Soliton-like solutions of the modified Camassa–Holm equation with variable coefficients

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Abstract

We study soliton- and peakon-like solutions of the modified Camassa–Holm equation with variable coefficients and a singular perturbation. This equation is a direct generalization of the well-known modified Camassa–Holm equation, which is an integrable system having both smooth and peaked soliton solutions, named peakons.

The novelty of this paper lies in the development of a general methodology for constructing asymptotic peakon-like solutions. We present a general scheme for finding approximations of any order and study their accuracy.

The results are illustrated by nontrivial examples of both soliton- and peakon-like solutions with global phase function. The proposed technique can be used for studying wave-like solutions of other equations with variable coefficients and small dispersion.

Keywords: modified Camassa–Holm equation, soliton solution, peakon solution, shallow water models, singular perturbation, WKB approximations, fluid dynamics.
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1. Introduction

Nonlinear partial differential equations are widely used to describe the propagation of waves at the surface in liquids. The Boussinesq equation [1]

$$u_{tt} - c^2 u_{xx} - \alpha u_x^2 - \alpha u u_{xx} - \beta u_{xxxx} = 0, \tag{1}$$

where $c, \alpha > 0$ and $\beta \neq 0$, and the celebrated Korteweg–de Vries (KdV) equation [2]

$$u_t + u u_x + u_{xxx} = 0 \tag{2}$$

are two early models that were successfully used to capture the dispersive features of water wave propagation. These equations were studied with various methods and approaches, including analytical, numerical, and algebraic–geometric techniques. The intensive research on the KdV equation motivated the development of the theory of completely integrable infinite-dimensional dynamical systems and of new concepts, such as that of *soliton* [3, 4]. Solitons have become a central object of modern mathematical physics because they are an important peculiarity of many integrable nonlinear evolution equations [5, 6, 7]. Soliton solutions describe wave processes localized in space, propagating with a speed depending on the wave amplitude and interacting according to the nonlinear superposition principle [8, 9].

The Boussinesq equation has only one-soliton solutions and is not an integrable system, whereas the Korteweg–de Vries equation has one-, two-, and multi-soliton solutions and is completely integrable. Subsequently, several new nonlinear partial differential equations with solitons were found, although not all of them are integrable systems. In this connection, we can mention the regularized long-wave equation, which is also known as the Benjamin–Bona–Mahony (BBM) equation [10, 11, 12]

$$u_t + u_x + \alpha u u_x - u_{xxt} = 0 \tag{3}$$

or the time regularized long–wave equation [13] also called the Joseph–Egri equation [14]

$$u_t + u_x + \alpha u u_x + u_{xtt} = 0. \tag{4}$$

These “KdV–like“ equations, along with others of a similar nature, emerged during the search for potential alternatives to the original KdV equation in the study of shallow water wave phenomena [15, 16, 17]. For example, one motivation of the BBM equation is that its solutions have better stability properties at high wave numbers.

In parallel, intense studies were carried out in the direction of looking for new classes of integrable equations. A remarkable illustration of this fact is the Camassa–Holm (CH) equation, which was discovered twice: the first time it appeared in a work by Fokas and Fuchssteiner [18] as a member of a new large class of completely integrable nonlinear equations. Its hydrodynamic relevance was put in evidence only a few years later, when the same equation was reproposed by Camassa and Holm [19] as an asymptotic model of the free-surface Euler equations. Subsequently, the CH equation, along with other Camassa–Holm-type equations, was found to be effective in modeling turbulent flows [20] and axisymmetric pipe flows [21, 22].

The CH equation was studied through a range of techniques and approaches including the inverse scattering transform [23, 24], Hirota’s bilinear method [25], Bäcklund transformation [26], numerical methods [27, 28], etc.

After the paper by Camassa and Holm, this equation immediately attracted considerable interest: it turned out to be better suited than the KdV equation in modeling the propagation of waves of larger amplitude, for which the nonlinear effects are often predominant and wave-breaking effects can appear during the evolution. Many papers are devoted to the study of wave breaking criteria. Here we just mention the early papers [19, 29], and the ”local-in-space” blowup criterion [30], that encompasses the previous ones, and also shows that there is a deep connection between solitons and wave breaking effects: if the initial profile decays faster than Camassa–Holm’s peaked solitons at the spatial infinity, than blowup of the solution will occur after some time. The CH equation can be written in the following form [19]:

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{5}$$

This paper focuses on the modified version of the above equation, commonly referred to in the literature as the modified Camassa–Holm (mCH) equation, that is also often used for describing wave processes in shallow water (see, e.g., [16]):

$$u_t - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + uu_{xxx}. \tag{6}$$

Another example of an integrable system associated with (5) that is used to describe traveling waves in shallow water is the two-component CH equation, which was successfully analyzed in detail in [31, 32] using the phase plane method.

Like the CH equation (5), the mCH equation (6) is a completely integrable dynamical system and was studied by numerous methods, including the Riemann–Hilbert approach [33]. Several results are available for the stability of peakon solutions [34, 35], the well-posedness [36], the existence of global solutions, as well as blowup phenomena [37].

Both the CH equation and the mCH equation have solutions with different properties, including soliton solutions, multi-soliton solutions, periodic solutions, and so on. For example, the soliton solutions to the mCH equation can be written as [38]:

$$u(x,t) = \frac{8\mu^2}{\left(\mu + A\cosh\left(2t - x\right) - A\sinh\left(2t - x\right)\right)^2} - \frac{8\mu}{\mu + A\cosh\left(2t - x\right) - A\sinh\left(2t - x\right)}, \tag{7}$$

where A, μ are reals and $\mu A > 0$.

In addition to the general properties of soliton systems, the CH and mCH equations (5), (6) have the specificity of admitting peaked solitons, called peakons. Similar features are also inherent in the famous Degasperis–Procesi equation [39, 40] associated with (5).

Peakons have a peak at their crest, so their derivative at that point is discontinuous and has opposite signs on the left and right. Because of such a singularity, peakons are solutions of the corresponding equations in a weak sense, but otherwise they enjoy the usual properties of smoother soliton solutions. In particular, they describe localized waves that interact without collision and have a velocity that depends on the wave amplitude. The peakon solutions of the CH equation (5) are given [19] by the expression

$$u(x,t) = c\,\exp\left(-|x-ct|\right), \tag{8}$$

and describe moving waves with a speed $c > 0$ equal to the height of the peakon. If $c < 0$, then the wave moves to the left with a downward peak, and it is sometimes called an anti-peakon. Detailed analysis of the complex dynamics of peaked solitons associated with the CH-equation can be found in [41].

For describing wave processes in liquids with heterogeneous characteristics, the above equations are no longer sufficient: to this purpose, more general versions with variable coefficients have been proposed. In this extended case, the exact form of the solutions is not known, as most traditional analytical methods lose their effectiveness because of the variable coefficients. Consequently, it is reasonable to seek approximate solutions that are similar to the exact solutions of the corresponding equations with constant coefficients.

In situations characterized by a small dispersion in the medium or the influence of an external force, the effective techniques of asymptotic analysis have demonstrated their utility [5, 42, 43]. In this regard it is worth noting some papers. In [44, 45, 46], Lax and Levermore effectively employed asymptotic analysis to examine the weak limit of the solution to the initial value problem for the KdV equation, as the dispersion tends to zero. In [47], Miura and Kruskal developed the nonlinear Wentzel–Kramers–Brillioun (WKB) method to study nearly-periodic solutions of the KdV equation with a small dispersion.

The WKB method has proven to be useful for constructing soliton-like solutions of KdV-like equations with variable coefficients. For instance, asymptotic one phase and multi-phase soliton-like solutions were developed for the Korteweg–de Vries equation with variable coefficients (vcKdV) and singular perturbations [48, 49, 50]. The characteristics of these solutions are similar to those of the classical KdV equation with constant coefficients. Nontrivial examples of such systems are presented in [51, 52, 53].

Analogously, for the singularly perturbed BBM equation with variable coefficients, soliton-like solutions were found in [54]. Due to the absence of multi-soliton solutions to the BBM equation with constant coefficients, the so-called asymptotic Σ-soliton solutions to the singularly perturbed BBM equation with variable coefficients were obtained [55]. The concept of the asymptotic Σ-soliton solution is based on the idea of splitting multi–soliton solutions into a set of one-soliton solutions at large values of the independent variables.

The present paper deals with the modified Camassa–Holm equation with variable coefficients (vcmCH) and a singular perturbation of the form [42]:

$$a(x,t,\varepsilon)u_t - \varepsilon^2 u_{xxt} + b(x,t,\varepsilon)u^2 u_x - 2\varepsilon^2 u_x u_{xx} - \varepsilon^2 u u_{xxx} = 0. \tag{9}$$

Here, ε is a small parameter. We assume that the functions $a(x,t,\varepsilon)$ and $b(x,t,\varepsilon)$ with $(x,t) \in \mathbb{R} \times [0;T]$ for some $T > 0$ can be presented as

$$a(x,t,\varepsilon) = \sum_{k=0}^{N} \varepsilon^k a_k(x,t) + O(\varepsilon^{N+1}), \quad b(x,t,\varepsilon) = \sum_{k=0}^{N} \varepsilon^k b_k(x,t) + O(\varepsilon^{N+1}), \tag{10}$$

and $a_0(x,t)\,b_0(x,t) \neq 0$ for all $(x,t) \in \mathbb{R} \times [0;T]$.

In the sequel, we use the following notation from asymptotic analysis [56]: if Ψ is a function defined in $\mathbb{R} \times [0;T]$ and depending on a small parameter $\varepsilon > 0$, then $\Psi(x,t,\varepsilon) = O(\varepsilon^N)$ for integer $N \geq 0$ means that for any bounded and closed set $K \subset \mathbb{R}^n \times [0;T]$, there exists a positive value C, possibly depending on the set K, and $\varepsilon_0 > 0$, independent of (x,t), such that the inequality $|\Psi(x,t,\varepsilon)| \leq C\,\varepsilon^N$ holds for all $\varepsilon \in (0,\varepsilon_0)$ and all $(x,t) \in K$.

