 0, \\ \limsup\limits_{r\to 0+} r^{-2\sigma}\displaystyle\int_{|\xi|\le r}|\widehat{U}_0(\xi)|^2 d\xi < \infty \end{cases} \Longleftrightarrow U_0\in \dot{\mathcal{B}}^{-\sigma}_{2,\infty} \Longleftrightarrow (1+t)^{-\frac{\sigma}{2\alpha}} \lesssim \|e^{t\mathcal{L}}U_0\|_{L^2} \lesssim (1+t)^{-\frac{\sigma}{2\alpha}}. \tag{1.19}$$

He also discussed the application to the decay of Leray-Hopf's weak solutions to the incompresible Navier-Stokes equations.

Generally speaking, those parabolic arguments in [2,3,38] cannot be directly applied to the compressible Navier-Stokes system (1.9) due to its hyperbolic nature. We need to investigate the precise pointwise behavior of solutions to the linear hyperbolic-parabolic mixed system

$$\begin{cases} \partial_t a + \operatorname{div} u = 0, \\ \partial_t u - \mathcal{A}u + \nabla a = 0, \\ (a,u)|_{t=0} = (a_0,u_0)(x) \triangleq U_0(x). \end{cases} \tag{1.20}$$

Let $\{\mathcal{G}(t)\}_{t\ge 0}$ be the semi-group associated with (1.20). Observe that there is the following key pointwise estimate at low frequencies ($|\xi|\ll 1$):

$$e^{-\max\{\frac{\nu}{2},\mu\}|\xi|^2 t}(|\widehat{a}_0(\xi)|+|\widehat{u}_0(\xi)|) \lesssim |\widehat{\mathcal{G}(t)U_0}(\xi)| \lesssim e^{-\min\{\frac{\nu}{2},\mu\}|\xi|^2 t}(|\widehat{a}_0(\xi)|+|\widehat{u}_0(\xi)|),$$

which enables us to obtain sufficient and necessary conditions for sharp decay estimates of solutions to (1.20) under the the low-frequency assumption in terms of $\dot{B}^{\sigma_1}_{2,\infty}$ or $\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ (see Proposition 3.1). Furthermore, we also perform Schonbek's Fourier splitting methods (see [44]) and establish the equivalence between the low-frequency assumption $\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ and the theory of decay characters (see Proposition 3.2).

To establish the optimal time-decay bounds of the solution to the nonlinear problem (1.9), we will adapt to the compressible Navier-Stokes equations (1.9) well known Wiegner's argument from incompressible flows (see [49]): namely we compute faster time-decay rates of the nonlinear terms compared with that of the solution to the linear problem in L^p-type Besov spaces (see Proposition 4.2). Here, the major difficulty lies in nonconservative terms, for example, $u\cdot\nabla u$ and $I(a)\mathcal{A}u$, which cannot provide faster time-decay rates. To overcome the obstacle, as in [24], we consider the following Navier-Stokes system in terms of the momentum formulation:

$$\begin{cases} \partial_t a + \operatorname{div} m = 0, \\ \partial_t m - \mathcal{A} m + \nabla a = -\operatorname{div} F, \end{cases} \tag{1.21}$$

where the nonlinear terms are given by

$$F \triangleq (1 - I(a)) m \otimes m + H(a)\mathbb{I}_d + \mu \nabla (I(a)m) + (\mu + \lambda) \operatorname{div} (I(a)m) \tag{1.22}$$

with $H(a) \triangleq P(1+a) - P(1) - P'(1)a$. Let (a_L, m_L) be the corresponding solution to the linearized problem of (1.21). Precisely, one has

$$\begin{cases} \partial_t a_L + \operatorname{div} m_L = 0, \\ \partial_t m_L - \mathcal{A} m_L + \nabla a_L = 0, \\ (a_L, m_L)(x, 0) = (a_0, m_0)(x) \triangleq (a_0, \rho_0 u_0)(x). \end{cases} \tag{1.23}$$

It should be noted that due to the smallness condition (1.7) and product laws for hybrid norms, $(a_0, u_0)^\ell \in \dot{B}^{\sigma_1}_{2,\infty}$ (resp. $\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$) if and only if $(a_0, m_0)^\ell \in \dot{B}^{\sigma_1}_{2,\infty}$ (resp. $\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$). Therefore, our key ingredient is to perform time-weighted estimates on the difference $(\widetilde{a}, \widetilde{m}) \triangleq (a - a_L, m - m_L)$ satisfying the difference system

$$\begin{cases} \partial_t \widetilde{a} + \operatorname{div} \widetilde{m} = 0, \\ \partial_t \widetilde{m} - \mathcal{A} \widetilde{m} + \nabla \widetilde{a} = -\operatorname{div} F, \\ (\widetilde{a}, \widetilde{m})(x, 0) = (0, 0). \end{cases} \tag{1.24}$$

Indeed, by Duhamel's principle, the structure of conservation law in (1.9) allows to the improvement of time-decay rates of $(\widetilde{a}, \widetilde{m})$ up to $\frac{1}{2}$-order in low frequencies. In order to remove the smallness of $\|(a_0, u_0)^\ell\|_{\dot{B}^{\sigma_1}_{2,\infty}}$ as in [17, 52], we take advantage of the decay of linearized system and decompose the nonlinear terms in F as the sum of the linear part and the error part, for example,

$$m \otimes m = m_L \otimes m_L + \widetilde{m} \otimes m_L + m \otimes \widetilde{m}.$$

Note that the time-decay rates of $m_L \otimes m_L$ (quadratic) are fast and are given by linear analysis, and $\widetilde{m} \otimes m_L + m \otimes \widetilde{m}$ can be bounded by the faster decay estimates of the difference with a small quantity from (1.7). On the other hand, when we handle the high-frequency part of $(\widetilde{a}, \widetilde{m})$, one has to overcome the difficulty coming from the higher order term $\mathcal{A}(I(a)m)$ in $(1.24)_2$ as it may cause a loss of one derivative on a. For that end, we have to resort to the weighted L^p-energy estimate of (a, u), which, together with the product law on $m = u + au$, implies the desired decay estimate of m. These new observations enable us to establish refined time-weighted estimates for $(\widetilde{a}, \widetilde{m})$ in the Fourier semi-group framework. Furthermore, by combing the decay of (a_L, m_L) with the faster decay of $(\widetilde{a}, \widetilde{m})$, one can establish the upper and lower bounds of (a, m) (1.14), which depends mainly on non L^p standard product laws and the elaborate use of Sobolev embeddings.

Finally, we prove the necessary part of the low-frequency assumption in terms of $\dot{B}^{\sigma_1}_{2,\infty}$ on the upper and lower bounds for decay rates. For that purpose, we develop inverse Wiegner's argument from incompressible flows (see Skalák [46]) to the compressible Navier-Stokes equations (1.9) in the framework of L^p-type Besov spaces. It can be shown that the solution (a_L, m_L) to (1.23) has the same decay rates as the global-in-time solution (a, u) constructed in Theorem 1.1.

The rest of the paper unfolds as follows. In Section 2, we briefly recall the Littlewood-Paley decomposition, Besov spaces and Chemin-Lerner spaces. Section 3 is devoted to the sharp time-decay characterization for the linear compressible Navier-Stokes equations. In Section 4, we establish Wiegner's argument for nonlinear compressible Navier-Stokes equations and deduce the two-sided bounds for decay rates. In Section 5, we develop the inverse Wiegner's argument and justify the implication of low-frequency assumptions. Appendix 6 collects some useful lemmas for non standard product laws and composition of functions that will be used throughout the text.

Notations. For simplicity, C denotes a generic positive constant that may change from line to line. $A \lesssim B$ ($A \gtrsim B$) means that both $A \leq CB$ ($A \geq CB$), while $A \sim B$ means that both $A \lesssim B$ and $A \gtrsim B$. For Banach space X, $p \in [1, \infty]$ and $T > 0$, the notation $L^p(0, T; X)$ or $L^p_T(X)$ designates the set of measurable functions $f : [0, T] \to X$ with $t \mapsto \|f(t)\|_X$ in $L^p(0, T)$, endowed with the norm $\|\cdot\|_{L^p_T(X)} \triangleq \|\|\cdot\|_X\|_{L^p(0,T)}$. Let $\mathcal{F}(f) = \widehat{f}$ and $\mathcal{F}^{-1}(f) = \check{f}$ be the Fourier transform of f and its inverse, and $\Lambda^\sigma f \triangleq \mathcal{F}^{-1}\big(|\xi|^\sigma \mathcal{F}(f)\big)(\sigma \in \mathbb{R})$. In addition, we write $\langle t \rangle = \sqrt{1 + t^2}$, and for any $s > 0$, $s-$ means that $s - \varepsilon$ for all $\varepsilon > 0$.

2 Preliminary

For the convenience of reader, we recall the Littewood-Paley decomposition, Besov spaces and Chemin-Lerner spaces in this section. The reader is referred to Chapters 2 and 3 in [1] or [15] for more details.

Choose a smooth radial non-increasing function $\chi(\xi)$ compactly supported in $B(0, \frac{4}{3})$ and satisfying $\chi(\xi) = 1$ in $B(0, \frac{3}{4})$. Then $\varphi(\xi) \triangleq \chi(\frac{\xi}{2}) - \chi(\xi)$ satisfies

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \cdot) = 1, \quad \text{Supp } \varphi \subset \{\xi \in \mathbb{R}^d \mid \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}.$$

For any $j \in \mathbb{Z}$, define the homogeneous dyadic blocks $\dot{\Delta}_j$ by

$$\dot{\Delta}_j u \triangleq \mathcal{F}^{-1}\big(\varphi(2^{-j} \cdot)\mathcal{F}(u)\big) = 2^{jd} h(2^j \cdot) \star u, \qquad h \triangleq \mathcal{F}^{-1}\varphi.$$

Let $\mathcal{P}$ be the class of all polynomials on $\mathbb{R}^d$ and $\mathcal{S}'_h \triangleq \mathcal{S}'/\mathcal{P}$ stand for the tempered distributions on $\mathbb{R}^d$ modulo polynomials. One can get

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \quad \text{in} \quad \mathcal{S}'_h, \quad \forall u \in \mathcal{S}'_h, \qquad \dot{\Delta}_j \dot{\Delta}_l u = 0, \quad \text{if} \quad |j - l| \geq 2.$$

With the help of those dyadic blocks, we give the definition of homogeneous Besov spaces and mixed space-time Besov spaces as follow. For $s\in\mathbb{R}$ and $1\leq p,r\leq\infty$, the homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by

$$\dot{B}^s_{p,r}\triangleq\big\{u\in\mathcal{S}'_h\mid \|u\|_{\dot{B}^s_{p,r}}\triangleq\|\{2^{js}\|\dot{\Delta}_ju\|_{L^p}\}_{j\in\mathbb{Z}}\|_{l^r}<\infty\big\}.$$

For $T>0$, $s\in\mathbb{R}$ and $1\leq\varrho,r,q\leq\infty$, we recall a class of mixed space-time Besov spaces $\widetilde{L}^\varrho(0,T;\dot{B}^s_{p,r})$ that were initiated by Chemin and Lerner in [10]:

$$\widetilde{L}^\varrho(0,T;\dot{B}^s_{p,r})\triangleq\big\{u\in L^\varrho(0,T;\mathcal{S}'_h)\mid\|u\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})}\triangleq\|\{2^{js}\|\dot{\Delta}_ju\|_{L^\varrho_T(L^p)}\}_{j\in\mathbb{Z}}\|_{l^r}<\infty\big\}.$$

By the Minkowski inequality, it holds that

$$\|u\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})}\leq\|u\|_{L^\varrho_T(\dot{B}^s_{p,r})}\quad\text{if } r\geq\varrho\quad(\text{resp. }\|u\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})}\geq\|u\|_{L^\varrho_T(\dot{B}^s_{p,r})}\quad\text{if } r\leq\varrho),$$

where $\|\cdot\|_{L^\varrho_T(\dot{B}^s_{p,r})}$ is the usual Lebesgue-Besov norm. Moreover, we denote

$$\mathcal{C}_b(\mathbb{R}_+;\dot{B}^s_{p,r})\triangleq\big\{u\in\mathcal{C}(\mathbb{R}_+;\dot{B}^s_{p,r})\mid\|f\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{B}^s_{p,r})}<\infty\big\}.$$

In order to restrict Besov norms to the low frequency part and the high-frequency part, we often use the following notations for any $s\in\mathbb{R}$ and $p\in[1,\infty]$:

$$\begin{cases}\|u\|^\ell_{\dot{B}^s_{p,r}}\triangleq\|\{2^{js}\|\dot{\Delta}_ju\|_{L^p}\}_{j\leq j_0}\|_{\ell^r}, & \|u\|^h_{\dot{B}^s_{p,r}}\triangleq\|\{2^{js}\|\dot{\Delta}_ju\|_{L^p}\}_{j\geq j_0-1}\|_{\ell^r},\\ \|u\|^\ell_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})}\triangleq\|\{2^{js}\|\dot{\Delta}_ju\|_{L^\varrho_T(L^p)}\}_{j\leq j_0}\|_{\ell^r}, & \|u\|^h_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})}\triangleq\|\{2^{js}\|\dot{\Delta}_ju\|_{L^\varrho_T(L^p)}\}_{j\geq j_0-1}\|_{\ell^r},\end{cases}$$

where j_0 is called threshold between low frequencies and high frequnencies which was chosen in [8,11,21].

Denote by u^ℓ (resp. u^h) the low-frequency (high-frequency) part of $u\in\mathcal{S}'_h$ as follows:

$$u^\ell\triangleq\sum_{j\leq j_0-1}\dot{\Delta}_ju,\qquad u^h\triangleq u-u^\ell=\sum_{j\geq j_0}\dot{\Delta}_ju.$$

It is easy to check for any $s'>0$ that

$$\begin{cases}\|u^\ell\|_{\dot{B}^s_{p,r}}\lesssim\|u\|^\ell_{\dot{B}^s_{p,r}}\lesssim\|u\|^\ell_{\dot{B}^{s-s'}_{p,\infty}}, & \|u^h\|_{\dot{B}^s_{p,1}}\lesssim\|u\|^h_{\dot{B}^s_{p,r}}\lesssim\|u\|^h_{\dot{B}^{s+s'}_{p,r}},\\ \|u^\ell\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})}\lesssim\|u\|^\ell_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})}\lesssim\|u\|^\ell_{L^\varrho_T(\dot{B}^{s-s'}_{p,\infty})}, & \|u^h\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})}\lesssim\|u\|^h_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})}\lesssim\|u\|^h_{L^\varrho_T(\dot{B}^{s+s'}_{p,r})}.\end{cases}\tag{2.1}$$

3 Two-sided bounds of decay for the linear compressible Navier-Stokes equations

In this section, we are interested in establishing the theory of decay characters for the linear compressible Navier-Stokes system

$$\begin{cases}\partial_ta+\operatorname{div}u=0,\\ \partial_tu-\mathcal{A}u+\nabla a=0,\\ (a,u)(x,0)=(a_0,u_0)(x).\end{cases}\tag{3.1}$$

Denote by $\Omega \triangleq \Lambda^{-1}\,\mathrm{curl}\,u$ the incompressible part of u and by $v \triangleq \Lambda^{-1}\mathrm{div}\,u$ the compressible part of u. Therefore, we see that Ω satisfies the heat equation

$$\partial_t\Omega - \mu\Delta\Omega = 0, \qquad \Omega(x,0) = \Lambda^{-1}\,\mathrm{curl}\,u_0(x). \tag{3.2}$$

On the other hand, one can get the hyperbolic-parabolic mixed system for (a, v):

$$\begin{cases} \partial_t a + \Lambda v = 0, \\ \partial_t v - \nu\Delta v - \Lambda a = 0, \end{cases} \tag{3.3}$$

with $\nu = \lambda + 2\mu$ and $(a, v)(x, 0) \triangleq (a_0, \Lambda^{-1}\mathrm{div}\,u_0)(x)$.

It should be noted that the theory of decay characters developed in [2,3,38] is not applicable to (3.3) in general due to the dispersion form in hyperbolic part, even though (3.2) is a pure heat equation. Indeed, we have the following pointwise estimates of (a, u) to the system (3.1) in Fourier spaces.

Lemma 3.1. *Let (a, u) satisfy System (3.1). It holds that*

$$\begin{cases} |\widehat{a}(\xi,t)| + |\widehat{u}(\xi,t)| \lesssim e^{-R_* t}(|\widehat{a}_0(\xi)| + |\widehat{u}_0(\xi)|), & \text{if } \ |\xi| \geq \dfrac{2}{\nu}, \\ |\widehat{a}(\xi,t)| + |\widehat{u}(\xi,t)| \lesssim e^{-\min\{\frac{\nu}{2},\mu\}|\xi|^2 t}(|\widehat{a}_0(\xi)| + |\widehat{u}_0(\xi)|), & \text{if } \ |\xi| \leq \dfrac{2}{\nu} \end{cases} \tag{3.4}$$

for $R_* \triangleq \min\{\frac{\nu}{2},\mu\}\frac{4}{\nu^2} > 0$ and

$$|\widehat{a}(\xi,t)| + |\widehat{u}(\xi,t)| \gtrsim e^{-\max\{\frac{\nu}{2},\mu\}|\xi|^2 t}(|\widehat{a}_0(\xi)| + |\widehat{u}_0(\xi)|), \quad \text{if } \ |\xi| \leq \eta, \tag{3.5}$$

where $\eta > 0$ is sufficiently small.

Proof. Taking the Fourier transform to (3.2) with respect to the space variable yields

$$|\widehat{\Omega}(\xi,t)| = e^{-\mu|\xi|^2 t}|\widehat{\Omega}_0(\xi)|, \quad \xi \in \mathbb{R}^d. \tag{3.6}$$

On the other hand, we have the following explicit expression for the Green matrix $\mathcal{G}$ of system (3.3) ([24]):

$$\widehat{\mathcal{G}}(\xi,t) \triangleq \begin{pmatrix} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} & -\big(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\big)|\xi| \\ \big(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\big)|\xi| & \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \end{pmatrix}$$

with the eigenvalues

$$\lambda_\pm(\xi) = \begin{cases} -\frac{\nu}{2}|\xi|^2 \pm \mathrm{i}\sqrt{|\xi|^2 - \frac{\nu^2}{4}|\xi|^4}, & \text{if } \ |\xi| \leq \frac{2}{\nu}, \\ -\frac{\nu}{2}|\xi|^2 \pm \sqrt{\frac{\nu^2}{4}|\xi|^4 - |\xi|^2}, & \text{if } \ |\xi| \geq \frac{2}{\nu}. \end{cases}$$

The upper bound $(3.4)_1$ in high frequencies $|\xi| \geq 2/\nu$ is classical (see for example [8,24,33]). We omit details for brevity.

In low frequencies $|\xi|\leq 2/\nu$, we write $b\triangleq\sqrt{|\xi|^2-\frac{\nu^2}{4}|\xi|^4}$. The direct computation gives

$$\begin{cases}\dfrac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}=e^{-\frac{\nu}{2}|\xi|^2}\dfrac{\sin(bt)}{b},\\ \dfrac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}=e^{-\frac{\nu}{2}|\xi|^2}\Big(\cos(bt)+\dfrac{\nu}{2}\dfrac{\sin(bt)}{b}|\xi|^2\Big),\\ \dfrac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}=e^{-\frac{\nu}{2}|\xi|^2}\Big(\cos(bt)-\dfrac{\nu}{2}\dfrac{\sin(bt)}{b}|\xi|^2\Big).\end{cases}$$

Therefore, we obtain

$$\widehat{a}(\xi,t)=\frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}\widehat{a}_0(\xi)-\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}|\xi|\widehat{v}_0(\xi)=e^{-\frac{\nu}{2}|\xi|^2t}\widehat{a}^*(\xi,t)\tag{3.7}$$

and

$$\widehat{v}(\xi,t)=\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}|\xi|\widehat{a}_0(\xi)+\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}\widehat{v}_0(\xi)=e^{-\frac{\nu}{2}|\xi|^2t}\widehat{v}^*(\xi,t),\tag{3.8}$$

where $a_0^*(x)$ and $v_0^*(x)$ are defined by

$$\begin{cases}\widehat{a}^*(\xi,t)\triangleq\Big(\cos(bt)+\dfrac{\nu}{2}\dfrac{\sin(bt)}{b}|\xi|^2\Big)\widehat{a}_0(\xi)-\dfrac{\sin(bt)}{b}|\xi|\widehat{v}_0(\xi),\\ \widehat{v}^*(\xi,t)\triangleq\dfrac{\sin(bt)}{b}|\xi|\widehat{a}_0(\xi)+\Big(\cos(bt)-\dfrac{\nu}{2}\dfrac{\sin(bt)}{b}|\xi|^2\Big)\widehat{v}_0(\xi).\end{cases}$$

Thus, the upper bound (3.4) in low frequencies can be derived from (3.6), (3.7) and (3.8) directly. Next, we turn to prove (3.5) from below in low frequencies.

