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Far field asymptotics of solutions to convection equation with anomalous diffusion

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Abstract. The initial value problem for the conservation law $\partial_t u + (-\Delta)^{\alpha/2}u + \nabla \cdot f(u) = 0$ is studied for $\alpha \in (1, 2)$ and under natural polynomial growth conditions imposed on the nonlinearity. We find the asymptotic expansion as $|x| \to \infty$ of solutions to this equation corresponding to initial conditions, decaying sufficiently fast at infinity.

1. Introduction

We study properties of solutions to the initial value problem for the multidimensional conservation law with the anomalous diffusion

$$\partial_t u + (-\Delta)^{\alpha/2}u + \nabla \cdot f(u) = 0, \quad x \in \mathbb{R}^d, t > 0, \tag{1.1}$$

$$u(x, 0) = u_0. \tag{1.2}$$

Here, we always impose the standing assumption $1 < \alpha < 2$. Moreover, we assume that the C^1- vector field $f(u) = \big(f_1(u), \dots, f_d(u)\big)$ is of a polynomial growth, namely, it satisfies the usual estimates

$$|f(u)| \le C|u|^q \quad \text{and} \quad |f(u) - f(v)| \le C|u - v|(|u|^{q-1} + |v|^{q-1}) \tag{1.3}$$

for some constants $C > 0$, $q > 1$ and for all $u, v \in \mathbb{R}$ (in fact, assumption (1.3) can be slightly relaxed in some parts of our considerations, cf. Remark 2.3, below).

Linear evolution problems involving fractional Laplacian describing *the anomalous diffusion* (or α-stable Lévy diffusion) have been extensively studied in the mathematical and physical literature (see, e.g., [11]). The probabilistic interpretation of nonlinear evolution problems with an anomalous diffusion, obtained recently by Jourdain, Méléard, and

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Woyczyński [9], motivated us to study (1.1)–(1.2). The authors of [9] considered a class of nonlinear integro-differential equations involving a fractional power of the Laplacian and a nonlocal quadratic nonlinearity represented by a singular integral operator. They associated with the equation a nonlinear singular diffusion and proved *propagation of chaos* to the law of this diffusion for the related interacting particle systems. In particular, due to the probabilistic origin of (1.1)–(1.2), the function $u(\cdot, t)$ should be interpreted as the density of a probability distribution for every $t > 0$, if the initial datum is so.

Although, the motivation for this paper comes from the probability theory, our methods are purely analytic. Hence, if $X(t)$ is the symmetric α-stable Lévy process, its density of the probability distribution $p_\alpha(x, t)$ is the fundamental solution of the linear equation

$$\partial_t v + (-\Delta)^{\alpha/2} v = 0, \tag{1.4}$$

hence, p_α can be computed *via* the Fourier transform $\widehat{p}_\alpha(\xi, t) = e^{-t|\xi|^\alpha}$. In particular,

$$p_\alpha(x, t) = t^{-d/\alpha} P_\alpha\big(xt^{-1/\alpha}\big),$$

where P_α is the inverse Fourier transform of $e^{-|\xi|^\alpha}$ (see [8, Ch. 3] for more details). It is well known that for every $\alpha \in (0, 2)$ the function P_α is smooth, nonnegative, and satisfies the estimates

$$0 < P_\alpha(x) \le C(1 + |x|)^{-(\alpha+d)} \qquad \text{and} \qquad |\nabla P_\alpha(x)| \le C(1 + |x|)^{-(\alpha+d+1)} \tag{1.5}$$

for a constant C and all $x \in \mathbb{R}^d$. Moreover,

$$P_\alpha(x) = c_0|x|^{-(\alpha+d)} + O\Big(|x|^{-(2\alpha+d)}\Big), \qquad \text{as } |x| \to \infty, \tag{1.6}$$

and

$$\nabla P_\alpha(x) = -c_1\, x|x|^{-(\alpha+d+2)} + O\Big(|x|^{-(2\alpha+d+1)}\Big), \qquad \text{as } |x| \to \infty, \tag{1.7}$$

where

$$c_0 = \alpha 2^{\alpha-1}\pi^{-(d+2)/2} \sin(\alpha\pi/2)\Gamma\left(\frac{\alpha+d}{2}\right)\Gamma\left(\frac{\alpha}{2}\right),$$

and

$$c_1 = 2\pi\alpha 2^{\alpha-1}\pi^{-(d+4)/2} \sin(\alpha\pi/2)\Gamma\left(\frac{\alpha+d+2}{2}\right)\Gamma\left(\frac{\alpha}{2}\right).$$

We refer to [3] for a proof of the formula (1.6) with the explicit constant c_0. The optimality of the estimate of the lower order term in (1.6) is due Kolokoltsov [10, Eq. (2.13)], where higher order expansions of P_α are also computed. The proof of the asymptotic expression (1.7) and the value of c_1 can be deduced from (1.6) using an identity by Bogdan and Jakubowski [4, Eq. (11)].

The asymptotic formula (1.6) for the kernel P_α plays an important role in the theory of α-stable processes. The main goal in this work is to present a method which allows to derive analogous asymptotic expansions as $|x| \to \infty$ of solutions to the Cauchy problem (1.1)–(1.2). In the next section, we recall several properties of solutions to (1.1)–(1.2) and we state our main results: Theorems 2.1 and 2.4. In Section 3, we gather technical space-time estimates of solutions to (1.1)–(1.2). The proofs of Theorems 2.1 and 2.4 are contained in Section 4.

Notation. The L^p-norm of a Lebesgue measurable, real-valued function v defined on $\mathbb{R}^d$ is denoted by $\|v\|_p$. In the following, we use the weighted L^∞ space

$$L^\infty_\vartheta = \{v \in L^\infty(\mathbb{R}^d) \ : \ \|v\|_{L^\infty_\vartheta} \equiv \operatorname{ess\,sup}_{x\in\mathbb{R}^d} |v(x)|(1+|x|)^\vartheta < \infty\}, \tag{1.8}$$

for any $\vartheta \geq 0$, and its homogeneous counterpart

$$\dot{L}^\infty_\vartheta = \{v \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d\backslash\{0\}) \ : \ \|v\|_{\dot{L}^\infty_\vartheta} \equiv \operatorname{ess\,sup}_{x\in\mathbb{R}^d} |v(x)||x|^\vartheta < \infty\}.$$

The constants (always independent of x) will be denoted by the same letter C, even if they may vary from line to line. Sometimes, we write, e.g., $C = C(T)$ when we want to emphasize the dependence of C on a parameter T.