Equation (9) generalizes the mCH equation (6). Let us recall that peakon and soliton solutions of the latter are written, respectively, as [57]

$$u(x,t) = 2\,\sinh^{-2}\left(\frac{|x-2t|}{2} + \operatorname{arccoth}\sqrt{2}\right), \tag{11}$$

and

$$u(x,t) = -2\cosh^{-2}\left(\frac{x-2t}{2}\right). \tag{12}$$

The main aim of this paper is to describe peakon-like solutions to the vcmCH equation (9). Since in some cases peakon solutions can be found as the limit of soliton solutions [58], the problem of constructing soliton-like solutions of equation (9) is also considered. To attack these problems, we will apply the nonlinear WKB method and make use of an appropriate modification of the basic ideas previously introduced to find soliton-like solutions of the KdV–like equations [48] with variable coefficients, as well as step-like solutions of the Burgers equation with a singular perturbation [59]. For these two problems, we will outline the main steps of the algorithm for determining the asymptotic solutions. We will derive the equations for the terms of the asymptotic expansions and establish the solvability of these equations in appropriate functional spaces.

The main result of this paper is the obtaining soliton- and peakon-like solutions of the mCH with variable coefficients (9).

We stress the fact that the peakon- and soliton-like solutions do not coincide, which is natural, but their discontinuity curve is the same. This discontinuity curve is determined by a first-order ordinary differential equation, rather than a second-order equation, as in the case of the KdV and the BBM equations [48, 54].

The peakon-like solutions obtained for the vcmCH equation (9), in the special case where $a(x,t,\varepsilon)$ and $b(x,t,\varepsilon)$ are constant, reduce to the exact peakon-like solutions of the mCH equation (6). This demonstrates that the peakon-like solutions found for the vcmCH equation should be interpreted as a deformation of the peakon solutions for the mCH equation (6), resulting from the introduction of variable coefficients.

The paper is organized as follows. In Section 2, we present preliminary remarks and formulate auxiliary notions, among which is the definition of an asymptotic soliton-like solution.

In Section 3, we describe in detail a technique for finding soliton-like solution to (9). We discuss the procedure for recursively finding the terms of the asymptotic expansions. In particular, we specify the conditions under which the terms appearing in the singular part of the expansion hold the requirement for the definition of the asymptotic soliton-like solution. We also discuss the accuracy with which the obtained solution satisfies the given equation. In the last part of Section 3 we illustrate an example of the application of the general technique.

Section 4 focuses on implementing a similar program for peakon-like solutions. We present here a nontrivial example of a peakon-like solution for the vcmCH equation.

2. Main definitions and form of the solutions

We denote by $\overline{C}_0^{\infty}(\mathbb{R})$ the space of infinitely differentiable functions $u\colon \mathbb{R}\to\mathbb{R}$, satisfying the relation

$$\frac{d^n u}{dx^n}(x) \to 0 \quad \text{as } |x|\to+\infty,$$

for any nonnegative integer n.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space, i.e., the space of infinitely differentiable functions on $\mathbb{R}$ such that for all integers $m,n\geq 0$ the condition

$$\sup_{x\in\mathbb{R}}\left| x^m \frac{d^n u}{dx^n}(x)\right| < +\infty$$

holds.

By $\mathrm{H}_s(\mathbb{R})$, $s\in\mathbb{R}$, we denote the Sobolev space [60, 61], i.e., the space of tempered distributions $\mathcal{S}^*(\mathbb{R})$, whose Fourier transforms $F[g](\xi)$ satisfy the condition

$$||g||_s^2 = \int\limits_{-\infty}^{+\infty} (1+|\xi|^2)^s \; |F[g](\xi)|^2\, d\xi < +\infty. \tag{13}$$

The definitions of the spaces G and G_0 below are taken from [48, 62]. We denote by G the space of infinitely differentiable functions $f\colon \mathbb{R}\times[0;T]\times\mathbb{R}\to\mathbb{R}$ satisfying the two following conditions:

1) For any nonnegative integers n, p, q and r

$$\lim_{\tau\to+\infty} \tau^n \frac{\partial^p}{\partial x^p}\frac{\partial^q}{\partial t^q}\frac{\partial^r}{\partial\tau^r} f(x,t,\tau) = 0,$$

uniformly with respect to $(x,t)\in K$, in any compact set $K\subset\mathbb{R}\times[0;T]$.

2) There exists a differentiable function $f^-: \mathbb{R} \times [0;T] \to \mathbb{R}$ such that, for any nonnegative integers n, p, q and r

$$\lim_{\tau\to-\infty} \tau^n \frac{\partial^p}{\partial x^p} \frac{\partial^q}{\partial t^q} \frac{\partial^r}{\partial \tau^r} \big(f(x,t,\tau) - f^-(x,t)\big) = 0,$$

uniformly with respect to $(x,t) \in K$, in any compact set $K \subset \mathbb{R} \times [0;T]$.

Let G_0 be the subspace of G, consisting of all functions $f \in G$ such that

$$\lim_{\tau\to-\infty} f(x,t,\tau) = 0,$$

uniformly with respect to the variables $(x,t) \in K$, in any compact set $K \subset \mathbb{R} \times [0;T]$.

We denote by $\widetilde{G}$ the space of infinitely differentiable functions $g: [0;T] \times \mathbb{R} \to \mathbb{R}$, satisfying the two following conditions:

1) For any nonnegative integers n, p and q

$$\lim_{\tau\to+\infty} \tau^n \frac{\partial^p}{\partial t^p} \frac{\partial^q}{\partial \tau^q} g(t,\tau) = 0,$$

uniformly with respect to $t \in [0;T]$.

2) There exists a differentiable function $g^-: [0;T] \to \mathbb{R}$ such that for any nonnegative integers n, p and q

$$\lim_{\tau\to-\infty} \tau^n \frac{\partial^p}{\partial t^p} \frac{\partial^q}{\partial \tau^q} \big(g(t,\tau) - g^-(t)\big) = 0,$$

uniformly with respect to $t \in [0;T]$.

Let $\widetilde{G}_0$ be the subspace of $\widetilde{G}$, consisting of all functions $g: [0;T] \times \mathbb{R} \to \mathbb{R}$, such that

$$\lim_{\tau\to-\infty} g(t,\tau) = 0,$$

uniformly with respect to $t \in [0;T]$.

The definition of an asymptotic soliton-like function is given below [48, 62].

Definition 1. *A function $u = u(x,t,\varepsilon)$, where $(x,t) \in \mathbb{R} \times [0;T]$, and $\varepsilon > 0$ is a small parameter, is called an asymptotic one-phase soliton-like function if for any integer $N \geq 0$ it can be represented in the form*

$$u(x,t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j \left[u_j(x,t) + V_j(x,t,\tau)\right] + O(\varepsilon^{N+1}), \quad \tau = \frac{x - \varphi(t)}{\varepsilon}, \tag{14}$$

where $\varphi \in C^\infty([0;T])$ is a scalar function, $u_j \in C^\infty(\mathbb{R} \times [0;T])$, for $j = 0, 1, \dots, N$, nontrivial function $V_0 \in G_0$, and $V_j \in G$, for $j = 1, \dots, N$.

An asymptotic one-phase soliton-like solution of equation (9) is searched in the form (14). The function φ is called *a phase function* and will be defined as a solution of the first-order differential equation that is found while constructing the asymptotic solution. For a given asymptotic soliton-like solution u as in (14), the curve determined by the equation $x - \varphi(t) = 0$ is called its *discontinuity curve* [48, 54].

The regular part $U_N(x,t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j u_j(x,t)$ of asymptotic solution (14) can be considered as a background function, while the singular part $V_N(x,t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j V_j(x,t,\tau)$ is required to reflect the soliton properties of the asymptotic solution. This leads us to impose appropriate functional constraints on the singular terms $V_j(x,t,\tau)$, $j = 0, 1, \dots, .$

3. The soliton-like solutions

Move on to the problem of constructing asymptotic soliton-like solutions of equation (9). We consider the case of zero background, i.e. we assume that the function $U_N(x,t,\varepsilon) \equiv 0$. So, the solutions are searched as

$$u(x,t,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j V_j\,(x,t,\tau) + O(\varepsilon^{N+1}), \quad \tau = \frac{x-\varphi(t)}{\varepsilon}. \tag{15}$$

The first term in expansion (15), i.e., the function $V_0 = V_0(x,t,\tau)$ is constructed as a solution of the third-order ordinary differential equation in the τ-variable, with parameters $(x,t) \in \mathbb{R}\times[0;T]$ (we drop below, for simplicity, the dependence on (x,t)):

$$-a_0\varphi'\,\frac{\partial V_0}{\partial\tau} + \varphi'\,\frac{\partial^3 V_0}{\partial\tau^3} + b_0\,V_0^2\,\frac{\partial V_0}{\partial\tau} - \frac{\partial}{\partial\tau}\left(\frac{\partial V_0}{\partial\tau}\right)^2 - V_0\,\frac{\partial^3 V_0}{\partial\tau^3} = 0. \tag{16}$$

The above ODE originates from the requirement that $(x,t) \mapsto V_0\left(x,t,\dfrac{x-\varphi(t)}{\varepsilon}\right)$ solves equation (9) in asymptotical sense. This means that we substitute expression (15) into equation (9), multiply the resulting equation by ε, and then take the limit as $\varepsilon \to 0$. Analogously, proceeding step by step for $k = 1,\ldots,N$, inserting $\sum_{j=0}^{k} \varepsilon^j V_j\left(x,t,\dfrac{x-\varphi(t)}{\varepsilon}\right)$ in equation (9) and equalizing to zero the linear combination of all terms with the same power of ε, we get an ODE for $V_j = V_j(x,t,\tau)$, with $j = 1,\ldots,N$. Namely,

$$-a_0\varphi'\,\frac{\partial V_j}{\partial\tau} + \varphi'\,\frac{\partial^3 V_j}{\partial\tau^3} + b_0\frac{\partial}{\partial\tau}\left(V_0^2\,V_j\right) - V_0\,\frac{\partial^3 V_j}{\partial\tau^3} - \frac{\partial^3 V_0}{\partial\tau^3}\,V_j = F_j, \tag{17}$$

where the functions $F_j = F_j(x,t,\tau)$ (with $j = 1,\ldots,N$) can be computed recursively, after functions $V_1,\ldots,V_{j-1}$ are determined in the previous step.

