LARGE SELF-SIMILAR SOLUTIONS TO OBERBECK–BOUSSINESQ SYSTEM WITH NEWTONIAN GRAVITATIONAL FIELD

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ABSTRACT. The Navier-Stokes system for an incompressible fluid coupled with the equation for a heat transfer is considered in the whole three dimensional space. This system is invariant under a suitable scaling. Using the Leray–Schauder theorem and compactness arguments, we construct self-similar solutions to this system without any smallness assumptions imposed on homogeneous initial conditions.

1. Introduction

The Oberbeck–Boussinesq system is a mathematical model of a stratified flow, where the fluid is assumed to be incompressible and convecting by a diffusive quantity creating positive and negative buoyancy force. The diffusive quantity in this model is identified with the deviation of temperature from its equilibrium value and the resulting system has the following form

$$\begin{aligned} &\partial_t u + u\cdot\nabla u + \nabla p = \Delta u + \theta\nabla G + f, \quad x\in\mathbb{R}^3,\ t\in\mathbb{R}_+ \\ &\nabla\cdot u = 0, \\ &\partial_t\theta + u\cdot\nabla\theta = \Delta\theta, \end{aligned} \tag{1.1}$$

with the unknown fluid velocity $u = u(x,t)$, the temperature $\theta = \theta(x,t)$ and the pressure $p = p(x,t)$. In first equation of system (1.1), the symbol ∇G denotes a gravitational force acting on the fluid and $f = f(x,t)$ is a given external force. The equations in system (1.1) should contain important physical parameters such as is the viscosity coefficient, the heat conductivity coefficient, the fluid density, the reference temperature, the specific heat at constant pressure, the coefficient of thermal expansion of the fluid. However, since these physical constants do not play any role in this work, we put all of them equal to one, for simplicity of the exposition.

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As noticed by Feireisl and Schonbek [12], it is customary to take $\nabla G = g(0,0,-1)$ (where the constant g represents the acceleration rate caused by Earth's gravity) as the gravitational force acting on the fluid which is a reasonable approximation provided the fluid occupies a bounded domain, where the gravitational field can be taken constant. On the other hand, from a physical point of view, for system (1.1) in the whole space $\mathbb{R}^3$, it is better to consider the gravitational potential of the following form

$$G(x) = \int_{\mathbb{R}^3} \frac{1}{|x-y|} m(y)\, dy,$$

where m denotes the mass density of the object acting on the fluid by means of gravitation. Moreover, if the size of the object is negligible, we may choose $G(x) = |x|^{-1}$.

The Oberbeck–Boussinesq approximation (1.1) can be derived as a singular limit of the full Navier-Stokes–Fourier system with suitable boundary conditions and with the Mach and the Froude numbers tending to zero and when the family of domains on which the primitive problems are stated converges to the whole space $\mathbb{R}^3$, see [12, 26] and the references therein. Several authors studied properties solutions to system (1.1) with $\nabla G = g(0,0,-1)$ on the whole space $\mathbb{R}^3$, see *e.g* [5, 6, 10, 18, 24] and references therein. On the other hand, this work is devoted to system (1.1) with singular gravitational force $\nabla G(x) = \nabla |x|^{-1}$.

The Oberbeck-Boussinesq model enjoys the following scaling property: if (u,θ) is a solution of system (1.1) with

$$G(x) = \frac{1}{|x|} \quad \text{and} \quad f(x,t) = \frac{1}{(\sqrt{2t})^3} F\left(\frac{x}{\sqrt{2t}}\right), \tag{1.2}$$

then

$$u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t), \qquad \theta_\lambda(x,t) := \lambda \theta(\lambda x, \lambda^2 t) \tag{1.3}$$

is also a solution of the same system for each $\lambda > 0$. A *self-similar solution* to system (1.1)-(1.2) is, by definition, a solution which is left invariant by this rescaling: $(u,p,\theta) = (u_\lambda, \theta_\lambda)$ for every $\lambda > 0$. Equivalently, by choosing $\lambda = 1/\sqrt{2t}$, self-similar solutions are those that can be written in the form

$$u(x,t) = \frac{1}{\sqrt{2t}} U\Big(\frac{x}{\sqrt{2t}}\Big), \qquad \theta(x,t) = \frac{1}{\sqrt{2t}} \Theta\Big(\frac{x}{\sqrt{2t}}\Big), \tag{1.4}$$

with $U(x) = u(x,1/2)$ and $\Theta(x) = \theta(x,1/2)$.

If system (1.1) is supplemented with an initial condition

$$u(x,0) = u_0(x), \qquad \theta(x,0) = \theta_0(x), \tag{1.5}$$

in the case of a self-similar solution, it has to be also invariant under the scaling $(u_{0,\lambda}, \theta_{0,\lambda})(x) = \lambda(u_0, \theta_0)(\lambda x)$, which means that these are homogeneous function of degree -1. In this work, we construct self-similar solutions of system (1.1)-(1.2) with arbitrary, not necessarily small initial conditions (1.5) which are homogeneous of degree -1 and essentially bounded on the unit sphere of $\mathbb{R}^3$. The main result of this work is stated in the following theorem.

Theorem 1.1. *Let* $(u_0, \theta_0) \in \mathbf{L}^\infty_{\mathrm{loc}}(\mathbb{R}^3\backslash\{0\})$ *be homogeneous of degree* -1*, with* $\nabla \cdot u_0 = 0$*. Let the external force* $f(x,t)$ *be of the form as in* (1.2) *with the profile* $F \in L^p(\mathbb{R}^3)^3$ *with some* $p \in [6/5, 2]$*. Then there exists a self-similar solution* (u, p) *of system* (1.1)-(1.2)*. This solution has the following properties:*

- $(u, \theta) \in C_w([0, \infty), \mathbf{L}^{3,\infty}(\mathbb{R}^3))$*;*
- *for all* $t > 0$

$$\begin{aligned} &\|u(t) - e^{t\Delta}u_0\|_2 + \|\theta(t) - e^{t\Delta}\theta_0\|_2 \le Ct^{1/4}, \\ &\|\nabla u(t) - \nabla e^{t\Delta}u_0\|_2 + \|\nabla\theta(t) - \nabla e^{t\Delta}\theta_0\|_2 \le Ct^{-1/4}. \end{aligned} \tag{1.6}$$

Theorem 1.1 extends to the Oberbeck–Boussinesq system (1.1)-(1.2) the well known existence result of large (*i.e.* with no size restriction on the initial data) forward self-similar solutions to the Navier–Stokes equations, first established by Jia and Šverák [15].

Let us briefly review related results on self-similar solutions to the Navier-Stokes system and the Oberbeck–Boussinesq system. If initial data are sufficiently small, unique mild solutions to the Cauchy problem either for the Navier-Stokes system or the Oberbeck–Boussinesq system can be obtained via the contraction mapping argument applied to integral formulations of these problems. If the considered space has a scaling invariant norm and contains homogeneous initial conditions, the uniqueness property ensures that obtained solutions are self-similar. This approach was first applied to the Cauchy problem for the Navier-Stokes system by Giga-Miyakawa [14] and refined by Kato [19] and Cannone-Meyer-Planchon [8,9], see also *e.g.* [4,7,17,27] and references therein. Methods of constructing small self-similar solutions corresponding to small initial conditions were then applied to other models including the Oberbeck–Boussinesq, see *e.g.* [11,13,18].