Since b is real when $|\xi|\leq 2/\nu$, we have

$$\begin{aligned}|\widehat{a}^*(\xi,t)|^2=\widehat{a}^*(\xi,t)\overline{\widehat{a}^*(\xi,t)}&=\Big(\cos(bt)+\frac{\nu}{2}\frac{\sin(bt)}{b}|\xi|^2\Big)^2|\widehat{a}_0(\xi)|^2+\frac{|\sin(bt)|^2}{b^2}|\xi|^2|\widehat{v}_0(\xi)|^2\\&-\frac{\nu}{2}\frac{|\sin(bt)|^2}{b^2}|\xi|^3\Big(\overline{\widehat{a}_0(\xi)}\widehat{v}_0(\xi)+\widehat{a}_0(\xi)\overline{\widehat{v}_0(\xi)}\Big)-\frac{\cos(bt)\sin(bt)}{b}|\xi|\Big(\overline{\widehat{a}_0(\xi)}\widehat{v}_0(\xi)+\widehat{a}_0(\xi)\overline{\widehat{v}_0(\xi)}\Big)\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}|\widehat{v}^*(\xi,t)|^2=\widehat{v}^*(\xi,t)\overline{\widehat{v}^*(\xi,t)}&=\frac{|\sin(bt)|^2}{b^2}|\xi|^2|\widehat{a}_0(\xi)|^2+\Big(\cos(bt)-\frac{\nu}{2}\frac{\sin(bt)}{b}|\xi|^2\Big)^2|\widehat{v}_0(\xi)|^2\\&-\frac{\nu}{2}\frac{|\sin(bt)|^2}{b^2}|\xi|^3\Big(\overline{\widehat{a}_0(\xi)}\widehat{v}_0(\xi)+\widehat{a}_0(\xi)\overline{\widehat{v}_0(\xi)}\Big)+\frac{\cos(bt)\sin(bt)}{b}|\xi|\Big(\overline{\widehat{a}_0(\xi)}\widehat{v}_0(\xi)+\widehat{a}_0(\xi)\overline{\widehat{v}_0(\xi)}\Big).\end{aligned}\tag{3.10}$$

Owing to the fact that $b=|\xi|+O(|\xi|^3)$ as $|\xi|\to 0$, we see that the last terms on the right-hand side of (3.9) and (3.10) are of zero-order with respect to the variable ξ, which turn out to be some obstacles to get the dissipative estimate (3.5). To handle the difficulty, special assumptions were imposed in earlier works (e.g., [26, 27, 33]). In the present paper, it is observed that the two "bad" terms standing for the hyperbolic dispersion effect could be cancelled if one adds (3.9) and (3.10) together. Indeed, using

$|\cos(bt)|^2+\frac{|\sin(bt)|^2}{b^2}|\xi|^2=1+O(|\xi|^4)$, we have

$$
\begin{aligned}
|\widehat{a}^*(\xi,t)|^2+|\widehat{v}^*(\xi,t)|^2=&\Big(|\cos(bt)|^2+\frac{|\sin(bt)|^2}{b^2}|\xi|^2+\nu\frac{\sin(bt)\cos(bt)}{b}|\xi|^2+\frac{\nu^2}{4}\frac{|\sin(bt)|^2}{b^2}|\xi|^4\Big)|\widehat{a}_0(\xi)|^2\\
&+\Big(|\cos(bt)|^2+\frac{|\sin(bt)|^2}{b^2}|\xi|^2-\nu\frac{\sin(bt)\cos(bt)}{b}|\xi|^2+\frac{\nu^2}{4}\frac{|\sin(bt)|^2}{b^2}|\xi|^4\Big)|\widehat{v}_0(\xi)|^2\\
&-\nu\frac{|\sin(bt)|^2}{b^2}|\xi|^3\Big(\overline{\widehat{a}_0(\xi)}\widehat{v}_0(\xi)+\widehat{a}_0(\xi)\overline{\widehat{v}_0(\xi)}\Big)\\
\geq&(1-C_\nu|\xi|)(|\widehat{a}_0(\xi)|^2+|\widehat{v}_0(\xi)|^2)-C_\nu|\xi||\widehat{a}_0(\xi)||\widehat{v}_0(\xi)|\\
\geq&\frac{1}{2}(|\widehat{a}_0(\xi)|^2+|\widehat{v}_0(\xi)|^2)
\end{aligned}
$$

for $|\xi|\leq\eta\triangleq\min\{1/3C_\nu,2/\nu\}$. Therefore, (3.5) is followed by (3.6) and the fact that $|\widehat{v}(\xi,t)|^2+|\widehat{\Omega}(\xi,t)|^2\sim|\widehat{u}(\xi,t)|^2$. □

The pointwise estimates (3.4)-(3.5) indicate that the total energy of (a,u) to (3.1) behaves like that of heat kernel from above and below in low frequencies, which motivates us to establish a sharp decay characterization for (3.1) in terms of the Besov regularity. First of all, we establish the following sufficient and necessary conditions for the upper and lower bounds of decay of solutions to (3.1).

Proposition 3.1. *Let $\sigma,\sigma_1\in\mathbb{R}$ such that $\sigma>\sigma_1$. Assume that (a,u) satisfies System (3.1) and $(a_0,u_0)\in\dot{B}^{\sigma}_{2,1}$. For any given time $t_L\geq0$, the following properties hold:*

(1) *The solution (a,u) has upper bounds of time-decay estimate*

$$\|(a,u)(t)\|_{\dot{B}^{\sigma}_{2,1}}\lesssim\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)},\quad t>t_L, \tag{3.11}$$

if and only if $(a_0,u_0)\in\dot{B}^{\sigma_1}_{2,\infty}$;

(2) *The solution (a,u) has upper and lower bounds of time-decay estimate*

$$\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}\lesssim\|(a,u)(t)\|_{\dot{B}^{\sigma}_{2,1}}\lesssim\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)},\quad t>t_L, \tag{3.12}$$

if and only if $(a_0,u_0)\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$.

Proof. We first justify (3.11). Under the additional condition $(a_0,u_0)\in\dot{B}^{\sigma_1}_{2,\infty}$, it follows from (3.4) that

$$
\begin{aligned}
\|(a,u)(t)\|_{\dot{B}^{\sigma}_{2,1}}\lesssim&\sum_{j\leq[\log_2\frac{2}{\nu}]}e^{-\frac{9}{16}\min\{\frac{\nu}{2},\mu\}2^{2j}t}2^{j\sigma}\|\dot{\Delta}_j(a_0,u_0)\|_{L^2}+\sum_{j\geq[\log_2\frac{2}{\nu}]+1}e^{-R_*t}2^{j\sigma}\|\dot{\Delta}_j(a_0,u_0)\|_{L^2}\\
\lesssim&t^{-\frac{1}{2}(\sigma-\sigma_1)}\|(a_0,u_0)\|_{\dot{B}^{\sigma_1}_{2,\infty}}+e^{-R_*t}\|(a_0,u_0)\|_{\dot{B}^{\sigma}_{2,1}}\\
\lesssim&\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)},\quad t>1,
\end{aligned}
$$

where we used the fact

$$\sup_{t>0}\sum_{j\in\mathbb{Z}}t^{\frac{1}{2}(\sigma-\sigma_1)}2^{j(\sigma-\sigma_1)}e^{-\frac{9}{16}\min\{\frac{\nu}{2},\mu\}2^{2j}t}<\infty$$

for $\sigma > \sigma_1$.

On the other hand, since $(a_0, u_0) \in \dot{B}^{\sigma}_{2,1}$, one can get from (3.4) and Parseval's theorem that

$$\|(a,u)(t)\|_{\dot{B}^{\sigma}_{2,1}} \lesssim \|(a_0,u_0)\|_{\dot{B}^{\sigma}_{2,1}} \lesssim \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}, \qquad 0 < t \leq 1. \tag{3.13}$$

Therefore, the upper bound (3.11) follows.

Conversely, assume that (a, u) satisfies (3.11) for $t > t_L$. In fact, by virtue of (3.13), (3.11) holds for $t > 0$. The Fourier transform $(\widehat{a}_0, \widehat{u}_0)$ can be represented by

$$(\widehat{a}_0,\widehat{u}_0)(\xi) = \frac{2\max\{\frac{\nu}{2},\mu\}}{\Gamma(\frac{1}{2}(\sigma-\sigma_1)+1)} \int_0^\infty t^{\frac{1}{2}(\sigma-\sigma_1)}|\xi|^{\sigma-\sigma_1+2} e^{-2\max\{\frac{\nu}{2},\mu\}|\xi|^2 t}(\widehat{a}_0,\widehat{u}_0)(\xi)dt,$$

where $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}ds$. This implies for any integer $j \leq [\log_2 2/\nu]$ that

$$\begin{aligned}\|\dot{\Delta}_j(a_0,u_0)\|_{L^2} &\sim \|\varphi(2^{-j}|\xi|)(\widehat{a}_0,\widehat{u}_0)\|_{L^2}\\ &\lesssim \int_0^\infty t^{\frac{1}{2}(\sigma-\sigma_1)} 2^{(-\sigma_1+2)j} e^{-\frac{9}{16}\max\{\frac{\nu}{2},\mu\}2^{2j}t} \||\xi|^{\sigma} e^{-\max\{\frac{\nu}{2},\mu\}|\xi|^2 t}(\widehat{a}_0,\widehat{u}_0)\|_{L^2}dt.\end{aligned} \tag{3.14}$$

In view of (3.5), (3.11) and Parseval's theorem, it holds that

$$\||\xi|^{\sigma}e^{-\max\{\frac{\nu}{2},\mu\}|\xi|^2 t}(\widehat{a}_0,\widehat{u}_0)\|_{L^2} \lesssim \||\xi|^{\sigma}(\widehat{a},\widehat{u})(t)\|_{L^2} \sim \|(a,u)(t)\|_{\dot{H}^{\sigma}} \lesssim \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}. \tag{3.15}$$

Substituting (3.15) into (3.14) and using the fact that $\|(a_0,u_0)\|_{\dot{B}^{\sigma}_{2,1}} < \infty$, we get

$$\begin{aligned}\|(a_0,u_0)\|_{\dot{B}^{\sigma_1}_{2,\infty}} &\lesssim \sup_{j\leq[\log_2\frac{2}{\nu}]} 2^{\sigma_1 j}\|\dot{\Delta}_j(a_0,u_0)\|_{L^2} + \sup_{j\geq[\log_2\frac{2}{\nu}]+1} 2^{\sigma_1 j}\|\dot{\Delta}_j(a_0,u_0)\|_{L^2}\\ &\lesssim \int_0^\infty 2^{2j}e^{-\frac{9}{16}\max\{\frac{\nu}{2},\mu\}2^{2j}t}dt + \Big(\frac{2}{\nu}\Big)^{-\sigma+\sigma_1} \sum_{j\geq[\log_2\frac{2}{\nu}]+1} 2^{\sigma j}\|\dot{\Delta}_j(a_0,u_0)\|_{L^2} \lesssim 1.\end{aligned} \tag{3.16}$$

Next, we turn to prove the two-sided bounds (3.12). Assume $(a_0, u_0) \in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$. The upper bound in (3.12) follows directly from (3.11). In order to derive the lower bound, it follows from the definition $\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ as in (1.11) that there exists two constants $c, M > 0$ and a sequence $\{j_k\}_{k=1,2,...}$ such that

$$j_k \to -\infty \text{ as } k\to\infty, \quad |j_k - j_{k+1}| \leq M, \quad 2^{\sigma_1 j_k}\|\dot{\Delta}_{j_k}(a_0,u_0)\|_{L^2} \geq c, \quad k = 1,2,... \tag{3.17}$$

Without loss of generality, we assume that j_k, $k = 1, 2, ...,$ is less than $[\log_2 \eta]$. It follows from Parseval's theorem and (3.5) that

$$\begin{aligned}\|(a,u)(t)\|_{\dot{B}^{\sigma}_{2,1}} &\geq \sum_{j\leq[\log_2\eta]} 2^{\sigma j}\|\dot{\Delta}_j(a,u)(t)\|_{L^2}\\ &= \sum_{j\leq[\log_2\eta]} 2^{\sigma j}\|\varphi(2^{-j}\cdot)(\widehat{a},\widehat{u})(t)\|_{L^2}\\ &\gtrsim \sum_{j\leq[\log_2\eta]} e^{-\frac{64}{9}\max\{\frac{\nu}{2},\mu\}2^{2j}t} 2^{\sigma j}\|\dot{\Delta}_j(a_0,u_0)\|_{L^2}.\end{aligned} \tag{3.18}$$

For all $t > t_L$, since j_k tends to $-\infty$ as $k \to \infty$, we are able to find a maximal integer j_{k_0} satisfying $j_{k_0} \leq -\frac{1}{2}\log_2(1+t_L+t)$. Then we have $j_{k_0} > -M - \frac{1}{2}\log_2(1+t_L+t)$; otherwise, from (3.17) another integer j_{k_0-1} fulfills $j_{k_0-1} \leq j_{k_0} + M \leq -\frac{1}{2}\log_2(1+t_L+t)$ which contradicts with the maximality of j_{k_0}. Therefore, it follows from (3.17) and the fact $2^{j_{k_0}} \sim \langle t\rangle^{-\frac{1}{2}}$ that

$$
\begin{aligned}
&\sum_{j\leq[\log_2 \eta]} e^{-\frac{64}{9}\max\{\frac{\nu}{2},\mu\}2^{2j}t} 2^{\sigma j}\|\dot{\Delta}_j(a_0,u_0)\|_{L^2} \\
&\qquad \gtrsim e^{-\frac{64}{9}\max\{\frac{\nu}{2},\mu\}2^{2j_{k_0}}t} 2^{(\sigma-\sigma_1)j_{k_0}} 2^{\sigma_1 j_{k_0}}\|\dot{\Delta}_{j_{k_0}}(a_0,u_0)\|_{L^2} \\
&\qquad \gtrsim 2^{(\sigma-\sigma_1)j_{k_0}} \\
&\qquad \gtrsim \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)},
\end{aligned}
$$

from which one can deduce the lower bound in (3.12).

Conversely, if we assume that (a,u) satisfy the two-sided bounds (3.12) for $t > t_L$. The upper bound in (3.12) implies that $(a_0,u_0) \in \dot{B}^{\sigma_1}_{2,\infty}$. It suffices to construct a sequence $\{j_k\}_{k=1,2,...}$ such that $j_k \to -\infty$, $|j_k - j_{k+1}| \leq M$, and $2^{\sigma_1 j_k}\|\dot{\Delta}_{j_k}(a_0,u_0)\| \geq c$. For that end, we deduce from the high-frequency bound in (3.4) and (3.12) that

$$
\begin{aligned}
\sum_{j\leq[\log_2 \frac{2}{\nu}]} 2^{\sigma j}\|\dot{\Delta}_j(a,u)(t)\|_{L^2} &= \|(a,u)(t)\|_{\dot{B}^{\sigma}_{2,1}} - \sum_{j\geq[\log_2 \frac{2}{\nu}]+1} 2^{\sigma j}\|\dot{\Delta}_j(a,u)(t)\|_{L^2} \\
&\gtrsim \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)} - e^{-R_* t}\sum_{j\geq[\log_2 \frac{2}{\nu}]+1} 2^{\sigma j}\|\dot{\Delta}_j(a_0,u_0)\|_{L^2} \\
&\gtrsim \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}, \qquad t \gg 1,
\end{aligned}
$$

which, together with the low-frequency bound (3.4), implies that there exists a suitably large time $t_* > t_L$ and a constant $\eta_* > 0$ independent of time such that

$$
\sum_{j\in\mathbb{Z}} e^{-\frac{9}{16}\min\{\frac{\nu}{2},\mu\}2^{2j}t} 2^{\sigma j} t^{\frac{1}{2}(\sigma-\sigma_1)}\|\dot{\Delta}_j(a_0,u_0)\|_{L^2} > \eta_* > 0, \qquad t \geq t_*. \tag{3.19}
$$

In particular, (3.19) holds true with $t = t_* + k$ for all $k = 1,2,...$, and then we define

$$
j_{1,k} \triangleq -\Big[\frac{1}{2}\log_2(t_*+k)\Big].
$$

Making use of (3.19) and the fact $2^{-2j_{1,k}-2} \leq t_* + k \leq 2^{-2j_{1,k}}$, we get

$$
\sum_{j\in\mathbb{Z}} e^{-\frac{9}{16}\min\{\frac{\nu}{2},\mu\}2^{2(j-j_{1,k})-2}} 2^{(\sigma-\sigma_1)(j-j_{1,k})} 2^{\sigma_1 j}\|\dot{\Delta}_j(a_0,u_0)\|_{L^2} > \eta_* > 0.
$$

Shifting the index $j - j_{1,k}$ to j', we deduce that

$$
\sum_{j'\in\mathbb{Z}} e^{-\frac{9}{16}\min\{\frac{\nu}{2},\mu\}2^{2j'-2}} 2^{j'(\sigma-\sigma_1)} \Big(2^{(j'+j_{1,k})\sigma_1}\|\dot{\Delta}_{j'+j_{1,k}}(a_0,u_0)\|_{L^2}\Big) > \eta_* > 0. \tag{3.20}
$$

Due to the fact that $e^{-\frac{9}{16}\min\{\frac{\nu}{2},\mu\}2^{2j'-2}} 2^{j'(\sigma-\sigma_1)} \in l^1(\mathbb{Z})$ holds for $\sigma > \sigma_1$, there exists a sufficiently large integer $J > 0$ such that

$$
\sum_{|j'|>J} e^{-\frac{9}{16}\min\{\frac{\nu}{2},\mu\}2^{2j'-2}} 2^{j'(\sigma-\sigma_1)} < \frac{\eta_*}{2\|(a_0,u_0)\|_{\dot{B}^{\sigma_1}_{2,\infty}} + 1}.
$$

Consequently, we have

$$\sum_{|j'|>J} e^{-\frac{9}{16}\min\{\frac{\nu}{2},\mu\}2^{2j'-2}} 2^{j'(\sigma-\sigma_1)}\Big(2^{(j'+j_{1,k})\sigma_1}\|\dot{\Delta}_{j'+j_{1,k}}(a_0,u_0)\|_{L^2}\Big) < \frac{\eta_*}{2}. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\sum_{|j'|\le J} e^{-\frac{9}{16}\min\{\frac{\nu}{2},\mu\}2^{2j'-2}} 2^{j'(\sigma-\sigma_1)}\Big(2^{(j'+j_{1,k})\sigma_1}\|\dot{\Delta}_{j'+j_{1,k}}(a_0,u_0)\|_{L^2}\Big) > \frac{\eta_*}{2} > 0. \tag{3.22}$$

For every given $j_{1,k}$, let $j_{2,k}\in[-J,J]$ be the integer such that

$$2^{(j_{2,k}+j_{1,k})\sigma_1}\|\dot{\Delta}_{j_{2,k}+j_{1,k}}(a_0,u_0)\|_{L^2} = \max_{|j'|\le J} 2^{(j'+j_{1,k})\sigma_1}\|\dot{\Delta}_{j'+j_{1,k}}(a_0,u_0)\|_{L^2}.$$

If we define

$$j_k \triangleq j_{1,k}+j_{2,k}, \qquad k=1,2,...,$$

then it follows from (3.22) and the definitions of $j_{1,k}$, $j_{2,k}$ that $j_k\to-\infty$ as $k\to\infty$,

$$|j_k-j_{k+1}|\le 2J+\frac{1}{2}\log_2(1+\frac{1}{t_*})+1 \quad \text{and} \quad 2^{\sigma_1 j_k}\|\dot{\Delta}_{j_k}(a_0,u_0)\|_{L^2}\ge \frac{\eta_*}{4J}e^{-\frac{64}{9}\min\{\frac{\nu}{2},\mu\}2^{2J}}.$$

This implies that $(a_0,u_0)\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$. The proof of Proposition 3.1 is complete. □

Furthermore, it is shown that $(a_0,u_0)\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ is equivalent to those conditions on the theory of decay characters developed by [2,3,38]. Precisely, we have the following proposition.

Proposition 3.2. *Let $\sigma,\sigma_1\in\mathbb{R}$ such that $\sigma>\sigma_1$. Assume that (a,u) satisfy System (3.1) and $(a_0,u_0)\in\dot{B}^{\sigma}_{2,1}$. Then the following two statements are equivalent:*

(1) $(a_0,u_0)\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$;

(2) (a_0,u_0) *satisfies*

$$\begin{cases} P_{\sigma_1}(a_0,u_0)_+ \triangleq \limsup\limits_{r\to0^+} r^{-2(\sigma-\sigma_1)}\displaystyle\int_{\{|\xi|\le r\}}|\xi|^{2\sigma}(|\widehat{a}_0(\xi)|^2+|\widehat{u}_0(\xi)|^2)d\xi<\infty,\\ P_{\sigma_1}(a_0,u_0)_- \triangleq \liminf\limits_{r\to0^+} r^{-2(\sigma-\sigma_1)}\displaystyle\int_{\{|\xi|\le r\}}|\xi|^{2\sigma}(|\widehat{a}_0(\xi)|^2+|\widehat{u}_0(\xi)|^2)d\xi>0. \end{cases} \tag{3.23}$$

Proof. We first prove that $(a_0,u_0)\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ implies (3.23). For $r>0$, let the integer $j=[\log_2 r]$ such that $2^j\le r<2^{j+1}$. Owing to $(a_0,u_0)\in\dot{B}^{\sigma_1}_{2,\infty}$, we have

$$\begin{aligned} &r^{-2(\sigma-\sigma_1)j}\int_{\{|\xi|\le r\}}|\xi|^{2\sigma}(|\widehat{a}_0(\xi)|^2+|\widehat{u}_0(\xi)|^2)d\xi\\ &\le 2^{-2(\sigma-\sigma_1)j}\int_{\{|\xi|\le 2^{j+1}\}}|\xi|^{2\sigma}(|\widehat{a}_0(\xi)|^2+|\widehat{u}_0(\xi)|^2)d\xi\\ &\le 2^{-2(\sigma-\sigma_1)j}\sum_{j'\le j+1}2^{2j'(\sigma-\sigma_1)}\sup_{j'\le j+1}2^{2j'\sigma_1}\|\dot{\Delta}_{j'}(a_0,u_0)\|^2_{L^2}\\ &\lesssim \|(a_0,u_0)\|^2_{\dot{B}^{\sigma_1}_{2,\infty}}, \end{aligned}$$

which implies $P_{\sigma_1}(a_0,u_0)_+<\infty$.