2. Main results

It is well known (see [1, 5, 6]) that given $u_0 \in L^1(\mathbb{R}^d)$ and $1 < \alpha \leq 2$, the initial value problem (1.1)-(1.2) has the unique solution $u \in C([0,\infty), L^1(\mathbb{R}^d))$. Moreover, this solution satisfies

$$u \in C((0,\infty), W^{1,p}(\mathbb{R}^d))$$

for every $p \in [1,\infty]$ and the following estimates hold true (see [1, Cor. 3.2])

$$\|u(t)\|_p \leq Ct^{-\frac{d}{\alpha}\left(1-\frac{1}{p}\right)}\|u_0\|_1 \tag{2.1}$$

for all $t > 0$ and C independent of t and of u_0. Under the additional assumption $u_0 \in L^p(\mathbb{R}^d)$, the corresponding solution satisfies $u \in C([0,\infty), L^p(\mathbb{R}^d))$ together with the estimate

$$\|u(t)\|_p \leq \|u_0\|_p. \tag{2.2}$$

Below, in Proposition 3.3, we complete these preliminary results providing the estimates of solutions to (1.1)–(1.2) in weighted L^∞-spaces. In particular, if $u_0 \in L^\infty_{\alpha+d}$ (cf. (1.8)), then the corresponding solution of (1.1)–(1.2) satisfies $u \in C([0,T], L^\infty_{\alpha+d})$ for every $T > 0$. Such a result was already obtained in the one dimensional case, see [7, Sect. 2]. In Section 3, we state and prove its multidimensional counterpart for the completeness of the exposition. We complement this result with additional estimates for the gradient of the solution, which will be useful in the proofs of asymptotic formulas in Section 4.

Let us recall that, when studying the large time behavior of solutions for the problem (1.1)–(1.2), an important role is played by the *critical exponent*

$$\widetilde{q} \equiv 1 + \frac{\alpha - 1}{d}.$$

Indeed, using the terminology of [2] the behavior of solutions as $t \to \infty$ is *genuinely non-linear* when $q = \widetilde{q}$, is *weakly non-linear* when $q > \widetilde{q}$, and is (expected to be) *hyperbolic* when $1 < q < \widetilde{q}$.

In this paper, in the supercritical case $q > \widetilde{q}$, as well as for $q = \widetilde{q}$ provided $\|u_0\|_1$ is sufficiently small, we will improve the space-time estimates of [7, Sect. 2], showing that

$$|u(x,t)| \le C p_\alpha(x, 1+t), \tag{2.3}$$

for all $x \in \mathbb{R}^d$, $t > 0$, and $C > 0$ independent of x, t. Under the additional assumption that $\nabla u_0 \in L^\infty_{\alpha+d+1}$, we will also prove that

$$\|\nabla u(t)\|_{L^\infty_{\alpha+d+1}} \le C(1+t), \tag{2.4}$$

see Theorem 3.5, below. In other words, $\nabla u(x,t)$ has the same space-time decay profiles as $\nabla p_\alpha(x, 1+t)$ (cf. the second inequality in (1.5)).

Furthermore, we make evidence of the *second critical exponent*, namely,

$$q^* \equiv 1 + \frac{1}{\alpha + d},$$

playing an important role in the study of the pointwise behavior of solutions as $|x| \to \infty$. The following theorem explains the role of q^*, showing that any decaying solution has a precise spatial asymptotic profile. Here, we denote by $S_\alpha(t)u_0(x) = p_\alpha(t) * u_0(x)$ the solution of the linear equation (1.4) supplemented with the initial datum u_0.

THEOREM 2.1. *Assume that $\alpha \in (1,2)$, and let $u = u(x,t)$ be the solution of (1.1)–(1.2) with the nonlinearity satisfying (1.3), and with $u_0 \in L^\infty_{\alpha+d}$.*

(i) Then, for all $t > 0$, $x \in \mathbb{R}^d$,

$$u(x,t) = S_\alpha(t)u_0(x) + \frac{c_1 x}{|x|^{\alpha+d+2}} \cdot \int_0^t \int (t-s) f(u(y,s))\, dy\, ds + O\Big(\max\big\{|x|^{-q(\alpha+d)}\,;\ |x|^{-(\alpha+d+2)}\big\}\Big), \qquad \text{as } |x| \to \infty, \tag{2.5}$$

uniformly in any time interval $t \in [0, T]$, $T > 0$. *This conclusion is interesting only when the last term on the right hand side of* (2.5) *is the lower order term as* $|x| \to \infty$*: this happens when* $q > q^*$.

(ii) The conclusion at the point (i) can be improved under the additional assumption $\nabla u_0 \in L^\infty_{\alpha+d+1}$, *replacing the third term on the right hand side of* (2.5) *by*

$$O\big(\max\big\{|x|^{-q(\alpha+d)-1}\,;\ |x|^{-(\alpha+d+2)}\big\}\big), \qquad \text{as } |x| \to \infty.$$

Now, this conclusion is interesting also for $1 < q \le q^*$.

(iii) If u *satisfies inequality* (2.3) *for all* $x \in \mathbb{R}^d$, $t > 0$, *and* $C > 0$ *independent of* x, t, *then one can make precise the behavior for large* t *of the remainder term in relation* (2.5), *replacing it by*

$$O\big((1+t)^N \max\big\{|x|^{-q(\alpha+d)}\,;\ |x|^{-(\alpha+d+2)}\big\}\big), \quad \text{as } |x| \to \infty,$$

uniformly in $t \in [0, \infty)$, *for some exponent* $N = N(\alpha, q, d) \le 3$, *independent on* u_0.

If, in addition, the solution satisfies inequality (2.4), *the conclusion at the point (ii) can be improved replacing the remainder term by*

$$O\big((1+t)^N \max\big\{|x|^{-q(\alpha+d)-1}\,;\ |x|^{-(\alpha+d+2)}\big\}\big)$$

for some exponent $N = N(\alpha, q, d) \le 3$, *independent on* u_0, *and the convergence as* $|x| \to \infty$ *holds true uniformly in* $t \in [0, \infty)$.

It follows from the Duhamel formula that the solution of the Cauchy problem (1.1)–(1.2) satisfies the integral equation

$$u(t) = S_\alpha(t)u_0 - \int_0^t \nabla S_\alpha(t-s) \cdot f(u)(s)\, ds. \tag{2.6}$$

Hence, it is possible to give a heuristic explanation of the role of space-critical exponent $q = q^*$, simply, by looking at the integrand of the second term on the right hand side of (2.6). Indeed, the kernel of ∇S_α behaves as $|x|^{-(\alpha+d+1)}$ as $|x| \to \infty$ (cf. (1.6)), whereas $|f(u(x,t))| \le C|x|^{-q(d+\alpha)}$ for $u(t) \in L^\infty_{\alpha+d}$. Then, it is natural to expect that the large space asymptotics is influenced by the competition between these two decay rates as $|x| \to \infty$. In fact, the proof of Theorem 2.1 (given in Section 4) consists in finding the asymptotic expansion of the second term on the right hand side of (2.6) and the equality between these two decay rates occurs precisely when $q = q^*$.

REMARK 2.2. It is worth observing that this type of asymptotic expansion of solutions to convection-diffusion equations is specific of the fractional nature of the diffusion operator

$(-\Delta)^{\alpha/2}$ and is caused by the algebraic decay of the fundamental solution $p_\alpha(x,t)$. For the viscous Burgers equation, or for multidimensional diffusion-convection equations with standard dissipation (*i.e.*, with the usual Laplacian) Theorem 2.1 remains valid, but it is not interesting because the coefficient c_1 vanishes in the limit case $\alpha = 2$.