For large initial conditions, the contraction mapping argument no longer works and a question on the existence of large self-similar solutions remained open until the seminal paper by Jia and Šverák [15] who constructed self-similar solutions of the three dimensional Navier-Stokes system, supplemented with a homogeneous not necessarily small initial condition which is Hölder continuous outside of the origin. In their construction, they used the theory of local-Leray solutions in L^2_{uloc} developed by Lemarié-Rieusset [22] and they obtained a local Hölder estimate for local-Leray solutions near $t = 0$, assuming minimal control of initial data. That estimate enables them to prove *a priori* estimates of self-similar solutions, and then to show their existence by the Leray-Schauder degree theorem. Results by Jia and Šverák [15] were then extended either to discretely self-similar solutions of the Navier-Stokes system or to the Navier-Stokes system in a half-space or to the fractional Navier-Stokes system in the works [1–3, 20, 21, 25].

Now let us describe the strategy of proving Theorem 1.1. Substituting expressions (1.4) into system (1.1)-(1.2) allows us to eliminate the time variable and we are led to construct a solution (U, P, Θ) to the following elliptic system

$$\begin{aligned} -\Delta U - U - (x\cdot\nabla)U + (U\cdot\nabla)U + \nabla P &= \Theta\,\nabla(|\cdot|^{-1}) + F, \\ \nabla\cdot U &= 0, \\ -\Delta\Theta - \Theta - (x\cdot\nabla)\Theta + \nabla(\Theta U) &= 0. \end{aligned} \tag{1.7}$$

In this construction, we use solutions to the heat equation given as convolutions of the Gauss-Weierstrass kernel with homogeneous initial data which are themselves of self-similar form: in particular, by our assumptions on u_0 and θ_0, we have

$$e^{t\Delta}u_0(x) = \frac{1}{\sqrt{2t}}U_0\left(\frac{x}{\sqrt{2t}}\right) \quad \text{and} \quad e^{t\Delta}\theta_0(x) = \frac{1}{\sqrt{2t}}\Theta_0\left(\frac{x}{\sqrt{2t}}\right), \tag{1.8}$$

where the self-similar profiles

$$U_0 := e^{\Delta/2}u_0 \quad \text{and} \quad \Theta_0 := e^{\Delta/2}\theta_0 \tag{1.9}$$

have properties recalled below in Proposition 2.1. Then, rather than studying directly system (1.7), we will consider the perturbations

$$V = U - U_0 \quad \text{and} \quad \Psi = \Theta - \Theta_0,$$

which satisfy the elliptic system

(1.10)
$$-\Delta V - V - (x\cdot\nabla)V + (V+U_0)\cdot\nabla(V+U_0) + \nabla P = (\Psi+\Theta_0)\,\nabla(|\cdot|^{-1}) + F,$$
$$\nabla\cdot V = 0,$$
$$-\Delta\Psi - \Psi - (x\cdot\nabla)\Psi + \nabla\cdot\big((\Psi+\Theta_0)(V+U_0)\big) = 0.$$

In the next section, we will construct solutions of system (1.10) in the Sobolev space $H^1(\mathbb{R}^3)^4$, see Theorem 2.6 below, and we deduce Theorem 1.1 as a direct corollary. Our strategy of studying system (1.10) is closely inspired by the paper of Korobkov and Tsai [20], where they established a similar result for the Navier–Stokes equations in the half-space. In that approach, we first solve system (1.10) supplemented with the Dirichlet boundary condition in a ball by using the Leray–Schauder theorem. Then, we obtain a solution in the whole space by passing with the radius of the ball to infinity and using an H^1-estimate of the sequence of solutions which is independent of the radius of the ball. The singular nature of the forcing term arising from the temperature variations and the coupling bring a few new technical difficulties. We overcome them by means of an approximation procedure and suitable *a priori* estimates, whose derivation do not appear to be so standard (see, *e.g.* the contradiction argument contained in the proofs of Propositions 2.2 and 2.5). Assuming that the components of the initial data are $L^\infty_{\rm loc}(\mathbb{R}^3\backslash\{0\})$ and not, e.g., only $L^2_{\rm uloc}$, considerably simplifies the presentation. This makes possible to provide a proof which is more elementary than that presented in [15, 23] in the case of the Navier–Stokes equations with external forces, despite the Oberbeck–Boussinesq (1.1)-(1.2) system is more general.

As the usual practice, we skip any comment about the pressure in this introduction, because it disappears in the weak formulation of system (1.1) and because it can be obtained in the well-known way by applying the divergence operator to first equation in system (1.1). Here, we only mention that the pressure corresponding to the self-similar solution constructed in Theorem 1.1 is self-similar of the form $p(x,t) = (2t)^{-1}P\Big(x/\sqrt{2t}\Big)$.

Notations. The symbol $\|\cdot\|_p$ denotes the usual Lebesgue L^p-norm. The space $L^{p,q}(\Omega)$ are the Lorentz spaces. If Ω is a domain of $\mathbb{R}^3$, then we denote by $C^\infty_{0,\sigma}(\Omega)^3$ the space of smooth, divergence-free vector fields, with support contained in Ω. We denote by $\mathbf{H}(\Omega)$ the closure of $C^\infty_{0,\sigma}(\Omega)^3\times C^\infty_0(\Omega)$ in the Sobolev space $H^1(\Omega)^4$. In general, we adopt bold symbols for function spaces of 4-dimensional vector-valued functions. For example, $\mathbf{L}^p(\Omega) = L^p(\Omega)^3 \times L^p(\Omega)$ and $\mathbf{C}^1(\Omega) = C^1(\Omega)^3 \times C^1(\Omega)$. Constants in estimates below are denoted by the same letter C, even if they vary from line to line.

2. Analysis of the perturbed elliptic system

We begin by establishing a few simple properties of self-similar solutions to the heat equation that will be useful in the sequel.

Proposition 2.1. *Let $(u_0, \theta_0) \in \mathbf{L}^\infty_{\rm loc}(\mathbb{R}^3 \backslash \{0\})$ be homogeneous of degree -1 with $\nabla \cdot u_0 = 0$ and consider the corresponding self-similar solutions of the heat equation $e^{t\Delta}u_0$ and $e^{t\Delta}\theta_0$ written in the form* (1.8)*. Then, the profiles U_0 and Θ_0 satisfy the following equations*

$$
\begin{aligned}
&U_0 + x\cdot\nabla U_0 + \Delta U_0 = 0, \qquad \nabla\cdot U_0 = 0,\\
&\Theta_0 + x\cdot\nabla \Theta_0 + \Delta\Theta_0 = 0
\end{aligned}
\tag{2.1}
$$

and the estimates

$$
\begin{aligned}
&|U_0(x)| + |\Theta_0(x)| \le C(1+|x|)^{-1},\\
&|\nabla U_0(x)| + |\nabla\Theta_0(x)| \le C(1+|x|)^{-1},
\end{aligned}
\tag{2.2}
$$

for all $x \in \mathbb{R}^3$ and some constant $C > 0$ independent on x.