Moreover, from our assumption we can take $\{j_k\}_{k=1,2,...}$ as in the definition of $\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$. For any $0<r<2^{j_1}$, let j_k be the largest integer of the sequence $\{j_k\}_{k=1,2,...}$ such that $2^{j_k}\leq r$. Then we have $2^{j_k+M}\geq r$; otherwise, from $\|j_k-j_{k+1}\|_{l^\infty}\leq M$ we would find another integer $j_l\in[j_k,j_k+M]$ such that $2^{j_l}\leq r$, which contradicts the maximality of j_k. Hence, we have $2^{j_k}\leq r\leq 2^{j_k+M}$. Consequently,

$$
\begin{aligned}
&r^{-2(\sigma-\sigma_1)j_k}\int_{\{|\xi|\leq r\}}|\xi|^{2\sigma}(|\widehat{a}_0(\xi)|^2+|\widehat{u}_0(\xi)|^2)d\xi\\
&\gtrsim 2^{-2(\sigma-\sigma_1)j_k}\int_{\{\frac{3}{4}2^{j_k}\leq|\xi|\leq\frac{8}{3}2^{j_k}\}}2^{2\sigma j_k}(|\widehat{a}_0(\xi)|^2+|\widehat{u}_0(\xi)|^2)d\xi\\
&\gtrsim 2^{2\sigma_1 j_k}\|\dot{\Delta}_{j_k}(a_0,u_0)\|_{L^2}^2\gtrsim 1.
\end{aligned}
$$

This indicates $P_{\sigma_1}(a_0,u_0)_->0$.

Conversely, if (3.23) holds, then there exists some constants $r_0,c_1,c_2>0$ such that for any $0<r\leq r_0$, it holds that

$$
0<c_1r^{2(\sigma-\sigma_1)}\leq\int_{\{|\xi|\leq r\}}|\xi|^{2\sigma}(|\widehat{a}_0(\xi)|^2+|\widehat{u}_0(\xi)|^2)d\xi\leq c_2r^{2(\sigma-\sigma_1)}. \tag{3.24}
$$

In order to show $(a_0,u_0)\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$, it suffices to prove the two-sided bounds of decay estimates of the solution (a,u) to (3.1) under the condition (3.24). To do this, we perform Schonbek's Fourier splitting methods as in [44] to the compressible Navier-Stokes equations (3.1). Applying the operator $\dot{S}_\nu\Lambda^\sigma$ with the low-frequency cut-off $\dot{S}_\nu z\triangleq\sum_{j\leq[\log_2\frac{2}{\nu}]}\dot{\Delta}_jz$ to (3.1), we get

$$
\begin{cases}
\partial_t\dot{S}_\nu\Lambda^\sigma a+\mathrm{div}\,\dot{S}_\nu\Lambda^\sigma u=0,\\
\partial_t\dot{S}_\nu\Lambda^\sigma u-\mathcal{A}\dot{S}_\nu\Lambda^\sigma u+\nabla\dot{S}_\nu\Lambda^\sigma a=0.
\end{cases} \tag{3.25}
$$

Multiplying the first equation of (3.25) by $\dot{S}_\nu\Lambda^\sigma a$, the second one by $\dot{S}_\nu\Lambda^\sigma u$, adding the resulting equations together, then integrating it over $\mathbb{R}^d$, we have

$$
\frac{1}{2}\frac{d}{dt}\|\dot{S}_\nu\Lambda^\sigma(a,u)(t)\|_{L^2}^2+\int_{\mathbb{R}^d}\Big(\mu|\nabla\dot{S}_\nu\Lambda^\sigma u|^2+(\mu+\lambda)(\mathrm{div}\,\dot{S}_\nu\Lambda^\sigma u)^2\Big)dx=0. \tag{3.26}
$$

To capture the dissipation of $\dot{S}_\nu\Lambda^\sigma a$, it follows from (3.25) that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}\dot{S}_\nu\Lambda^\sigma u\cdot\nabla\dot{S}_\nu\Lambda^\sigma a\;dx+\|\nabla\dot{S}_\nu\Lambda^\sigma a\|_{L^2}^2\\
&\quad-\|\mathrm{div}\,\dot{S}_\nu\Lambda^\sigma u\|_{L^2}^2-\int_{\mathbb{R}^d}\mathcal{A}\dot{S}_\nu\Lambda^\sigma u\cdot\nabla\dot{S}_\nu\Lambda^\sigma a\;dx=0.
\end{aligned} \tag{3.27}
$$

Define

$$
\mathcal{L}(t)\triangleq\frac{1}{2}\|\dot{S}_\nu\Lambda^\sigma(a,u)(t)\|_{L^2}^2+\varepsilon\int_{\mathbb{R}^d}\dot{S}_\nu\Lambda^\sigma u\cdot\nabla\dot{S}_\nu\Lambda^\sigma a\;dx
$$

for some constant $\varepsilon>0$. We are able to choose ε sufficiently small such that

$$
\mathcal{L}(t)\sim\|\dot{S}_\nu(a,u)(t)\|_{\dot{H}^\sigma}^2. \tag{3.28}
$$

Furthermore, combining with (3.26)-(3.27), we obtain

$$\frac{d}{dt}\mathcal{L}(t)+c_*\|\nabla\dot{S}_\nu(a,u)\|_{\dot{H}^\sigma}^2\leq 0, \tag{3.29}$$

where $c_*>0$ is a uniform constant. For any function $\mathcal{R}(t)$, the classical Fourier splitting idea [44] is then used to deduce the estimate

$$\begin{aligned}\|\nabla\dot{S}_\nu(a,u)\|_{\dot{H}^\sigma}^2&\geq\int_{\{|\xi|\geq\mathcal{R}(t)\}}|\xi|^{2+2\sigma}(|\widehat{\dot{S}_\nu a}|^2+|\widehat{\dot{S}_\nu u}|^2)d\xi\\&\geq\mathcal{R}^2(t)\Big(\int_{\mathbb{R}^d}|\xi|^{2\sigma}(|\widehat{\dot{S}_\nu a}|^2+|\widehat{\dot{S}_\nu u}|^2)d\xi-\int_{\{|\xi|\leq\mathcal{R}(t)\}}|\xi|^{2\sigma}(|\widehat{\dot{S}_\nu a}|^2+|\widehat{\dot{S}_\nu u}|^2)d\xi\Big)\\&\geq\mathcal{R}^2(t)\mathcal{L}(t)-\mathcal{R}^{2\sigma}(t)\int_{\{|\xi|\leq\mathcal{R}(t)\}}|\xi|^{2\sigma}(|\widehat{\dot{S}_\nu a}|^2+|\widehat{\dot{S}_\nu u}|^2)d\xi.\end{aligned}$$

This, together with (3.29), leads to

$$\frac{d}{dt}\mathcal{L}(t)+c_*\mathcal{R}^2(t)\mathcal{L}(t)\lesssim\mathcal{R}^2(t)\int_{\{|\xi|\leq\mathcal{R}(t)\}}|\xi|^{2\sigma}(|\widehat{a}(\xi,t)|^2+|\widehat{u}(\xi,t)|^2)d\xi. \tag{3.30}$$

For some sufficiently large constant β, choosing now

$$\mathcal{R}(t)=\frac{\beta}{c_*}\langle t\rangle^{-\frac{1}{2}}\leq\min\{\frac{2}{\nu},r_0\}\quad\text{for}\quad t>t_1^*=\Big(\frac{\beta}{c_*\min\{\frac{2}{\nu},r_0\}}\Big)^2.$$

Hence, it follows from (3.4) and (3.24) that

$$\begin{aligned}\int_{\{|\xi|\leq\mathcal{R}(t)\}}|\xi|^{2\sigma}(|\widehat{a}(\xi,t)|^2+|\widehat{u}(\xi,t)|^2)d\xi&\lesssim\int_{\{|\xi|\leq\mathcal{R}(t)\}}|\xi|^{2\sigma}(|\widehat{a}_0(\xi)|^2+|\widehat{u}_0(\xi)|^2)d\xi\\&\lesssim\mathcal{R}^{2(\sigma-\sigma_1)}(t)\sim\langle t\rangle^{-(\sigma-\sigma_1)},\qquad t>t_1^*.\end{aligned} \tag{3.31}$$

Multiplying (3.30) by the factor $\langle t\rangle^\beta$ with $\beta>\sigma-\sigma_1+1$, furthermore, we obtain

$$\frac{d}{dt}\Big(\langle t\rangle^\beta\mathcal{L}(t)\Big)\lesssim\langle t\rangle^{\beta-1-(\sigma-\sigma_1)}. \tag{3.32}$$

Then integrating (3.32) over $[t_1^*,t]$ yields

$$\langle t\rangle^\beta\|\dot{S}_\nu(a,u)(t)\|_{\dot{H}^\sigma}^2\lesssim\langle t_1^*\rangle^\beta\|\dot{S}_\nu(a,u)(t_1^*)\|_{\dot{H}^\sigma}^2+\langle t\rangle^{\beta-(\sigma-\sigma_1)}.$$

On the other hand, one deduces from the pointwise estimates (3.4) and Parseval's theorem that

$$\|\dot{S}_\nu(a,u)(t_1^*)\|_{\dot{H}^\sigma}^2\lesssim\|(a_0,u_0)\|_{\dot{H}^\sigma}^2.$$

Therefore, we obtain

$$\|\dot{S}_\nu(a,u)(t)\|_{\dot{H}^\sigma}\lesssim\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)},\quad t>0.$$

Consequently, noticing that the exponential decay property in (3.4) at high frequencies, the upper bound of decay follows that

$$\begin{aligned}\|(a,u)(t)\|_{\dot{H}^\sigma}&\leq\|\dot{S}_\nu(a,u)(t)\|_{\dot{H}^\sigma}+\|(\mathrm{Id}-\dot{S}_\nu)(a,u)(t)\|_{\dot{H}^\sigma}\\&\lesssim\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}+e^{-R_*t}\|(a_0,u_0)\|_{\dot{B}^\sigma_{2,1}}\\&\lesssim\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}.\end{aligned} \tag{3.33}$$

Finally, performing the same procedure leading to (3.14)-(3.16), we arrive at $(a_0, u_0) \in \dot{B}^{\sigma_1}_{2,\infty}$.

On the other hand, it follows from (3.5), (3.24) and Parseval's theorem that

$$\begin{aligned}\|(a,u)(t)\|^2_{\dot{B}^{\sigma}_{2,1}} &\gtrsim \|\Lambda^{\sigma}(a,u)(t)\|^2_{L^2}\\ &\gtrsim e^{-2\max\{\frac{\nu}{2},\mu\}r^2t}\int_{\{|\xi|\leq r\}}|\xi|^{2\sigma}(|\widehat{a}_0(\xi)|^2+|\widehat{u}_0(\xi)|^2)d\xi \\ &\gtrsim r^{2(\sigma-\sigma_1)}\sim\langle t\rangle^{-(\sigma-\sigma_1)},\end{aligned} \tag{3.34}$$

where we have chosen $r = r_0\langle t\rangle^{-\frac{1}{2}} \leq r_0$. According to Proposition 3.1 and (3.34), we therefore prove $(a_0, u_0) \in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$. The proof of Proposition 3.2 is complete. □

By employing a similar argument in the proof of Propositions 3.1-3.2, one can present the sharp decay characterization with the Besov regularity for a large class of dissipative systems (including incompressible Stokes flows) studied in [2,3,38].

Corollary 3.1. *Let $\sigma, \sigma_1 \in \mathbb{R}$ such that $\sigma > \sigma_1$. Assume that U satisfies System (1.17) and $U_0 \in \dot{B}^{\sigma}_{2,1}$. Then the following three statements are equivalent:*

(1) $U_0 \in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$;

(2) U_0 *satisfies*

$$\begin{cases} P_{\sigma_1}(U_0)_+ \triangleq \limsup\limits_{r\to0^+} r^{-2(\sigma-\sigma_1)}\displaystyle\int_{\{|\xi|\leq r\}}|\xi|^{2\sigma}|\widehat{U}_0(\xi)|^2d\xi<\infty,\\ P_{\sigma_1}(U_0)_- \triangleq \liminf\limits_{r\to0^+} r^{-2(\sigma-\sigma_1)}\displaystyle\int_{\{|\xi|\leq r\}}|\xi|^{2\sigma}|\widehat{U}_0(\xi)|^2d\xi>0;\end{cases}$$

(3) *For any $t_L \geq 0$, U has upper and lower bounds of time-decay:*

$$\langle t\rangle^{-\frac{1}{2\alpha}(\sigma-\sigma_1)} \lesssim \|U(t)\|_{\dot{B}^{\sigma}_{2,1}} \lesssim \langle t\rangle^{-\frac{1}{2\alpha}(\sigma-\sigma_1)}, \quad t > t_L.$$

4 Sufficient condition

It suffices to show that the solution constructed in Theorem 1.1 satisfies (1.13) (resp. (1.14)) if and only if $(a_0, u_0) \in \dot{B}^{\sigma_1}_{2,\infty}$ (resp. $(a_0, u_0) \in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$), since (1.12) is the direct consequence of Lemma 5.1 in [51]. In this section, we shall develop Wiegner's argument from incompressible Navier-Stokes equations to compressible Navier-Stokes equations, and prove the "if" part. Compared with the classical works [17,52], the additional smallness of low frequencies is no longer needed in the Fourier semi-group framework.

4.1 Wiegner's argument for compressible Navier-Stokes equations

Our argument depends on the momentum formulation of compressible Navier-Stokes equations (1.1) and is to establish the decay estimate of difference $(\widetilde{a},\widetilde{m}) \triangleq (a-a_L, m-m_L)$, where (a_L,m_L) is the solution to the linear problem (1.23) subject to the initial data (a_0,m_0) with $m_0=(1+a_0)u_0$. First of all, we have the following sharp decay characterization of (a_L,m_L).

Proposition 4.1. *Let p satisfy (1.8). It holds that*

$$\begin{aligned}\|(a_L,m_L)\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|e^{R\tau}(\nabla a_L,m_L)\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|e^{R\tau}m_L\|^{h}_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})}\\ +\|(a_L,m_L)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}+\|(a_L,\nabla m_L)\|^{h}_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim \mathcal{X}_{p,0},\end{aligned} \tag{4.1}$$

where $R>0$ is some constant and $\mathcal{X}_{p,0}$ is defined by (1.7). Moreover, if assume that $\sigma_0\leq\sigma_1<\frac{d}{2}-1$ and $\sigma>\sigma_1$, then for any $t_L\geq 0$ the following decay properties hold:

$$\|(a_L,m_L)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}}\lesssim\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)},\quad t>t_L, \tag{4.2}$$

if and only if $(a_0,u_0)^{\ell}\in\dot{B}^{\sigma_1}_{2,\infty}$;

$$\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}\lesssim\|(a_L,m_L)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}}\lesssim\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)},\quad t>t_L, \tag{4.3}$$

if and only if $(a_0,u_0)^{\ell}\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$.

Proof. By employing the same procedure as in [15] (see pages 1882-1884), one can arrive at

$$\begin{aligned}&\|(a_L,m_L)\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|(\nabla a_L,m_L)\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|(a_L,m_L)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}+\|(a_L,\nabla m_L)\|^{h}_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\ &\lesssim\|(a_0,m_0)\|^{\ell}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|(\nabla a_0,m_0)\|^{h}_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\triangleq\mathcal{X}^*_{p,0}.\end{aligned} \tag{4.4}$$

In addition, the L^p energy method developed in [17] implies that there exists a generic constant $R>0$ such that

$$\|e^{R\tau}(\nabla a_L,m_L)\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\lesssim\mathcal{X}^*_{p,0}. \tag{4.5}$$

To establish gain of regularity and decay altogether for the high frequencies of momentum, we reformulate the second equation in (1.23) as follows

$$\partial_t(\chi(t)m_L)-\mathcal{A}(\chi(t)m_L)=\chi'(t)m_L-\chi(t)\nabla a_L, \tag{4.6}$$

where $\chi(t)\in C^1(\mathbb{R}_+)$ satisfies $\chi(t)=t$ for $0\leq t\leq\frac{1}{2}$ and $\chi(t)=e^{Rt}$ for $t>1$. Then it follows from the maximal regularity estimate for Lamé semi-group in Lemma 6.11 that

$$\|\chi(t)m_L\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\|e^{R\tau}(m_L,\nabla a_L)|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\lesssim\mathcal{X}^*_{p,0}, \tag{4.7}$$

where we used the fact $\chi(t)m_L|_{t=0}=0$. Hence, in order to get (4.1), it only need to show that

$$\mathcal{X}^*_{p,0}\lesssim \mathcal{X}_{p,0}. \tag{4.8}$$

Indeed, from Lemma 6.3 and Bernstein's inequality, we arrive at

$$\|m_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}}\lesssim \|u_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}}+\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}}\|u_0\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\lesssim \mathcal{X}_{p,0},$$

where the boundedness of $\mathcal{X}_{p,0}$ in (1.7) is used. On the other hand, bounding $\|m_0\|^\ell_{\dot B^{\frac{d}{2}-1}_{2,1}}$ is a little bit complicated and follows from the similar strategy as in [15]. To this end, we employ the following two inequalities:

$$\|T_f g\|_{\dot B^{s-1+\frac{d}{2}-\frac{d}{p}}_{2,1}}\lesssim \|f\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\|g\|_{\dot B^s_{p,1}}\quad \text{if } d\geq 2 \text{ and } \frac{d}{d-1}\leq p\leq \min(4,d^*), \tag{4.9}$$

$$\|R(f,g)\|_{\dot B^{s-1+\frac{d}{2}-\frac{d}{p}}_{2,1}}\lesssim \|f\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\|g\|_{\dot B^s_{p,1}}\quad \text{if } s>1-\min\Big(\frac{d}{p},\frac{d}{p'}\Big) \text{ and } 1\leq p\leq 4 \tag{4.10}$$

with $1/p+1/p'=1$ and $d^*\triangleq \frac{2d}{d-2}$. By using Bony's para-product decomposition, one has

$$a_0u_0=T_{u_0}a_0+R(u_0,a_0)+T_{a_0}u_0^\ell+T_{a_0}u_0^h. \tag{4.11}$$

Thanks to (4.9) and (4.10) with $s=\frac{d}{p}$, one can get

$$\|T_{u_0}a_0\|^\ell_{\dot B^{\frac{d}{2}-1}_{2,1}}\lesssim \|u_0\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}},\quad \|R(a_0,u_0)\|^\ell_{\dot B^{\frac{d}{2}-1}_{2,1}}\lesssim \|u_0\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}}.$$

Since T maps $L^\infty\times \dot B^{\frac{d}{2}-1}_{2,1}$ to $\dot B^{\frac{d}{2}-1}_{2,1}$, we have

$$\|T_{a_0}u_0^\ell\|^\ell_{\dot B^{\frac{d}{2}-1}_{2,1}}\lesssim \|a_0\|_{L^\infty}\|u_0^\ell\|_{\dot B^{\frac{d}{2}-1}_{2,1}}\lesssim \|a_0\|_{\dot B^{\frac{d}{p}}_{p,1}}\|u_0\|^\ell_{\dot B^{\frac{d}{2}-1}_{2,1}}.$$

In order to handle the last term on the right-side of (4.11), we observe that owing to the spectral cut-off, there exists a universal integer N_0 such that

$$\Big(T_{a_0}u_0^h\Big)^\ell=\dot S_{k_0+1}\Big(\sum_{|j-k_0|\leq N_0}\dot S_{j-1}a_0\dot\Delta_j u_0^h\Big).$$

Hence $\|T_{a_0}u_0^h\|^\ell_{\dot B^{\frac{d}{2}-1}_{2,1}}\approx 2^{k_0(\frac{d}{2}-1)}\sum_{|j-k_0|\leq N_0}\|\dot S_{j-1}a_0\dot\Delta_j u_0^h\|_{L^2}$. If $2\leq p\leq \min(d,d^*)$ then one may use for $|j-k_0|\leq N_0$,

$$2^{k_0(\frac{d}{2}-1)}\|\dot S_{j-1}a_0\dot\Delta_j u_0^h\|_{L^2}\lesssim \|\dot S_{j-1}a_0\|_{L^d}\Big(2^{j(\frac{d}{d^*}-1)}\|\dot\Delta_j u_0^h\|_{L^{d^*}}\Big)\lesssim \|a_0\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\|u_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}},$$

and if $d\leq p\leq 4$, then it holds that

$$\begin{aligned}2^{k_0(\frac{d}{2}-1)}\|\dot S_{j-1}a_0\dot\Delta_j u_0^h\|_{L^2}&\lesssim \Big(2^{j\frac{d}{4}}\|\dot S_{j-1}a_0\|_{L^4}\Big)\Big(2^{j(\frac{d}{4}-1)}\|\dot\Delta_j u_0^h\|_{L^4}\Big)\\&\lesssim \Big(2^{j(\frac{d}{p}-1)}\|a_0\|_{L^p}\Big)\Big(2^{j(\frac{d}{p}-1)}\|\dot\Delta_j u_0^h\|_{L^p}\Big)\lesssim \|a_0\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\|u_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}}.\end{aligned}$$

Hence, the inequality (4.8) follows directly by combining above estimates.