REMARK 2.3. The conclusion (i) of Theorem 2.1 remains valid under more general assumptions on the nonlinearity. What we really need is that f is a C^1-vector field such that $|f(u)| \le c(R)|u|^q$ for some $q > 1$, a continuous nondecreasing function $c(\cdot)$ on $[0,\infty)$, and all $|u| \le R$. For the part (ii), we need also a similar condition for f', namely, $|f'(u)| \le c_1(R)|u|^{q-1}$ for $|u| \le R$. On the other hand, the present form of Theorem 2.1. iii is no longer valid for such more general nonlinearities. Our more stringent assumption (1.3) allows us to present the essential ideas avoiding uninteresting technicalities in the proofs, in particular, separating the cases of large and small u in our estimates. Moreover, such an assumption is well suited for studying self-similar solutions.

For the homogeneous nonlinear term $\nabla \cdot f(u) = b \cdot \nabla(u|u|^{q-1})$ with a fixed $b \in \mathbb{R}^d$ and with the time-critical exponent $q = \widetilde{q}$, the authors of [2] constructed a family of self-similar solutions $u_M = u_M(x,t)$ of equation (1.1). Those functions satisfy the scaling relation

$$u_M(x,t) = t^{d/\alpha} U_M(xt^{-1/\alpha}) \quad \text{where} \quad U_M(x) = u_M(x,1) \tag{2.7}$$

for all $x \in \mathbb{R}^d$ and $t > 0$. Moreover, each of them is the unique solution of the initial value problem

$$\partial_t u + (-\Delta)^{\alpha/2} u + b \cdot \nabla(u|u|^{(\alpha-1)/d}) = 0 \tag{2.8}$$

$$u(x,0) = M\delta_0 \tag{2.9}$$

for $\alpha \in (1,2)$ and $M > 0$, where δ_0 is the Dirac delta. We refer the reader to [2] for more information concerning solutions of problem (2.8)-(2.9).

In this paper, we complete results from [2] providing space-time estimates of those self-similar solutions. First, in Corollary 3.6 below, we establish, for sufficiently small $M > 0$, the estimate

$$0 \le u_M(x,t) \le C p_\alpha(x,t) \quad \text{for all } x \in \mathbb{R}^d \text{ and } t > 0, \tag{2.10}$$

We conjecture that such estimate remains true without the smallness assumption imposed on M. Inequality (2.10) plays a crucial role in the proof of the following asymptotic expansion of the self-similar kernel U_M.

THEOREM 2.4. *Assume that* $1 < \alpha < 2$ *and* $\widetilde{q} > q^*$. *Let* u_M *be a self-similar solution of* (2.8)–(2.9), *satisfying the estimate* (2.10). *Then the self-similar profile* $U_M(x) =$

$u_M(x, 1)$ *has the following behavior as* $|x| \to \infty$*:*

$$U_M(x) = MP_\alpha(x) + \frac{c_1\alpha^2}{\alpha+1}\|U_M\|_{\widetilde{q}}^{\widetilde{q}}\frac{b\cdot x}{|x|^{\alpha+d+2}} + O(\max\{|x|^{-\widetilde{q}(\alpha+d)};\ |x|^{-(\alpha+d+2)}\}). \tag{2.11}$$

The asymptotic expansion of solutions to (1.1) stated in (2.5) and in (2.11) can be viewed as the true counterparts of the well-known result for the α-stable distribution recalled in (1.6).

3. Preliminary space-time estimates

We begin this section by the study of the solution of the linear problem

$$\partial_t v + (-\Delta)^{\alpha/2} v = 0, \qquad v(x, 0) = v_0 \tag{3.1}$$

denoted by

$$v(x, t) = S_\alpha(t)v_0(x) = p_\alpha(\cdot, t) * v_0(x).$$

The following lemma contains a direct generalization to $\mathbb{R}^d$ of estimates from [7, Lemma 1.40]. By this reason, we sketch its proof only.

LEMMA 3.1. *Assume that* $v_0 \in L^\infty_{\alpha+d}$. *There exists* $C > 0$ *independent of* v_0 *and* t *such that*

$$\|S_\alpha(t)v_0\|_\infty \le C\min\{t^{-d/\alpha}\|v_0\|_1, \|v_0\|_\infty\}, \tag{3.2}$$

$$\|S_\alpha(t)v_0\|_{L^\infty_{\alpha+d}} \le C(1+t)\|v_0\|_{L^\infty_{\alpha+d}}, \tag{3.3}$$

$$\|\nabla S_\alpha(t)v_0\|_{L^\infty_{\alpha+d}} \le Ct^{-1/\alpha}\|v_0\|_{L^\infty_{\alpha+d}} + Ct^{1-1/\alpha}\|v_0\|_1, \tag{3.4}$$

Proof. Estimate (3.2) results immediately from the Young inequality applied to the convolution $S_\alpha(t)v_0 = p_\alpha(t) * v_0$, due to the identities

$$\|p_\alpha(t)\|_1 = 1, \quad \|p_\alpha(t)\|_\infty = t^{-d/\alpha}\|P_\alpha\|_\infty \quad \text{for all } t > 0.$$

Since $|v_0(x)| \le C(1+|x|)^{-(d+\alpha)}$, by the asymptotic properties of the kernel $p_\alpha(x, 1) = P_\alpha(x)$ (cf. (1.6)), we immediately obtain $|v_0(x)| \le Cp_\alpha(x, 1)$ for all $x \in \mathbb{R}^d$ and a constant $C > 0$ independent of x. Consequently, by the semigroup property, we conclude

$$\|S_\alpha(t)v_0\|_{L^\infty_{\alpha+d}} \le C\|S_\alpha(t)p_\alpha(1)\|_{L^\infty_{\alpha+d}} = C\|p_\alpha(t+1)\|_{L^\infty_{\alpha+d}} \le C(1+t).$$

Now, replacing v_0 by $v_0/\|v_0\|_{L^\infty_{\alpha+d}}$ we obtain (3.3).

To prove (3.4), we use the pointwise estimate

$$(1+|x|)^{\alpha+d} \le C(1+|y|)^{\alpha+d} + C|x-y|^{\alpha+d},$$

valid for all $x, y \in \mathbb{R}$ and a constant $C > 0$, and we apply the Young inequality. We get

$$\|\nabla S_\alpha(t)v_0\|_{L^\infty_{\alpha+d}} \le C\|\nabla p_\alpha(t)\|_1\|v_0\|_{L^\infty_{\alpha+d}} + C\|\nabla p_\alpha(t)\|_{\dot L^\infty_{\alpha+d}}\|v_0\|_1$$

and (3.4) immediately follows. □

Under an additional information on the gradient of v_0, we can obtain analogous estimates for $\nabla S_\alpha(t)v_0$. In order to give a precise statement, let us introduce the space

$$E_{\alpha+d} \equiv \left\{v \in W^{1,\infty}_{loc}(\mathbb{R}^d)\ :\ \|v\|_{E_{\alpha+d}} \equiv \|v\|_{L^\infty_{\alpha+d}} + \|\nabla v\|_{L^\infty_{\alpha+d+1}} < \infty\right\}. \tag{3.5}$$

LEMMA 3.2 *Assume that* $v_0 \in E_{\alpha+d}$. *There exists* $C > 0$ *independent of* v_0 *and* t *such that*

$$\|\nabla S_\alpha(t)v_0\|_\infty \le C\min\left\{t^{-(d+1)/\alpha}\|v_0\|_1\,;\ t^{-1/\alpha}\|v_0\|_\infty\,;\ \|\nabla v_0\|_\infty\right\}, \tag{3.6}$$

$$\|S_\alpha(t)v_0\|_{E_{\alpha+d}} \le C(1+t)\|v_0\|_{E_{\alpha+d}}\,, \tag{3.7}$$

$$\|\nabla S_\alpha(t)v_0\|_{E_{\alpha+d}} \le Ct^{-1/\alpha}\|v_0\|_{E_{\alpha+d}} + Ct^{1-1/\alpha}\|v_0\|_1 \tag{3.8}$$

for all $t > 0$.