We recall that the solutions to equations (16), (17) have to belong to the spaces G_0, G respectively. Besides, while searching the functions V_j, for $j = 0,1,\ldots,N$, we have also to find a phase function $\varphi = \varphi(t)$ defining a discontinuity curve $\Gamma = \{(x,t) \in \mathbb{R}\times[0;T] : x = \varphi(t)\}$.

Taking into account these remarks we may study system (16), (17) as follows. Firstly, we assume that the function $\varphi = \varphi(t)$ is known. Then, equations (16), (17) are considered in the context of their restriction to the discontinuity curve Γ, treating the variable t as a parameter. In this way, the function $v_0 = v_0(t,\tau) = V_0(x,t,\tau)\big|_{x=\varphi(t)}$ can be found in explicit form. Secondly, we prove that $v_0 = v_0(t,\tau)$ is a rapidly decreasing function with respect to the variable τ, i.e. $v_0 \in \widetilde{G}_0$.

Then using property $V_1 \in G$, we find the solution $v_1(t,\tau) = V_1(x,t,\tau)\big|_{x=\varphi(t)}$ in explicit form. Moreover, we receive necessary and sufficient conditions for the existence of the solution as a rapidly decreasing function as $\tau \to +\infty$. Later, the conditions are used to deduce a nonlinear ODE for the phase function $\varphi = \varphi(t)$.

It should be mentioned that if the function $V_0 \in G_0$ then the function $v_0(t,\tau) = V_0(x,t,\tau)\big|_{x=\varphi(t)} \in \widetilde{G}_0$, and if the function $V_1 \in G$ then the function $v_1(t,\tau) = V_1(x,t,\tau)\big|_{x=\varphi(t)} \in \widetilde{G}$.

Now, let us consider the algorithm in detail. Denote, for $j = 0,1,\ldots,N$,

$$v_j = v_j(t,\tau) = V_j(x,t,\tau)\big|_{x=\varphi(t)}.$$

From (9), (16), (17), it follows that the functions $v_j(t,\tau)$, $j = 0,1,\ldots,N$, satisfy differential equations:

$$(\,\varphi' - v_0\,)\,\frac{\partial^3 v_0}{\partial\tau^3} - a_0(\varphi,t)\,\varphi'\,\frac{\partial v_0}{\partial\tau} + b_0(\varphi,t)\,v_0^2\,\frac{\partial v_0}{\partial\tau} - \frac{\partial}{\partial\tau}\left(\frac{\partial v_0}{\partial\tau}\right)^2 = 0, \tag{18}$$

and

$$(\,\varphi' - v_0\,)\,\frac{\partial^3 v_j}{\partial\tau^3} - a_0(\varphi,t)\,\varphi'\,\frac{\partial v_j}{\partial\tau} + b_0(\varphi,t)\,\frac{\partial}{\partial\tau}\left(v_0^2\,v_j\,\right) - 2\,\frac{\partial}{\partial\tau}\left(\frac{\partial v_0}{\partial\tau}\,\frac{\partial v_j}{\partial\tau}\right) - \frac{\partial^3 v_0}{\partial\tau^3}\,v_j = \mathcal{F}_j, \quad j = 1,\ldots,N, \tag{19}$$

where $\mathcal{F}_j = \mathcal{F}_j(t,\tau)$ are recurrently defined after calculation of the functions $V_0(x,t,\tau)\big|_{x=\varphi(t)}$, $V_1(x,t,\tau)\big|_{x=\varphi(t)}$, $\ldots$, $V_{j-1}(x,t,\tau)\big|_{x=\varphi(t)}$, for $j=1,\ldots,N$. Here and below, we simplify the notation by writing φ instead of $\varphi(t)$. In particular, for $j=1$ we find

$$\mathcal{F}_1(t,\tau) = \frac{\partial^3 v_0}{\partial\tau^2\partial t} - a_0(\varphi,t)\frac{\partial v_0}{\partial t} + \left[\tau a_{0x}(\varphi,t) + a_1(\varphi,t)\right]\varphi'\frac{\partial v_0}{\partial\tau} - \left[\tau\, b_{0x}(\varphi,t) + b_1(\varphi,t)\right] v_0^2\frac{\partial v_0}{\partial\tau}. \tag{20}$$

3.1. The main term

Let us proceed to equation (18). Despite the fact that the equation is nonlinear, a particular solution $v_0(t,\tau)$ can be found in explicit form, for an appropriate choice of the phase function φ. Firstly, by integrating equation (18) with respect to τ we obtain

$$[\varphi' - v_0]\; v_{0\tau\tau} - \frac{1}{2}\left(v_{0\tau}\right)^2 - a_0(\varphi,t)\,\varphi'\,v_0 + \frac{1}{3}\,b_0(\varphi,t)\,v_0^3 = C_1(t), \tag{21}$$

where the constant of integration $C_1(t)$ is chosen as $C_1(t)\equiv 0$ since $v_0\in\widetilde{G}_0$.

A solution to equation (21) is taken in the form [38]:

$$v_0(t,\tau) = A_0 + \frac{A_1}{\Psi} + \frac{A_2}{\Psi^2}, \tag{22}$$

where the function $\Psi = \Psi(\tau)$ is supposed to be represented as

$$\Psi = -\frac{\mu}{\lambda} + A\cosh(\lambda\tau) - A\sinh(\lambda\tau), \tag{23}$$

with the values $\lambda\neq 0$, A_0, A_1, A_2 that are determined below, and arbitrary real μ, A. This implies that the function Ψ satisfies the first-order ODE

$$\Psi' + \lambda\Psi + \mu = 0.$$

Substituting expressions (22), (23) into equation (21) provides us with a system of algebraic relations for the values λ, A_0, A_1 and A_2 of the form:

$$-\varphi' a_0(\varphi,t)A_0 + \frac{1}{3}\,b_0(\varphi,t)A_0^3 = 0,$$

next

$$-\left[a_0(\varphi,t)A_1 - \lambda^2 A_1\right]\varphi' + b_0(\varphi,t)A_0^2 A_1 - \lambda^2 A_0 A_1 = 0,$$

and

$$-\left[a_0(\varphi,t)A_2 - 3\lambda\mu A_1 - 4\lambda^2 A_2\right]\varphi' + b_0(\varphi,t)A_0 A_1^2 + b_0(\varphi,t)A_0^2 A_2 - \frac{3}{2}\lambda^2 A_1^2 - A_0\left(3\lambda\mu A_1 + 4\lambda^2 A_2\right) = 0.$$

The other relations that one obtains are:

$$\left[10\lambda\mu A_2 + 2\mu^2 A_1\right]\varphi' + \frac{1}{3}b_0(\varphi,t)A_1^3 + 2b_0(\varphi,t)A_0 A_1 A_2 - 4\lambda\mu A_1^2 - 7\lambda^2 A_1 A_2 - A_0\left(10\lambda\mu A_2 + 2\mu^2 A_1\right) = 0,$$

and

$$6\mu^2\varphi' A_2 + b_0(\varphi,t)A_0 A_2^2 + b_0(\varphi,t)A_1^2 A_2 - \frac{5}{2}\mu^2 A_1^2 - 17\lambda\mu A_1 A_2 - 6\lambda^2 A_2^2 - 6\mu^2 A_0 A_2 = 0.$$

The last useful relations are:

$$b_0(\varphi,t)A_1 A_2^2 - 10\mu^2 A_1 A_2 - 14\lambda\mu A_2^2 = 0, b_0(\varphi,t)A_2^3 - 24\mu^2 A_2^2 = 0.$$

The above equalities lead us to set, in (22)

$$A_0 = 0, \quad A_1 = \frac{24\lambda\mu}{b_0(\varphi,t)}, \quad A_2 = \frac{24\mu^2}{b_0(\varphi,t)},$$

and

$$\lambda^2 = a_0(\varphi, t), \tag{24}$$

where function φ is a solution to the first-order ODE:

$$\frac{d\varphi}{dt} = 6\,\frac{a_0(\varphi, t)}{b_0(\varphi, t)}. \tag{25}$$

Relation (24) implies $a_0(\varphi(t), t) \geq 0$ for all $t \in [0; T]$. Because equation (25) is nonlinear and in general its solution exists on a finite interval, we suppose that the function $\varphi = \varphi(t)$ is defined at least on the interval $[0; T]$ for some $T > 0$. This could be seen as a limitation of the method, because, on one hand, solitons are by definition global solutions and, on the other hand, by the above restriction, asymptotic soliton-like solutions that we are constructing, are, a priori, not global in time. But in fact, under suitable conditions on the coefficients a_0 and b_0, one can easily ensure, by the general ODE theory, that the solution to (25) is globally defined in time. This happens, for example, when both a_0 and b_0 are in $C^1(\mathbb{R} \times \mathbb{R}_+)$, bounded together with their derivatives in $\mathbb{R} \times \mathbb{R}_+$, and if $b_0 \geq \gamma$ for some $\gamma > 0$. Indeed, under these conditions the map $(t, \varphi) \mapsto 6\, a_0(\varphi, t)/b_0(\varphi, t)$ is globally Lipschitz with respect to the φ-variable.