Proof. From our assumptions we deduce that the map $x \mapsto |x|(|u_0(x)| + |\theta_0(x)|)$ is in $L^\infty(\mathbb{R}^3)$. Let us denote by $G_t(x) = (4\pi t)^{-3/2}\exp(-|x|^2/(4t))$ the heat kernel. We do have $U_0 = G_{1/2} * u_0$, $\Theta_0 = G_{1/2} * \theta_0$ and $\nabla U_0 = (\nabla G_{1/2}) * u_0$, $\nabla\Theta_0 = (\nabla G_{1/2}) * \theta_0$. But $(\nabla G_{1/2}, G_{1/2})$ belong to the space $\mathbf{L}^{3/2,1}(\mathbb{R}^3)$ (and to any other Lorentz space $L^{p,q}(\mathbb{R}^3)$, for $1 < p, q \le \infty$). As $(u_0, \theta_0) \in \mathbf{L}^{3,\infty}(\mathbb{R}^3)$, the Young inequality for Lorentz spaces (see [22, Ch. 2]) implies that $(U_0, \Theta_0) \in \mathbf{L}^\infty(\mathbb{R}^3)$. From a simple convolution estimate, relying on the fast decay of $G_{1/2}$ and $\nabla G_{1/2}$ at the spatial infinity, we easily deduce estimates (2.2). Moreover, as u_0 is divergence-free, it results that $\nabla \cdot U_0 = 0$. The two other equations in (2.1) are well known and follow from the scaling invariance of the heat equation. $\square$

2.1. **A priori estimates for a perturbed elliptic system in bounded domains.** First, we construct solutions to system (1.10) considered in a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary. For technical reasons, we introduce a smooth, bounded function $\rho \in C_b(\mathbb{R}^3)$ which will be used in the next section to cut off the singularity at zero of the potential $|\cdot|^{-1}$. In view of the application of the Leray-Schauder theorem, our first goal is to derive *a priori* estimates independent of $\lambda \in [0,1]$ of solutions to the system

(2.3)
$$\begin{aligned} -\Delta V + \nabla P &= \lambda\Big(V + x\cdot\nabla V + F_0 + F_1(V) + (\Psi+\Theta_0)\,\rho\nabla(|\cdot|^{-1}) + F\Big),\\ \nabla\cdot V &= 0, \qquad\qquad x\in\Omega\\ -\Delta\Psi &= \lambda\Big(\Psi + x\cdot\nabla\Psi - \nabla\cdot\big((\Psi+\Theta_0)(V+U_0)\big)\Big), \end{aligned}$$

where, for simplicity of notation, we set

$$F_0 := -U_0\cdot\nabla U_0, \quad \text{and} \quad F_1(V) := -(U_0+V)\cdot\nabla V + V\cdot\nabla U_0.$$

and which we supplement with the Dirichlet boundary conditions

$$V = 0 \qquad \text{and} \qquad \Psi = 0 \qquad\qquad \text{on} \quad \partial\Omega. \tag{2.4}$$

Proposition 2.2. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary and let $\rho\in C_b(\mathbb{R}^3)$. Assume that $(U_0,\Theta_0)\in \mathbf{L}^\infty(\mathbb{R}^3)$ does satisfy the decay estimates* (2.2) *and $F\in L^{6/5}(\Omega)^3$. Let $\lambda\in[0,1]$ and $(V,\Psi)\in\mathbf{H}(\Omega)$ be solutions to problem* (2.3)-(2.4)*. Then there exists a constant $C_0 = C_0(\Omega, F, \rho, U_0, \Theta_0)$, independent on λ, such that*

$$\int_\Omega \Big(|V|^2 + \Psi^2 + |\nabla V|^2 + |\nabla\Psi|^2\Big) \le C_0. \tag{2.5}$$

Proof. Step 1. Multiplying third equation in (2.3) by Ψ and integrating on Ω, after noticing that

$$\int_\Omega \nabla\cdot(\Psi(V+U_0))\Psi = 0$$

because both V and U_0 are divergence-free, we get

$$\int_\Omega |\nabla\Psi|^2 + \frac{\lambda}{2}\int_\Omega \Psi^2 + \lambda\int_\Omega \nabla\cdot[\Theta_0(V+U_0)]\Psi = 0. \tag{2.6}$$

The latter integral on the left-hand side can be estimated by

$$\frac12\int_\Omega |\nabla\Psi|^2 + \|\Theta_0\|_\infty^2\int_\Omega |V|^2 + \int_\Omega |\Theta_0 U_0|^2.$$

The latter integral is finite because, by (2.2), we have got $|\Theta_0 U_0|\in L^2(\mathbb{R}^3)$. Thus, as $0\le\lambda\le1$, we get the estimate

$$\frac12\int_\Omega |\nabla\Psi|^2 + \frac{\lambda}{2}\int_\Omega \Psi^2 \le \lambda\Big(\|\Theta_0\|_\infty^2\int_\Omega |V|^2 + \int_\Omega |\Theta_0 U_0|^2\Big). \tag{2.7}$$

By the Poincaré inequality and by estimate (2.7), we deduce that, in order to establish (2.5), it is sufficient to prove that

$$\int_\Omega |\nabla V|^2 \le C_1 \tag{2.8}$$

for some $C_1 = C_1(\Omega, c, \rho, U_0, \Theta_0) > 0$ independent on $\lambda \in [0,1]$.

Multiplying first equation of system (2.3) by V, after some integration by parts, we get

$$\begin{aligned}\int_\Omega |\nabla V|^2 + \frac{\lambda}{2}\int_\Omega |V|^2 =& \lambda\Big[\int_\Omega (F_0 + F_1(V))\cdot V - \int_\Omega (V\cdot\nabla U_0)\cdot V \\ &+ \int_\Omega \big((\Psi+\Theta_0)\rho\nabla|\cdot|^{-1}\big)\cdot V + \int_\Omega F\cdot V\Big].\end{aligned} \tag{2.9}$$

In the above identity, the convergence of the integral $\int_\Omega \big((\Psi+\Theta_0)\rho\nabla|\cdot|^{-1}\big)\cdot V$ deserves an explanation: the fact that $\Psi\rho\nabla|\cdot|^{-1}\cdot V$ and $\Theta_0\rho\nabla|\cdot|^{-1}\cdot V$ are both integrable on Ω follows from estimates (2.11)–(2.12) below. The other integrals in (2.9) are obviously convergent because of our assumptions on (U_0, Θ_0).

But rather than working directly with the energy inequality (2.9), we proceed by contradiction: assume that there exist a sequence $(\lambda_k) \subset [0,1]$ and a sequence of solutions $(V_k, \Psi_k) \subset \mathbf{H}(\Omega)$ to problem (2.3)-(2.4), with λ_k instead of λ, such that

$$J_k := \Big(\int_\Omega |\nabla V_k|^2\Big)^{1/2} \to +\infty.$$

We also set

$$L_k := \Big(\int_\Omega |\nabla \Psi_k|^2\Big)^{1/2}$$

and we introduce the normalized functions

$$\widehat{V}_k = \frac{V_k}{J_k} \quad \text{and} \quad \widehat{\Psi}_k = \frac{\Psi_k}{L_k},$$

so that $(\widehat{V}_k, \widehat{\Psi}_k)$ is a bounded sequence in $\mathbf{H}(\Omega)$. After extracting a suitable subsequence we can assume that $(\widehat{V}_k, \widehat{\Psi}_k) \to (\widetilde{V}, \widetilde{\Psi})$ weakly in $\mathbf{H}(\Omega)$ and strongly in $\mathbf{L}^p(\Omega)$, for $p \in [2,6)$. We can also assume that $\lambda_k \to \lambda_0$, for some $\lambda_0 \in [0,1]$.

Step 2. Excluding the case: $\limsup_{k\to+\infty} J_k/L_k < \infty$.