Proof. Estimate (3.6) is the straightforward application of the L^1-L^∞ convolution inequalities. In order to prove (3.7) using the radial symmetry of $p_\alpha(\cdot, t)$, we see that, for all $R > 0$, $\int_{B_R} \nabla p(y,t)\,dy = 0$, where B_R denotes the ball centered at the origin and of radius R. Hence,

$$\nabla S_\alpha(t)v_0(x) = \int\limits_{|y|\le|x|/2} \left[v_0(x-y) - v_0(x)\right]\nabla p_\alpha(y,t)\,dy + \int\limits_{|y|\ge|x|/2} v_0(x-y)\nabla p_\alpha(y,t)\,dy.$$

This decomposition shows that, for some constant $C > 0$, the quantity $|\nabla S_\alpha(t)v_0(x)|$ can be bounded from above by

$$C|x|^{-(\alpha+d+1)}\|\nabla v_0\|_{L^\infty_{\alpha+d+1}}\int\limits_{\mathbb{R}^d}|y|\,|\nabla p(y,t)|\,dy + Ct|x|^{-(\alpha+d+1)}\int\limits_{\mathbb{R}^d}|v_0(y)|\,dy,$$

which implies

$$\|\nabla S_\alpha(t)v_0\|_{\dot L^\infty_{\alpha+d+1}} \le C(\|\nabla v_0\|_{\dot L^\infty_{\alpha+d+1}} + t\|v_0\|_1) \le C(1+t)\|v_0\|_{E_{\alpha+d}}. \tag{3.9}$$

Now, estimate (3.7) follows from (3.3), (3.6) and from the bound for the homogeneous norm (3.9).

Let us prove (3.8). By (3.4) and the inequality

$$\|\nabla^2 S_\alpha(t)v_0\|_\infty \le \|\nabla S_\alpha(t)\|_1\|\nabla v_0\|_\infty \le Ct^{-1/\alpha}\|\nabla v_0\|_\infty,$$

we see that we only have to establish the following estimate in the homogeneous space $\dot{L}^\infty_{\alpha+d+1}$

$$\|\nabla^2 S_\alpha(t)v_0\|_{\dot{L}^\infty_{\alpha+d+1}} \le Ct^{-1/\alpha}\|v_0\|_{E_{\alpha+d}} + Ct^{1-1/\alpha}\|v_0\|_1. \tag{3.10}$$

To prove (3.10), we consider the decomposition

$$\nabla^2 S_\alpha(t)v_0(x) = (J_1 + J_2 + J_3)(x,t),$$

where

$$J_1(x,t) \equiv \int\limits_{|y|\le|x|/2} [v_0(x-y) - v_0(x)]\nabla^2 p_\alpha(y,t)\,dy,$$

$$J_2(x,t) \equiv \int\limits_{|y|\ge|x|/2} v_0(x-y)\nabla^2 p_\alpha(y,t)\,dy,$$

$$J_3(x,t) \equiv -v_0(x) \int\limits_{|y|\ge|x|/2} \nabla^2 p_\alpha(y,t)\,dy$$

(note that $\int_{\mathbb{R}^d} \nabla^2 p_\alpha(y,t)\ dy = 0$). From the well known estimate (see [10])

$$|\nabla^2 P_\alpha(x)| \le C(1+|x|)^{-(\alpha+d+2)}, \tag{3.11}$$

we deduce $\int_{\mathbb{R}^d} |y|\,|\nabla^2 p_\alpha(y,t)|\,dy \le Ct^{-1/\alpha}$. Then, the application of the Taylor formula in the integral defining J_1 yields

$$|J_1(x,t)| \le Ct^{-1/\alpha}|x|^{-(\alpha+d+1)}\|\nabla v_0\|_{L^\infty_{\alpha+d+1}}.$$

To deal with the terms J_2 and J_3, we use two different pointwise estimates of $\nabla^2 p_\alpha(x,t)$ resulting from (3.11):

$$|\nabla^2 p_\alpha(x,t)| \le Ct^{-(d+2)/\alpha}(1+|x|t^{-1/\alpha})^{-(\alpha+d+2)} \le Ct^{1-1/\alpha}|x|^{-(\alpha+d+1)}$$

and

$$|\nabla^2 p_\alpha(x,t)| \le Ct^{-1/\alpha}|x|^{-(d+1)},$$

which imply

$$\begin{aligned} |J_2(x,t)| &\le \sup_{|y|\ge|x|/2} |\nabla^2 p_\alpha(y,t)| \int\limits_{|y|\ge|x|/2} |v_0(x-y)|\ dy \\ &\le Ct^{1-1/\alpha}|x|^{-(\alpha+d+1)}\|v_0\|_1 \end{aligned}$$

and

$$|J_3(x,t)| \le C|x|^{-(\alpha+d)}\|v_0\|_{L^\infty_{\alpha+d}} \int\limits_{|y|\ge|x|/2} |\nabla^2 p_\alpha(y,t)|\,dy$$
$$\le Ct^{-1/\alpha}|x|^{-(\alpha+d+1)}\|v_0\|_{L^\infty_{\alpha+d}}.$$

Combining all these inequalities yields (3.10). □

We are in a position to construct solutions of the Cauchy problem (1.1)–(1.2) in the weighted space $L^\infty_{\alpha+d}$.

PROPOSITION 3.3. *(i) Let* $\alpha \in (1,2)$ *and* $q > 1$. *Assume that* u *is a solution of the Cauchy problem* (*1.1*)–(*1.2*) *with the nonlinearity satisfying* (*1.3*). *If* $u_0 \in L^\infty_{\alpha+d}$, *then*

$$u \in C([0,T], L^\infty_{\alpha+d}) \quad \textit{for each } T > 0. \tag{3.12}$$

(ii) Under the more stringent assumption $u_0 \in E_{\alpha+d}$, *cf.* (*3.5*), *we have also*

$$u \in L^\infty([0,T], E_{\alpha+d}) \quad \textit{for each } T > 0. \tag{3.13}$$

Proof. In order to prove (3.12), it suffices to show that the nonlinear operator

$$T(u)(t) = S_\alpha(t)u_0 - \int_0^t \nabla S_\alpha(t-\tau) f(u(\tau))\,d\tau$$

has the fixed point in the space

$$X_T = \{u \in C([0,T], L^\infty_{\alpha+d}) \ :\ \sup_{t\in[0,T]} \|u(t)\|_{L^\infty_{\alpha+d}} < \infty\}.$$