So, according to (22), (23) the function $v_0 = v_0(t, \tau)$ for asymptotic solution (15) is written as follows:

$$v_0(t, \tau) = \frac{24\lambda^2\mu}{b_0(\varphi, t)} \left(\frac{\mu}{(-\mu + A\lambda\cosh(\lambda\tau) - A\lambda\sinh(\lambda\tau))^2} + \frac{1}{-\mu + A\lambda\cosh(\lambda\tau) - A\lambda\sinh(\lambda\tau)} \right), \tag{26}$$

with arbitrary reals μ, A and $\lambda^2 = a_0(\varphi, t)$, $\varphi = \varphi(t)$, $t \in [0; T]$.

Under assumptions $\lambda > 0$, $\mu < 0$, $A > 0$, formula (26) can be transformed into the following form through straightforward calculations:

$$v_0(t, \tau) = -6\,\frac{a_0(\varphi, t)}{b_0(\varphi, t)} \cosh^{-2}\left(\sqrt{a_0(\varphi, t)}\,\frac{\tau}{2} + C_0 \right), \tag{27}$$

where C_0 does not depend on τ.

It is clear that function (27) belongs to the space of rapidly decreasing functions with respect to τ. In this case, we can extend the function $v_0 = v_0(t, \tau)$ to a function $V_0 \in G_0$, such that $V_0(x, t, \tau)\big|_{x=\varphi(t)} = v_0(t, \tau)$. The obvious way is to define $V_0(x, t, \tau) := V_0(\varphi(t), t, \tau) = v_0(t, \tau)$, i.e. to choose V_0 constant with respect to x. In this way, we do have $V_0 \in G_0$.

Thus, the main term of the asymptotic one-phase soliton-like solution to the vcmCH equation with singular perturbation (9) is found in the form

$$V_0(x, t, \tau) = -6\,\frac{a_0(\varphi(t), t)}{b_0(\varphi(t), t)} \cosh^{-2}\left(\sqrt{a_0(\varphi(t), t)}\;\frac{\tau}{2} + C_0 \right). \tag{28}$$

Remark 1. *Formula (28) gives a soliton-like function. In the case* $a(x, t, \varepsilon) = a_0(x, t) = 1$, $b(x, t, \varepsilon) = b_0(x, t) = 3$ *we have* $\varphi(t) = 2t$, $\tau = (x - 2t)/\varepsilon$. *Thus, the main term (28) of asymptotic soliton-like solution (15) completely coincides with the exact soliton solution of the mCH equation (6) given by formula (12).*

3.2. The higher terms

Let us move on to equations for the higher terms $v_j(t, \tau)$, $j = 1, \ldots, N$. We suppose that the function $\varphi = \varphi(t)$, $t \in [0; T]$, is known and we treat the variable t as a parameter. After integrating equation (19) with respect to the variable τ we come to the second-order inhomogeneous ODE:

$$(\varphi' - v_0)v_{j\tau\tau} - v_{0\tau}\, v_{j\tau} + \left(-a_0(\varphi, t)\varphi' + b_0(\varphi, t)v_0^2 - v_{0\tau\tau}\right) v_j = \Phi_j, \tag{29}$$

where

$$\Phi_j = \Phi_j(t, \tau) = \int\limits_{-\infty}^{\tau} \mathcal{F}_j(t, \xi)\, d\xi + E_j(t), \quad j = 1, \ldots, N,$$

and $E_j(t)$, $j = 1, \dots, N$, is a constant of integration.

In particular,

$$\Phi_1(t,\tau) = \left[-a_0(\varphi,t)\frac{d}{dt}\left(\frac{A}{\alpha}\right) - \frac{A}{\alpha}\,\varphi'\,a_{0x}(\varphi,t) + \frac{8}{45}\frac{A^3}{\alpha}\,b_{0x}(\varphi,t)\right](\tanh\kappa - 1) \tag{30}$$

$$+A\left[\varphi'\tau a_{0x}(\varphi,t) + \varphi' a_1(\varphi,t)\right]\cosh^{-2}\kappa + \left[-2\frac{d}{dt}(\alpha A) + \frac{4}{45}\frac{A^3}{\alpha}b_{0x}(\varphi,t)\right]\cosh^{-2}\kappa\tanh\kappa$$

$$+\frac{1}{15}\frac{A^3}{\alpha}b_{0x}(\varphi,t)\cosh^{-4}\kappa\tanh\kappa - 6\alpha A\,\kappa_t\cosh^{-4}\kappa - \frac{1}{3}A^3\left[b_1(\varphi,t) + \tau\,b_{0x}(\varphi,t)\right]\cosh^{-6}\kappa,$$

where

$$\alpha = \alpha(t) = \frac{\sqrt{a_0(\varphi,t)}}{2}, \quad A = A(t) = -6\frac{a_0(\varphi,t)}{b_0(\varphi,t)}, \quad \kappa = \kappa(t,\tau) = \alpha(t)\tau + C_0, \tag{31}$$

the function $\varphi = \varphi(t)$, for $t \in [0;T]$, is a solution to ODE (25) and $C_0 = C_0(t)$ is an arbitrary value.

Now, let us introduce the differential operator

$$L = L\left(t,\tau,\frac{d}{d\tau}\right) := \rho(t,\tau)\frac{d^2}{d\tau^2} - v_{0\tau}\frac{d}{d\tau} + \left[-a_0(\varphi,t)\varphi' + b_0(\varphi,t)v_0^2 - v_{0\tau\tau}\right], \tag{32}$$

where $\rho(t,\tau) = \varphi'(t) - v_0(t,\tau)$, $t \in [0;T]$, and rewrite linear differential equation (29) in the operator form

$$Lv = \Phi. \tag{33}$$

Recall that $v = v(t,\tau)$, $\Phi = \Phi(t,\tau)$, and $t \in [0;T]$ is a parameter.

The coefficients of the differential operator L in (33) depend only on the values $a_0(\varphi(t),t)$, $b_0(\varphi(t),t)$, $t \in [0;T]$. Thus, it is completely determined by the conditions of problem (9) under consideration.

We use operator equation (33) to find conditions under which differential equation (29) has a solution from the space $\widetilde{G}$. To do it, we apply the results of the theory pseudodifferential operators [63], in particular from [64, 65].

3.3. Solvability of operator equation (33) in the space $\mathcal{S}(\mathbb{R})$

The following theorem is true.

Theorem 1. *Let the following conditions be fulfilled:*

1. *For all $t \in [0;T]$, $a_0(\varphi(t),t) > 0$;*
2. *The function $\tau \mapsto \Phi(t,\tau)$ belongs to $\mathcal{S}(\mathbb{R})$ for all $t \in [0;T]$.*

Then, for any $t \in [0;T]$ equation (33) has a solution $v(t,\cdot)$ in the space $\mathcal{S}(\mathbb{R})$ if and only if the function Φ satisfies the orthogonality condition of the form

$$\int_{-\infty}^{+\infty} \Phi_\tau(t,\tau)v_0(t,\tau)\,d\tau = 0, \quad t \in [0;T], \tag{34}$$

where the function v_0 is defined by formula (27).

While proving Theorem 1 we use some notations which we now remind. For any function $h \in \mathcal{S}(\mathbb{R})$ its Fourier transform is given as

$$F[h](\xi) = \int_{-\infty}^{+\infty} e^{-i\xi x}h(x)\,dx.$$

Due to the properties of the Fourier transform for any differential operator

$$p\left(x,\frac{d}{dx}\right) = \sum_{k=0}^{n} a_k(x)\frac{d^k}{dx^k}, \quad x \in \mathbb{R},$$

it is possible to define its action on a function $h \in \mathcal{S}(\mathbb{R})$ as

$$p\left(x, \frac{d}{dx}\right) h(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} e^{ix\xi}\, p(x,\xi) F[h](\xi)\, d\xi, \tag{35}$$

where

$$p\,(x,\xi) = \sum_{k=0}^{n} a_k(x)(-i\xi)^k.$$

In the sequel, we will use the following notation [63]. Let $S^{\,m}$, $m \in \mathbb{N}$, be a set of symbols $p = p(x,\xi) \in \mathrm{C}^\infty(\mathbb{R}^2)$ such that for any $k, l \in \mathbb{N} \cup \{0\}$ the inequality

$$\left| \frac{\partial^{k+l}}{\partial \xi^k \partial x^l}\, p(x,\xi) \right| \le C_{kl}\, (1 + |\xi|)^{m-k}, \quad (x,\xi) \in \mathbb{R}^2,$$

holds, with $k, l \in \mathbb{N} \cup \{0\}$ and C_{kl} are some constants independent on (x,ξ).

Let $S_0^m \subset S^{\,m}$ be the set of symbols p that satisfy the condition

$$|p(x,\xi)| \le M(x)\, (1 + |\xi|)^m\,,$$

where the value $M(x) \to 0$ as $|x| \to +\infty$.

It is worth recalling the following theorem.

Theorem 2. [64] *Let $p \in S^{\,m}$ be a symbol such that*

$$\frac{\partial^{\,l} p(x,\xi)}{\partial x^l} \in S_0^m, \quad l = 1, 2, \ldots,$$

and the inequality

$$\lim_{(x,\xi)\to\infty} \frac{|p(x,\xi)|}{(1 + |\xi|)^m} > 0 \tag{36}$$

holds. Then the differential operator

$$p\left(x, \frac{d}{dx}\right) : H_{s+m}(\mathbb{R}) \to H_s(\mathbb{R})$$

defined by formula (35) is Noetherian for any $s \in \mathbb{R}$.

Proof of Theorem 1. A scheme of proving the theorem is based on ideas of papers [66, 67]. Firstly, let us show that the differential operator $L\colon \mathrm{H}_{s+2}(\mathbb{R}) \to \mathrm{H}_s(\mathbb{R})$ is the Noether operator for any $s \in \mathbb{R}$. Next, we prove the solvability operator equation (33) in the Schwartz space $\mathcal{S}(\mathbb{R})$.