If by contradiction, $\limsup_{k\to+\infty} J_k/L_k < \infty$, then after a new extraction of a subsequence, we can assume that there exists $\gamma \ge 0$ such that

$$J_k/L_k \to \gamma.$$

In fact, $\gamma > 0$ by estimate (2.7). Moreover, as $J_k \to +\infty$, we must have $L_k \to +\infty$. Equation (2.6) holds with (V_k, Ψ_k) instead of (V, Ψ), namely

$$\int_\Omega |\nabla \Psi_k|^2 + \frac{\lambda_k}{2} \int_\Omega |\Psi_k|^2 + \lambda_k \int_\Omega \nabla \cdot [\Theta_0 (V_k + U_0)] \Psi_k = 0.$$

After dividing term-by-term by L_k^2 and taking the limit as $k \to +\infty$, using that $\widehat{\Psi}_k \to \widetilde{\Psi}$ weakly in $H_0^1(\Omega)$, $\widehat{V}_k \to \widetilde{V}$ strongly in $L^2(\Omega)^3$ and $\Theta_0 \in L^\infty(\Omega)$ as well as the fact that $\Theta_0 U_0 \in L^2(\mathbb{R}^3)$, implies

$$\frac{1}{L_k^2} \Big| \int_\Omega \nabla \cdot (\Theta_0 U_0) \Psi_k \Big| \leq \frac{C}{L_k} \|\Theta_0 U_0\|_{L^2(\Omega)} \to 0.$$

Hence, we get the equation

$$1 + \frac{\lambda_0}{2} \int_\Omega |\widetilde{\Psi}|^2 = -\lambda_0 \gamma \int_\Omega \nabla \cdot (\Theta_0 \widetilde{V})\, \widetilde{\Psi}. \tag{2.10}$$

The weak formulation of the third equation of (2.3), written for (V_k, Ψ_k) and λ_k, gives, for all $\chi \in C_0^\infty(\Omega)$,

$$\int_\Omega \nabla \Psi_k \cdot \nabla \chi = \lambda_k \Big[\int_\Omega [\Psi_k + x \cdot \nabla \Psi_k] \chi - \int_\Omega \nabla \cdot [(\Psi_k + \Theta_0)(V_k + U_0)] \chi \Big].$$

Let us divide this identity by L_k^2 and let $k \to +\infty$. All the terms $\frac{1}{L_k^2} \int_\Omega$ which are linear with respect to $\{V_k, \Psi_k\}$ tend to zero. Hence, in the limit, we find

$$\lambda_0 \gamma \int_\Omega \nabla \cdot (\widetilde{\Psi} \widetilde{V}) \chi = 0 \qquad \text{for all } \chi \in C_0^\infty(\Omega).$$

But $\lambda_0 \gamma \neq 0$ by equation (2.10), so

$$\int_\Omega \nabla \cdot (\widetilde{\Psi} \widetilde{V}) \chi = 0 \qquad \text{for all } \chi \in C_0^\infty(\Omega).$$

This in turn implies that

$$\widetilde{V} \cdot \nabla \widetilde{\Psi} = 0.$$

But then

$$\int_\Omega \nabla \cdot (\Theta_0 \widetilde{V}) \widetilde{\Psi} = -\int_\Omega \Theta_0 \widetilde{V} \cdot \nabla \Psi = 0$$

which contradicts equation (2.10). This excludes that $\limsup_{k \to +\infty} J_k / L_k < \infty$.

Step 3. We reduced ourselves to the case $\limsup_{k \to +\infty} J_k / L_k = +\infty$. After extracting a new subsequence, we can assume that $L_k / J_k \to 0$. Equation (2.9) holds true for (V_k, Ψ_k) and λ_k instead of (V, Ψ) and λ. Let us divide it by J_k^2 and study the limit of each term, as $k \to +\infty$. We have

$$\frac{1}{J_k^2} \int_\Omega |\nabla V_k|^2 = 1, \qquad \frac{1}{J_k^2} \int_\Omega |V_k|^2 \to \int_\Omega |\widetilde{V}|^2, \qquad \frac{1}{J_k^2} \int_\Omega F_0 \cdot V_k \to 0,$$

and

$$\frac{1}{J_k^2}\int_\Omega F\cdot V_k \to 0, \qquad \frac{1}{J_k^2}\int_\Omega (V_k\cdot\nabla U_0)\cdot V_k \to \int_\Omega (\widetilde{V}\cdot\nabla U_0)\cdot\widetilde{V},$$

because $|\nabla U_0|\in L^2(\Omega)$ and $\widehat{V}_k\to\widetilde{V}$ strongly in $L^p(\Omega)^3$ for $p\in[2,6)$. For the next term, we rely on the Hardy inequality: as Ψ_k and V_k belong to $H^1_0(\Omega)$ we can write

$$\begin{aligned}\frac{1}{J_k^2}\Big|\int_\Omega \Psi_k\rho\nabla\big(|\cdot|^{-1}\big)\cdot V_k\Big| &\le \frac{C}{J_k^2}\Big\|\frac{\Psi_k}{|\cdot|}\Big\|_{L^2(\Omega)}\Big\|\frac{V_k}{|\cdot|}\Big\|_{L^2(\Omega)}\\ &\le \frac{C}{J_k^2}\|\nabla\Psi_k\|_{L^2(\Omega)}\|\nabla V_k\|_{L^2(\Omega)}\\ &\le \frac{C}{J_k}\|\nabla\Psi_k\|_{L^2(\Omega)} = C\frac{L_k}{J_k}\to 0.\end{aligned} \tag{2.11}$$

The function Θ_0 does not belong to $H^1_0(\Omega)$, so we deal in a slightly different way with the term of (2.9) containing Θ_0. First of all, since $\Theta_0\in L^{3,\infty}\cap L^\infty(\mathbb{R}^3)$, by the real interpolation of Lorentz spaces, we get $\Theta_0\in L^{6,1}(\mathbb{R}^3)$. (See *e.g.*, [22, Ch. 2]). Then, by the Hölder inequality in Lorentz spaces (see, *e.g.*, again [22, Ch. 2]),

$$\begin{aligned}\frac{1}{J_k^2}\Big|\int_\Omega \Theta_0\rho\nabla\big(|\cdot|^{-1}\big)\cdot V_k\Big| &\le \frac{C}{J_k^2}\Big\|\Theta_0\Big\|_{L^{6,1}(\Omega)}\|\nabla(|\cdot|^{-1})\|_{L^{3/2,\infty}(\Omega)}\|V_k\|_{L^6(\Omega)}\\ &\le \frac{C}{J_k^2}\|\nabla V_k\|_{L^2(\Omega)} = \frac{C}{J_k}\to 0.\end{aligned} \tag{2.12}$$

The above calculations lead to the identity

$$1+\frac{\lambda_0}{2}\int_\Omega|\widetilde{V}|^2 = -\lambda_0\int_\Omega(\widetilde{V}\cdot\nabla U_0)\cdot\widetilde{V}, \tag{2.13}$$

which implies $\lambda_0\neq 0$. So, for large enough k, we have $\lambda_k\neq 0$ and we can normalize the pressure putting

$$\widehat{P_k} := \frac{P_k}{\lambda_k J_k^2}.$$

We now go back to first equation in (2.3), written for (V_k,Ψ_k) and λ_k instead of (V,Ψ) and λ. Dividing by $\lambda_k J_k^2$ we obtain

$$\begin{aligned}\widehat{V}_k\cdot\nabla\widehat{V}_k+\nabla\widehat{P}_k =& \frac{1}{J_k}\Bigg(\frac{\Delta\widehat{V}_k}{\lambda_k}+\widehat{V}_k + x\cdot\nabla\widehat{V}_k+\frac{F_0}{J_k}\\ &- U_0\cdot\nabla\widehat{V}_k - \widehat{V}_k\cdot\nabla U_0 + \Big(\frac{L_k}{J_k}\widehat{\Psi}_k+\frac{\Theta_0}{J_k}\Big)\rho\nabla(|\cdot|^{-1})+\frac{F}{J_k}\Bigg).\end{aligned}$$

More precisely, we consider the weak formulation of the above equation: testing with an arbitrary solenoidal vector field $\eta\in C^\infty_{0,\sigma}(\mathbb{R}^3)^3$, after integrating on Ω and letting

$k \to +\infty$, the terms obtained in the right-hand side vanish, in the limit. Indeed, for the two last terms, following the calculations in (2.12), we obviously have

$$\frac{1}{J_k^2}\int_\Omega \big(\Theta_0\rho\nabla(|\cdot|^{-1}) + F\big)\cdot\eta \to 0.$$