Under the assumption (1.7), we claim that $(a_0, m_0)^\ell \in \dot{B}^{\sigma_1}_{2,\infty} \Longleftrightarrow (a_0, u_0)^\ell \in \dot{B}^{\sigma_1}_{2,\infty}$ for $\sigma_0 \le \sigma_1 < \frac{d}{2}-1$. Indeed, it follows from $m_0 = u_0 + a_0u_0$ that

$$\|m_0^\ell\|_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \|u_0^\ell\|_{\dot{B}^{\sigma_1}_{2,\infty}} + \|a_0u_0\|^\ell_{\dot{B}^{\sigma_1}_{2,\infty}}. \tag{4.12}$$

It is convenient to decompose the product a_0u_0 in terms of low-frequency and high-frequency parts: $a_0u_0 = a_0u_0^\ell + a_0u_0^h$. According to Lemma 6.7, we arrive at

$$\|a_0u_0^\ell\|^\ell_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \|a_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|u_0^\ell\|_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \Big(\|a_0^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a_0^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)\|u_0^\ell\|_{\dot{B}^{\sigma_1}_{2,\infty}}. \tag{4.13}$$

Bounding $a_0u_0^h$ is divided into cases $2 \le p \le d$ and $p > d$. If $2 \le p < d$, then (6.8) with $\sigma = \frac{d}{p} - 1$ yields

$$\|a_0u_0^h\|^\ell_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \|a_0u_0^h\|^\ell_{\dot{B}^{\sigma_0}_{2,\infty}} \lesssim \Big(\|a_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|a_0^\ell\|_{L^{p^*}}\Big)\|u_0^h\|_{\dot{B}^{1-\frac{d}{p}}_{p,1}}, \tag{4.14}$$

since $\sigma_0 \le \sigma_1$. In the limit case $p = d$, one can get by the Sobolev embedding that

$$\|a_0u_0^h\|^\ell_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \|a_0u_0^h\|^\ell_{\dot{B}^{\sigma_0}_{2,\infty}} \lesssim \|a_0u_0^h\|_{L^{\frac{d}{2}}} \le \|a_0\|_{L^d}\|u_0^h\|_{L^d} \lesssim \|a_0\|_{\dot{B}^0_{d,1}}\|u_0^h\|_{\dot{B}^0_{d,1}}. \tag{4.15}$$

Furthermore, combining (4.14)-(4.15) and using the embeddings $\dot{B}^{\frac{d}{2}}_{2,1} \hookrightarrow L^{p^*}$ and $\dot{B}^{\frac{d}{p}-1}_{p,1} \hookrightarrow \dot{B}^0_{d,1}$, we obtain

$$\|a_0u_0^h\|^\ell_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \Big(\|a_0^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a_0^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)\|u_0^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}, \tag{4.16}$$

due to the fact $\frac{d}{2} - 1 \le \frac{d}{p}$ and $1 - \frac{d}{p} \le \frac{d}{p} - 1$. If $p > d$, applying (6.8) with $\sigma = 1 - \frac{d}{p}$ once again implies that

$$\|a_0u_0^h\|^\ell_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \Big(\|a_0\|_{\dot{B}^{1-\frac{d}{p}}_{p,1}} + \|a_0^\ell\|_{L^{p^*}}\Big)\|u_0^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}. \tag{4.17}$$

By using the embedding $\dot{B}^{\frac{d}{2}}_{2,1} \hookrightarrow \dot{B}^{\frac{d}{p}}_{2,1} \hookrightarrow L^{p_*}$ in low frequencies and the fact $\frac{d}{2} - 1 < 1 - \frac{d}{p} \le \frac{d}{p}$ owing to $p > d$, we obtain

$$\|a_0u_0^h\|^\ell_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \Big(\|a_0^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a_0^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)\|u_0^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}. \tag{4.18}$$

Together with (4.12), (4.13), (4.16) and (4.18), we conclude that the “if" part of this claim is true. Conversely, the proof of “only if" part follows from the similar procedure if noticing that $u_0 = m_0 + I(a_0)m_0$ and using the composite estimate in Lemma 6.9.

Furthermore, it can be shown that $(a_0, m_0)^\ell \in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ if and only if $(a_0, u_0)^\ell \in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$. If $(a_0, m_0)^\ell \in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$, then it follows from (4.12)-(4.13), (4.16) and (4.18) that

$$
\begin{aligned}
&2^{\sigma_1 j_k}\|\dot{\Delta}_{j_k}(a_0,u_0)^\ell\|_{L^2}\\
&\quad\geq 2^{\sigma_1 j_k}\|\dot{\Delta}_{j_k}(a_0,m_0)^\ell\|_{L^2}-\|a_0u_0\|^\ell_{\dot{B}^{\sigma_1}_{2,\infty}}\\
&\quad\geq 2^{\sigma_1 j_k}\|\dot{\Delta}_{j_k}(a_0,m_0)^\ell\|_{L^2}-C(\|a_0^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|a_0^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}})(\|u_0^\ell\|_{\dot{B}^{\sigma_1}_{2,\infty}}+\|u_0^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}})\\
&\quad\geq c_0-C\mathcal{X}_{p,0}-C\mathcal{X}_{p,0}^2\geq\frac{c_0}{2}>0,
\end{aligned}
$$

which implies that $(a_0,u_0)^\ell\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$, where the sequence $\{j_k\}_{k=1,2,...}$ comes from the definition (1.11) of $\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ and the smallness assumption in (1.7) has been used. Similarly, one can prove that $(a_0,u_0)^\ell\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ implies that $(a_0,m_0)^\ell\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$.

Applying the low-frequency cut-off operator $\dot{S}_{j_0}$ to (1.23) gives

$$
\begin{cases}
\partial_t a_L^\ell+\operatorname{div} m_L^\ell=0,\\
\partial_t m_L^\ell-\mathcal{A}m_L^\ell+\nabla a_L^\ell=0,\\
(a_L^\ell,m_L^\ell)(x,0)=(a_0^\ell,m_0^\ell)(x).
\end{cases}
\tag{4.19}
$$

Note that $\|(a_0,m_0)^\ell\|_{\dot{B}^{\sigma}_{2,1}}\lesssim\|(a_0,m_0)^\ell\|_{\dot{B}^{\sigma_1}_{2,\infty}}$ with $\sigma>\sigma_1$, the upper bound (4.2) and two-sided bounds (4.3) hold for $t>t_L$, respectively, according to Proposition 3.1. □

From Theorem 1.1, we see that the Cauchy problem (1.21) with initial data $(a,m)|_{t=0}=(a_0,(1+a_0)u_0)$ admits the global-in-time unique solution (a,m) with $m=(1+a)u$ satisfying

$$
\begin{aligned}
(a,m)^\ell\in\mathcal{C}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{2}+1}_{2,1}),\quad &a^h\in\mathcal{C}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{p}}_{p,1})\cap L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{p}}_{p,1}),\\
&m^h\in\mathcal{C}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{p}-1}_{p,1})\cap\tilde{L}^2(\mathbb{R}_+;\dot{B}^{\frac{d}{p}}_{p,1}).
\end{aligned}
\tag{4.20}
$$

For the case of compressible fluids (1.21), we get the following analogue of Wiegner's theorem (see [49]).

Proposition 4.2. *Assume that the initial data satisfy* $(a_0,u_0)^\ell\in\dot{B}^{\sigma_1}_{2,\infty}$ *with* $\sigma_0\leq\sigma_1<\frac{d}{2}-1$. *Then the difference* $(\widetilde{a},\widetilde{m})\triangleq(a-a_L,m-m_L)$ *fulfills the time-weighted inequality*

$$
\widetilde{\mathcal{D}}_p(t)\lesssim 1
\tag{4.21}
$$

for $t>0$, *where the difference functional* $\widetilde{\mathcal{D}}_p(t)$ *is defined as*

$$
\widetilde{\mathcal{D}}_p(t)\triangleq\sup_{\sigma_1<\sigma<\frac{d}{2}}\|\langle\tau\rangle^{\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}(\widetilde{a},\widetilde{m})\|^\ell_{L^\infty_t(\dot{B}^{\sigma}_{2,1})}+\|\langle\tau\rangle^{\alpha_*}(\nabla\widetilde{a},\widetilde{m})\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|\tau^{\alpha_*}\widetilde{m}\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})}
$$

with $\alpha_*=\frac{1}{2}(\frac{d}{2}-\sigma_1+\sigma_2)-$ *and the number* $\sigma_2\in(0,1]$ *given by*

$$
\sigma_2=\begin{cases}
1, & \textit{if}\quad \sigma_1<\sigma\leq\frac{d}{2}-1,\quad \sigma_1<\frac{d}{2}-2,\\
1-, & \textit{if}\quad \sigma_1<\sigma\leq\frac{d}{2}-1,\quad \sigma_1=\frac{d}{2}-2,\\
\frac{d}{2}-1-\sigma_1, & \textit{if}\quad \sigma_1<\sigma\leq\frac{d}{2}-1,\quad \frac{d}{2}-2<\sigma_1<\frac{d}{2}-1,\\
\min\{\frac{1}{2},(\frac{d}{2}-1-\sigma_1)-\}, & \textit{if}\quad \frac{d}{2}-1<\sigma<\frac{d}{2}.
\end{cases}
\tag{4.22}
$$

4.1.1 Bounds for the low frequencies

Let us keep in mind that due to product laws on $m=u+au$, the global solution (a,m) in Theorem 1.1 satisfies

$$\|a\|_{\widetilde{L}^\infty_t(\dot B^{\frac{d}{p}}_{p,1})}+\|m\|_{\widetilde{L}^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|m\|_{\widetilde{L}^2_t(\dot B^{\frac{d}{p}}_{p,1})}\lesssim \mathcal{X}_{p,0}\ll 1 \quad \text{for all} \quad t>0. \tag{4.23}$$

As shown by Proposition 4.1, the assumption $(a_0,u_0)^\ell\in \dot B^{\sigma_1}_{2,\infty}$ is equivalent to the upper bound of decay of solutions to (1.23):

$$\|(a_L,m_L)^\ell(t)\|_{\dot B^{\sigma}_{2,1}}\lesssim \langle t\rangle^{-\frac12(\sigma-\sigma_1)},\quad \sigma>\sigma_1,\quad t>0. \tag{4.24}$$

In what follows, we shall use repeatedly that for $0\le\gamma_1\le\gamma_2$,

$$\int_0^t\langle t-\tau\rangle^{-\gamma_1}\langle\tau\rangle^{-\gamma_2}d\tau\lesssim\begin{cases}\langle t\rangle^{-\gamma_1}, & \text{if}\quad \gamma_2>1,\\ \langle t\rangle^{-(\gamma_1-)}, & \text{if}\quad \gamma_2=1,\\ \langle t\rangle^{-\gamma_1-\gamma_2+1}, & \text{if}\quad \gamma_2<1.\end{cases} \tag{4.25}$$

Apply $\dot\Delta_j$ to the difference system (1.24). It follows from Lemma 3.1 and Duhamel's principle that

$$\|\dot\Delta_j(\widetilde a,\widetilde m)(t)\|_{L^2}\lesssim\int_0^t e^{-R_3 2^{2j}(t-\tau)}2^j\|\dot\Delta_jF\|_{L^2}d\tau \tag{4.26}$$

for $j\le j_0\in\mathbb{Z}$ and $R_3=\max\{R_*,\frac{4}{\nu^2 2^{2j_0}}\min\{\frac1\nu,\mu\}\}$. It is easy to see that ($\sigma'\in(0,1]$)

$$\begin{aligned}t^{\frac{\sigma-\sigma_1+\sigma'}{2}}&\sum_{j\le j_0}2^{j(\sigma-\sigma_1+\sigma')}2^{j(\sigma_1+1-\sigma')}e^{-R_3 2^{2j}t}\|\dot\Delta_jF\|_{L^2}\\ &\lesssim\|F\|^\ell_{\dot B^{\sigma_1+1-\sigma'}_{2,\infty}}\sum_{j\in\mathbb{Z}}(\sqrt{t}2^j)^{\sigma-\sigma_1+\sigma'}e^{-R_3 2^{2j}t}\lesssim\|F\|^\ell_{\dot B^{\sigma_1+1-\sigma'}_{2,\infty}}\end{aligned}$$

and

$$\sum_{j\le j_0}2^{j(\sigma-\sigma_1+\sigma')}2^{j(\sigma_1+1-\sigma')}e^{-R_3 2^{2j}t}\|\dot\Delta_jF\|_{L^2}\lesssim\|F\|^\ell_{\dot B^{\sigma_1+1-\sigma'}_{2,\infty}}\sum_{j\le j_0}2^{j(\sigma-\sigma_1+\sigma')}\lesssim\|F\|^\ell_{\dot B^{\sigma_1+1-\sigma'}_{2,\infty}}$$

for $\sigma-\sigma_1+\sigma'>0$, where we used the series inequalities ($s>0$):

$$\sum_{j\le j_0}2^{js}\le C_s,\qquad \sup_{t\ge0}\sum_{j\in\mathbb{Z}}t^{\frac s2}2^{js}e^{-c_0 2^{2j}t}\le C_s.$$

Consequently, we get

$$\|(\widetilde a,\widetilde m)(t)\|^\ell_{\dot B^{\sigma}_{2,1}}\lesssim\int_0^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+\sigma')}\|F\|^\ell_{\dot B^{\sigma_1+1-\sigma'}_{2,\infty}}d\tau,\qquad \sigma>\sigma_1. \tag{4.27}$$

Regarding the integral on right-hand side of (4.27), we consider cases $0<t\le2$ and $t>2$ separately.

Lemma 4.1. *Let p satisfy (1.8) and $\sigma_0 \le \sigma_1 < \frac{d}{2} - 1$. It holds that*

$$\begin{aligned} &\int_0^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma')}\|F\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma'}_{2,\infty}}d\tau \\ \lesssim\quad & \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}(\mathcal{X}_{p,0}+\mathcal{X}^2_{p,0})(\|(a,m)^{\ell}\|_{L^\infty_t(\dot{B}^{\sigma_1}_{2,\infty})}+\mathcal{X}_{p,0}) \end{aligned} \tag{4.28}$$

for $0 < t \le 2$ and $\sigma', \sigma_2 \in (0, 1]$.

Proof. The case $0 < t \le 2$ implies that $\langle t\rangle \approx 1$ and $\langle t-\tau\rangle \approx 1$ for $0 \le \tau \le t \le 2$. Set

$$F = F^{\ell} + F^h$$

with

$$F^{\ell} = F^{\ell}_1 + F^{\ell}_2 + F^{\ell}_3, \qquad F^h = F^h_1 + F^h_2 + F^h_3,$$

where

$$F^{\ell}_1 = (1-I(a))m\otimes m^{\ell}, \quad F^{\ell}_2 = (P''(1)+G(a))aa^{\ell}\mathbb{I}_d, \quad F^{\ell}_3 = \mu\nabla(I(a)m^{\ell}) + (\mu+\lambda)\mathrm{div}\,(I(a)m^{\ell})\mathbb{I}_d,$$

$$F^h_1 = (1-I(a))m\otimes m^h, \quad F^h_2 = (P''(1)+G(a))aa^h\mathbb{I}_d, \quad F^h_3 = \mu\nabla(I(a)m^h) + (\mu+\lambda)\mathrm{div}\,(I(a)m^h)\mathbb{I}_d.$$

Here $G(a)$ satisfies $G(0) = 0$ and $(P''(1) + G(a))a^2 = P(1 + a) - P(1) - P'(1)a$. Due to $\frac{d}{2} - 1 \le \frac{d}{p}$, it follows from the lemmas 6.7 and 6.9 that

$$\begin{aligned} \|F^{\ell}_1\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma'}_{2,\infty}} &\lesssim \|F^{\ell}_1\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \|F^{\ell}_1\|^{\ell}_{\dot{B}^{\sigma_1+\frac{d}{p}-\frac{d}{2}}_{2,\infty}} \lesssim (1+\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\|m\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|m^{\ell}\|_{\dot{B}^{\sigma_1+\frac{d}{p}-\frac{d}{2}+1}_{2,\infty}} \\ &\lesssim (1+\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\|m\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|m^{\ell}\|_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim (\mathcal{X}_{p,0}+\mathcal{X}^2_{p,0})\|m^{\ell}\|_{L^\infty_t(\dot{B}^{\sigma_1}_{2,\infty})}. \end{aligned}$$

Similarly, one also has

$$\|F^{\ell}_2\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma'}_{2,\infty}} \lesssim (1+\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|a^{\ell}\|_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim (\mathcal{X}_{p,0}+\mathcal{X}^2_{p,0})\|a^{\ell}\|_{L^\infty_t(\dot{B}^{\sigma_1}_{2,\infty})}$$

and

$$\|F^{\ell}_3\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma'}_{2,\infty}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|m^{\ell}\|_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \mathcal{X}_{p,0}\|m^{\ell}\|_{L^\infty_t(\dot{B}^{\sigma_1}_{2,\infty})}.$$

To limit the term with $m\otimes m^h$, we use a similar procedure leading to (4.16) and (4.18). More precisely, if $2 \le p < d$, then (6.8) with $\sigma = \frac{d}{p} - 1$ yields

$$\|m\otimes m^h\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \Big(\|m\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|m^{\ell}\|_{L^{p^*}}\Big)\|m^h\|_{\dot{B}^{1-\frac{d}{p}}_{p,1}}. \tag{4.29}$$

In the limit case $p = d$, one can get

$$\|m\otimes m^h\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \|m\otimes m^h\|^{\ell}_{\dot{B}^{\sigma_0}_{2,\infty}} \lesssim \|m\otimes m^h\|_{L^{\frac{d}{2}}} \lesssim \|m\|_{\dot{B}^0_{d,1}}\|m^h\|_{\dot{B}^0_{d,1}}. \tag{4.30}$$

Employing the embeddings $\dot{B}^{\frac{d}{p}}_{2,1} \hookrightarrow L^{p^*}$ and $\dot{B}^{\frac{d}{p}-1}_{p,1} \hookrightarrow \dot{B}^{0}_{d,1}$ in (4.29)-(4.30) gives

$$\|m\otimes m^h\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \Big(\|m^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|m^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\Big)\|m^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \lesssim \mathcal{X}^2_{p,0}, \tag{4.31}$$

due to the fact that $\frac{d}{2}-1 \leq \frac{d}{p}$ and $1-\frac{d}{p} \leq \frac{d}{p}-1$. If $p>d$, applying (6.8) with $\sigma = 1-\frac{d}{p}$ again implies that

$$\|m\otimes m^h\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \Big(\|m\|_{\dot{B}^{1-\frac{d}{p}}_{p,1}} + \|m^\ell\|_{L^{p^*}}\Big)\|m^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}. \tag{4.32}$$

By using the embedding $\dot{B}^{1+\sigma_0}_{2,1} \hookrightarrow \dot{B}^{1-\frac{d}{p}}_{p,1}$ and $\frac{d}{2}-1 < 1+\sigma_0$ owing to $p>d$, we obtain

$$\|m\otimes m^h\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \Big(\|m^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|m^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)\|m^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \lesssim \mathcal{X}^2_{p,0}. \tag{4.33}$$

On the other hand, using Lemma 6.9 and Bony's decomposition, we follow from those lines of bounding (4.11) and arrive at

$$\|(I(a)m)^\ell\|_{L^{p^*}} \lesssim \|(I(a)m)^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big(\|m^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|m^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\Big), \tag{4.34}$$

and one can thus bound the term corresponding to $I(a)m\otimes m^h$ as $m\otimes m^h$. Consequently, we deduce that

$$\|F^h_1\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma'}_{2,\infty}} \lesssim \|F^h_1\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim (1+\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\Big(\|m^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|m^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)\|m^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \lesssim (1+\mathcal{X}_{p,0})\mathcal{X}^2_{p,0}. \tag{4.35}$$

In order to bound the term with F^h_2, we mimic the proof of (4.35) and get

$$\|F^h_2\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma'}_{2,\infty}} \lesssim (1+\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\Big(\|a^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)\|a^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \lesssim (1+\mathcal{X}_{p,0})\mathcal{X}^2_{p,0}. \tag{4.36}$$

Using the composition inequality in Lebesgue spaces and the embeddings $\dot{B}^{\frac{d}{p}}_{2,1} \hookrightarrow \dot{B}^{\sigma_0}_{p,1} \hookrightarrow L^{p^*}$, we get

$$\|I(a)^\ell\|_{L^{p^*}} \lesssim \|a\|_{L^{p^*}} \lesssim \|a^\ell\|_{\dot{B}^{\frac{d}{p}}_{2,1}} + \|a^h\|_{\dot{B}^{\sigma_0}_{p,1}} \lesssim \|a^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}. \tag{4.37}$$

Consequently, we have

$$\|F^h_3\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma'}_{2,\infty}} \lesssim \Big(\|a^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)\|m^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \lesssim \mathcal{X}^2_{p,0}. \tag{4.38}$$

Therefore, the proof of Lemma 4.1 is complete. □

For the nontrivial case $t>2$, we shall proceed differently depending on whether $\sigma_1 < \sigma \leq \frac{d}{2}-1$ or $\frac{d}{2}-1 < \sigma < \frac{d}{2}$. For the case $\sigma_1 < \sigma \leq \frac{d}{2}-1$, we choose $\sigma'=1$ in (4.27) and have the following lemma.