As usual, we work in the ball $B(0,R) = \{u \in C([0,T], L^\infty_{\alpha+d}) \ :\ \sup_{t\in[0,T]} \|u(t)\|_{L^\infty_{\alpha+d}} \le R\}$, where $R = M\|u_0\|_{L^\infty_{\alpha+d}}$ and $M > 0$ is a large constant, and $T > 0$. Combining inequality (3.4) with assumption (1.3) we get

$$\|\nabla S_\alpha(t) f(u)\|_{L^\infty_{\alpha+d}} \le Ct^{-1/\alpha}\|\,|u|^q\|_{L^\infty_{\alpha+d}} + Ct^{1-1/\alpha}\|u\|_q^q$$
$$\le Ct^{-1/\alpha}(1+t)\|u\|_\infty^{q-1}\|u\|_{L^\infty_{\alpha+d}}. \tag{3.14}$$

Applying now inequalities (3.3)-(3.14) we can estimate, for $u \in B(0,R)$,

$$\|T(u)(t)\|_{L^\infty_{\alpha+d}} \le C(1+t)\|u_0\|_{L^\infty_{\alpha+d}}$$
$$+ CR^{q-1}\int_0^t (t-\tau)^{-1/\alpha}(1+(t-\tau))\|u(\tau)\|_{L^\infty_{\alpha+d}}\,d\tau$$
$$\le R/2 + CM^{q-1}\|u_0\|^{q-1}_{L^\infty_{\alpha+d}} R\,t^{1-1/\alpha}(1+t)$$
$$\le R,$$

provided that $0 \le t \le T$ and

$$T \le C \min\{1, \|u_0\|_{L^\infty_{\alpha+d}}^{-\alpha(q-1)/(\alpha-1)}\},$$

with $C > 0$ small enough.

In the same way, for all $u, \tilde{u} \in B(0, R)$,

$$\|T(u)(t) - T(\tilde{u})(t)\|_{L^\infty_{\alpha+d}} \le CR^{q-1} \int_0^t (t-\tau)^{-1/\alpha}(1 + (t-\tau))\|u(\tau) - \tilde{u}(\tau)\|_{L^\infty_{\alpha+d}}\, d\tau.$$

The Banach fixed point theorem now guarantees the existence of a local-in-time solution. In the next step, such solution must be extended globally-in-time. The argument is standard: we fix $T > 0$ arbitrarily large and using that $\|u(t)\|_\infty \le C$ on $[0, T]$ (see inequality (2.2)), we show that $\|u(t)\|_{L^\infty_{\alpha+d}}$ does not blow up on $[0, T]$. Indeed for some constants $C_1, C_2, \ldots,$ depending on T, for $0 \le t \le T$ we have

$$\|u(t)\|_{L^\infty_{\alpha+d}} \le C_1 + C_2 \int_0^t (t-\tau)^{-1/\alpha}\|u(\tau)\|_{L^\infty_{\alpha+d}}\, d\tau.$$

Iterating this inequality and applying Fubini's theorem we get

$$\|u(t)\|_{L^\infty_{\alpha+d}} \le C_3 + C_4 \int_0^t (t-\tau)^{1-2/\alpha}\|u(\tau)\|_{L^\infty_{\alpha+d}}\, d\tau.$$

We repeat this argument until we obtain the integrand factor $(t-\tau)$ with a positive exponent; here, only a finite number if iterations are needed, since $\alpha > 1$. This leads to $\|u(t)\|_{L^\infty_{\alpha+d}} \le C_5 + C_6 \int_0^t \|u(\tau)\|_{L^\infty_{\alpha+d}}\, d\tau$ and finally to $\|u(t)\|_{L^\infty_{\alpha+d}} \le C_5 \exp(C_6 t)$ by the classical Gronwall lemma.

To prove of (3.13) under the stronger assumption $u_0 \in E_{\alpha+d}$, one could proceed in the same way, replacing the space $L^\infty_{\alpha+d}$ with $E_{\alpha+d}$ (and using the estimates of Lemma 3.2). However, this argument would require additional restrictions, such as inequalities of the form $|f'(u) - f'(v)| \le C|u-v|(|u|^{q-2} + |v|^{q-2})$, which are not fulfilled for some nonlinearities satisfying (1.3) with $q < 2$.

Let us proceed in a slightly different way. First of all we have, by [2, 6], $\nabla u(t) \in L^\infty([0, T], L^\infty(\mathbb{R}^d))$ for all $T > 0$. We rewrite the integral equation (2.6) in the following way

$$\begin{aligned} \nabla u(x,t) &= \nabla S_\alpha(t)u_0(x) \\ &\quad - \int_0^t \left(\int_{|y|\le|x|/2} + \int_{|y|\ge|x|/2} \right) \nabla p_\alpha(x-y, t-s)\nabla f(u(y,s))\, dy\, ds. \end{aligned} \tag{3.15}$$

It follows from condition (1.3) that $|f'(u)| \le C|u|^{q-1}$, hence, for every u satisfying (3.12) we have

$$|\nabla f(u(y,s))| \le C(1+|y|)^{-(q-1)(\alpha+d)}|\nabla u(y,s)| \le C(1+|y|)^{-(q-1)(\alpha+d)}, \tag{3.16}$$

for a positive constant $C = C(T)$ and all $y \in \mathbb{R}^d$, $s \in [0,T]$. Combining (3.16) with (3.7) and with the decay estimate $|\nabla p_\alpha(x,t)| \le Ct|x|^{-(\alpha+d+1)}$, we get from (3.15) the preliminary inequality

$$\begin{aligned} |\nabla u(x,t)| \le C(1+|x|)^{-(\alpha+d+1)} + C(1+|x|)^{-(\alpha+d+1)+q_1} \\ + C(1+|x|)^{-(q-1)(\alpha+d)} \end{aligned} \tag{3.17}$$

for some constant $C = C(T) > 0$, all $x \in \mathbb{R}^d$, $t \in [0,T]$, and with $q_1 = d$. Since $q > 1$, now we can use this inequality to improve the estimate in (3.16). This allows us to replace q_1 with some $0 \le q_2 < q_1$ and to improve also the estimate of the third term in (3.17). After finitely many iterations of this argument (more and more iterations are needed when q approaches 1), we get $|\nabla u(x,t)| \le C(T)(1+|x|)^{-(\alpha+d+1)}$ for all $x \in \mathbb{R}^d$ and $t \in [0,T]$. □

Let us now recall a singular version of the Gronwall lemma. This fact seems to be well-known, we state it, however, in the form which is the most suitable for our application and we prove it for the completeness of the exposition.

LEMMA 3.4. *Assume that a nonnegative and locally bounded function $h = h(t)$ satisfies the inequality*

$$h(t) \le C_1(1+t) + C_2 \int_0^t (t-\tau)^{-a}(1+\tau)^{-b} h(\tau)\, d\tau \tag{3.18}$$

for some $a \in (0,1)$, $b > 0$, positive constants C_1 and C_2, and all $t \ge 0$. If $a+b > 1$, then $h(t) \le C(1+t)$ for all $t \ge 0$ and C independent of t. The same conclusion holds true in the limit case $a+b = 1$ under the weaker assumption

$$h(t) \le C_1(1+t) + C_2 \int_0^t (t-\tau)^{-a}\tau^{-b} h(\tau)\, d\tau \tag{3.19}$$

provided C_2 is sufficiently small.