Moreover, we also have

$$\frac{L_k}{J_k^2}\int \widehat{\Psi}_k\rho\nabla(|\cdot|^{-1})\cdot\eta \to 0$$

because $L_k/J_k^2 \to 0$, ρ is a bounded function, and $|\int_\Omega \widehat{\Psi}_k\nabla(|\cdot|^{-1})\cdot\eta|$ can be bounded uniformly with respect to k applying the Hardy inequality as in (2.11). The terms obtained testing with η the other terms on the right-hand side also vanish (because $\widehat{V}_k$ is bounded in $H^1_0(\Omega)^3$ and $|U_0|$ and $|\nabla U_0|$ are both in $L^\infty(\Omega)$, and because $J_k \to +\infty$). But $\int_\Omega \nabla\widehat{P}_k\cdot\eta = 0$, therefore, we find in the limit

$$\int_\Omega (\widetilde{V}\cdot\nabla\widetilde{V})\cdot\eta = 0, \qquad \text{for all } \eta \in C^\infty_{0,\sigma}(\mathbb{R}^3)^3. \tag{2.14}$$

This means that $\widetilde{V} \in H^1_0(\Omega)$ is a stationary solution of the Euler equations. At this stage, the proof can be finished exactly as in the paper by Korobkov and Tsai [20]: there exists $\widetilde{P} \in L^3(\Omega)$, such that $\|\nabla\widetilde{P}\|_{L^{3/2}(\Omega)} < \infty$, satisfying

$$\begin{cases} \widetilde{V}\cdot\nabla\widetilde{V} = -\nabla\widetilde{P} & \text{in } \Omega\\ \nabla\cdot\widetilde{V} = 0 & \text{in } \Omega\\ \widetilde{V} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, going back to (2.13), and using once more that $U_0, \nabla U_0$ are in $L^\infty(\mathbb{R}^3)$, we find

$$\begin{aligned} 1 + \frac{\lambda_0}{2}\int_\Omega |\widetilde{V}|^2 &= -\lambda_0\int_\Omega (\widetilde{V}\cdot\nabla U_0)\cdot\widetilde{V} = \lambda_0\int_\Omega (\widetilde{V}\cdot\nabla\widetilde{V})\cdot U_0\\ &= -\lambda_0\int_\Omega \nabla\widetilde{P}\cdot U_0 = -\lambda_0\int_\Omega \nabla\cdot(PU_0)\\ &= 0. \end{aligned}$$

The last equality relies on a classical result on the stationary Euler equation [16, Lemma 4], implying that in addition to the above properties, the pressure P can be taken additionally such that $P(x) \equiv 0$ a.e. on $\partial\Omega$, with respect to the two-dimensional Hausdorff measure. From the last equality we get a contradiction. □

2.2. **Existence of solutions to the perturbed elliptic system in bounded domains.** Let Ω be a bounded domain with a smooth boundary. As before, we take $\rho \in C_b(\mathbb{R}^3)$, but here we additionally assume that the support of ρ does not contain the origin, in a such way that $\rho\nabla(|\cdot|^{-1}) \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Let us define the linear map L_ρ and the nonlinear map N,

$$L_\rho(V,\Psi) := \Big(V + x\cdot\nabla V - U_0\cdot\nabla V + V\cdot\nabla U_0 + \Psi\rho\nabla(|\cdot|^{-1}),$$

$$\Psi + x\cdot\nabla\Psi - \nabla\cdot\big(\Psi U_0 + \Theta_0 V\big)\Big)$$

$$N(V,\Psi) := \Big(-U_0\cdot\nabla U_0 - V\cdot\nabla V, \nabla\cdot(\Psi V + \Theta_0 U_0)\Big).$$

We also introduce the following nonlinear map

$$G_\rho(V,\Psi) := L_\rho(V,\Psi) + N(V,\Psi). \tag{2.15}$$

In this way, our system (2.3), in the case $\lambda = 1$, can be rewritten as

$$(-\Delta V + \nabla P - F, -\Delta\Psi) = G_\rho(V,\Psi). \tag{2.16}$$

We endow the dual space $\mathbf{H}(\Omega)'$ with the usual norm of dual Banach spaces.

Lemma 2.3. *Let Ω be a bounded domain with a smooth boundary and $\rho \in C_b(\mathbb{R}^3)$, such that $0 \notin \mathrm{supp}(\rho)$. The nonlinear map G_ρ is well defined as a map $G_\rho\colon \mathbf{H}(\Omega) \to \mathbf{L}^{3/2}(\Omega)$ and is compact as a map $G_\rho\colon \mathbf{H}(\Omega) \to \mathbf{H}(\Omega)'$.*

Proof. Notice that $G_\rho\colon \mathbf{H}(\Omega) \to \mathbf{L}^{3/2}(\Omega)$ is well defined. Indeed, if $(V,\Psi) \in \mathbf{H}(\Omega) \subset \mathbf{L}^6(\Omega)$, then the components of $V\cdot\nabla V$ and $\nabla\cdot(\Psi V)$ do belong to $L^{3/2}(\Omega)$. In the same way, using the conditions (2.2) on U_0 and Θ_0, one easily checks that all the other terms defining $G_\rho(V,\Psi)$ belong also to $L^{3/2}(\Omega)$ (or even to a smaller space). The presence of the function ρ cutting out the singularity of $\nabla(|\cdot|^{-1})$ near the origin is important here.

As the Sobolev embedding $\mathbf{H}(\Omega) \subset \mathbf{L}^6(\Omega)$ is continuous, by the previous considerations the map $G_\rho\colon \mathbf{H}(\Omega) \to \mathbf{L}^{3/2}(\Omega)$ is continuous. Moreover, every function $f \in \mathbf{L}^{3/2}(\Omega)$ can be identified to an element of $\mathbf{H}(\Omega)'$ through the usual duality $h \mapsto \int_\Omega f\cdot h$, where $h \in \mathbf{H}(\Omega)$. Adopting this identification, it results that the map $G_\rho\colon \mathbf{H}(\Omega) \to \mathbf{H}(\Omega)'$ is continuous.

Let us prove that, in fact, $G_\rho\colon \mathbf{H}(\Omega) \to \mathbf{H}'(\Omega)$ is a compact operator. For every $(V,\Psi), (\widetilde{V},\widetilde{\Psi}) \in \mathbf{H}(\Omega)$, denoting

$$v := \widetilde{V} - V, \quad \text{and} \quad \psi := \widetilde{\Psi} - \Psi,$$

we have,

$$N(\widetilde{V},\widetilde{\Psi}) - N(V,\Psi) = \Big(-(v+V)\cdot\nabla v - v\cdot\nabla V, \nabla\cdot((\psi+\Psi)v+\psi V)\Big).$$

Now, if (V_k,Ψ_k) is a bounded sequence in $\mathbf{H}(\Omega)$, with $\|(V_k,\Psi_k)\|_{\mathbf{H}(\Omega)} \le R$, then there exists $(\widetilde{V},\widetilde{\Psi}) \in \mathbf{H}(\Omega)$ such that, after extraction of a subsequence,

$$(v_k,\psi_k) := (\widetilde{V}-V_k, \widetilde{\Psi}-\Psi_k) \to 0 \text{ weakly in } \mathbf{H}(\Omega) \text{ and strongly in } \mathbf{L}^3(\Omega).$$