Lemma 4.2. *Let p satisfy (1.8) and $\sigma_0 \leq \sigma_1 < \frac{d}{2}-1$. It holds that*

$$\begin{aligned}\int_0^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+1)}\|F\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}}\,d\tau \lesssim{}& \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}(\mathcal{X}_{p,0}+\mathcal{X}^2_{p,0})(\|(a,m)^\ell\|_{L^\infty_1(\dot{B}^{\sigma_1}_{2,\infty})}+\mathcal{X}_{p,0})\\ &+\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}\Big((1+\mathcal{X}_{p,0})(\|(a_L,m_L,a,m)^\ell\|_{L^\infty_t(\dot{B}^{\sigma_1+\sigma_2}_{2,\infty})}+\mathcal{X}_{p,0})\widetilde{\mathcal{D}}_p(t)\Big)\end{aligned} \tag{4.39}$$

for $t>2$ *and* $\sigma_1<\sigma\le \frac{d}{2}-1$, *where*

$$\sigma_2=\begin{cases}1, & \text{if} \quad \sigma_1<\frac{d}{2}-2,\\ 1-, & \text{if} \quad \sigma_1=\frac{d}{2}-2,\\ \frac{d}{2}-1-\sigma_1, & \text{if} \quad \frac{d}{2}-2<\sigma_1<\frac{d}{2}-1.\end{cases}$$

Proof. For $t>2$, we write

$$\begin{aligned}&\int_0^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\|F\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}\,d\tau\\ =\;&\int_0^1\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\|F\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}\,d\tau+\int_1^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\|F\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}\,d\tau.\end{aligned}\tag{4.40}$$

It follows from the same computations in Lemma 4.1 that

$$\int_0^1\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\|F\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}\,d\tau\lesssim\langle t\rangle^{-\frac12(\sigma-\sigma_1+\sigma_2)}(\mathcal{X}_{p,0}+\mathcal{X}_{p,0}^2)(\|(a,m)^{\ell}\|_{L^\infty_1(\dot B^{\sigma_1}_{2,\infty})}+\mathcal{X}_{p,0}).$$

To handle with the second integral on $[1,t]$ for $t>2$, we decompose F in terms of linear part and difference part of solutions:

$$F=F_L+\widetilde F_L+\widetilde F$$

with

$$\begin{aligned}F_L&\triangleq(1-I(a))m_L\otimes m_L+(P''(1)+G(a))a_L^2\mathbb{I}_d\\ &\quad+\mu\nabla((1+\tilde I(a))a_Lm_L)+(\mu+\lambda)\mathrm{div}\,((1+\tilde I(a))a_Lm_L)\triangleq F_{1L}+F_{2L}+F_{3L},\end{aligned}\tag{4.41}$$

$$\begin{aligned}\widetilde F_L&\triangleq(1-I(a))\widetilde m\otimes m_L+(P''(1)+G(a))\widetilde a a_L\mathbb{I}_d\\ &\quad+\mu\nabla((1+\tilde I(a))\widetilde a m_L)+(\mu+\lambda)\mathrm{div}\,((1+\tilde I(a))\widetilde a m_L)\triangleq\widetilde F_{1L}+\widetilde F_{2L}+\widetilde F_{3L},\end{aligned}\tag{4.42}$$

$$\begin{aligned}\widetilde F&\triangleq(1-I(a))m\otimes\widetilde m+(P''(1)+G(a))a\widetilde a\mathbb{I}_d\\ &\quad+\mu\nabla((1+\tilde I(a))a\widetilde m)+(\mu+\lambda)\mathrm{div}\,((1+\tilde I(a))a\widetilde m),\end{aligned}\tag{4.43}$$

where $\tilde I(a)$ is a smooth function $\tilde I$ vanishing at zero and satisfies $I(a)=(1+\tilde I(a))a$.

We first claim that

$$\|F_L\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}\lesssim(1+\mathcal{X}_{p,0}+\mathcal{X}_{p,0}^2)\langle t\rangle^{-\frac12(\frac d2-\sigma_1)},\quad t>1.\tag{4.44}$$

Indeed, decompose $m_L\otimes m_L=m_L^{\ell}\otimes m_L^{\ell}+m_L^h\otimes m_L^{\ell}+m_L\otimes m_L^h$. It follows from Lemma 6.3 that

$$\|m_L^{\ell}\otimes m_L^{\ell}\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}\lesssim\|m_L^{\ell}\|_{\dot B^{\frac d2-1}_{2,1}}\|m_L^{\ell}\|_{\dot B^{\sigma_1+1}_{2,\infty}},\tag{4.45}$$

where we used the fact that $\sigma_1<\frac d2-1$ and $\sigma_1+\frac d2\ge d-\frac{2d}{p}\ge0$. Thanks to Lemma 6.5, we get

$$\begin{aligned}\|m_L^h\otimes m_L^{\ell}\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}&\lesssim\|m_L^h\otimes m_L^{\ell}\|^{\ell}_{\dot B^{\frac dp-\frac d2+\sigma_1}_{2,\infty}}\\ &\lesssim\|m_L^h\|_{\dot B^{\frac dp-\frac d2+\sigma_1}_{p,1}}\|m_L^{\ell}\|_{\dot B^{\frac dp}_{2,1}}\lesssim\|m_L^h\|_{\dot B^{\frac dp-1}_{p,1}}\|m_L^{\ell}\|_{\dot B^{\frac d2-1}_{2,1}}.\end{aligned}\tag{4.46}$$

To handle the term corresponding to $m_L \otimes m_L^h$, we observe that applying Lemma 6.8 and tracking those lines from (4.29) to (4.33) yields

$$\|m_L \otimes m_L^h\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \Big(\|m_L^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|m_L^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)\|m_L^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}. \tag{4.47}$$

Hence, in view of (4.1) and (4.2), we deduce that

$$\|m_L \otimes m_L\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim (1+\mathcal{X}_{p,0})\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}, \quad t>1. \tag{4.48}$$

Similarly, we write

$$I(a)m_L \otimes m_L = I(a)m_L^\ell \otimes m_L^\ell + I(a)m_L^h \otimes m_L^\ell + I(a)m_L \otimes m_L^h.$$

Now, arguing as for proving (4.45), it easily follows from Lemmas 6.5 and 6.9 that

$$\|I(a)m_L^\ell \otimes m_L^\ell\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \|I(a)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|m_L^\ell \otimes m_L^\ell\|_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|m_L^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \|m_L^\ell\|_{\dot{B}^{\sigma_1+1}_{2,\infty}}. \tag{4.49}$$

Note also that if $\sigma_0 < \sigma_1 < \frac{d}{2}-1$ and $\frac{d}{2}-1 \leq \frac{d}{p}$, as (4.46), we have

$$\begin{aligned}\|I(a)m_L^h \otimes m_L^\ell\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} &\lesssim \|I(a)m_L^h\|_{\dot{B}^{\frac{d}{p}-\frac{d}{2}+\sigma_1}_{p,1}} \|m_L^\ell\|_{\dot{B}^{\frac{d}{p}}_{2,1}} \\ &\lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|m_L^h\|_{\dot{B}^{\frac{d}{p}-\frac{d}{2}+\sigma_1}_{p,1}} \|m_L^\ell\|_{\dot{B}^{\frac{d}{p}}_{2,1}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|m_L^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \|m_L^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}},\end{aligned} \tag{4.50}$$

where we used the second item of Lemma 6.3 with $\sigma_0+\sigma_1>0$. If $\sigma_1=\sigma_0$, then by (6.4) it holds that

$$\|I(a)m_L^h \otimes m_L^\ell\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \|I(a)m_L^h\|_{\dot{B}^{1-\frac{d}{p}}_{p,1}} \|m_L^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|m_L^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \|m_L^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}. \tag{4.51}$$

Keep in mind that (4.34), one can bound the term corresponding to $I(a)m_L \otimes m_L^h$ as $m_L \otimes m_L^h$. Precisely,

$$\|I(a)m_L \otimes m_L^h\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \Big(\|m_L^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|m_L^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)\|m_L^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}. \tag{4.52}$$

Consequently, by combining (4.49)-(4.51) and (4.1)-(4.2), we obtain

$$\|I(a)m_L \otimes m_L\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim (\mathcal{X}_{p,0}+\mathcal{X}_{p,0}^2)\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}, \quad t>1. \tag{4.53}$$

Bounding F_{2L} and F_{3L} follows from the same arguments as F_{1L}. As a matter of fact, it can be shown that

$$\begin{aligned}\|F_{2L}\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} &\lesssim (1+\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\Big(\|a_L^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|a_L^\ell\|_{\dot{B}^{\sigma_1+1}_{2,\infty}} + \|a_L^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a_L^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a_L^h\|^2_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big) \\ &\lesssim (1+\mathcal{X}_{p,0})^2\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}\end{aligned}$$

and

$$\begin{aligned}&\|F_{3L}\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} \\ &\lesssim (1+\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\Big(\|a_L^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|m_L^\ell\|_{\dot{B}^{\sigma_1+1}_{2,\infty}} + \|a_L^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|m_L^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + (\|a_L^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|a_L^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\|m_L^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\Big) \\ &\lesssim (1+\mathcal{X}_{p,0})^2\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}.\end{aligned}$$

Therefore, the claim (4.44) is proved.

Regarding $\widetilde{F}_L$, our aim is to show that

$$\|(\widetilde{F}_L,\widetilde{F})\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}}\lesssim(1+\mathcal{X}_{p,0})(\|(a,m,a_L,m_L)^{\ell}\|_{\dot{B}^{\sigma_1+\sigma_2}_{2,\infty}}+\mathcal{X}_{p,0})\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}\widetilde{\mathcal{D}}_p(t) \tag{4.54}$$

for $t>1$, where $\sigma_2\in(0,1]$ is to be confirmed.

Firstly, we write $m_L\otimes\widetilde{m}=m_L^{\ell}\otimes\widetilde{m}^{\ell}+m_L^h\otimes\widetilde{m}^{\ell}+m_L\otimes\widetilde{m}^h$. Owing to $\sigma_1+\sigma_2\leq\sigma_1+1<\frac{d}{2}$, it follows from the third item of Lemma 6.3 and the definition of $\widetilde{\mathcal{D}}_p(t)$ that

$$\|m_L^{\ell}\otimes\widetilde{m}^{\ell}\|_{\dot{B}^{\sigma_1}_{2,\infty}}\lesssim\|m_L^{\ell}\|_{\dot{B}^{\sigma_1+\sigma_2}_{2,\infty}}\|\widetilde{m}^{\ell}\|_{\dot{B}^{\frac{d}{2}-\sigma_2}_{2,1}}\lesssim\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}\|m_L^{\ell}\|_{\dot{B}^{\sigma_1+\sigma_2}_{2,\infty}}\widetilde{\mathcal{D}}_p(t). \tag{4.55}$$

Arguing as (4.46), we have

$$\|m_L^h\otimes\widetilde{m}^{\ell}\|_{\dot{B}^{\sigma_1}_{2,\infty}}\lesssim\|m_L^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|\widetilde{m}^{\ell}\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\lesssim\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}\mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t), \tag{4.56}$$

where we used (4.1) and that $\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-1-\sigma_1+\sigma_2)}e^{-Rt}\leq\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}$ in the last inequality. From (4.47), (4.1)-(4.2), we get

$$\|m_L\otimes\widetilde{m}^h\|_{\dot{B}^{\sigma_1}_{2,\infty}}\lesssim(\|m_L^{\ell}\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|m_L^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\|\widetilde{m}^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\lesssim\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}\mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t) \tag{4.57}$$

for $t>1$. Together with (4.55)-(4.57), we thus get

$$\|m_L\otimes\widetilde{m}\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}}\lesssim\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}(\|m_L^{\ell}\|_{\dot{B}^{\sigma_1+\sigma_2}_{2,\infty}}+\mathcal{X}_{p,0})\widetilde{\mathcal{D}}_p(t),\quad t>1. \tag{4.58}$$

Bounding the composite term with $I(a)m_L\otimes\widetilde{m}$ follows essentially from (4.49)-(4.51) and yields

$$\|I(a)\widetilde{m}\otimes m_L\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}}\lesssim\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}(\|m_L^{\ell}\|_{\dot{B}^{\sigma_1+\sigma_2}_{2,\infty}}+\mathcal{X}_{p,0})\mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t),\quad t>1. \tag{4.59}$$

Similarly, we have

$$\begin{aligned}\|\widetilde{F}_{2L}\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}}&\lesssim(1+\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\Big(\|a_L^{\ell}\|_{\dot{B}^{\sigma_1+\sigma_2}_{2,\infty}}\|\widetilde{a}^{\ell}\|_{\dot{B}^{\frac{d}{2}-\sigma_2}_{2,1}}+\|a_L^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\widetilde{a}^{\ell}\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\&\quad+(\|a_L^{\ell}\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|a_L^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\|\widetilde{a}^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\Big)\\&\lesssim\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}(1+\mathcal{X}_{p,0})(\|a_L^{\ell}\|_{\dot{B}^{\sigma_1+\sigma_2}_{2,\infty}}+\mathcal{X}_{p,0})\widetilde{\mathcal{D}}_p(t)\end{aligned} \tag{4.60}$$

and

$$\begin{aligned}\|\widetilde{F}_{3L}\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}}&\lesssim(1+\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\Big(\|m_L^{\ell}\|_{\dot{B}^{\sigma_1+\sigma_2}_{2,\infty}}\|\widetilde{a}^{\ell}\|_{\dot{B}^{\frac{d}{2}-\sigma_2}_{2,1}}+\|m_L^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\widetilde{a}^{\ell}\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\&\quad+(\|m_L^{\ell}\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|a_L^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\|\widetilde{a}^h\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\Big)\\&\lesssim\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}(1+\mathcal{X}_{p,0})(\|m_L^{\ell}\|_{\dot{B}^{\sigma_1+\sigma_2}_{2,\infty}}+\mathcal{X}_{p,0})\widetilde{\mathcal{D}}_p(t),\quad t>1.\end{aligned} \tag{4.61}$$

In addition, those terms in $\widetilde{F}$ can be treated along the same lines as $\widetilde{F}_L$, and is thus omitted. Consequently, (4.54) holds. It follows from (4.44) and (4.54) that

$$\begin{aligned}&\int_1^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+1)}\|F\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}}d\tau\\&\quad\lesssim \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}\Big((1+\mathcal{X}_{p,0})(\|(a_L,m_L,a,m)^{\ell}\|_{L^\infty_t(\dot{B}^{\sigma_1+\sigma_2}_{2,\infty})}+\mathcal{X}_{p,0})\widetilde{\mathcal{D}}_p(t)\Big),\end{aligned}$$

where σ_2 is given by (4.22). In fact, we performed the following inequality that due to (4.25),

$$\int_0^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+1)}\langle\tau\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}d\tau\lesssim\begin{cases}\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+1)}, & \text{if } \ \frac{1}{2}(\frac{d}{2}-\sigma_1)>1,\\ \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+1)-}, & \text{if } \ \frac{1}{2}(\frac{d}{2}-\sigma_1)=1,\\ \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\frac{d}{2}-1-\sigma_1)}, & \text{if } \ \frac{1}{2}(\frac{d}{2}-\sigma_1)<1\end{cases}$$

for $\sigma_1<\sigma\le\frac{d}{2}-1$. The proof of Lemma 4.2 is complete. □

The case $\frac{d}{2}-1<\sigma<\frac{d}{2}$ requires more elaborate estimates. In (4.27), we take $\sigma'=\sigma_2$ on the part of F_L and $\sigma'=1$ on the part of $\widetilde{F}_L+\widetilde{F}$ and prove the following lemma.

Lemma 4.3. *Let p satisfy (1.8) and $\sigma_0\le\sigma_1<\frac{d}{2}-1$. For all $t>2$ and $\frac{d}{2}-1<\sigma<\frac{d}{2}$, it holds that*

$$\begin{aligned}&\int_0^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}\|F_L\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma_2}_{2,\infty}}d\tau+\int_0^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+1)}\|(\widetilde{F}_L,\widetilde{F})\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}}d\tau\\&\lesssim \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}(\mathcal{X}_{p,0}+\mathcal{X}^2_{p,0})(\|(a,m)^{\ell}\|_{L^\infty_1(\dot{B}^{\sigma_1}_{2,\infty})}+\mathcal{X}_{p,0})\\&\quad+\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}\Big((1+\mathcal{X}_{p,0})(\|(a_L,m_L,a,m)^{\ell}\|_{L^\infty_t(\dot{B}^{\sigma_1+\sigma_3}_{2,\infty})}+\mathcal{X}_{p,0})\widetilde{\mathcal{D}}_p(t)\Big),\end{aligned}$$

where F_L, $\widetilde{F}_L$ and $\widetilde{F}$ are defined by (4.41), (4.42) and (4.43), respectively, and $\sigma_2,\sigma_3>0$ are given by

$$\sigma_2=\min\Big\{\frac{1}{2},\Big(\frac{d}{2}-1-\sigma_1\Big)-\Big\},\quad \sigma_3=\min\Big\{\frac{d}{2}-\sigma,\frac{d}{2}-1-\sigma_1-\sigma_2\Big\}.$$

Proof. We deal with the first term on the left-hand side of (4.62). Since the integral on $[0,1]$ can be handled similarly as in Lemma 4.1, we deal with the following integral with $t>2$ only:

$$\int_1^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}\|F_L\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma_2}_{2,\infty}}d\tau,$$

where $\sigma_2\in(0,1)$ is to be confirmed. According to Lemmas 4.1 and 6.3, we arrive at

$$\|m_L^{\ell}\otimes m_L^{\ell}\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma_2}_{2,\infty}}\lesssim\|m_L^{\ell}\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|m_L^{\ell}\|_{\dot{B}^{\sigma_1+1-\sigma_2}_{2,\infty}}\lesssim\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1+1-\sigma_2)}. \tag{4.62}$$

Here we noticed that $\sigma_1\le\sigma_1+1-\sigma_2<\frac{d}{2}$ and $\sigma_1+1-\sigma_2+\frac{d}{2}\ge d-\frac{2d}{p}+1-\sigma_2\ge 0$. Combining with (4.46)-(4.47), one can get

$$\|m_L\otimes m_L\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma_2}_{2,\infty}}\lesssim(1+\mathcal{X}_{p,0})\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1+1-\sigma_2)},\quad t>1. \tag{4.63}$$

Let us next look at the composite term with $I(a)m_L \otimes m_L$, which resorts to the more elaborate analysis. We consider cases $\sigma_1+1-\sigma_2<\frac{d}{2}-1$ and $\frac{d}{2}-1\leq\sigma_1+1-\sigma_2<\frac{d}{2}$ separately. The case $\sigma_1+1-\sigma_2<\frac{d}{2}-1$ implies that $\sigma_1+1-\sigma_2<\frac{d}{p}$. Note that $\sigma_1+1-\sigma_2+\frac{d}{p}>0$, it follows from Lemmas 6.6 and 6.9 that

$$\|I(a)m_L^\ell\otimes m_L^\ell\|^\ell_{\dot B^{\sigma_1+1-\sigma_2}_{2,\infty}}\lesssim\|a\|_{\dot B^{\frac{d}{p}}_{p,1}}\|m_L^\ell\otimes m_L^\ell\|_{\dot B^{\sigma_1+1-\sigma_2}_{2,\infty}}\lesssim\langle t\rangle^{-\frac12(\frac{d}{2}-\sigma_1+1-\sigma_2)}. \tag{4.64}$$

If $\frac{d}{2}-1\leq\sigma_1+1-\sigma_2<\frac{d}{2}$, thanks to $\frac{d}{2}-1\leq\frac{d}{p}$, we end up with

$$\begin{aligned}&\|I(a)m_L^\ell\otimes m_L^\ell\|^\ell_{\dot B^{\sigma_1+1-\sigma_2}_{2,\infty}}\lesssim\|I(a)m_L^\ell\otimes m_L^\ell\|^\ell_{\dot B^{\frac{d}{2}-1}_{2,\infty}}\\ \lesssim\;&\|a\|_{\dot B^{\frac{d}{p}}_{p,1}}\|m_L^\ell\otimes m_L^\ell\|_{\dot B^{\frac{d}{2}-1}_{2,1}}\lesssim\|a\|_{\dot B^{\frac{d}{p}}_{p,1}}\|m_L^\ell\|_{\dot B^{\frac{d}{2}-1}_{2,1}}\|m_L^\ell\|_{\dot B^{\frac{d}{2}}_{2,1}}\lesssim\langle t\rangle^{-\frac12(\frac{d}{2}-\sigma_1)}\|a\|_{\dot B^{\frac{d}{p}}_{p,1}}.\end{aligned} \tag{4.65}$$

Furthermore, due to the decomposition $a=a_L+\widetilde a$, it follows from Proposition 4.1 and the definition of $\widetilde{\mathcal{D}}_p(t)$ that

$$\|a\|_{\dot B^{\frac{d}{p}}_{p,1}}\leq\|a_L\|_{\dot B^{\frac{d}{p}}_{p,1}}+\|\widetilde a\|_{\dot B^{\frac{d}{p}}_{p,1}}\lesssim\langle t\rangle^{-\frac12(\frac{d}{2}-\sigma_1)}(1+\widetilde{\mathcal{D}}_p(t)). \tag{4.66}$$

Noticing that $\frac12(\frac{d}{2}-\sigma_1+1-\sigma_2)<\frac12(\frac{d}{2}-\sigma_1+1)<\frac{d}{2}-\sigma_1$, from (4.65)-(4.66) we obtain

$$\|I(a)m_L^\ell\otimes m_L^\ell\|^\ell_{\dot B^{\sigma_1+1-\sigma_2}_{2,\infty}}\lesssim\langle t\rangle^{-\frac12(\frac{d}{2}-\sigma_1+1-\sigma_2)}(1+\widetilde{\mathcal{D}}_p(t)). \tag{4.67}$$

To handle those terms involving $I(a)m_L^h\otimes m_L^\ell$ and $I(a)m_L\otimes m_L^h$, by repeating the procedure leading to (4.50)-(4.52), we conclude that

$$\|I(a)m_L\otimes m_L\|^\ell_{\dot B^{\sigma_1+1-\sigma_2}_{2,\infty}}\lesssim\langle t\rangle^{-\frac12(\frac{d}{2}-\sigma_1+1-\sigma_2)}(1+\widetilde{\mathcal{D}}_p(t)),\quad t>1. \tag{4.68}$$

The nonlinear terms F_{2L} and F_{3L} can be essentially estimated at the same way. Consequently, one can arrive at

$$\begin{aligned}&\int_1^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+\sigma_2)}\|F_L\|^\ell_{\dot B^{\sigma_1+1-\sigma_2}_{2,\infty}}d\tau\\ &\lesssim(1+\mathcal{X}_{p,0}+\widetilde{\mathcal{D}}_p(t))\int_0^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+\sigma_2)}\langle\tau\rangle^{-\frac12(\frac{d}{2}-\sigma_1+1-\sigma_2)}d\tau\\ &\lesssim\langle t\rangle^{-\frac12(\sigma-\sigma_1+\sigma_2)}(1+\mathcal{X}_{p,0}+\widetilde{\mathcal{D}}_p(t)),\end{aligned} \tag{4.69}$$

if choosing

$$\sigma_2\triangleq\min\Big\{\frac12,\Big(\frac{d}{2}-1-\sigma_1\Big)-\Big\},$$

which leads to that $\sigma-\sigma_1+\sigma_2\leq\frac{d}{2}-\sigma_1+1-\sigma_2$ and $\frac12(\frac{d}{2}-\sigma_1+1-\sigma_2)>1$ for $\frac{d}{2}-1<\sigma<\frac{d}{2}$.