So, let us consider a symbol of the differential operator L having a form

$$p(t,\tau,\xi) = -\rho(t,\tau)\,\xi^2 + i\xi v_{0\tau} + \left[-\varphi' a_0 + b_0 v_0^2 - v_{0\tau\tau} \right], \tag{37}$$

where $a_0 = a_0(\varphi, t)$, $b_0 = b_0(\varphi, t)$, $\varphi = \varphi(t)$, and $t \in [0; T]$ is treated as a parameter, $v_0 = v_0(t,\tau)$ is given via formula (27).

The symbol (37) obeys to the inequality

$$\left| \frac{\partial^{k+l}}{\partial \xi^k \partial \tau^l}\; p(t,\tau,\xi) \right| \le C_{kl}(1 + |\xi|)^2$$

with some bounded values $C_{kl} = C_{kl}(t)$, $k, l \in \mathbb{N} \cup \{0\}$.

Moreover,

$$\frac{\partial^{\,l}}{\partial \tau^l}\; p(t,\tau,\xi) \in S_0^2, \quad l = 1, 2, \ldots.$$

Because of formulae (25), (27) for all $\tau \in \mathbb{R}$ we have

$$\rho(t,\tau) = 6\,\frac{a_0(\varphi,t)}{b_0(\varphi,t)}\,\left[1+\cosh^{-2}\left(\sqrt{a_0(\varphi,t)}\,\frac{\tau}{2}+C_0\right)\right] \neq 0, \tag{38}$$

where $\varphi = \varphi(t)$, $t \in [0;T]$, and condition (36) of Theorem 2 holds for symbol (37) for all $t \in [0;T]$.

Thus, for any $s \in \mathbb{R}$ the operator $L\colon \mathrm{H}_{s+2}(\mathbb{R}) \to \mathrm{H}_s(\mathbb{R})$ satisfies all conditions of Theorem 2, and is Noetherian, i.e., normally solvable operator.

Denote by L^* an operator being adjoint to the operator L, given by formula (32). First, we study the case of a nontrivial kernel of L^*. Next we consider the case with the trivial kernel of L^*.

In the first case, due to the normal solvability of the operator L, differential equation (33) is solvable in $\mathrm{H}_{s+2}(\mathbb{R})$ if and only if the orthogonality condition [66]

$$\langle \Phi, \ker L^* \rangle = 0 \tag{39}$$

holds.

Applying Sobolev embedding theorems for the spaces $\mathrm{H}_s(\mathbb{R})$, $s \in \mathbb{R}$, we deduce the inclusion $v_0^* \in \overline{\mathrm{C}}{}_0^\infty(\mathbb{R})$ for any element $v_0^* \in \ker L^*$. As a consequence of the orthogonality condition (39), one easily obtains that solution $v(x)$ of equation (33) belongs to the space $\bigcap_{s\in\mathbb{R}} H^s(\mathbb{R})$ [64]. Applying again Sobolev embedding theorems, we get $v \in \overline{\mathrm{C}}{}_0^\infty(\mathbb{R})$.

Now let us demonstrate that $v \in \mathcal{S}(\mathbb{R})$. Indeed, the function $v \in \overline{\mathrm{C}}{}_0^\infty(\mathbb{R})$ and it satisfies an ordinary differential equation

$$\frac{d^2\,v}{d\tau^2} - a_0(\varphi,t)v = f, \tag{40}$$

where $\varphi = \varphi(t)$, and $t \in [0;T]$ is treated as a parameter,

$$f = f(t,\tau) = \frac{1}{\rho(t,\tau)}\left[v_{0\tau}\,v_\tau - \left(-a_0(\varphi,t)v_0 + b_0(\varphi,t)v_0^2 - v_{0\tau\tau}\right)v + \Phi\right], \tag{41}$$

the function $v_0 = v_0(t,\tau)$ is given via formula (27).

On the other hand, differential equation (40) is equivalent to relation (29) because of inequality (38). Remind that equation (29) is written in an operator form as (33).

It is obvious that $f \in \mathcal{S}(\mathbb{R})$ with respect to the variable $\tau \in \mathbb{R}$ since its every term in (41) belongs to the space of rapidly decreasing functions in variable τ accordingly properties of the function $v_0 = v_0(t,\tau)$ and condition 2 of Theorem 1. Remind also the inclusion $\overline{\mathrm{C}}{}_0^\infty(\mathbb{R}) \subset \mathcal{S}^*(\mathbb{R})$.

So, due to properties of elliptic pseudodifferential operators with polynomial coefficients [65] we come to the conclusion that any solution to equation (33) from the space $\mathcal{S}^*(\mathbb{R})$ belongs to the space $\mathcal{S}(\mathbb{R})$. As a result, we obtain that $v \in \mathcal{S}(\mathbb{R})$. The last property allows us to consider the action of the operator L^* as an automorphism of the space $\mathcal{S}(\mathbb{R})$.

Let us proceed to clarifying the orthogonality condition (39). The operator $L^* : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is written as

$$L^* = \frac{d^2}{d\tau^2}\,\rho(t,\tau) + \frac{d}{d\tau}v_{0\tau} - a_0(\varphi,t)\varphi' + b_0(\varphi,t)v_0^2 - v_{0\tau\tau}\,.$$

According to equation (18), the function $v_{0\tau}(t,\tau)$ belongs to the kernel of the operator $L^*\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$. Another solution to the equation

$$L^* v = 0$$

can be written making use Abel's formula

$$w_0(t,\tau) = v_{0\tau}(t,\tau)\int\limits_{\tau_0}^{\tau} \frac{d\xi}{\rho(t,\xi)\,v_{0\xi}^2(t,\xi)}, \quad \tau_0 \in [-\infty;+\infty).$$

Considering the Wronskian for the functions $v_0(t,\tau)$ and $w_0(t,\tau)$ as variable τ tends to infinity we deduce that $w_0 \notin S(\mathbb{R})$. Thus, the dimension of the kernel of the operator $L^*\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ equals to 1. It allows us to represent the orthogonality condition (39) in the form:

$$\int\limits_{-\infty}^{+\infty} \Phi(t,\tau) v_{0\tau}(t,\tau)\, d\tau = 0, \quad t \in [0;T]. \tag{42}$$

Summarizing the arguments above, we conclude that equation (33) has a solution in the space $\mathcal{S}(\mathbb{R})$ if and only if the orthogonality condition (42) is satisfied. Due to the property $\Phi \in \mathcal{S}(\mathbb{R})$, we finally get condition (34).

Now let us study the case of the trivial kernel of L^*. Then equation (33) has a solution in the space $\mathrm{H}_{s+2}(\mathbb{R})$ for any $\Phi \in \mathcal{S}(\mathbb{R})$ because $L\colon \mathrm{H}_{s+2}(\mathbb{R}) \to \mathrm{H}_s(\mathbb{R})$ is the Noether operator. In addition, from the above arguing it also follows that if the kernel of the operator $L^*\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is trivial, then equation (33) has a solution in the space $\mathcal{S}(\mathbb{R})$ for any $\Phi \in \mathcal{S}(\mathbb{R})$. Theorem 1 is proved. □

3.4. Solvability of differential equation (19) in the space $\widetilde{G}$

Now consider equation (19) for the function $v_j = v_j(t,\tau)$, $j = 1,\dots,N$. We have the following lemmas.

Lemma 1. *Let* $a_0(\varphi(t),t) > 0$ *for all* $t \in [0;T]$, *and the function* $\mathcal{F}_j \in \widetilde{G}_0$, $j = 1,\dots,N$. *Then equation (19) has a solution* $v_j \in \widetilde{G}$, $j = 1,\dots,N$, *if and only if the function* $\mathcal{F}_j$, $j = 1,\dots,N$, *satisfies the orthogonality condition of the form:*

$$\int\limits_{-\infty}^{+\infty} \mathcal{F}_j(t,\tau) v_0(t,\tau)\, d\tau = 0, \quad t \in [0;T], \quad j = 1,\dots,N, \tag{43}$$

where the function $v_0(t,\tau)$ *is defined via formula (27).*

Proof. First, we show that the solutions v_j of equation (19) can be represented as

$$v_j(t,\tau) = \nu_j(t)\eta_j(t,\tau) + \psi_j(t,\tau), \tag{44}$$

where

$$\nu_j(t) = -\frac{1}{a_0(\varphi(t),t)\,\varphi'(t)} \lim_{\tau\to-\infty} \Phi_j(t,\tau), \tag{45}$$

$$\Phi_j(t,\tau) = \int\limits_{-\infty}^{\tau} \mathcal{F}_j(t,\xi) d\xi + E_j(t), \tag{46}$$

$\eta_j \in \widetilde{G}$ and additionally $\lim_{\tau\to-\infty} \eta_j(t,\tau) = 1$, and $\psi_j \in \widetilde{G}_0$, $j = 1,\dots,N$. Here the value $E_j(t)$ does not depend on the variable τ and it can be found from formula (46) using condition

$$\lim_{\tau\to+\infty} \Phi_j(t,\tau) = 0.$$

To prove relation (44) we integrate equation (19) in τ in limits from $-\infty$ to τ and we obtain the operator equation

$$Lv_j = \Phi_j, \quad j = 1,\dots,N, \tag{47}$$

where the operator L is given by formula (32).

By virtue of formulae (44), (47), for all $t \in [0;T]$ the function $\tau \mapsto \psi_j(t,\tau)$, $j = 1,\dots,N$, has to satisfy the inhomogeneous equation

$$L\psi_j = \Phi_j - \nu_j L\eta_j, \tag{48}$$

where $\Phi_j - \nu_j L\eta_j \in \mathcal{S}(\mathbb{R})$, $j = 1,\dots,N$.