Proof. If $a+b=1$, we deduce from (3.19) the following inequality

$$h(t) \le C_1(1+t) + C_2 K(a,b) \sup_{0\le\tau\le t} h(\tau),$$

where

$$K(a,b) = \int_0^t (t-\tau)^{-a}\tau^{-b}\, d\tau = \int_0^1 (1-s)^{-a}s^{-b}\, ds.$$

Consequently, $\sup_{0\le\tau\le t} h(\tau) \le \frac{C_1}{1-C_2K(a,b)}(1+t)$ provided $C_2 < 1/K(a,b)$.

In the case $a+b > 1$, using (3.18), we write $b = b_1 + \eta$ with $a + b_1 = 1$ and $\eta > 0$, and we fix $t_1 > 0$ such that $C_2(1+t_1)^{-\eta} < 1/K(a,b_1)$. Now, splitting the integral in (3.18) at t_1 yields

$$h(t) \le C(1+t) + C_2K(a,b_1)(1+t_1)^{-\eta} \sup_{0\le\tau\le t} h(\tau)$$

for some $C > 0$ independent of t. The conclusion of Lemma 3.4 now follows. □

If the exponent q in the assumptions on the nonlinearity (1.3) is larger than the time-critical value $\widetilde{q}$, we can improve the space decay estimates from Proposition 3.3 through the following space-time decay result.

THEOREM 3.5. *(i) Let $\alpha \in (1,2)$. Assume that $u = u(x,t)$ is a solution of the Cauchy problem (1.1)–(1.2), where the nonlinearity f satisfies (1.3) with $q > \widetilde{q} = 1 + (\alpha-1)/d$ and $u_0 \in L^\infty_{\alpha+d}$. There exists $C > 0$ (depending on u_0 but independent of x, t) such that*

$$|u(x,t)| \le Cp_\alpha(x, 1+t) \quad \textit{for all} \quad x \in \mathbb{R}^d \quad \textit{and} \quad t > 0. \tag{3.20}$$

The same conclusion holds true for $q = \widetilde{q}$ provided $\|u_0\|_1$ is sufficiently small.

(ii) Under the more stringent assumption $u_0 \in E_{\alpha+d}$ we have also

$$\|\nabla u(t)\|_{L^\infty_{\alpha+d+1}} \le C(1+t). \tag{3.21}$$

Proof. First recall that by estimates (2.1) and (2.2) with $p = \infty$, the solution satisfies

$$\|u(t)\|_\infty \le C(1+t)^{-d/\alpha}.$$

Hence, to establish (3.20), it suffices to prove

$$\|u(t)\|_{L^\infty_{\alpha+d}} \le C(1+t). \tag{3.22}$$

Indeed, the inequality

$$g(x,t) \equiv \min\left\{(1+t)^{-d/\alpha};\ \frac{1+t}{(1+|x|)^{\alpha+d}}\right\} \le Cp_\alpha(x, t+1).$$

is the consequence of the elementary estimate

$$g(x,t) \le (1+t)^{-d/\alpha} \min\{1\,;\ |x(1+t)^{-1/\alpha}|^{-\alpha-d}\}$$

and the asymptotic formula (1.6) (implying, in particular, that $\min\{1\,;\,|x|^{-\alpha-d}\} \le CP_\alpha(x)$ for all $x \in \mathbb{R}^d$ and a constant $C > 0$).

In the proof of (3.22), we use the integral equation (2.6), hence we begin by the preliminary estimate (resulting from (3.4) and from the hypothesis (1.3))

$$\|\nabla S_\alpha(t-\tau) f(u(\tau))\|_{L^\infty_{\alpha+d}} \le C(t-\tau)^{-1/\alpha}\|u(\tau)\|_\infty^{q-1}\|u(\tau)\|_{L^\infty_{\alpha+d}} + C(t-\tau)^{1-1/\alpha}\|u(\tau)\|_q^q.$$

Moreover, since by (2.1) and (2.2) with $p = q$, the solution satisfies the decay estimate

$$\|u(\tau)\|_q^q \le C(1+\tau)^{-d(q-1)/\alpha}, \tag{3.23}$$

we have the following inequalities

$$\int_0^t (t-\tau)^{1-1/\alpha}\|u(\tau)\|_q^q\, d\tau \le C\int_0^t (t-\tau)^{1-1/\alpha}(1+\tau)^{-d(q-1)/\alpha}\, d\tau \le C(1+t)$$

which are valid for $1/\alpha + d(q-1)/\alpha \ge 1$.

Consequently, after computing the $L^\infty_{\alpha+d}$-norm of equation (2.6) and using estimate (3.3) we arrive at

$$\|u(t)\|_{L^\infty_{\alpha+d}} \le C(1+t) + C\int_0^t (t-\tau)^{-1/\alpha}(1+\tau)^{-d(q-1)/\alpha}\|u(\tau)\|_{L^\infty_{\alpha+d}}\, d\tau. \tag{3.24}$$

In the time-critical case $1/\alpha + d(q-1)/\alpha = 1$ (i.e. for $q = \widetilde{q}$) we proceed analogously, however, now we use the estimate

$$\|u(\tau)\|_\infty \le C\tau^{-d/\alpha}\|u_0\|_1 \tag{3.25}$$

with a constant C independent of u_0 and t. Hence, we obtain the following counterpart of inequality (3.24)

$$\|u(t)\|_{L^\infty_{\alpha+d}} \le C(1+t) + C\|u_0\|_1^{q-1}\int_0^t (t-\tau)^{-1/\alpha}\tau^{-d(q-1)/\alpha}\|u(\tau)\|_{L^\infty_{\alpha+d}}\, d\tau. \tag{3.26}$$

Finally, the singular Gronwall lemma (Lemma 3.4) applied to inequalities (3.24) and (3.26) completes the proof of (3.20).

To prove inequality (3.21) one should follow exactly the same argument as for the proof of (3.22), putting everywhere $E_{\alpha+d}$-norms instead of the corresponding $L^\infty_{\alpha+d}$-norms, and applying Lemma 3.2 instead of Lemma 3.1. □

We conclude this section with estimates of self-similar solutions to problem (2.8)–(2.9).

COROLLARY 3.6. *If the constant $M > 0$ in (2.9) is sufficiently small, then the corresponding solution of problem (2.8)–(2.9) satisfies*

$$0 \le u_M(x,t) \le C p_\alpha(x,t), \qquad \textit{for all } x \in \mathbb{R}^d,\ t > 0, \tag{3.27}$$

with $C = C(M, \alpha, d) > 0$ independent of x and t.