Then, we bound the second term on the left-hand side of (4.62) concerning the integral with the difference $(\widetilde a,\widetilde m)$. Without loss of generality, we only estimate

$$\int_1^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\|(\tilde F_L,\tilde F)\|^\ell_{\dot B^{\sigma_1}_{2,\infty}}d\tau.$$

Let $\sigma_3\in(0,1)$ be sufficiently small (to be confirmed). Applying the product law in Lemma 6.3 gives

$$\|m_L^\ell\otimes\widetilde{m}^\ell\|^\ell_{\dot B^{\sigma_1}_{2,\infty}}\lesssim\|m_L^\ell\|_{\dot B^{\sigma_1+\sigma_3}_{2,\infty}}\|\widetilde{m}^\ell\|_{\dot B^{\frac d2-\sigma_3}_{2,1}}\lesssim\langle t\rangle^{-\frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)}\|m_L^\ell\|_{\dot B^{\sigma_1+\sigma_3}_{2,\infty}}\widetilde{\mathcal{D}}_p(t), \tag{4.70}$$

where we noticed that $\sigma_1+\sigma_3<\frac d2$ and $\sigma_1+\frac d2\geq 0$. Similar to (4.56)-(4.57), we obtain

$$\|m_L^h\otimes\widetilde{m}^\ell\|^\ell_{\dot B^{\sigma_1}_{2,\infty}}\lesssim\|m_L^h\|_{\dot B^{\frac dp-1}_{p,1}}\|\widetilde{m}^\ell\|_{\dot B^{\frac d2-1}_{2,1}}\lesssim\langle t\rangle^{-\frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)}\mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t) \tag{4.71}$$

and

$$\|m_L\otimes\widetilde{m}^h\|^\ell_{\dot B^{\sigma_1}_{2,\infty}}\lesssim(\|m_L^\ell\|_{\dot B^{\frac d2-1}_{2,1}}+\|m_L^h\|_{\dot B^{\frac dp}_{p,1}})\|\widetilde{m}^h\|_{\dot B^{\frac dp-1}_{p,1}}\lesssim\langle t\rangle^{-\frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)}\mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t),\quad t>1. \tag{4.72}$$

Likewise, we see that, using those inequalities for composition in Lemma 6.9,

$$\|I(a)m_L\otimes\widetilde{m}\|^\ell_{\dot B^{\sigma_1}_{2,\infty}}\lesssim\langle t\rangle^{-\frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)}(\|m_L^\ell\|_{\dot B^{\sigma_1+\sigma_3}_{2,\infty}}+\mathcal{X}_{p,0})\mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t),\quad t>1. \tag{4.73}$$

Bounding $\widetilde{F}_{2L},\widetilde{F}_{3L}$ and $\widetilde{F}$ can be proceeded along the same lines as $\widetilde{F}_{1L}$. The details are left to the interested reader. Thus, we conclude that

$$\begin{aligned}&\int_1^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\|(\tilde F_L,\tilde F)\|^\ell_{\dot B^{\sigma_1}_{2,\infty}}d\tau\\&\quad\lesssim\int_0^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\langle\tau\rangle^{-\frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)}d\tau\Big((1+\mathcal{X}_{p,0})(\|(a_L,m_L,a,m)^\ell\|_{\dot B^{\sigma_1+\sigma_3}_{2,\infty}}+\mathcal{X}_{p,0})\Big)\widetilde{\mathcal{D}}_p(t).\end{aligned}$$

It follows from (4.25) that

$$\begin{aligned}&\int_0^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\langle\tau\rangle^{-\frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)}d\tau\\&\lesssim\begin{cases}\langle t\rangle^{-\frac12(\sigma-\sigma_1+1)}, & \text{if }\ \frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)>1,\\ \langle t\rangle^{-\frac12(\sigma-\sigma_1+1)-}, & \text{if }\ \frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)=1,\\ \langle t\rangle^{-\frac12(\sigma-\sigma_1+\frac d2-1-\sigma_1-\sigma_3)}, & \text{if }\ \frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)<1,\end{cases}\end{aligned}$$

if $\frac12(\sigma-\sigma_1+1)\leq\frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)$ or that

$$\begin{aligned}&\int_0^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\langle\tau\rangle^{-\frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)}d\tau\\&\lesssim\begin{cases}\langle t\rangle^{-\frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)}, & \text{if }\ \frac12(\sigma-\sigma_1+1)>1,\\ \langle t\rangle^{-\frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)-}, & \text{if }\ \frac12(\sigma-\sigma_1+1)=1,\\ \langle t\rangle^{-\frac12(\sigma-\sigma_1+\frac d2-1-\sigma_1-\sigma_3)}, & \text{if }\ \frac12(\sigma-\sigma_1+1)<1,\end{cases}\end{aligned}$$

if $\frac12(\frac d2-\sigma_1+\sigma_2-\sigma_3)\leq\frac12(\sigma-\sigma_1+1)$. Recalling that $\sigma_2=\min\{\frac12,(\frac d2-1-\sigma_1)-\}<\frac d2-1-\sigma_1$ and $\frac d2-1<\sigma<\frac d2$, it is shown that above two integrals can be both controlled by $\langle t\rangle^{-\frac12(\sigma-\sigma_1+\sigma_2)}$ provided that $\sigma_3>0$ is chosen small enough such that

$$\sigma_3=\min\Big\{\frac d2-\sigma,\frac d2-1-\sigma_1-\sigma_2\Big\}.$$

Hence, the proof of Lemma 4.3 is finished. □

Combining those time-weighted estimates in Lemmas 4.1-4.3, we conclude from (4.27) that

$$
\begin{aligned}
&\sup_{\sigma_1<\sigma<\frac{d}{2}} \|\langle\tau\rangle^{\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}(\widetilde{a},\widetilde{m})\|^{\ell}_{L^\infty_t(\dot{B}^{\sigma}_{2,1})}\\
&\lesssim (\mathcal{X}_{p,0}+\mathcal{X}_{p,0}^2)(\|(a,m)^\ell\|_{L^\infty_t(\dot{B}^{\sigma_1}_{2,\infty})}+\mathcal{X}_{p,0})\\
&\quad+\Big((1+\mathcal{X}_{p,0})(\|(a_L,m_L,a,m)^\ell\|_{L^\infty_t(\dot{B}^{\sigma_1+\sigma_*}_{2,\infty})}+\mathcal{X}_{p,0})\Big)\widetilde{\mathcal{D}}_p(t),
\end{aligned}
\tag{4.74}
$$

where the exponent $\sigma_*>0$ is given by $\sigma_*=\sigma_2$ for $\sigma_1<\sigma\leq\frac{d}{2}-1$ and $\sigma_*=\sigma_3$ for $\frac{d}{2}-1<\sigma<\frac{d}{2}$.

4.1.2 Bounds for the high frequencies

To achieve the high-frequency estimates of $(\widetilde{a},\widetilde{m})$ in Proposition 4.2, it is natural to look at the difference system (1.24) with the nonlinear term $\operatorname{div}F$. The problem is that $\operatorname{div}F$ (for example $\mathcal{A}(I(a)u)$) will cause a loss of one derivative. In the critical regularity framework, however, one cannot afford any loss of regularity for the high frequency part of a. To overcome the difficulty, we use the decomposition

$$
\widetilde{a}=a-a_L,\quad \widetilde{m}=(1+a)u-m_L,
$$

which implies that it suffices to estimate (a,u) instead of $(\widetilde{a},\widetilde{m})$. The proof is proceeded into two steps. Firstly, we consider the system (1.9)-(1.10) and track the decay exponent for high frequencies according to the definition of $\widetilde{\mathcal{D}}_p(t)$, by the energy approach in terms of *effective velocity* $w=\nabla(-\Delta)^{-1}(a-\operatorname{div}u)$ that has been successfully developed by Haspot [21] to prove Theorem 1.1 (see also [15]). Precisely, we have the following lemma about weighted estimate of (a,u) in high frequencies.

Lemma 4.4. *If p satisfy (1.8), then it holds that*

$$
\begin{aligned}
&\|\langle\tau\rangle^{\alpha_*}(\nabla a,u)\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|\tau^{\alpha_*}u\|^{h}_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})}\\
&\lesssim 1+\mathcal{X}_{p,0}+\mathcal{X}_{p,0}^2+\|\tau^{\frac{1}{2}(\frac{d}{2}-\sigma_1)-}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})}+\mathcal{X}_{p,0}\Big(\|\tau^{\alpha_*}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})}+\widetilde{\mathcal{D}}_p(t)\Big)
\end{aligned}
\tag{4.75}
$$

with $\alpha_*=\frac{1}{2}(\frac{d}{2}-\sigma_1+\sigma_2)-$.

Proof. We shall modify the L^p time-weighted energy argument performed in [17] slightly. With the aid of the effective velocity, one can end up with

$$
\begin{aligned}
&\|\langle\tau\rangle^{\alpha_*}(\nabla a,u)\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\\
&\lesssim \|(\nabla a_0,u_0)\|^{h}_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\sum_{j\geq j_0-1}2^{j(\frac{d}{p}-1)}\sup_{0\leq\tau\leq t}\Big(\langle\tau\rangle^{\alpha_*}\int_0^\tau e^{-c(\tau-s)}Z_j(s)ds\Big)
\end{aligned}
\tag{4.76}
$$

with $Z_j=Z_j^1+\cdots+Z_j^5$ and

$$
\begin{aligned}
&Z_j^1\triangleq\|\dot{\Delta}_j(au)\|_{L^p},\qquad Z_j^2\triangleq\|\nabla\dot{\Delta}_j(a\operatorname{div}u)\|_{L^p},\qquad Z_j^3\triangleq\|\operatorname{div}u\|_{L^\infty}\|\nabla\dot{\Delta}_j a\|_{L^p},\\
&Z_j^4\triangleq\sum_{i,k}\|[u^i,\dot{\Delta}_j]\partial^2_{ik}a\|_{L^2},\quad Z_j^5\triangleq\|\dot{\Delta}_j g\|_{L^p}.
\end{aligned}
$$

Without loss of generality, one can assume that $t > 2$. First, let us handle the time-weighted integral in (4.76) for $0 \le \tau \le 2$. It is easy to see that

$$I_1 \triangleq \sup_{0\le\tau\le2}\Big(\langle\tau\rangle^{\alpha_*}\int_0^\tau e^{-c(\tau-s)}Z_j(s)ds\Big) \le \int_0^2 Z_j(s)ds.$$

For the integral with $2 \le \tau \le t$, it is convenient to split it into two parts: $[0,1]$ and $[1,\tau]$. It is also simple to handle:

$$I_2 \triangleq \sup_{2\le\tau\le t}\Big(\langle\tau\rangle^{\alpha_*}\int_0^1 e^{-c(\tau-s)}Z_j(s)ds\Big) \le \sup_{2\le\tau\le t}\Big(\langle\tau\rangle^{\alpha_*}e^{-\frac{c}{2}\tau}\Big)\int_0^1 Z_j(s)ds \le \int_0^1 Z_j(s)ds$$

since $2 \le \tau \le t$ and $0 \le s \le 1$. On the other hand, note that $s \approx 1+s \approx \langle s\rangle$, the integral on the part $[1,\tau]$ can be dealt with as follows:

$$\begin{aligned}I_3 &\triangleq \sup_{2\le\tau\le t}\Big(\langle\tau\rangle^{\alpha_*}\int_1^\tau e^{-c(\tau-s)}Z_j(s)ds\Big)\\ &\lesssim \sup_{2\le\tau\le t}\sup_{1\le s\le\tau}(s^{\alpha_*}Z_j(s))\langle\tau\rangle^{\alpha_*}\int_1^\tau e^{-c(\tau-s)}s^{-\alpha_*}ds \lesssim \sup_{1\le\tau\le t}(\tau^{\alpha_*}Z_j(\tau)).\end{aligned}$$

Consequently, by employing Lemmas 6.9 and 6.10, we obtain

$$\begin{aligned}&\sum_{j\ge j_0-1}2^{j(\frac{d}{p}-1)}(I_1+I_2)\\ &\lesssim \|a\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|a\|^2_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|u\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim \mathcal{X}^2_{p,0},\end{aligned}\tag{4.77}$$

where the interpolation inequalities are also used. Next, we focus on the nontrivial case

$$\sum_{j\ge j_0-1}2^{j(\frac{d}{p}-1)}I_3 \lesssim \sum_{j\ge j_0-1}2^{j(\frac{d}{p}-1)}\sup_{1\le\tau\le t}(\tau^{\alpha_*}Z_j(\tau)).$$

We shall use repeatedly the following inequalities:

$$\|\langle\tau\rangle^{\frac12(\frac{d}{2}-\sigma_1)-}a_L\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim 1+\mathcal{X}_{p,0};\quad \|\langle\tau\rangle^{\alpha_*}\widetilde{a}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim \widetilde{\mathcal{D}}_p(t).\tag{4.78}$$

Indeed, it follows from the embedding, the definition of $\widetilde{\mathcal{D}}_p$ and tilde norms that

$$\|\langle\tau\rangle^{\frac12(\frac{d}{2}-\sigma_1)-}a_L\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim \|\langle\tau\rangle^{\frac12(\frac{d}{2}-\sigma_1)-}a_L^\ell\|_{L^\infty_t(\dot{B}^{\frac{d}{2}-}_{2,1})}+\|e^{R\tau}a_L\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim 1+\mathcal{X}_{p,0}$$

and

$$\|\langle\tau\rangle^{\alpha_*}\widetilde{a}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim \|\langle\tau\rangle^{\alpha_*}\widetilde{a}\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{2}-}_{2,1})}+\|\langle\tau\rangle^{\alpha_*}\widetilde{a}\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim \widetilde{\mathcal{D}}_p(t).$$

Noticing that $\alpha_* < \frac{d}{2}-\sigma_1$, it is clear that

$$\begin{aligned}&\sum_{j\ge j_0-1}2^{j(\frac{d}{p}-1)}\sup_{1\le\tau\le t}(\tau^{\alpha_*}Z^1_j(\tau)) = \|\tau^{\alpha_*}au\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}-1}_{p,1})}\\ &\lesssim \|\tau^{\alpha_*}a_Lu\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^{\alpha_*}\widetilde{a}u\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}-1}_{p,1})}\\ &\lesssim \|\langle\tau\rangle^{\frac12(\frac{d}{2}-\sigma_1)-}a_L\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|\tau^{\frac12(\frac{d}{2}-\sigma_1)-}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})}+\|u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}-1}_{p,1})}\|\langle\tau\rangle^{\alpha_*}\widetilde{a}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\ &\lesssim \|\tau^{\frac12(\frac{d}{2}-\sigma_1)-}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})}+\mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t),\end{aligned}\tag{4.79}$$

where we used the decomposition $au = a_L u + \widetilde{a}u$ and (4.78). Furthermore, it follows from the embedding $\dot{B}^{\frac{d}{p}}_{p,1} \hookrightarrow L^\infty$ and Lemma 6.10 that

$$\begin{aligned}&\sum_{j\geq j_0-1} 2^{j(\frac{d}{p}-1)} \sup_{1\leq\tau\leq t}(\tau^{\alpha_*}(Z_j^2+Z_j^3+Z_j^4)(\tau))\\&\lesssim \|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|\tau^{\alpha_*}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})} \lesssim \mathcal{X}_{p,0}\|\tau^{\alpha_*}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})}.\end{aligned} \tag{4.80}$$

Next, let us pay attention to the term Z_j^5 associated with $g = -u\cdot\nabla u - k(a)\nabla a - I(a)\mathcal{A}u$. It follows from product laws in Lemma 6.3 adapted to the tilde spaces that

$$\|\tau^{\alpha_*}u\cdot\nabla u\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|u\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|\tau^{\alpha_*}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})} \lesssim \mathcal{X}_{p,0}\|\tau^{\alpha_*}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})}$$

and

$$\|\tau^{\alpha_*}I(a)\mathcal{A}u\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|\tau^{\alpha_*}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})} \lesssim \mathcal{X}_{p,0}\|\tau^{\alpha_*}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})}.$$

Regarding to the pressure term with $k(a)\nabla a$, we use the following decomposition:

$$k(a)\nabla a = k(a_L)\nabla a_L + (k(a)-k(a_L))\nabla a_L + k(a)\nabla\widetilde{a}.$$

Then Lemma 6.3, Lemma 6.9 and (4.78) ensure that

$$\|\tau^{\alpha_*}k(a_L)\nabla a_L\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|\langle\tau\rangle^{\frac{1}{2}(\frac{d}{2}-\sigma_1)-}a_L\|^2_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim 1+\mathcal{X}_{p,0}^2,$$

$$\|\tau^{\alpha_*}(k(a)-k(a_L))\nabla a_L\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|a_L\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|\langle\tau\rangle^{\alpha_*}\widetilde{a}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim \mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t)$$

and

$$\|\tau^{\alpha_*}k(a)\nabla\widetilde{a}\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|\langle\tau\rangle^{\alpha_*}\widetilde{a}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim \mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t).$$

Therefore, we obtain

$$\sum_{j\geq j_0-1} 2^{j(\frac{d}{p}-1)} \sup_{1\leq\tau\leq t}(\tau^{\alpha_*}Z_j^5(\tau)) \lesssim 1+\mathcal{X}_{p,0}^2+\mathcal{X}_{p,0}\Big(\|\langle\tau\rangle^{\alpha_*}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})}+\widetilde{\mathcal{D}}_p(t)\Big). \tag{4.81}$$

Putting all above estimates (4.77)-(4.81) together, it thus follows from (4.76) that

$$\begin{aligned}\|\langle\tau\rangle^{\alpha_*}(\nabla a,u)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} &\lesssim 1+\mathcal{X}_{p,0}+\mathcal{X}_{p,0}^2+\|\tau^{\frac{1}{2}(\frac{d}{2}-\sigma_1)-}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})}\\&\quad+\mathcal{X}_{p,0}\Big(\|\tau^{\alpha_*}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})}+\widetilde{\mathcal{D}}_p(t)\Big).\end{aligned} \tag{4.82}$$

Finally, we are going to establish gain of regularity and decay altogether for the high frequencies of u, which strongly depends on the parabolic maximal regularity for the Lamé semi-group (see Lemma 6.11). It follows from the velocity equation in (1.9) that

$$\begin{cases}\partial_t(tu)-\mathcal{A}(tu) = u - t\nabla a + tg,\\ tu|_{t=0}=0\end{cases}$$

with $0 \le t \le 1$. Consequently, the regularity property in Lemma 6.11, standard product laws and composite estimates enable us to get

$$\|tu\|^h_{\widetilde{L}^\infty(0,1;\dot{B}^{\frac{d}{p}+1}_{p,1})} \lesssim \|(u,\nabla a)\|^h_{\widetilde{L}^\infty(0,1;\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|tg\|^h_{\widetilde{L}^\infty(0,1;\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \mathcal{X}_{p,0} + \mathcal{X}_{p,0}^2 + \mathcal{X}_{p,0}\|tu\|^h_{\widetilde{L}^\infty(0,1;\dot{B}^{\frac{d}{p}+1}_{p,1})},$$

which, together with $\mathcal{X}_{p,0} \ll 1$, leads to

$$\sup_{t\in[0,1]} \|u(t)\|^h_{\dot{B}^{\frac{d}{p}+1}_{p,1}} \lesssim \mathcal{X}_{p,0}. \tag{4.83}$$

To obtain decay estimates of u for $t > 1$, we reformulate the velocity equation in (1.9) as follows

$$\partial_t(\tau^{\alpha_*}u) - \mathcal{A}(\tau^{\alpha_*}u) = \alpha_*\tau^{\alpha_*-1}u - \tau^{\alpha_*}(\nabla a - g)$$

with $1 \le \tau \le t$. Thus, employing Lemma 6.11 again implies that

$$\begin{aligned}\|\tau^{\alpha_*}u\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})} \lesssim{}& \|u(1)\|_{\dot{B}^{\frac{d}{p}+1}_{p,1}} + \|\tau^{\alpha_*-1}u\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\tau^{\alpha_*}a\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})}\\ &+ \|\tau^{\alpha_*}g\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}-1}_{p,1})}.\end{aligned} \tag{4.84}$$

Due to that fact $\tau \ge 1$, we see that

$$\|\tau^{\alpha_*-1}u\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|\langle\tau\rangle^{\alpha_*}u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}, \quad \|\tau^{\alpha_*}a\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim \|\langle\tau\rangle^{\alpha_*}a\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

Bounding the norm $\|\tau^{\alpha_*}g\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}-1}_{p,1})}$ is exactly the same as (4.81), and one thus arrive at (4.75) readily. □

Secondly, we establish several calculus inequalities to deduce the desired high-frequency decay of $(\widetilde{a},\widetilde{m})$ in Proposition 4.2.