So, according to Theorem 1 equation (48) has a solution in the space $\widetilde{G}_0$ if and only if the following orthogonality condition

$$\int_{-\infty}^{+\infty} (\Phi_j - \nu_j L\eta_j)\, v_{0\tau} d\tau = 0, \quad j = 1, \dots, N, \tag{49}$$

holds. Finally, from (49), (46), (18) and (47) by integration, we obtain condition (34). □

Remark 2. *In the case $j = 1$ the orthogonality condition (43) implies the relation:*

$$a_0(\varphi, t) \frac{d}{dt}\left(\frac{A^2}{\alpha}\right) + 6\, \frac{a_0(\varphi, t)}{b_0(\varphi, t)}\, \frac{A^2}{\alpha}\, a_{0x}(\varphi, t) + \frac{4}{5} A^3 \frac{d}{dt}\left(\alpha A^2\right) - \frac{12}{35}\, \frac{A^4}{\alpha}\, b_{0x}(\varphi, t) = 0, \tag{50}$$

where the functions $\alpha = \alpha(t)$, $A = A(t)$ are defined by formula (31), and the function φ is a solution of differential equation (25).

Condition (50) implies certain restrictions on the coefficients $a_0(x,t)$, $b_0(x,t)$ of equation (9) under which its asymptotic soliton-like solutions can be constructed. The orthogonality condition (43) as $j > 1$, provides us with similar relations for higher terms of asymptotic expansions for the coefficients of equation (9). In particular cases, these relations can be essentially simplified, as in the case of the KdV equation [51]. For example, if in (9) we put $a_0(x,t) = a_0(x)$, $b_0(x,t) = b_0(x)$, conditions (25), (43) are satisfied when the equality

$$52\, b_0'(\varphi(t)) a_0(\varphi(t)) = 35\, a_0'(\varphi(t)) b_0(\varphi(t))$$

holds.

Lemma 2. *Let the conditions of Lemma 1 be true. Then $v_j \in \widetilde{G}_0$, $j = 1, \dots, N$, if and only if the condition*

$$\lim_{\tau \to -\infty} \Phi_j(t, \tau) = 0, \quad j = 1, \dots, N, \tag{51}$$

is true.

Proof. This statement follows from the representation (44). Really, the relation (51) means that $E_j(t) = 0$, $j = 1, \dots, N$, in (46). This equality with formulas (44)–(46) yields the conclusion $v_j \in \widetilde{G}_0$. □

Remark 3. *In particular case $j = 1$, relation (51) becomes as*

$$a_0(\varphi, t)\, \frac{d}{dt}\left(\frac{A}{\alpha}\right) - \frac{A^2}{\alpha}\, a_{0x}(\varphi, t) - \frac{8}{45}\, \frac{A^3}{\alpha}\, b_{0x}(\varphi, t) = 0, \tag{52}$$

where the functions $\alpha = \alpha(t)$, $A = A(t)$, $t \in [0; T]$, are defined by formula (31) and the function $\varphi = \varphi(t)$, $t \in [0; T]$, is a solution of differential equation (25).

The last formula provides us with an additional condition for the main coefficients of expansions (10). It should be compatible with condition (50). This takes place, for example, for equation (62), which is a specific case of the vcmCH equation (9).

3.5. Constructing the higher terms

Finally, the function $V_j(x, t, \tau)$, $j = 0, 1, \dots, N$, is determined outside of the discontinuity curve Γ. At the beginning let us remark that since $v_0 \in \widetilde{G}_0$ we can put

$$V_0(x, t, \tau) = v_0(t, \tau). \tag{53}$$

Taking into consideration formulae (44), which provide us with the values on Γ, we define $V_j(x, t, \tau)$, $j = 1, \dots, N$, by extending $v_j(t, \tau)$, $j = 1, \dots, N$, from the curve Γ, in a manner that depends on the properties of the function $v_j(t, \tau)$, $j = 1, \dots, N$. When prolonging $v_j(t, \tau)$, $j = 1, \dots, N$, it should be considered two cases. Firstly, suppose that condition (51) holds, i.e. $v_j(t,\tau) \in \widetilde{G}_0$. This case is similar to that of the function $v_0(t,\tau)$. It means that the prolongation of the function $v_j(t,\tau)$, $j = 1, \dots, N$, from Γ to its neighborhood can be expressed as

$$V_j(x,t,\tau) = v_j(t,\tau). \tag{54}$$

In opposite case, when it is not true that condition (51) is satisfied, we make use of representation (44) and the prolongation is realized as

$$V_j(x,t,\tau) = u_j^-(x,t)\eta_j(t,\tau) + \psi_j(t,\tau), \tag{55}$$

where the functions $\eta_j(t,\tau)$, $\psi_j(t,\tau)$, $j = 1, \dots, N$, are defined via formulae (44), (45) while the function $u_j^-(x,t)$, $j = 1, \dots, N$, is a solution to the Cauchy problem

$$\Lambda u_j^-(x,t) = f_j^-(x,t), \tag{56}$$

$$u_j^-(x,t)\big|_{\Gamma} = \nu_j(t), \tag{57}$$

with differential operator

$$\Lambda = a_0(x,t)\frac{\partial}{\partial t}. \tag{58}$$

In particular, the first right-hand side functions in (56) are written as

$$f_1^-(x,t) = 0, \quad f_2^-(x,t) = -a_1(x,t)\frac{\partial u_1^-}{\partial t}, \quad f_3^-(x,t) = -a_1(x,t)\frac{\partial u_2^-}{\partial t} - b_0(x,t){u_1^-}^2\frac{\partial u_1^-}{\partial x}.$$

The differential equation (56) is deduced after substituting the representation (55) into equation (9) and limiting as variable τ tends to $-\infty$. The initial condition (57) follows from the representation (44).

A general solution of equation (56) can be written as

$$u_j^-(x,t) = \int_0^t \frac{f_j^-(x,\xi)}{a_0(x,\xi)}\,d\xi + \chi_j(x), \tag{59}$$

where the function $\chi_j(x)$, $j = 1, \dots, N$, has to satisfy condition (57), i.e., the equality

$$\chi_j(\varphi(t)) = \nu_j(t) - \int_0^t \frac{f_j^-(\varphi(t),\xi)}{a_0(\varphi(t),\xi)}\,d\xi$$

is true.

Due to the assumption $a_0(x,t)\,b_0(x,t) \neq 0$ and equation (25), the function $t \mapsto \varphi(t)$, $t \in [0;T]$, is monotonic and has an inverse. Thus, we obtain the solution of problem (56), (57) in exact form as

$$u_j^-(x,t) = \nu_j \circ \varphi^{-1}(x) + \int_{\varphi^{-1}(x)}^t \frac{f_j^-(x,\xi)}{a_0(x,\xi)}\,d\xi,$$

which yields the higher terms given by (55).

So, the problem of finding the asymptotic soliton-like solution (15) is solved completely.

Note that the singular terms of the asymptotic solutions for equation (9) are represented by formula (15) in two ways depending on condition (51). However, both forms of asymptotic solutions satisfy the equation with the same precision. This is confirmed by the following theorems, formulated on the basis of the procedure for constructing singular terms in (15).

Theorem 3. *Assume the following conditions:*

1. *The functions $a_j(x,t)$, $b_j(x,t) \in C^\infty(\mathbb{R}\times[0;T])$, $j = 0, 1, \dots, N$, and*

$$a_0(x,t)\,b_0(x,t) \neq 0 \quad \textit{for all} \quad (x,t) \in \mathbb{R}\times[0;T];$$

2. *The inequality* $a_0(\varphi(t), t) > 0$ *is fulfilled for all* $t \in [0; T]$, *where the function* φ *satisfies equation (25);*
3. *The orthogonality conditions (43) are true;*
4. *Conditions (51) hold.*

Then the function

$$u_N(x, t, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j V_j(x, t, \tau), \quad \tau = \frac{x - \varphi(t)}{\varepsilon}, \tag{60}$$

satisfies equation (9) on the set $\mathbb{R} \times [0; T]$ *with an asymptotic accuracy* $O(\varepsilon^N)$ *and represents the* N*–th approximation for the asymptotic soliton-like solution to equation (9).*

Theorem 4. *Let the following assumptions be supposed:*

1. *The conditions* $1 - 3$ *of the Theorem 3 are true;*
2. *Problem (56), (57) has a solution in the set*

$$\{(x, t) \in \mathbb{R} \times [0; T] : x - \varphi(t) \leq 0\}.$$

Then the function

$$u_N(x, t, \varepsilon) = \sum_{j=0}^{N} \varepsilon^j V_j(x, t, \tau), \quad \tau = \frac{x - \varphi(t)}{\varepsilon}, \tag{61}$$

satisfies equation (9) with an asymptotic accuracy $O(\varepsilon^N)$ *on the set* $\mathbb{R} \times [0; T]$, *and it is the* N*–th approximation for the asymptotic soliton-like solution to equation (9).*

Proof. The proofs of Theorems 3 and 4 are similar to the proof of Theorem 1 [53]. These proofs are technical and somewhat cumbersome; therefore, we will not repeat them here, but will provide an outline of the main idea of the proof.

To prove Theorem 3, we examine a residual function and estimate it. In this context, we utilize the fact that the singular terms satisfy equations (18) and (19), and that they decrease rapidly with respect to the variable τ. We also apply the Taylor representation of the coefficients of the functions $a(x, t, \varepsilon)$, $b(x, t, \varepsilon)$ in the neighborhood of the curve Γ.

While proving Theorem 4 we additionally make use the property of the function $\eta_j(t, \tau)$ and $\psi_j(t, \tau)$, for $j = 1, 2, \ldots, N$, as elements of the spaces $\widetilde{G}$ and $\widetilde{G}_0$.

□

Remark 4. *The residual value for both functions (60) and (61), and the vcmCH equation (9) tends to zero as* $\tau \to \pm\infty$ *for any nonnegative integer* N.