Proof. Let us recall that the solution of (2.8)-(2.9) has been constructed in [2] as the limit of the rescaled functions $u^\lambda(x,t) \equiv \lambda^d u(\lambda x, \lambda^\alpha t)$, where $u = u(x,t)$ is the fixed solution of equation (2.8) supplemented with the nonnegative initial datum $u(\cdot, 0) = u_0 \in C_c^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} u_0(x)\, dx = M$. By Theorem 3.5, used in the critical case $q = \widetilde{q}$, the rescaled family u^λ satisfies

$$|u^\lambda(x,t)| \le C\lambda^d p_\alpha(\lambda x, 1 + \lambda^\alpha t) = C p_\alpha(x, \lambda^{-\alpha} + t) \tag{3.28}$$

for all $x \in \mathbb{R}^d$, $t > 0$, and a constant $C = C(M, \alpha, d)$ independent of x, t, λ, provided $M > 0$ is sufficiently small. Since $u^\lambda(x,t) \to U_M(x,t)$ as $\lambda \to \infty$ almost every where in (x,t) (see [2, Lemma 3.7]), passing to the limit in (3.28) we complete the proof of estimate (3.27). □

4. Asymptotic profiles

In this section, we derive the asymptotic expansions from Theorems 2.1 and 2.4. Let us recall that all positive constants, which appear here, are independent of x and t and are denoted by the same letter C.

Proof of Theorem 2.1. Let us consider the nonlinear term appearing in the integral equation (2.6),

$$\mathcal{N}(u)(t) \equiv \int_0^t \int \nabla_{\mathbb{R}^d} p_\alpha(x - y, t - s) f(u(y,s))(s)\, ds.$$

In order to find an asymptotics of $\mathcal{N}$ for large $|x|$, we define two remainder functions $\mathcal{R}(x,t)$ and $\mathcal{R}_1(x,t)$, through the relations

$$\begin{aligned} \mathcal{N}(u)(x,t) &= \int_0^t \int f(u(y,s)) \nabla p_\alpha(x, t-s)\, dy\, ds + \mathcal{R}_1(x,t) \\ &= -\frac{c_1 x}{|x|^{\alpha+d+2}} \int_0^t \int (t-s) f(u(y,s))\, dy\, ds - \mathcal{R}(x,t). \end{aligned} \tag{4.1}$$

Here, c_1 is the constant appearing in relation (1.7). Hence, it follows from the integral equation (2.6) that

$$u(x,t) = S_\alpha(t)u_0(x) + \frac{c_1\, x}{|x|^{\alpha+d+2}} \int_0^t \int_{\mathbb{R}^d} (t-s)\, f(u(y,s))\, dy\, ds + \mathcal{R}(x,t) \tag{4.2}$$

and it remains to estimate $\mathcal{R}(x,t)$.

Computing the difference of the two expressions of $\mathcal{N}$ in (4.1) we deduce a bound for $\mathcal{R} + \mathcal{R}_1$, implying

$$\begin{aligned} |\mathcal{R}(x,t)| \le |\mathcal{R}_1(x,t)| &+ C|x|^{-(2\alpha+d+1)} \\ &\int_0^t \int_{\mathbb{R}^d} \left[\nabla p_\alpha(x, t-s) + \frac{c_1\, x}{|x|^{\alpha+d+2}}(t-s) \right] |f(u(y,s))|\, dy\, ds. \end{aligned}$$

Now, the asymptotic expansion (1.7), the assumption (1.3), and the L^q-estimates (3.23) lead to

$$\begin{aligned} |\mathcal{R}(x,t)| &\le |\mathcal{R}_1(x,t)| + C|x|^{-(2\alpha+d+1)} \int_0^t \int_{\mathbb{R}^d} (t-s)^2 |f(u(y,s))|\, dy\, ds \\ &\le |\mathcal{R}_1(x,t)| + C|x|^{-(2\alpha+d+1)}\, t^2 \int_0^t (1+s)^{-(q-1)d/\alpha}\, ds. \end{aligned} \tag{4.3}$$

In order to estimate $\mathcal{R}_1$, we decompose it as $\mathcal{R}_1 = I_1 + \cdots + I_4$, where

$$\begin{aligned} I_1(x,t) &\equiv \int_0^t \int_{|y|\le|x|/2} [\nabla p_\alpha(x-y, t-s) - \nabla p_\alpha(x, t-s)] \cdot f(u(y,s))\, dy\, ds, \\ I_2(x,t) &\equiv -\int_0^t \left(\int_{|y|\ge|x|/2} f(u(y,s))\, dy \right) \nabla p_\alpha(x, t-s)\, ds, \\ I_3(x,t) &\equiv \int_0^t \int_{|y|\ge|x|/2,\ |x-y|\ge|x|/2} \nabla p_\alpha(x-y, t-s)\, f(u(y,s))\, dy\, ds, \\ I_4(x,t) &\equiv \int_0^t \int_{|y|\le|x|/2} \nabla p_\alpha(y, t-s)\, f(u(x-y,s))\, dy\, ds. \end{aligned}$$

In our next two estimates, we use the inequality (which is a consequence of the L^∞-bound of the solution, see (2.2))

$$|u(y,s)|^q \le C(1+|y|)^{-(\alpha+d)}(1+s)^{-(q-1)d/\alpha}\|u(s)\|_{L^\infty_{\alpha+d}}. \tag{4.4}$$

This leads to

$$\begin{aligned} |I_1(x,t)| &\le C|x|^{-(\alpha+d+2)} \int_0^t (t-s) \int_{|y|\le|x|/2} |y|\,|u(y,s)|^q\,dy\,ds \\ &\le C|x|^{-(\alpha+d+2)}\, t \int_0^t (1+s)^{-(q-1)d/\alpha}\|u(s)\|_{L^\infty_{\alpha+d}}\,ds. \end{aligned} \tag{4.5}$$

Here, we have applied also the Taylor formula and the bound (3.11).

The next two integrals can be bounded by the same quantity, indeed

$$\begin{aligned} |I_2(x,t)|+|I_3(x,t)| &\le C|x|^{-(\alpha+d+1)} \int_0^t (t-s) \int_{|y|\ge|x|/2} |u(y,s)|^q\,dy\,ds \\ &\le C|x|^{-(2\alpha+d+1)}\, t \int_0^t (1+s)^{-(q-1)d/\alpha}\|u(s)\|_{L^\infty_{\alpha+d}}\,ds. \end{aligned} \tag{4.6}$$

The estimate for the last term is

$$|I_4(x,t)| \le C|x|^{-q(\alpha+d)} \int_0^t (t-s)^{-1/\alpha}\|u(s)\|^q_{L^\infty_{\alpha+d}}\,ds. \tag{4.7}$$

Since we are assuming $\alpha > 1$, when we compare the exponents of $|x|$ in inequalities (4.3) and (4.5)-(4.6), we see that

$$|\mathcal{R}(x,t)| \le C\,\max\big\{|x|^{-q(\alpha+d)}\,;\ |x|^{-(\alpha+d+2)}\big\} \quad \text{for all } |x|\ge 1 \text{ and } t\in(0,T], \tag{4.8}$$

where $C = C(T) > 0$ is uniformly bounded with respect to x and t, in any time interval $t \in [0,T]$. Part (i) of Theorem 2.1 now follows.