Lemma 4.5. *If p satisfy (1.8), then it holds that*

$$\begin{cases}\|\langle\tau\rangle^{\alpha_*}\widetilde{m}\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|\langle\tau\rangle^{\alpha_*}u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + 1 + \mathcal{X}_{p,0} + \mathcal{X}_{p,0}^2 + \mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t),\\ \|\tau^{\alpha_*}\widetilde{m}\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim \|\tau^{\alpha_*}u\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})} + 1 + \mathcal{X}_{p,0} + \mathcal{X}_{p,0}^2 + \mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t),\\ \|\tau^{\alpha_*}u^\ell\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})} \lesssim 1 + \mathcal{X}_{p,0} + \mathcal{X}_{p,0}^2 + (1+\mathcal{X}_{p,0})\widetilde{\mathcal{D}}_p(t),\\ \|\tau^{\frac{1}{2}(\frac{d}{2}-\sigma_1)-}u\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim 1 + \mathcal{X}_{p,0} + \mathcal{X}_{p,0}^2 + (1+\mathcal{X}_{p,0})\widetilde{\mathcal{D}}_p(t).\end{cases} \tag{4.85}$$

Proof. Bounding the first and second inequalities in (4.85) are almost the same, which both depends on the decomposition

$$\widetilde{m} = (1+a)u - m_L = u - m_L + I(a)\widetilde{m} + I(a_L)m_L + (I(a)-I(a_L))m_L \quad \text{with} \quad I(a) = \frac{a}{1+a}. \tag{4.86}$$

Let us take a look at $(4.85)_1$ for example. It follows from Proposition 4.1 that

$$\|\langle\tau\rangle^{\alpha_*}m_L\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\leq\|e^{R\tau}m_L\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\lesssim\mathcal{X}_{p,0}. \tag{4.87}$$

The definition of $\widetilde{\mathcal{D}}_p$, product laws and composite estimates in Lemmas 6.3 and 6.9 allow to get

$$\begin{aligned}\|\langle\tau\rangle^{\alpha_*}I(a)\widetilde{m}\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}&\leq\|\langle\tau\rangle^{\alpha_*}I(a)\widetilde{m}^\ell\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\langle\tau\rangle^{\alpha_*}I(a)\widetilde{m}^h\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\\&\lesssim\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}(\|\langle\tau\rangle^{\alpha_*}\widetilde{m}^\ell\|_{L^\infty_t(\dot{B}^{\frac{d}{2}-}_{2,1})}+\|\langle\tau\rangle^{\alpha_*}\widetilde{m}^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})})\\&\lesssim\mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t).\end{aligned} \tag{4.88}$$

Similarly, by (4.78), we have

$$\begin{aligned}&\|\langle\tau\rangle^{\alpha_*}I(a_L)m_L\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\leq\|\langle\tau\rangle^{\alpha_*}I(a_L)m_L^\ell\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\langle\tau\rangle^{\alpha_*}I(a_L)m_L^h\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\\&\lesssim\|\langle\tau\rangle^{\frac12(\frac{d}{2}-\sigma_1)-}a_L\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big(\|\langle\tau\rangle^{\frac12(\frac{d}{2}-\sigma_1)-}m_L^\ell\|_{L^\infty_t(\dot{B}^{\frac{d}{2}-}_{2,1})}+\|\langle\tau\rangle^{\frac12(\frac{d}{2}-\sigma_1)-}m_L^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\Big)\\&\lesssim(1+\mathcal{X}_{p,0})^2.\end{aligned} \tag{4.89}$$

Also, it follows from (4.1), (4.78) and Lemma 6.9 that

$$\|\langle\tau\rangle^{\alpha_*}(I(a)-I(a_L))m_L\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\lesssim\|\langle\tau\rangle^{\alpha_*}\widetilde{a}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|m_L\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\lesssim\mathcal{X}_{p,0}\widetilde{\mathcal{D}}_p(t). \tag{4.90}$$

Therefore, combining with (4.86), (4.87), (4.88), (4.89) and (4.90) lead to $(4.85)_1$ directly.

To show $(4.85)_3$-$(4.85)_4$, it suffices to use the decomposition that $u=m-I(a)m=\widetilde{m}+m_L-I(a)(\widetilde{m}+m_L)$. Keeping in mind that $\sigma_2\leq1$, by Proposition 4.1, we arrive at

$$\begin{aligned}\|\langle\tau\rangle^{\alpha_*}u^\ell\|_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}+1}_{p,1})}&\lesssim\|\langle\tau\rangle^{\alpha_*}\widetilde{m}^\ell\|_{L^\infty(1,t;\dot{B}^{\frac{d}{2}-}_{2,1})}+\|\langle\tau\rangle^{\frac12(\frac{d}{2}+1-\sigma_1)-}m_L^\ell\|_{L^\infty(1,t;\dot{B}^{\frac{d}{2}+1-}_{2,1})}\\&\quad+\|\langle\tau\rangle^{\alpha_*}I(a)\widetilde{m}\|^\ell_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})}+\|\langle\tau\rangle^{\alpha_*}I(a)m_L\|^\ell_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})}\\&\lesssim1+\mathcal{X}_{p,0}+\mathcal{X}_{p,0}^2+(1+\mathcal{X}_{p,0})\widetilde{\mathcal{D}}_p(t).\end{aligned} \tag{4.91}$$

Bounding $(4.85)_4$ is totally similar, and thus those details can be omitted. □

Plugging (4.85) into (4.75), and remembering the smallness of $\mathcal{X}_{p,0}$ and the fact that the high-frequency part of a_L decays exponentially in the norm of $\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})$, we eventually conclude that

$$\|\langle\tau\rangle^{\alpha_*}(\nabla\widetilde{a},\widetilde{m})\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|\tau^{\alpha_*}\widetilde{m}\|^h_{\widetilde{L}^\infty(1,t;\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim1+\mathcal{X}_{p,0}\widetilde{\mathcal{D}}(t). \tag{4.92}$$

Finally, adding up (4.92) to (4.74), we conclude that there exists a constant $\sigma_*\in(0,1]$ such that

$$\widetilde{\mathcal{D}}_p(t)\lesssim1+\|(a,m)^\ell\|_{L^\infty_t(\dot{B}^{\sigma_1}_{2,\infty})}+(\|(a_L,m_L,a,m)^\ell\|_{L^\infty_t(\dot{B}^{\sigma_1+\sigma_*}_{2,\infty})}+\mathcal{X}_{p,0})\widetilde{\mathcal{D}}_p(t),\quad t>0. \tag{4.93}$$

As shown by the priori work [51] (see Lemma 5.1), there is the nonlinear evolution to the solution (a,u) at low frequencies:

$$\|(a,u)^{\ell}\|_{L^{\infty}_{t}(\dot{B}^{\sigma_1}_{2,\infty})}\leq C_0. \tag{4.94}$$

for $t>0$, where the constant C_0 depends on $\mathcal{X}_{p,0}$ and the norm $\|(a_0,u_0)^{\ell}\|_{\dot{B}^{\sigma_1}_{2,\infty}}$ with $\sigma_0\leq\sigma_1<\frac{d}{2}-1$. Arguing similarly as those lines (4.12), (4.13), (4.16) and (4.18), one can deduce from (4.94) that $\|(a,m)^{\ell}\|_{L^{\infty}_{t}(\dot{B}^{\sigma_1}_{2,\infty})}\leq C_0$, provided that Theorem 1.1 holds. Combining this with the interpolation (6.1), the fact $\sigma_1<\sigma_1+\sigma_*\leq\frac{d}{2}$ and $\|(a,m)^{\ell}\|_{L^{\infty}_{t}(\dot{B}^{\frac{d}{2}}_{2,1})}\leq\|(a,m)^{\ell}\|_{L^{\infty}_{t}(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim\mathcal{X}_{p,0}$, we deduce

$$\|(a,m)^{\ell}\|_{L^{\infty}_{t}(\dot{B}^{\sigma_1+\sigma_*}_{2,\infty})}\lesssim\|(a,m)^{\ell}\|^{\theta_*}_{L^{\infty}_{t}(\dot{B}^{\sigma_1}_{2,\infty})}\|(a,m)^{\ell}\|^{1-\theta_*}_{L^{\infty}_{t}(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim C_0^{\theta_*}\mathcal{X}_{p,0}^{1-\theta_*}<<1, \tag{4.95}$$

with $\theta_*\in(0,1)$ satisfying $\sigma_1\theta_*+(1-\theta)\frac{d}{2}=\sigma_1+\sigma_*$. Since $\|(a_L,m_L)^{\ell}\|_{\dot{B}^{\sigma}_{2,1}}$ is uniformly bounded for all $\sigma>\sigma_1$ due to (4.2), a similar interpolation argument implies that

$$\|(a_L,m_L)^{\ell}\|_{L^{\infty}_{t}(\dot{B}^{\sigma_1+\sigma_*}_{2,\infty})}<<1. \tag{4.96}$$

According to (4.93), (4.95) and (4.96), the time-weighted difference estimate (4.21) follows. This completes the proof of Proposition 4.2.

4.2 Lower and upper bounds for decay rates

The subsection is devoted to the proof of "if" part of Theorem 1.2. We first prove (1.13) under the assumption $(a_0,u_0)^{\ell}\in\dot{B}^{\sigma_1}_{2,\infty}$ with $\sigma_0\leq\sigma_1<\frac{d}{2}-1$. Indeed, (1.12) is exactly the same as (4.94), which has been shown by [51]. From Lemmas 4.4-4.5, we have

$$\|(a,u)(t)\|^{h}_{\dot{B}^{\frac{d}{p}}_{p,1}}\lesssim\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1+\sigma_2)-} \tag{4.97}$$

for $t>1$ and $\sigma_2\in(0,1]$ given by (4.22). Note that $\|(a,u)(t)\|^{h}_{\dot{B}^{\frac{d}{p}}_{p,1}}$ is bounded for $t>t_0$ with $t_0\in(0,1)$ due to Theorem 1.1 and (4.83). Thus, (4.97) holds true for $t>t_0$ with any $t_0>0$. So we only need to show the decay of the low-frequency part of (a,u). In fact, it follows from Propositions 4.1 and 4.2 that

$$\begin{aligned}\|(a,m)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}}&\leq\|(a_L,m_L)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}}+\|(a-a_L,m-m_L)(t)\|^{\ell}_{\dot{B}^{\sigma}_{2,1}}\\&\lesssim\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}+\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}\lesssim\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}\end{aligned} \tag{4.98}$$

for $\sigma_1<\sigma\leq\frac{d}{2}$ and $t>0$. The endpoint case $\sigma=\frac{d}{2}$ can be handled due to the low-frequency localization and the faster decay rates of the difference $(a-a_L,m-m_L)$. To derive the decay of u^{ℓ}, we resort to the decomposition that $u=m-I(a)m^{\ell}-I(a)m^{h}$ again. By employing product laws and composite in

Lemmas (6.4) and (6.9), we deduce that

$$
\begin{aligned}
\|I(a)m^\ell\|^\ell_{\dot B^\sigma_{2,1}} &\lesssim \|I(a)\|_{\dot B^{\frac{d}{p}}_{p,1}}\|m^\ell\|_{\dot B^\sigma_{2,1}}\\
&\lesssim (\|a^\ell\|_{\dot B^{\frac{d}{2}}_{2,1}}+\|a^h\|_{\dot B^{\frac{d}{p}}_{p,1}})\|m^\ell\|_{\dot B^\sigma_{2,1}} \lesssim \langle t\rangle^{-\frac12(\sigma-\sigma_1+\frac d2-\sigma_1)},\quad \sigma_1<\sigma\le \frac dp, \qquad (4.99)\\
\|I(a)m^\ell\|^\ell_{\dot B^\sigma_{2,1}} &\lesssim \|I(a)m^\ell\|^\ell_{\dot B^{\frac dp}_{2,1}} \lesssim \|a\|_{\dot B^{\frac dp}_{p,1}}\|m^\ell\|_{\dot B^{\frac dp}_{2,1}}\\
&\lesssim \langle t\rangle^{-\frac12(\frac dp-\sigma_1+\frac d2-\sigma_1)} \lesssim \langle t\rangle^{-\frac12(\sigma-\sigma_1)},\quad \frac dp\le\sigma\le\frac d2, \qquad (4.100)
\end{aligned}
$$

where (4.97) and (4.98) were used. Concerning $I(a)m^h$, we take advantage of the low-frequency cut-off and argue similarly as in (4.14)-(4.18) to get

$$
\|I(a)m^h\|^\ell_{\dot B^\sigma_{2,1}} \lesssim \|I(a)m^h\|^\ell_{\dot B^{\sigma_1}_{2,\infty}} \lesssim (\|a^\ell\|_{\dot B^{\frac d2-1}_{2,1}}+\|a^h\|_{\dot B^{\frac dp}_{p,1}})\|m^h\|_{\dot B^{\frac dp-1}_{p,1}} \lesssim \langle t\rangle^{-\frac12(\frac d2-\sigma_1+\sigma_2)}. \qquad (4.101)
$$

Together with (4.98)-(4.101), it is shown that

$$
\begin{aligned}
\|u^\ell(t)\|_{\dot B^\sigma_{2,1}} &\lesssim \|m^\ell(t)\|_{\dot B^\sigma_{2,1}}+\|I(a)m^\ell\|^\ell_{\dot B^\sigma_{2,1}}+\|I(a)m^h\|^\ell_{\dot B^\sigma_{2,1}}\\
&\lesssim \langle t\rangle^{-\frac12(\sigma-\sigma_1)},\qquad \sigma_1<\sigma\le\frac d2.
\end{aligned} \qquad (4.102)
$$

Hence, by (4.97), (4.98) and (4.102), we immediately get the upper bound (1.13).

Furthermore, we establish the two-sided decay (1.14) under the stronger assumption that $(a_0,u_0)^\ell\in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ with $\sigma_0\le\sigma_1<\frac d2-1$. It suffices to show the lower bound in (1.14), since $\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ is a subset of $\dot B^{\sigma_1}_{2,\infty}$. For that end, by virtue of Propositions 4.1 and 4.2, we arrive at

$$
\begin{aligned}
\|(a,m)^\ell(t)\|_{\dot B^\sigma_{2,1}} &\ge \|(a_L,m_L)^\ell(t)\|_{\dot B^\sigma_{2,1}}-\|(a-a_L,m-m_L)(t)\|^\ell_{\dot B^\sigma_{2,1}}\\
&\ge \frac{1}{C_0}\langle t\rangle^{-\frac12(\sigma-\sigma_1)}-C_0\langle t\rangle^{-\frac12(\sigma-\sigma_1+\sigma_2)}\\
&\ge \frac{1}{2C_0}\langle t\rangle^{-\frac12(\sigma-\sigma_1)}
\end{aligned} \qquad (4.103)
$$

for $\sigma_1<\sigma\le\frac d2$ and suitably large time $t>t_1$, where $C_0>1$ is chosen into a greater constant if necessary. The endpoint case $\sigma=\frac d2$ is due to faster rates of $(a-a_L,m-m_L)$ under some low-frequency cut-off. Since the product $I(a)m$ decays at faster rates (see (4.99)-(4.101)), one can get

$$
\|(a,u)^\ell(t)\|_{\dot B^\sigma_{2,1}} \ge \|(a,m)^\ell(t)\|_{\dot B^\sigma_{2,1}}-\|I(a)m\|^\ell_{\dot B^\sigma_{2,1}} \ge \frac{1}{4C_0}\langle t\rangle^{-\frac12(\sigma-\sigma_1)}
$$

for $\sigma_1<\sigma\le\frac d2$ and suitably large time $t>t_1$. This completes the proof of the two-sided time-decay estimate (1.14).

5 Necessary condition

The section is devoted to the proof of "only if" part of Theorem 1.2. That is, if the solution (a,u) admits the upper bounds (1.13) (resp., two-sided bounds (1.14)), then $(a_0,u_0)^\ell\in\dot B^{\sigma_1}_{2,\infty}$ (resp., $(a_0,u_0)^\ell\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$)

with $\sigma_0 \le \sigma_1 < \frac{d}{2}-1$. The crucial ingredient of this claim is to develop the inverse Wiegner's argument from incompressible Navier-Stokes equations (as shown by Skalák [46] and the first author et al. [5]) to compressible Navier-Stokes equations.

5.1 Inverse Wiegner's argument for compressible Navier-Stokes equations

Our aim is to derive the following result, which can be regarded as the analogue of inverse Wiegner's Theorem in [5,46].

Proposition 5.1. *Let $\sigma_0 \le \sigma_1 < \frac{d}{2}-1$. If the solution (a,u) to the Cauchy problem (1.9) satisfies (1.12) and (1.13), then $(\widetilde{a},\widetilde{m}) \triangleq (a-a_L, m-m_L)$ has faster decay rates at low frequencies:*

$$\|(\widetilde{a},\widetilde{m})(t)\|^{\ell}_{\dot{B}^{\sigma}_{2,1}} \lesssim \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)} \tag{5.1}$$

for $\sigma_1 < \sigma \le \frac{d}{2}$ and $t>0$, where $\sigma_2 \in (0,1]$ is defined in (4.22).

Proof. The proof follows from the similar procedure leading to Lemmas 4.1-4.3 in fact. We recall (4.27) that

$$\|(\widetilde{a},\widetilde{m})(t)\|^{\ell}_{\dot{B}^{\sigma}_{2,1}} \lesssim \int_0^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma')}\|F\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma'}_{2,\infty}}\,d\tau, \quad \sigma>\sigma_1 \tag{5.2}$$

for $\sigma' \in (0,1]$ and $t>0$. We focus on the integral on the right-hand side of (5.2) and consider cases $0<t\le 2$ and $t>2$ separately. The case $0<t\le 2$ implies that $\langle t\rangle \approx 1$ and $\langle t-\tau\rangle \approx 1$ for $0\le \tau\le t\le 2$. The nonlinear term F in (1.21) can be rewritten as

$$F=(1+a)u\otimes u+(P''(1)+G(a))a^2\mathbb{I}_d+\mu\nabla(au)+(\mu+\lambda)\mathrm{div}\,(au)\mathbb{I}_d$$

with $(P''(1)+G(a))a^2 = P(1+a)-P(1)-P'(1)a$ satisfying $G(0)=0$. It follows from the proof of Lemma 4.1 that

$$\begin{aligned}\|F\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma'}_{2,\infty}} \lesssim \|F\|^{\ell}_{\dot{B}^{\sigma_1}_{2,\infty}} &\lesssim (1+\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}})(\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|u\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}})\|(a,u)^{\ell}\|_{\dot{B}^{\sigma_1}_{2,\infty}}\\ &\quad+(1+\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}})(\|(a,u)^{\ell}\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|(a,u)^{h}\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\|(a,u)^{h}\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\\ &\lesssim (\mathcal{X}_{p,0}+\mathcal{X}_{p,0}^2)(\|(a,u)^{\ell}\|_{L^\infty_t(\dot{B}^{\sigma_1+1-\sigma'}_{2,\infty})}+\mathcal{X}_{p,0}).\end{aligned} \tag{5.3}$$

Hence, for $0<t\le 2$, we arrive at

$$\int_0^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma')}\|F\|^{\ell}_{\dot{B}^{\sigma_1+1-\sigma'}_{2,\infty}}d\tau \lesssim \mathcal{X}_{p,0}(1+\mathcal{X}_{p,0})^2\langle t\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1)}, \tag{5.4}$$

where we have used (1.12).

To handle the integral in (5.2) for $t>t_0$, we divide it into two cases $\sigma_1<\sigma\le\frac{d}{2}-1$ and $\frac{d}{2}-1<\sigma\le\frac{d}{2}$.