For any $\Phi \in \mathbf{H}(\Omega)$, we have, after some integration by parts,

$$\Big|\int_\Omega N(\widetilde{V},\widetilde{\Psi})\cdot\Phi - \int_\Omega N(V_k,\Psi_k)\cdot\Phi\Big| \le CR\big(\|v_k\|_3 + \|\psi_k\|_3\big)\|\Phi\|_{\mathbf{H}(\Omega)}$$

for some constant $C>0$ independent on k and Φ. It follows that

$$\begin{aligned}\|N(\widetilde{V},\widetilde{\Psi}) - N(V_k,\Psi_k)\|_{\mathbf{H}'(\Omega)} &:= \sup_{\|\Phi\|_{\mathbf{H}(\Omega)}=1} \Big|\int_\Omega [N(\widetilde{V},\widetilde{\Psi}) - N(V_k,\Psi_k)]\cdot\Phi\Big| \\ &\le CR\big(\|v_k\|_3 + \|\psi_k\|_3\big) \to 0.\end{aligned}$$

For the linear terms we have also,

$$\|L_\rho(V_k,\Psi_k) - L_\rho(\widetilde{V},\widetilde{\Psi})\|_{\mathbf{H}(\Omega)'} = \|L_\rho(v_k,\psi_k)\|_{\mathbf{H}(\Omega)'} \to 0,$$

as one can check using conditions (2.2). Here the condition $0 \notin \mathrm{supp}(\rho)$ is useful to prove that $\|\psi_k\,\rho\nabla(|\cdot|^{-1})\|_{H^1(\Omega)'} \to 0$, which is part for the previous claim; no other difficulty arises for the other terms of $L_\rho(v_k,\psi_k)$.

Hence,

$$G_\rho(V_k,\Psi_k) \to G_\rho(\widetilde{V},\widetilde{\Psi}) \qquad \text{strongly in } \mathbf{H}(\Omega)'.$$

□

Proposition 2.4. *Let Ω be a bounded domain with a smooth boundary and $\rho \in C_b^\infty(\mathbb{R}^3)$, such that $0 \notin \mathrm{supp}(\rho)$. Let (U_0,Θ_0) be as in (2.2) and assume that $F \in L^{6/5}(\Omega)^3$. Then the system*

(2.17)

$$\begin{aligned}-\Delta V + \nabla P &= V + x\cdot\nabla V + F_0 + F_1(V) + (\Psi+\Theta_0)\,\rho\nabla(|\cdot|^{-1}) + F \\ \nabla\cdot V &= 0, \qquad\qquad\qquad\qquad\qquad\qquad x\in\Omega, \\ -\Delta\Psi &= \Psi + x\cdot\nabla\Psi - \nabla\cdot\big((\Psi+\Theta_0)(V+U_0)\big)\end{aligned}$$

supplemented with the Dirichlet boundary conditions

$$V = 0, \qquad \Psi = 0 \qquad \text{on} \qquad \partial\Omega, \tag{2.18}$$

has a solution $(V,\Psi) \in \mathbf{H}(\Omega)$.

Proof. Let $T\colon \mathbf{H}(\Omega)' \to \mathbf{H}(\Omega)$ be the isomorphism given by the Riesz representation theorem for Hilbert spaces, where the Hilbert space $\mathbf{H}(\Omega)$ is endowed with the scalar product

$$\big((V,\Psi),(V',\Psi')\big) \mapsto \int_\Omega \nabla V\cdot\nabla V' + \int_\Omega \nabla\Psi\cdot\nabla\Psi'.$$

By assumption on F, and the usual identification of $L^{6/5}(\Omega)$ functions as elements of the dual of $H^1_0(\Omega)$, we have $(F,0)\in \mathbf{H}(\Omega)'$ (the fourth component is zero because we considered no forcing term in the equation of the temperature). The weak formulation of equation (2.16) reads

$$(V,\Psi) = T(G_\rho(V,\Psi)) + T((F,0)). \tag{2.19}$$

By Lemma 2.3, the nonlinear map $T\circ G_\rho\colon \mathbf{H}(\Omega)\to\mathbf{H}(\Omega)$ is compact. Hence, the map $(V,\Psi)\mapsto T(G_\rho(V,\Psi))+T((F,0))$ is compact on $\mathbf{H}(\Omega)$.

For every $\lambda\in[0,1]$, if $(V,\Psi)=\lambda(T\circ G_\rho)(V,\Psi)+\lambda T((F,0))$, *i.e.*, if (V,Ψ) is a solution of (2.3), then $\|(V,\Psi)\|_{\mathbf{H}(\Omega)}\le C_0$, where C_0 is the constant, independent on λ, obtained in Proposition 2.2. The Leray-Schauder fixed-point theorem (see *e.g.* [23, p.529]) then implies that the map $(V,\Psi)\mapsto T(G_\rho(V,\Psi))+T((F,0))$ has a fixed point $(V_\rho,\Psi_\rho)\in\mathbf{H}(\Omega)$, such that $\|(V_\rho,\Psi_\rho)\|_{\mathbf{H}(\Omega)}\le C_0$. □

2.3. **Existence of solutions of the perturbed elliptic system in the whole space.** In this subsection we choose, once for all, a cut-off function $\rho\in C^\infty_b(\mathbb{R}^3)$ such that, $\rho(x)=0$ if $|x|\le 1/2$, $0\le\rho(x)\le 1$ if $1/2\le|x|\le 1$ and $\rho(x)=1$ if $|x|\ge 1$. Then we set, for $x\in\mathbb{R}^3$,

$$\rho_k(x) := \rho(kx),$$

so that $0\le\rho_k(x)\le 1$ and $\rho_k\to 1$ a.e. in $\mathbb{R}^3$ as $k\to+\infty$.

Proposition 2.5. *Let* (U_0,Θ_0) *be as in* (2.2) *and* $F\in L^p(\mathbb{R}^n)^3$*, for some* $p\in[6/5,2]$*. Let* $k\in\mathbb{N}$*,* $k\ge 1$ *and* B_k *the open ball of* $\mathbb{R}^3$ *centered at the origin and of radius* k*. Let* $(V_k,\Psi_k)\in\mathbf{H}(B_k)$ *be a solution of problem* (2.17)-(2.18) *with* $\Omega=B_k$ *and* $\rho=\rho_k$*. Then there exists a constant* $C_1=C_1(F,U_0,\Theta_0)>0$*, independent on* k*, such that*

$$\int_{B_k}\Big(|V_k|^2+\Psi_k^2+|\nabla V_k|^2+|\nabla\Psi_k|^2\Big)\le C_1. \tag{2.20}$$

Proof. The proof has a similar structure to that of Proposition 2.2.

Step 1. First of all, by estimates (2.2) and (2.7), in the case $\Omega = B_k$ and $\lambda = 1$, we have

$$\int_{B_k} |\nabla\Psi_k|^2 + \int_{B_k} \Psi_k^2 \leq 2\Big(\|\Theta_0\|^2_{L^\infty(\mathbb{R}^3)} \int_{B_k} |V_k|^2 + \int_{\mathbb{R}^3} |\Theta_0 U_0|^2\Big). \tag{2.21}$$

Hence, it is sufficient to prove that

$$\int_{B_k} \big(\frac{1}{2}|V_k|^2 + |\nabla V_k|^2\big) \leq C_1. \tag{2.22}$$

With a slight change of notations with respect to Proposition 2.2, we now set

$$J_k := \Big(\int_{B_k} \big(\frac{1}{2}|V_k|^2 + |\nabla V_k|^2\big)\Big)^{1/2}, \quad \text{and} \quad L_k := \Big(\int_{B_k} \big(\frac{1}{2}\Psi_k^2 + |\nabla\Psi_k|^2\big)\Big)^{1/2}$$

and

$$\widehat{V}_k := \frac{V_k}{J_k}, \quad \widehat{P}_k = \frac{P_k}{J_k^2}, \quad \text{and} \quad \widehat{\Psi}_k := \frac{\Psi_k}{L_k}.$$