To establish Part (ii), we have only to improve the estimate of the integral (4.7). We can do it using, in a slightly deeper way, the properties of the fundamental solution $p_\alpha(x,t)$. In particular, its radial symmetry implies that

$$\int_{|y|\le|x|/2} \nabla p_\alpha(y,t-s)\,ds = 0,$$

so that

$$I_4(x,t) \equiv \int_0^t \int_{|y|\le|x|/2} \nabla p_\alpha(y,t-s)\cdot\big[f(u(x-y,s)) - f(u(x,s))\big]\,dy\,ds. \tag{4.9}$$

Owing to the more stringent assumption $u_0 \in E_{\alpha+d}$ and by Proposition 3.3, we deduce from the mean value theorem applied to $f(u)$ (recall that $|f'(u)| \le C|u|^{q-1}$)

$$|I_4| \le C|x|^{-q(\alpha+d)-1} \int_0^t \|u(s)\|^{q-1}_{L^\infty_{\alpha+d}} \|\nabla u(s)\|_{L^\infty_{\alpha+d+1}}\,ds. \tag{4.10}$$

Replacing inequality (4.7) with this new estimate shows that the bound (4.8) of the remainder term can be improved into

$$|\mathcal{R}(x,t)| \le C\ \max\big\{|x|^{-q(\alpha+d)-1}\,;\ |x|^{-(\alpha+d+2)}\big\} \quad \text{for all } |x|\ge 1 \text{ and } t\in(0,T]. \tag{4.11}$$

Hence, Part (ii) of Theorem 2.1 follows.

Let us prove assertion (iii). When the solution satisfies the additional estimate (2.3) (recall that, by Theorem 3.5, such an estimate holds true at least when either $q > \widetilde{q}$ or $q = \widetilde{q}$ and $\|u_0\|_1$ is small enough), we have $\|u(t)\|_{L^\infty_{\alpha+d}} \le C(1+t)$. In this case, it is easy to construct an exponent $N = N(\alpha, d, q)$ such that

$$|\mathcal{R}(x,t)| \le C(1+t)^N \max\big\{|x|^{-q(\alpha+d)}\,;\ |x|^{-(\alpha+d+2)}\big\} \quad \text{for all } |x|\ge 1,\ t>0. \tag{4.12}$$

Let us explain why $N \le 3$. It follows directly from (4.3) and from (4.5)–(4.7) that $N \le \max\{3\,;\ q+1-1/\alpha\}$. However, if $q > 2 + 1/\alpha$, then we can replace estimate (4.7) with

$$\begin{aligned} |I_4(x,t)| &\le C|x|^{-(\alpha+d+2)} \int_0^t (t-s)^{-1/\alpha} \|u(s)\|_\infty^{q-1-2/(\alpha+d)} \|u(s)\|^{1+2/(\alpha+d)}_{L^\infty_{\alpha+d}}\,ds \\ &\le C(1+t)^3|x|^{-(\alpha+d+2)}. \end{aligned} \tag{4.13}$$

If, moreover, ∇u satisfies the additional pointwise estimate (2.4) then we can precise in a similar way the bound (4.11). Namely, we can replace $C = C(T)$ in (4.11) with $C(1+t)^3$. Next, the proof of this claim relies either on inequality (4.10) if $1 < q \le 2$ or on the following new estimate of I_4 when $q > 2$

$$|I_4(x,t)| \le C|x|^{-(\alpha+d+2)} \int_0^t \|u(s)\|_\infty^{q-1-1/(\alpha+d)} \|u(s)\|^{1/(\alpha+d)}_{L^\infty_{\alpha+d}} \|\nabla u(s)\|_{L^\infty_{\alpha+d+1}}\,ds.$$

The estimates of the other terms remain unchanged. The proof of Theorem (2.1) is now complete. □

Proof of Theorem 2.4. Let u_M be a self-similar solution of (2.8)-(2.9), satisfying estimate (2.10). We consider the integrals I_1, I_2, I_3 and I_4 and also the remainder term $\mathcal{R}$, defined as in the proof of Part (i) of Theorem 2.1. We treat all these terms proceeding as before, but replacing everywhere estimate (4.4) with the estimate (deduced from (2.10))

$$|u_M(y,s)|^q \le Cs^{-dq/\alpha} P_\alpha^q(y/s^{1/\alpha}), \tag{4.14}$$

with $q = \widetilde{q}$ and $C = C(M)$, then making the change of variables $y \mapsto ys^{1/\alpha}$ in all the space integrals. After some simple computations, we arrive at

$$|\mathcal{R}(x,t)| \le Ct^{-d/\alpha}\left[\left(\frac{|x|}{t^{1/\alpha}}\right)^{-(2\alpha+d+1)} + \left(\frac{|x|}{t^{1/\alpha}}\right)^{-(\alpha+d+2)} + \left(\frac{|x|}{t^{1/\alpha}}\right)^{-\widetilde{q}(\alpha+d)}\right].$$

Recalling that $f(u) = bu^{\widetilde{q}}$, applying (4.2) to u_M we get

$$u_M(x,t) = Mp_\alpha(x,t) + t^{1+1/\alpha} \cdot \frac{c_1\alpha^2}{\alpha+1}\left(\int U_M(y)^{\widetilde{q}}\,dy\right)\frac{b\cdot x}{|x|^{\alpha+d+2}} + \mathcal{R}(x,t).$$

Now, passing to self-similar variables, we deduce that, for all $x \in \mathbb{R}^d$,

$$U_M(x) = MP_\alpha(x) + \frac{c_1\alpha^2}{\alpha+1}\|U_M\|_{\widetilde{q}}^{\widetilde{q}}\frac{b\cdot x}{|x|^{\alpha+d+2}} + \mathcal{R}_M(x),$$

where

$$\mathcal{R}_M(x) = O(\max\{|x|^{-\widetilde{q}(\alpha+d)};\ |x|^{-(\alpha+d+2)}\}), \qquad \text{as } |x| \to \infty.$$

Theorem (2.4) is now established. □

REMARK 4.1. We conclude observing that the above expression of the remainder term $\mathcal{R}_M(x)$ can be simplified distinguishing the two cases $d = 1$ and $d \ge 2$. Indeed, an elementary calculation shows that

1. In the one dimensional case $d = 1$ (hence, $\widetilde{q} = \alpha$, and the assumption $\widetilde{q} > q^*$ reads $\alpha > \sqrt{2}$), we have

$$\mathcal{R}_M(x) = \begin{cases} O\big(|x|^{-\alpha(\alpha+1)}\big) & \text{if } \sqrt{2} < \alpha \le \sqrt{3}, \\ O\big(|x|^{-(\alpha+3)}\big) & \text{if } \sqrt{3} \le \alpha < 2, \end{cases} \qquad \text{as} \quad |x| \to \infty.$$

2. For $d \ge 2$, it follows

$$\mathcal{R}_M(x) = O(|x|^{-\widetilde{q}(\alpha+d)}) \quad \text{as} \quad |x| \to \infty.$$

REMARK 4.2. Analogously, as in Theorem 2.1, one could remove the restriction $\widetilde{q} > q^*$ from Theorem 2.4, provided we have the additional weighted estimate

$$\|\nabla u_M(t)\|_{L^\infty_{\alpha+d+1}} \leq Ct. \tag{4.15}$$

We expect that inequality (4.15) can be proved using the scaling argument from the proof of Corollary 3.6, below. This reasoning would require, however, some improvements of estimates from [2]. We skip other details because the goal of this work was to present a method of deriving asymptotic expansions of solutions rather than to study the most general case.

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