Case 1: $\sigma_1<\sigma\le\frac{d}{2}-1$

We write (choosing $\sigma'=1$ in (4.27))

$$\begin{aligned}&\int_0^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\|F\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}d\tau\\ =\;&\int_0^{t_0}\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\|F\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}d\tau+\int_{t_0}^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\|F\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}d\tau. \end{aligned}\tag{5.5}$$

Arguing as (5.4) yields

$$\int_0^{t_0}\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\|F\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}d\tau\lesssim\mathcal{X}_{p,0}(1+\mathcal{X}_{p,0})^2\langle t\rangle^{-\frac12(\frac d2-\sigma_1)}.\tag{5.6}$$

On the other hand, employing the similar estimates as (4.45)-(4.52) gives that

$$\begin{aligned}\|F\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}&\lesssim(1+\|a\|_{\dot B^{\frac dp}_{p,1}})\|(a,u)^\ell\|_{\dot B^{\frac d2-1}_{2,1}}\|(a,u)^\ell\|_{\dot B^{\sigma_1+1}_{2,1}}\\&\quad+(1+\|a\|_{\dot B^{\frac dp}_{p,1}})(\|(a,u)^\ell\|_{\dot B^{\frac d2-1}_{2,1}}+\|(a,u)^h\|_{\dot B^{\frac dp}_{p,1}})\|(a,u)^h\|_{\dot B^{\frac dp-1}_{p,1}}\\&\lesssim(1+\mathcal{X}_{p,0})\langle t\rangle^{-\frac12(\frac d2-\sigma_1)}\end{aligned}\tag{5.7}$$

for $t>t_0$, where (1.13) and the fact that $\sigma_1<\frac d2-1$ were used. Consequently, it follows from (5.5) and (5.7) that

$$\int_{t_0}^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\|F\|^{\ell}_{\dot B^{\sigma_1}_{2,\infty}}d\tau\lesssim(1+\mathcal{X}_{p,0})\langle t\rangle^{-\frac12(\sigma-\sigma_1+\sigma_2)},$$

where we have performed the time-weighted inequality due to (4.25):

$$\begin{aligned}&\int_{t_0}^t\langle t-\tau\rangle^{-\frac12(\sigma-\sigma_1+1)}\langle\tau\rangle^{-\frac12(\frac d2-\sigma_1)}d\tau\\&\lesssim\begin{cases}\langle t\rangle^{-\frac12(\sigma-\sigma_1+1)}, & \text{if}\quad \frac12(\frac d2-\sigma_1)>1,\\ \langle t\rangle^{-\frac12(\sigma-\sigma_1+1)-}, & \text{if}\quad \frac12(\frac d2-\sigma_1)=1,\\ \langle t\rangle^{-\frac12(\sigma-\sigma_1+\frac d2-1-\sigma_1)}, & \text{if}\quad \frac12(\frac d2-\sigma_1)<1.\end{cases}\end{aligned}$$

Case 2: $\frac d2-1<\sigma\le\frac d2$

For brevity, we only deal with the integral on $[t_0,t]$ with $t>t_0$. We rewrite the nonlinear term F by

$$F=F_1^\ell+F_2^h$$

with

$$\begin{aligned}F_1^\ell&\triangleq(1+a)u^\ell\otimes u^\ell+(P''(1)+G(a))(a^\ell)^2\mathbb{I}_d+\mu\nabla(a^\ell u^\ell)+(\mu+\lambda)\mathrm{div}\,(a^\ell u^\ell)\mathbb{I}_d,\\ F_2^h&\triangleq(1+a)(u^h\otimes u^\ell+u\otimes u^h)+(P''(1)+G(a))(a^ha^\ell+aa^h)\mathbb{I}_d\\&\quad+\mu\nabla(u^ha^\ell+ua^h)+(\mu+\lambda)\mathrm{div}\,(u^ha^\ell+ua^h)\mathbb{I}_d.\end{aligned}$$

For those terms with low-frequencies, following from the line from (4.62), we choose $\sigma'=\sigma_2\in(0,1)$ in (5.2) and get

$$\begin{aligned}\|F_1^\ell\|^{\ell}_{\dot B^{\sigma_1+1-\sigma_2}_{2,\infty}}&\lesssim(1+\|a\|_{\dot B^{\frac dp}_{p,1}})\|(a,u)^\ell\|_{\dot B^{\sigma_1+1-\sigma_2}_{2,\infty}}\|(a,u)^\ell\|_{\dot B^{\frac d2}_{2,1}}\\&\lesssim\langle t\rangle^{-\frac12(\frac d2-\sigma_1+1-\sigma_2)}(1+\mathcal{X}_{p,0}),\quad t>t_0.\end{aligned}$$

The choice of $\sigma_2 \triangleq \min\left\{\frac{1}{2}, \left(\frac{d}{2}-1-\sigma_1\right)-\right\}$ indicates that $\sigma-\sigma_1+\sigma_2 \le \frac{d}{2}-\sigma_1+1-\sigma_2$ and $\frac{1}{2}(\frac{d}{2}-\sigma_1+1-\sigma_2)>1$ for $\frac{d}{2}-1<\sigma<\frac{d}{2}$. Consequently, we are led to

$$\begin{aligned}&\int_{t_0}^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}\|F_1^\ell\|^\ell_{\dot{B}^{\sigma_1+1-\sigma_2}_{2,\infty}}d\tau\\ &\quad\lesssim (1+\mathcal{X}_{p,0})\int_0^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}\langle\tau\rangle^{-\frac{1}{2}(\frac{d}{2}-\sigma_1+1-\sigma_2)}d\tau \lesssim (1+\mathcal{X}_{p,0})\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}.\end{aligned}$$

On the other hand, we take $\sigma'=1$ in (5.2) in order to bound those terms with high frequencies. Performing similar computations leading to (4.71)-(4.72) gives

$$\begin{aligned}\|F_2^h\|^\ell_{\dot{B}^{\sigma_1}_{2,\infty}} &\lesssim (1+\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\Big(\|(a,u)^h\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|(a,u)^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|(a,u)^h\|^2_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)\\ &\lesssim \langle t\rangle^{-\frac{1}{2}(d-1-\sigma_2)}(1+\mathcal{X}_{p,0}),\quad t>t_0,\end{aligned}$$

which enables us to obtain

$$\begin{aligned}&\int_{t_0}^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+1)}\|F_2^h\|^\ell_{\dot{B}^{\sigma_1}_{2,\infty}}d\tau\\ &\quad\lesssim (1+\mathcal{X}_{p,0})\int_{t_0}^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+1)}\langle t\rangle^{-\frac{1}{2}(d-1-2\sigma_1)}d\tau \lesssim (1+\mathcal{X}_{p,0})\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}.\end{aligned}$$

Indeed, owing to (4.25), we performed the following integral inequalities

$$\begin{aligned}&\int_0^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+1)}\langle\tau\rangle^{-\frac{1}{2}(d-1-2\sigma_1)}d\tau\\ &\quad\lesssim\begin{cases}\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+1)}, & \text{if } \frac{1}{2}(d-1-2\sigma_1)>1,\\ \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+1)-}, & \text{if } \frac{1}{2}(d-1-2\sigma_1)=1,\\ \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+d-2-2\sigma_1)}, & \text{if } \frac{1}{2}(d-1-2\sigma_1)<1,\end{cases}\end{aligned}$$

if $\frac{1}{2}(\sigma-\sigma_1+1)\le\frac{1}{2}(d-1-2\sigma_1)$ and

$$\begin{aligned}&\int_0^t \langle t-\tau\rangle^{-\frac{1}{2}(\sigma-\sigma_1+1)}\langle\tau\rangle^{-\frac{1}{2}(d-1-2\sigma_1)}d\tau\\ &\quad\lesssim\begin{cases}\langle t\rangle^{-\frac{1}{2}(d-1-2\sigma_1)}, & \text{if } \frac{1}{2}(\sigma-\sigma_1+1)>1,\\ \langle t\rangle^{-\frac{1}{2}(d-1-2\sigma_1)-}, & \text{if } \frac{1}{2}(\sigma-\sigma_1+1)=1,\\ \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+d-2-2\sigma_1)}, & \text{if } \frac{1}{2}(\sigma-\sigma_1+1)<1,\end{cases}\end{aligned}$$

if $\frac{1}{2}(\sigma-\sigma_1+1)>\frac{1}{2}(d-1-2\sigma_1)$, which are both controlled by $\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}$ due to the choice of σ_2 for $\sigma\in(\frac{d}{2}-1,\frac{d}{2}]$. Hence, the proof of Proposition 5.1 is complete. □

5.2 The implication of low-frequency assumptions

As the second step of Inverse Wiegner's argument is to show that the solution (a_L, m_L) to the linear problem (1.23), actually, has the same decay rates as the global-in-time solution (a,u) to the Cauchy problem (1.9) given by Theorem 1.1.

By employing similar estimates as (4.99)-(4.101), we see that $\|au(t)\|_{\dot{B}^{\sigma}_{2,1}}$ decays in time at the faster rate $\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\widetilde{\sigma})}$ with some $\widetilde{\sigma}>0$. This then gives, together with (1.13), that

$$
\begin{aligned}
\|m^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}} &\leq \|u^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}}+\|au(t)\|^{\ell}_{\dot{B}^{\sigma}_{2,1}}\\
&\lesssim \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}, \qquad t>t_0, \qquad \sigma_1<\sigma\leq \frac{d}{2}.
\end{aligned}
\tag{5.8}
$$

According to Proposition 5.1, we find that $(\widetilde{a},\widetilde{m})$ satisfies the faster decay (5.1). Furthermore, it follows from (1.13), (5.1) and (5.8) that

$$
\begin{aligned}
\|(a_L,m_L)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}} &\lesssim \|(a,m)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}}+\|(a-a_L,m-m_L)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}}\\
&\lesssim \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}+\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}\\
&\lesssim \langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}, \qquad t>t_0, \qquad \sigma_1<\sigma\leq \frac{d}{2}.
\end{aligned}
$$

Hence, the upper bound of decay estimates of (a_L,m_L) implies that $(a_0,u_0)^{\ell}\in \dot{B}^{\sigma_1}_{2,\infty}$ with $\sigma_0\leq\sigma_1<\frac{d}{2}-1$ with the aid of Proposition 4.1.

Next, we justify that $(a_0,u_0)^{\ell}\in \dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$ with $\sigma_0\leq\sigma_1<\frac{d}{2}-1$ provided that (a,u) satisfies (1.12) and (1.14). Notice that, for $\sigma_1<\sigma\leq\frac{d}{2}$,

$$
\|(a,m)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}}\geq \|(a,u)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}}-\|au(t)\|^{\ell}_{\dot{B}^{\sigma}_{2,1}}\geq \frac{1}{C_2}\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}, \qquad t>t_2 \tag{5.9}
$$

for suitably large $t_2>t_1$, where $C_2>1$ is chosen into a greater constant if necessary.

By using (5.1) and (5.9), we obtain

$$
\begin{aligned}
\|(a_L,m_L)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}} &\geq \|(a,m)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}}-\|(a-a_L,m-m_L)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}}\\
&\geq \frac{1}{C_2}\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}-C_3\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1+\sigma_2)}\\
&\geq \frac{1}{2C_2}\langle t\rangle^{-\frac{1}{2}(\sigma-\sigma_1)}
\end{aligned}
$$

for $t>t_3\triangleq\max\{t_2,(2C_2C_3)^{\frac{2}{\sigma_2}}\}$, where $C_3>0$ is some constant. Therefore, applying Proposition 4.1 again, we have $(a_0,u_0)^{\ell}\in\dot{\mathcal{B}}^{\sigma_1}_{2,\infty}$. This concludes the proof of Theorem 1.2.

6 Appendix

In the last section, we collect useful analysis tools which make the paper as self-contained as possible. The first lemma is devoted to the classical Bernstein's inequality.

Lemma 6.1. *Let $0<r<R, 1\leq p\leq q\leq\infty$ and $k\in\mathbb{N}$. Then for any function $u\in L^p$ and $\lambda_1>0$, it holds*

$$
\begin{cases}
\operatorname{Supp}\widehat{u}\subset\{\xi\in\mathbb{R}^d\mid |\xi|\leq\lambda_1 R\}\Rightarrow \|D^k u\|_{L^q}\lesssim \lambda_1^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p};\\
\operatorname{Supp}\widehat{u}\subset\{\xi\in\mathbb{R}^d\mid \lambda_1 r\leq|\xi|\leq\lambda_1 R\}\Rightarrow \|D^k u\|_{L^p}\sim \lambda_1^{k}\|u\|_{L^p}.
\end{cases}
$$

We state the interpolation inequality that is repeatedly used throughout the paper.

Lemma 6.2 ([1]). *The following real interpolation property is satisfied for* $1 \le p \le \infty$, $s_1 < s_2$ *and* $\theta \in (0,1)$:

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \lesssim \frac{1}{\theta(1-\theta)(s_2-s_1)} \|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}. \tag{6.1}$$

In addition, there are classical product estimates which play a fundamental role in bounding bilinear terms.

Lemma 6.3. *Let* $1 \le p, r \le \infty$. *Then*

$$\|FG\|_{\dot{B}^{s}_{p,r}} \lesssim \|F\|_{L^\infty} \|G\|_{\dot{B}^{s}_{p,r}} + \|G\|_{L^\infty} \|F\|_{\dot{B}^{s}_{p,r}}, \qquad \text{if } \ s > 0;$$

$$\|FG\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,1}} \lesssim \|F\|_{\dot{B}^{s_1}_{p,1}} \|G\|_{\dot{B}^{s_2}_{p,1}}, \qquad \text{if } \ s_1, s_2 \le \frac{d}{p} \ \text{and} \ \ s_1+s_2 > d\max\Big(0, \frac{2}{p}-1\Big);$$

$$\|FG\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,\infty}} \lesssim \|F\|_{\dot{B}^{s_1}_{p,1}} \|G\|_{\dot{B}^{s_2}_{p,\infty}}, \qquad \text{if } \ s_1 \le \frac{d}{p}, s_2 < \frac{d}{p} \ \text{and} \ \ s_1+s_2 \ge d\max\Big(0, \frac{2}{p}-1\Big).$$

In order to match different Lebesgue indices at low frequencies and high frequencies, non classical product estimates are further developed in the L^p framework (see [17, 52]). Precisely,

Lemma 6.4. *Let the real numbers* s_1, s_2, p_1 *and* p_2 *be such that*

$$s_1+s_2 > 0, \quad s_1 \le \frac{d}{p_1}, \quad s_2 \le \frac{d}{p_2}, \quad s_1 \ge s_2, \quad \frac{1}{p_1}+\frac{1}{p_2} \le 1.$$

Then it holds that

$$\|FG\|_{\dot{B}^{s_2}_{q,1}} \lesssim \|F\|_{\dot{B}^{s_1}_{p_1,1}} \|G\|_{\dot{B}^{s_2}_{p_2,1}} \quad \text{with} \quad \frac{1}{q} = \frac{1}{p_1}+\frac{1}{p_2}-\frac{s_1}{d}.$$

Additionally, for exponents $s > 0$ *and* $1 \le p_1, p_2, q \le \infty$ *satisfying*

$$\frac{d}{p_1}+\frac{d}{p_2}-d \le s \le \min\Big(\frac{d}{p_1}, \frac{d}{p_2}\Big) \quad \text{and} \quad \frac{1}{q} = \frac{1}{p_1}+\frac{1}{p_2}-\frac{s}{d},$$

one has

$$\|FG\|_{\dot{B}^{-s}_{q,\infty}} \lesssim \|F\|_{\dot{B}^{s}_{p_1,1}} \|G\|_{\dot{B}^{-s}_{p_2,\infty}}.$$

In particular, we have the following non product inequalities with respect to the regularity requirement in main results, which are employed in time-weighted energy methods.

Lemma 6.5. *Let* $\sigma_0 \le \sigma_1 < \frac{d}{2}-1$ *with* $\sigma_0 = \frac{d}{2}-\frac{2d}{p}$ *and* p *satisfy* (1.8). *It holds that*

$$\|FG\|_{\dot{B}^{\sigma_1}_{2,\infty}} \lesssim \|F\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|G\|_{\dot{B}^{\sigma_1}_{2,1}}, \tag{6.2}$$

$$\|FG\|_{\dot{B}^{\sigma_1+\frac{d}{p}-\frac{d}{2}}_{2,\infty}} \lesssim \|F\|_{\dot{B}^{\sigma_1+\frac{d}{p}-\frac{d}{2}}_{p,1}} \|G\|_{\dot{B}^{\frac{d}{p}}_{2,1}}. \tag{6.3}$$

In addition, we have

$$\|FG\|^{\ell}_{\dot{B}^{\sigma_0}_{2,\infty}} \lesssim \|F\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \|G\|_{\dot{B}^{1-\frac{d}{p}}_{p,1}} \tag{6.4}$$

for $2 \le p \le d$.

On the other hand, the third estimate in Lemma 6.3 can be also extended to the non classical form (see [51]).

Lemma 6.6. *Let the real numbers s_1, s_2, p_1 and p_2 be such that*

$$s_1+s_2\geq 0, \quad s_1\leq \frac{d}{p_1}, \quad s_2<\min\Big(\frac{d}{p_1},\frac{d}{p_2}\Big) \quad \text{and} \quad \frac{1}{p_1}+\frac{1}{p_2}\leq 1.$$

Then it holds that

$$\|FG\|_{\dot{B}^{s_1+s_2-\frac{d}{p_1}}_{p_2,\infty}}\lesssim \|F\|_{\dot{B}^{s_1}_{p_1,1}}\|G\|_{\dot{B}^{s_2}_{p_2,\infty}}. \tag{6.5}$$

Actually, we mainly employed the following product estimates.

Lemma 6.7. *Let $\sigma_0\leq\sigma_1<\frac{d}{2}-1$ and p satisfy (1.8). It holds that*

$$\|FG\|_{\dot{B}^{\sigma_1}_{2,\infty}}\lesssim \|F\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|G\|_{\dot{B}^{\sigma_1}_{2,\infty}}, \tag{6.6}$$

$$\|FG\|_{\dot{B}^{\sigma_1+\frac{d}{p}-\frac{d}{2}}_{2,\infty}}\lesssim \|F\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|G\|_{\dot{B}^{\sigma_1+\frac{d}{p}-\frac{d}{2}+1}_{2,\infty}}. \tag{6.7}$$

However, only resorting to Lemmas 6.4 and 6.6 is not enough to establish the desired decay estimates in particular in case of the oscillation case $p>d$, non standard product estimates with high frequencies are also needed (see [17]).

Lemma 6.8. *There exists a universal integer N_0 such that for any $2\leq p\leq 4$ and $s>0$, we have*

$$\|FG^h\|^{\ell}_{\dot{B}^{\sigma_0}_{2,\infty}}\leq C(\|F\|_{\dot{B}^{s}_{p,1}}+\|\dot{S}_{j_0+N_0}F\|_{L^{p^*}})\|G^h\|_{\dot{B}^{-s}_{p,\infty}} \tag{6.8}$$

$$\|F^hG\|^{\ell}_{\dot{B}^{\sigma_0}_{2,\infty}}\leq C(\|F^h\|_{\dot{B}^{s}_{p,1}}+\|\dot{S}_{j_0+N_0}F^h\|_{L^{p^*}})\,\|G\|_{\dot{B}^{-s}_{p,\infty}} \tag{6.9}$$

with $\sigma_0=\frac{d}{2}-\frac{2d}{p}$ and $\frac{1}{p^*}\triangleq\frac{1}{2}-\frac{1}{p}$, and C depending only on j_0, d and s.

System (1.9) also involves composite of functions (through $I(a)$ and $k(a)$) and they are bounded according to the following conclusion (see [1, 17]).

Lemma 6.9. *Let $F:\mathbb{R}\to\mathbb{R}$ be smooth with $F(0)=0$. For all $1\leq p,r\leq\infty$ and $s>0$, it holds that $F(u)\in\dot{B}^s_{p,r}\cap L^\infty$ for $u\in\dot{B}^s_{p,r}\cap L^\infty$, and*

$$\|F(u)\|_{\dot{B}^s_{p,r}}\leq C\,\|u\|_{\dot{B}^s_{p,r}}$$

with C depending only on $\|u\|_{L^\infty}$, F' (and higher derivatives), s, p and d.

In the case $s>-\min\left(\frac{d}{p},\frac{d}{p'}\right)$ then $u\in\dot{B}^s_{p,r}\cap\dot{B}^{\frac{d}{p}}_{p,1}$ implies that $F(u)\in\dot{B}^s_{p,r}\cap\dot{B}^{\frac{d}{p}}_{p,1}$, and

$$\|F(u)\|_{\dot{B}^s_{p,r}}\leq C(1+\|u\|_{\dot{B}^{\frac{d}{p}}_{p,1}})\|u\|_{\dot{B}^s_{p,r}}.$$

The following commutator estimates is useful to control nonlinearities in high frequencies ([1]):

Lemma 6.10. *Let $1 \le p \le \infty$ and $-\frac{d}{p} - 1 \le s \le 1 + \frac{d}{p}$. Then it holds*

$$\sum_{j\in\mathbb{Z}} 2^{js} \|[u, \dot{\Delta}_j]\partial_{x_i} a\|_{L^p} \lesssim \|u\|_{\dot{B}^{\frac{d}{p}+1}_{p,1}} \|a\|_{\dot{B}^s_{p,1}}, \qquad i = 1, 2, ..., d,$$

with the commutator $[A, B] \triangleq AB - BA$.

Finally, we present the endpoint maximal regularity property for the Lamé system below (see for instance [1]).

Lemma 6.11. *Let $T > 0$, $\mu > 0$, $2\mu + \lambda > 0$, $s \in \mathbb{R}$, $1 \le p, r \le \infty$ and $1 \le \varrho_2 \le \varrho_1 \le \infty$. Assume that $u_0 \in \dot{B}^s_{p,r}$ and $f \in \widetilde{L}^{\rho_2}(0, T; \dot{B}^{s-2+\frac{2}{\varrho_2}}_{p,r})$ hold. If u is a solution of*

$$\begin{cases} \partial_t u - \mu\Delta u - (\mu + \lambda)\nabla \mathrm{div}\, u = f, \qquad x \in \mathbb{R}^d, \quad t \in (0, T), \\ u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^d, \end{cases}$$

then u satisfies

$$\min\{\mu, 2\mu + \lambda\}^{\frac{1}{\varrho_1}} \|u\|_{\widetilde{L}^{\varrho_1}_T(\dot{B}^{s+\frac{2}{\varrho_1}}_{p,r})} \lesssim \|u_0\|_{\dot{B}^s_{p,r}} + \min\{\mu, 2\mu + \lambda\}^{\frac{1}{\varrho_1}-1} \|f\|_{\widetilde{L}^{\rho_2}_T(\dot{B}^{s-2+\frac{2}{\varrho_2}}_{p,r})}.$$

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