3.6. Example 1

Consider the vcmCH equation with singular perturbation of the form:

$$[1 + \varepsilon(x^2 + 1)]u_t - \varepsilon^2 u_{xxt} + u^2 u_x - 2\varepsilon^2 u_x u_{xx} - \varepsilon^2 u u_{xxx} = 0. \tag{62}$$

So, we have

$$a_0(x, t) = b_0(x, t) = 1, \quad a_1(x, t) = x^2 + 1, \quad b_1(x, t) = 0,$$
$$a_2(x, t) = b_2(x, t) = a_3(x, t) = b_3(x, t) = \ldots = 0.$$

The phase function $\varphi = \varphi(t)$ is a solution of equation (25) of the form:

$$\frac{d\varphi}{dt} = 6,$$

and under trivial initial condition it is given as $\varphi(t) = 6t$, $t \in \mathbb{R}$.

The main term of the asymptotic soliton-like solution is presented as

$$V_0(x,t,\tau) = v_0(t,\tau) = -6\cosh^{-2}\left(\frac{\tau}{2} + C_0\right), \quad \tau = \frac{x-6t}{\varepsilon}, \tag{63}$$

where C_0 is an arbitrary real.

Let us put $C_0 = 0$ and move on to the definition of the first singular term of the asymptotic soliton-like solution. In this case we find

$$\Phi_1(t,\tau) = -36(36t^2+1)\cosh^{-2}\frac{\tau}{2}.$$

By direct calculations, we find that $\Phi_1(t,\tau)$ satisfies the assumptions of Lemma 1 as well as condition (51) in Lemma 2. Thus, we can set $V_1(x,t,\tau) = v_1(t,\tau)$, where $v_1(t,\tau)$ is determined from (29) with $j = 1$ using the corresponding data:

$$V_1(x,t,\tau) = v_1(t,\tau) = (1+36t^2)\cosh^{-6}\frac{\tau}{2}\,\tanh\frac{\tau}{2} \tag{64}$$
$$\times\Bigg[\left(11270+216t^2\right)\tau + \left(4103,5+270t^2\right)\sinh\tau + \left(513,125+40,5t^2\right)\sinh 2\tau$$
$$+2\tau\cosh 2\tau - \sqrt{2}\left(507,5-162t^2\right)\operatorname{arctanh}\left(\frac{1}{\sqrt{2}}\tanh\frac{\tau}{2}\right)$$
$$-\frac{3\sqrt{2}}{2}\sinh^2\frac{\tau}{2}\,\cosh\tau\,\operatorname{arctanh}\left(\frac{1}{\sqrt{2}}\tanh\frac{\tau}{2}\right) - 8192\coth\frac{\tau}{2}\Bigg].$$

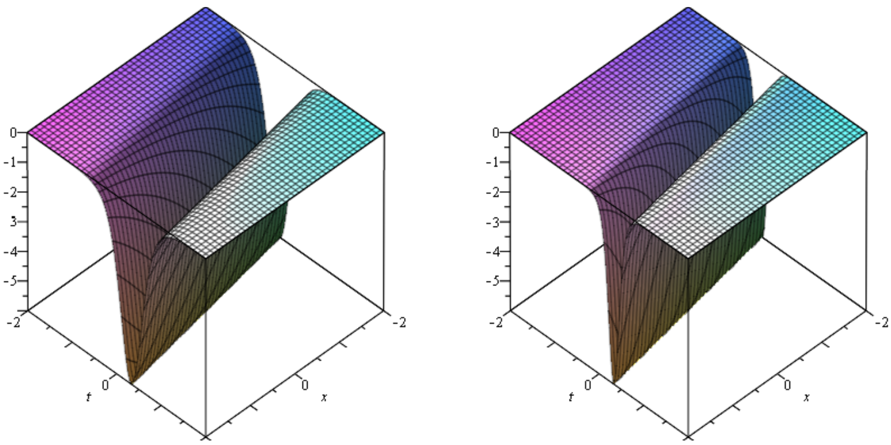

Figure 1: The main term $V_0(x,t,\tau)$ as $\varepsilon = 1$ (at the left) and $\varepsilon = 0.5$ (at the right).

It is clear that condition (51) is true, and the function $v_1 \in \widetilde{G}_0$. The first asymptotic approximation for soliton-like solution of equation (62) is global and it is given as

$$u_1(x,t,\varepsilon) = V_0(x,t,\tau) + \varepsilon V_1(x,t,\tau), \quad \tau = \frac{x-6t}{\varepsilon}, \tag{65}$$

where the functions $v_0(t,\tau)$, $v_1(t,\tau)$ are defined by (63), (64).

According to Theorem 3 function (65) satisfies equation (62) with an asymptotic accuracy $O(\varepsilon)$.

Graphs of the main and first terms of the asymptotic soliton-like solution as well as of the first approximation are presented on Fig. 1–3 for a parameter $\varepsilon = 1$ and $\varepsilon = 0.5$.

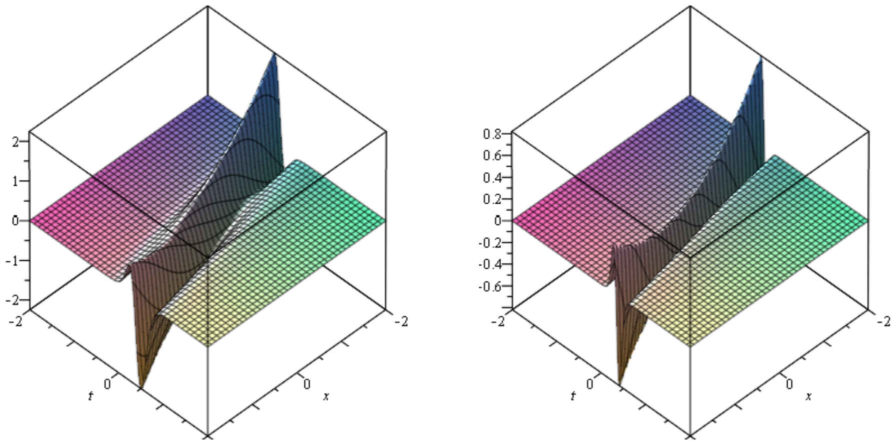

Figure 2: The term $\varepsilon V_1(x,t,\tau)$ as $\varepsilon = 1$ (at the left) and $\varepsilon = 0.5$ (at the right).

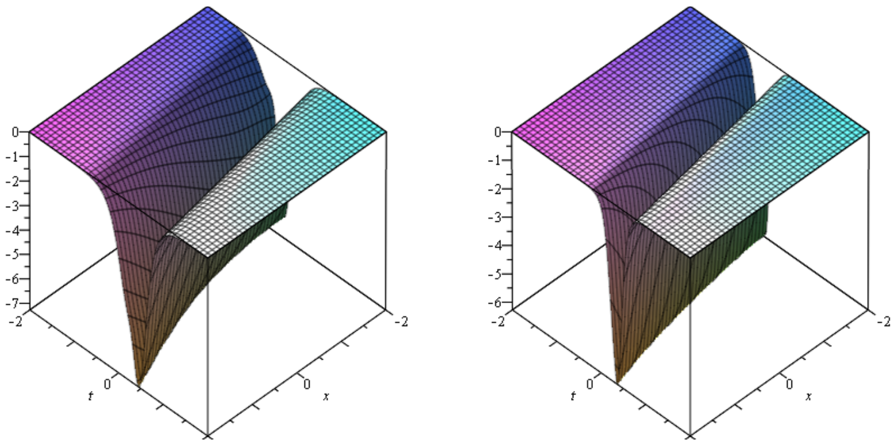

Figure 3: The first approximation of the soliton-like solution $u_1(x,t,\varepsilon)$ as $\varepsilon = 1$ (at the left) and $\varepsilon = 0.5$ (at the right).

4. The peakon-like solutions

Recall that the mCH equation (6) is well known for its soliton and peakon solutions, examples of which are given by formulae (11) and (12). Both soliton and peakon solutions representing solitary wave solutions are rapidly decreasing to a background function at infinity. It should be also remarked that soliton and peakon solutions differ in differentiability properties. In particular, whereas soliton solutions are described through functions that necessarily have inflection points, the peakon solutions are represented by functions that, like their derivatives, are monotone on any interval of their smoothness. It can be easily noticed for the peakon solution (8) of the CH equation.

Thus, the problem of constructing asymptotic peakon-like solutions of the vcmCH equation with singular perturbation (9) is natural, since these solutions have other properties than asymptotic soliton-like solutions. As a result, peakon-like solutions provide us with a new type of asymptotic solutions.

Due to the fact that the singular part of an asymptotic soliton-like solution reflects specific features of a soliton-like solution, here as above we suppose the regular part of the asymptotics to be zero. For the case of asymptotic soliton-like solutions, we propose definitions of suitable functional spaces that are modifications of the space G_0 introduced above in Section 2. For the problem under consideration, we take into account form of peakon solutions that have a peak at a point and as a result they possess a discontinuous first derivative at this point.

Now we move on to the definitions that are used in the sequel.

Let $G^+ = G^+(\mathbb{R} \times [0;T] \times \mathbb{R}_+)$ be the space of infinitely differentiable functions $f\colon \mathbb{R} \times [0;T] \times \mathbb{R}_+ \to \mathbb{R}$, such that for any nonnegative integers n, p, q and r

$$\lim_{\tau\to+\infty}\tau^n\frac{\partial^p}{\partial x^p}\frac{\partial^q}{\partial t^q}\frac{\partial^r}{\partial\tau^r}f(x,t,\tau)=0,\quad (x,t)\in K,$$

uniformly with respect to $(x,t)\in K$, in any compact set $K\subset\mathbb{R}\times[0;T]$. Here $\mathbb{R}_+=[0;+\infty)$.