Let us assume, by contradiction, that (2.22) does not hold. Thus, there exists a subsequnce of solutions $(V_k, \Psi_k) \in \mathbf{H}(B_k)$ of problem (2.17)-(2.18) with $\Omega = B_k$ and $\rho = \rho_k$ such that

$$J_k \to +\infty.$$

The boundedness of the sequence $(\widehat{V}_k, \widehat{\Psi}_k)$ in $\mathbf{H}(B_k)$ (or, more precisely, of the sequence obtained extending $(\widehat{V}_k, \widehat{\Psi}_k)$ to the whole $\mathbb{R}^3$ via the classical extension theorem for Sobolev spaces), implies that there exists $(\widetilde{V}, \widetilde{\Psi}) \in \mathbf{H}(\mathbb{R}^3)$, such that, after extraction of a subsequence,

$$(\widehat{V}_k, \widehat{\Psi}_k) \to (\widetilde{V}, \widetilde{\Psi})$$

weakly in $\mathbf{H}(\mathbb{R}^3)$ and strongly in $\mathbf{L}^p_{\text{loc}}(\mathbb{R}^3)$, for $2 \leq p < 6$. The divergence-free condition for $\widetilde{V}$ follows from the fact that, for every test function $\varphi \in C_0^\infty(\mathbb{R}^3)$, one has $\int \widetilde{V} \cdot \nabla\varphi = \lim_k \int \widetilde{V}_k \cdot \nabla\varphi = 0$.

Step 2. Excluding that $\limsup_{k\to+\infty} J_k/L_k < \infty$*.* Assume for the moment that $\limsup_{k\to+\infty} J_k/L_k < \infty$. So we must have also $L_k \to +\infty$. Then, after extraction, we have $J_k/L_k \to \gamma$, for some real $\gamma > 0$, because of inequality (2.21). But the following identity holds true, just like (2.6),

$$\int_{B_k} |\nabla\Psi_k|^2 + \frac{1}{2}\int_{B_k} \Psi_k^2 + \int_{B_k} \nabla\cdot[\Theta_0(V_k + U_0)]\Psi_k = 0. \tag{2.23}$$

Let us divide it by L_k^2 and take $k \to +\infty$. We claim that

$$\frac{1}{L_k^2}\int_{B_k} \nabla\cdot(\Theta_0 V_k)\Psi_k = \frac{1}{L_k^2}\int_{B_k} \nabla\Theta_0 \cdot V_k\Psi_k \to \gamma\int_{\mathbb{R}^3} (\nabla\Theta_0 \cdot \widetilde{V})\widetilde{\Psi}.$$

Indeed, let Ω be a bounded domain and k large enough so that $\Omega \subset B_k$. We make use of the fact that condition (2.2) implies $\nabla\Theta_0 \in L^4(\mathbb{R}^3)$, and that $H^1(\mathbb{R}^3)$ is continuously embedded in $L^{8/3}(\mathbb{R}^3)$. So we have

$$
\begin{aligned}
&\Big|\frac{1}{L_k^2}\int_{B_k}(\nabla\Theta_0\cdot V_k)\Psi_k - \gamma\int_{\mathbb{R}^3}(\nabla\Theta_0\cdot\widetilde{V})\widetilde{\Psi}\Big| \\
&\qquad \le \Big|\frac{J_k}{L_k}\int_{B_k}(\nabla\Theta_0\cdot\widehat{V}_k)\widehat{\Psi}_k - \gamma\int_{B_k}(\nabla\Theta_0\cdot\widetilde{V})\widetilde{\Psi}\Big| \\
&\qquad\qquad + \|\nabla\Theta_0\|_{L^4(B_k^c)}\|\widetilde{V}\|_{L^{8/3}(\mathbb{R}^3)}\|\widetilde{\Psi}\|_{L^{8/3}(\mathbb{R}^3)} \\
&\qquad \le \Big|\int_\Omega\Big(\frac{J_k}{L_k}(\nabla\Theta_0\cdot\widehat{V}_k)\widehat{\Psi}_k - \gamma(\nabla\Theta_0\cdot\widetilde{V})\widetilde{\Psi}\Big)\Big| + C\|\nabla\Theta_0\|_{L^4(\Omega^c)},
\end{aligned}
\tag{2.24}
$$

because $\widehat{V}_k$ and $\widehat{\Psi}_k$ can be extended to $\mathbb{R}^3$ and that such extensions are bounded in $H^1(\mathbb{R}^3)$, and so in $L^{8/3}(\mathbb{R}^3)$, with respect to k. The first term in the right-hand side tends to zero as $k \to +\infty$ by the compact embedding of $\mathbf{H}(\Omega)$ into $\mathbf{L}^{8/3}(\Omega)$. The second term can be taken as small as we wish, taking $\Omega = B_R$, $k > R$, choosing a radius $R > 0$ large enough.

Hence, we get from (2.23)

$$
1 = \gamma\int_{\mathbb{R}^3}\nabla\cdot(\Theta_0\widetilde{V})\widetilde{\Psi}. \tag{2.25}
$$

Let us consider an arbitrary test function $\chi \in C_c^\infty(\mathbb{R}^3)$ and k large enough so that the support of χ is contained in B_k. From the second equation of (2.17), written for (V_k, Ψ_k), we obtain

$$
\int_{B_k}\nabla\Psi_k\cdot\nabla\chi = \int_{B_k}[\Psi_k + x\cdot\nabla\Psi_k]\chi - \int_{B_k}\nabla\cdot[(\Psi_k+\Theta_0)(V_k+U_0)]\chi.
$$

Dividing by L_k^2 and letting $k \to +\infty$, all the integrals $\frac{1}{L_k^2}\int_{B_k}\dots$ with linear terms in V_k and Ψ_k in the above identity go to zero. On the other hand, proceedings as in (2.24),

$$
\frac{1}{L_k^2}\int_{B_k}\nabla\cdot(\Psi_k V_k)\chi \to \gamma\int_{\mathbb{R}^3}\nabla\cdot(\widetilde{\Psi}\widetilde{V})\chi.
$$

Then it follows that

$$
\gamma\int_{\mathbb{R}^3}\nabla\cdot(\widetilde{\Psi}\widetilde{V})\chi = 0, \qquad \text{for all } \chi \in \mathcal{D}(\mathbb{R}^3).
$$

But $\gamma \neq 0$ by (2.25), so

$$
\int_{\mathbb{R}^3}\nabla\cdot(\widetilde{\Psi}\widetilde{V})\chi = 0, \qquad \text{for all } \chi \in \mathcal{D}(\mathbb{R}^3).
$$

This in turn implies that $\widetilde{V} \cdot \nabla \widetilde{\Psi} = 0$. But then

$$\int_{\mathbb{R}^3} \nabla \cdot (\Theta_0 \widetilde{V}) \widetilde{\Psi} = -\int_{\mathbb{R}^3} \Theta_0 \widetilde{V} \cdot \nabla \Psi = 0$$

which contradicts (2.25). This excludes that $\limsup_{k\to+\infty} J_k/L_k < \infty$.

Step 3. We reduced ourselves to the case $\limsup_{k\to+\infty} J_k/L_k = \infty$. Multiplying equations (2.17) by V_k and integrating on B_k gives, in a similar way as we did in (2.9)

$$J_k^2 = \int_{B_k} F_0 V_k - \int_{B_k} (V_k \cdot \nabla U_0) \cdot V_k + \int_{B_k} \big((\Psi_k + \Theta_0)\rho_k \nabla\big(|\cdot|^{-1}\big) + F\big) \cdot V_k. \tag{2.26}$$

Applying the Hardy inequality (*c.f* (2.1)) we get, as $k \to +\infty$,

$$\begin{aligned}\Big|\frac{1}{J_k^2}\int_{B_k} (\Psi_k \rho_k \nabla(|\cdot|^{-1}) \cdot V_k\Big| &\le C\frac{L_k}{J_k}\Big\|\frac{\widehat{\Psi}_k}{|\cdot|}\Big\|_{L^2(B_k)}\Big\|\frac{\widehat{V}_k}{|\cdot|}\Big\|_{L^2(B_k)} \\ &\le C\frac{L_k}{J_k}\|\nabla\widehat{\Psi}_k\|_{L^2(B_k)}\|\nabla\widehat{V}_k\|_{L^2(B_k)} \to 0.\end{aligned}$$

Moreover,

$$\frac{1}{J_k^2}\int_{B_k} (V_k \cdot \nabla U_0) \cdot V_k \to \int_{\mathbb{R}^3} (\widetilde{V} \cdot \nabla U_0) \cdot \widetilde{V}$$

as one easily checks by reproducing the same calculations as in (2.24), using that $\nabla U_0 \in L^4(\mathbb{R}^3)$, by condition (2.2). Therefore, dividing equation (2.26) by J_k^2 and letting $k \to +\infty$ we find the identity

$$\int_{\mathbb{R}^3} (\widetilde{V} \cdot \nabla U_0) \cdot \widetilde{V} = -1. \tag{2.27}$$

Dividing by J_k^2 the first equation in (2.17) satisfied by (V_k, Ψ_k), we get

$$\begin{aligned}&\widehat{V}_k \cdot \nabla \widehat{V}_k + \nabla \widehat{P}_k \\ &\quad= \frac{1}{J_k}\Big(\Delta\widehat{V}_k + \widehat{V}_k + x\cdot\nabla\widehat{V}_k + \frac{F_0}{J_k} - U_0\cdot\nabla\widehat{V}_k - \widehat{V}_k\cdot\nabla U_0 \\ &\qquad\qquad + \frac{L_k}{J_k}\widehat{\Psi}_k\rho_k\nabla(|\cdot|^{-1}) + \frac{1}{J_k^2}\big(\Theta_0\rho_k\nabla(|\cdot|^{-1}) + F\big)\Big).\end{aligned}$$

Writing the weak formulation of the above equation, *i.e.*, testing with an arbitrary $\eta \in C^\infty_{0,\sigma}(\mathbb{R}^3)$ and using that $J_k \to +\infty$ and that L_k/J_k remains bounded as $k \to +\infty$, we deduce, in the limit:

$$\int_{\mathbb{R}^3} (\widetilde{V} \cdot \nabla \widetilde{V}) \cdot \eta = 0, \qquad \text{for all } \eta \in C^\infty_{0,\sigma}(\mathbb{R}^3)^3.$$

But $U_0 \in L^4_\sigma(\mathbb{R}^3)$ and $\widetilde{V}, \nabla\widetilde{V}$ are in $L^2(\mathbb{R}^3)$. Approximating U_0 in the L^4-norm by test functions implies

$$\int_{\mathbb{R}^3} (\widetilde{V} \cdot \nabla \widetilde{V}) \cdot U_0 = 0.$$

This is in contradiction with (2.27). □

Theorem 2.6. *Assume that U_0, and Θ_0 do satisfy* (2.2)*. Assume also that $F \in L^p(\mathbb{R}^3)^3$, for some $p \in [6/5, 2]$. Then elliptic system* (1.10) *possess at least a solution $(V, \Psi) \in \mathbf{H}(\mathbb{R}^3)$.*

Proof. Applying Proposition 2.4 with $\Omega = B_k$ and $\rho = \rho_k$ $(k = 1, 2, \ldots)$, we get a sequence of solutions $(V_k, \Psi_k) \in \mathbf{H}(B_k)$. By Proposition 2.5, such a sequence is bounded in the $\mathbf{H}(B_k)$-norm by a constant independent on k. This implies that there exist $(V, \Psi) \in \mathbf{H}(\mathbb{R}^3)$ and a subsequence, still denoted (V_k, Ψ_k), such that $(V_k, \Psi_k) \to (V, U)$ weakly in $\mathbf{H}(\Omega)$, for any bounded domain $\Omega \subset \mathbb{R}^n$.

It remains to prove that (V, Ψ) is a weak solution of the elliptic problem (1.10). Thus, we have to pass to the limit in all the terms of the variational formulation of problem (2.17)-(2.18) considered on the balls B_k. For example, for a test function $\chi \in C_0^\infty(\mathbb{R}^3)$, using $\rho_k \nearrow 1$ a.e. in $\mathbb{R}^3$ and the compact embedding $H^1_{\rm loc}(\mathbb{R}^3) \subset L^p_{\rm loc}(\mathbb{R}^3)$, $p \in [2, 6)$, we do get after extracting a new subsequence,

$$\int_{\mathbb{R}^3} \Psi_k \rho_k \nabla(|\cdot|^{-1})\chi \to \int_{\mathbb{R}^3} \Psi \nabla(|\cdot|^{-1})\chi.$$

Similar considerations prove that all the other terms also pass to the limit. This gives the result. □

3. Conclusion

This section is devoted to deduce the result of Theorem 1.1 from Theorem 2.6.

Proof of Theorem 1.1. Let (V, Ψ) be as in Theorem 2.6 and set

$$u(x,t) := \frac{1}{\sqrt{2t}}(U_0 + V)\Big(\frac{x}{\sqrt{2t}}\Big), \quad \text{and} \quad \theta(x,t) := \frac{1}{\sqrt{2t}}(\Theta_0 + \Psi)\Big(\frac{x}{\sqrt{2t}}\Big).$$

We have $(U_0, \Theta_0) \in \mathbf{L}^{3,\infty}(\mathbb{R}^3)$ by Proposition 2.1 and $(V, \Psi) \in \mathbf{H}(\mathbb{R}^3) \subset \mathbf{L}^{3,\infty}(\mathbb{R}^3)$. Then, from the scaling properties $(u, \theta) \in L^\infty(\mathbb{R}^+, \mathbf{L}^{3,\infty}(\mathbb{R}^3))$.

Let us now address the continuity with respect to t and we detail only the continuity property at 0, that is important to give a sense to the initial condition $(u_0, \theta_0)|_{t=0} = (u_0, \theta_0)$. We have

$$\frac{1}{\sqrt{2t}}(U_0, \Theta_0)\big(\frac{x}{\sqrt{2t}}\big) = e^{t\Delta}(u_0, \Theta_0) \to (u_0, \theta_0) \quad \text{as} \quad t \to 0+$$

in the weak-* topology of $\mathbf{L}^{3,\infty}(\mathbb{R}^3)$. Let us check that $\frac{1}{\sqrt{2t}}(V, \Psi)\big(\frac{x}{\sqrt{2t}}\big) \to 0$ in the same topology. Indeed, if $\varphi \in \mathbf{L}^{3,1}(\mathbb{R}^3)$, we can approximate it in the $\mathbf{L}^{3,1}$-norm with functions $\varphi_\epsilon \in \mathbf{L}^2 \cap \mathbf{L}^{3/2}(\mathbb{R}^3)$. Then is is enough to observe that

$$\Big|\int_{\mathbb{R}^3} \frac{1}{\sqrt{2t}}(V, \Psi)\big(\frac{\cdot}{\sqrt{2t}}\big) \cdot \varphi_\epsilon\Big| \le Ct^{1/4} \to 0 \quad \text{as} \quad t \to 0+ .$$

Therefore, $(u, \theta) \in C_w(\mathbb{R}^+, \mathbf{L}^{3,\infty}(\mathbb{R}^3))$.

Estimates (1.6) follow immediately from the fact that $(V, \Psi) = (U - U_0, \Theta - \Theta_0) \in \mathbf{H}^1$ and from the scaling properties of the L^2 and the $\dot{H}^1$-norms. □

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