Let $\widetilde{G}^+=\widetilde{G}^+([0;T]\times\mathbb{R}_+)$ be the space of infinitely differentiable functions $f\colon[0;T]\times\mathbb{R}_+\to\mathbb{R}$, such that for any nonnegative integers n, p and q

$$\lim_{\tau\to+\infty}\tau^n\frac{\partial^p}{\partial t^p}\frac{\partial^q}{\partial\tau^q}f(t,\tau)=0,\quad t\in[0;T].$$

We denote by $G^-=G^-(\mathbb{R}\times[0;T]\times\mathbb{R}_-)$ the space of infinitely differentiable functions $f\colon\mathbb{R}\times[0;T]\times\mathbb{R}_-\to\mathbb{R}$, such that for any nonnegative integers n, p, q and r

$$\lim_{\tau\to-\infty}\tau^n\frac{\partial^p}{\partial x^p}\frac{\partial^q}{\partial t^q}\frac{\partial^r}{\partial\tau^r}f(x,t,\tau)=0,\quad (x,t)\in K,$$

uniformly with respect to $(x,t)\in K$, in any compact set $K\subset\mathbb{R}\times[0;T]$. Here $\mathbb{R}_-=(-\infty;0]$.

Let $\widetilde{G}^-=\widetilde{G}^-([0;T]\times\mathbb{R}_-)$ be the space of infinitely differentiable functions $f\colon[0;T]\times\mathbb{R}_-\to\mathbb{R}$, such that for any nonnegative integers n, p and q

$$\lim_{\tau\to-\infty}\tau^n\frac{\partial^p}{\partial t^p}\frac{\partial^q}{\partial\tau^q}f(t,\tau)=0,\quad t\in[0;T].$$

Let $G^\pm=G^\pm(\mathbb{R}\times[0;T]\times\mathbb{R})$ be the space of continuous functions $f\colon\mathbb{R}\times[0;T]\times\mathbb{R}\to\mathbb{R}$, such that the function f can be written as

$$f=f(x,t,\tau)=\begin{cases} f^+(x,t,\tau), & (x,t,\tau)\in\mathbb{R}\times[0;T]\times\mathbb{R}_+,\\ f^-(x,t,\tau), & (x,t,\tau)\in\mathbb{R}\times[0;T]\times\mathbb{R}_-,\end{cases}$$

where $f^+\in G^+$ and $f^-\in G^-$.

We denote by $\widetilde{G}^\pm=\widetilde{G}^\pm([0;T]\times\mathbb{R})$ the space of continuous functions $f\colon[0;T]\times\mathbb{R}\to\mathbb{R}$, such that the function f can be written as

$$f=f(t,\tau)=\begin{cases} f^+(t,\tau), & (t,\tau)\in[0;T]\times\mathbb{R}_+,\\ f^-(t,\tau), & (t,\tau)\in[0;T]\times\mathbb{R}_-,\end{cases}$$

where $f^+\in\widetilde{G}^+$ and $f^-\in\widetilde{G}^-$.

We use the following definition of an asymptotic peakon-like function.

Definition 2. *A nontrivial function $u=u(x,t,\varepsilon)$, where $(x,t)\in\mathbb{R}\times[0;T]$ and ε is a small parameter, is called an asymptotic peakon-like function if for any integer $N\geq 0$ it can be represented as*

$$u(x,t,\varepsilon)=\sum_{j=0}^{N}\varepsilon^j V_j(x,t,\tau)+O(\varepsilon^{N+1}),\quad \tau=\frac{x-\varphi(t)}{\varepsilon},\tag{66}$$

where $\varphi(t)\in C^\infty([0;T])$ is a scalar function, and $V_j\in G^\pm$, for $j=0,1,\dots,N$.

As in the case of asymptotic soliton-like solutions, the function $x-\varphi(t)$ is called *a phase function* of the asymptotic peakon-like function $u(x,t,\varepsilon)$.

A curve determined by equation $x-\varphi(t)=0$ is called *a discontinuity curve* for the function $u(x,t,\varepsilon)$ [48, 54].

Let us move on to the description of an algorithm for constructing asymptotic peakon-like solutions of equation (9). The main idea resembles that of finding asymptotic soliton-like solutions for equation (9) presented above. It is lightly modified in comparison with the searched asymptotic solutions (66).

The principal problem is to find terms in the expansion (66). To solve it, we substitute the ansatz for asymptotic solutions into equation (9), we get differential equations for the singular terms of the series (66) and study these equations in a neighborhood of the discontinuity curve.

The next step is related to solving differential equations for the functions $v_j(t,\tau) = V_j(x,t,\tau)\big|_{x=\varphi(t)}$, $j = 0,1,\ldots,N$, in the functional spaces according to Definition 2.

Finally, to obtain the coefficients of expansion (66), we prolong the functions $v_j(t,\tau)$, $j = 0,1,\ldots,N$, analogously to the procedure of prolonging the terms of asymptotic soliton-like solutions in the case $v_j \in \widetilde{G}_0$, $j = 1,\ldots,N$, see Section 3.5.

4.1. The main term

Let us consider the main term $V_0(x,t,\tau)$ of the asymptotic peakon-like solution of equation (9). It can be explicitly found despite tedious calculations. In fact, the function $v_0(t,\tau) = V_0(x,t,\tau)\big|_{x=\varphi(t)}$ is a solution of the second-order differential equation of the form (21). Let denote

$$C_1(t) = -g = -g(t), \quad t \in [0;T],$$

and rewrite differential equation (21) as a system:

$$y = \frac{dv_0}{d\tau}, \tag{67}$$

$$\rho_1(t,\tau)\,\frac{dy}{d\tau} - a_0(\varphi,t)\,\varphi'\,v_0 + \frac{1}{3}b_0(\varphi,t)v_0^3 - \frac{1}{2}y^2 = -g, \tag{68}$$

where $\rho_1(t,\tau) = \varphi'(t) - v_0(t,\tau)$.

Now we find the first integral of system (67), (68). This can be done as follows. Equation (68) implies the equation

$$\frac{dy}{d\tau} = \frac{6a_0(\varphi,t)\,\varphi'\,v_0 - 2b_0(\varphi,t)v_0^3 + 3y^2 - 6g}{6\rho_1(t,\tau)},$$

which, when combined with (67), leads to the total differential equation

$$\left[6\,a_0(\varphi,t)\,\varphi'\,v_0 - 2b_0(\varphi,t)v_0^3 + 3y^2 - 6g\right]dv_0 - 6\rho_1(t,\tau)y\,dy = 0. \tag{69}$$

In a standard way [68], we calculate the first integral of equation (69) that is given as

$$H(v_0,y) = 6\,a_0(\varphi,t)\,\varphi'\,v_0^2 - b_0(\varphi,t)v_0^4 + 6y^2v_0 - 12gv_0 - 6\,\varphi'\,y^2, \tag{70}$$

where $\varphi = \varphi(t)$, and $t \in [0;T]$ plays the role of a parameter here.

Equating the function $H(v_0,y)$ to a constant provides us with an ODE for the function $v_0(t,\tau)$. Recall that we are interested in a particular solution of equation (21) and therefore we can choose a constant for the first integral in a suitable form. We apply an idea that was previously used to search for the asymptotic soliton-like solution of the singularly perturbed vcKdV equation [48].

Let us represent the relation $H(v_0,y) = C$ as

$$y^2 = Q(v_0), \tag{71}$$

where

$$Q(v_0) = \frac{b_0(\varphi,t)v_0^4 - 6a_0(\varphi,t)\,\varphi'\,v_0^2 + 12gv_0 + C}{6\,(v_0 - \varphi')}.$$

If we take

$$C = -b_0(\varphi,t)\,(\varphi')^4 + 6a_0(\varphi,t)\,(\varphi')^3 - 12g\varphi',$$

then the function $Q(v_0)$ becomes a cubic polynomial. We can write it as

$$Q(v_0) = \frac{1}{6}\,b_0(\varphi,t)(v_0 - \alpha_1)^2(v_0 - \alpha_2)$$

under the assumptions that the values α_1 and α_2 satisfy the system

$$2\alpha_1 + \alpha_2 = -\varphi', \tag{72}$$

$$\alpha_1^2 + 2\alpha_1\alpha_2 = \frac{1}{b_0(\varphi,t)} \left[b_0(\varphi,t)\,(\varphi')^2 - 6a_0(\varphi,t)\varphi' \right], \tag{73}$$

$$\alpha_1^2\alpha_2 = \frac{1}{b_0(\varphi,t)} \left[-b_0(\varphi,t)\,(\varphi')^3 + 6a_0(\varphi,t)\,(\varphi')^2 - 12g \right]. \tag{74}$$

System (72)–(74) has the particular solution

$$\alpha_1 = 0, \quad \alpha_2 = -\varphi'.$$

It implies the differential equation

$$\frac{d\varphi}{dt} = 6\,\frac{a_0(\varphi,t)}{b_0(\varphi,t)} \tag{75}$$

for the function φ, and the relation

$$\frac{6a_0(\varphi,t) - b_0(\varphi,t)\varphi'}{12}\,(\varphi')^2 = 0.$$

So, the function $v_0(t,\tau)$ satisfies the first-order ODE

$$\left(\frac{d\,v_0}{d\,\tau}\right)^2 = \frac{1}{6}\,b_0(\varphi,t)\,(v_0+\varphi')\,v_0^2, \tag{76}$$

which, under condition $b_0(\varphi(t),t) > 0$, $t \in [0;T]$, gives

$$\frac{d\,v_0}{d\,\tau} = \pm\frac{v_0}{\sqrt{6}}\,\sqrt{b_0(\varphi,t)(v_0+\varphi')}. \tag{77}$$

Formula (77) implies the equality

$$\int\limits_{\varphi'}^{v_0} \frac{d\,v_0}{v_0\,\sqrt{v_0+\varphi'}} = \pm \int\limits_0^{\tau} \sqrt{\frac{1}{6}\,b_0(\varphi,t)}\,d\tau,$$

which is equivalent to

$$\operatorname{arccoth}\frac{\sqrt{v_0+\varphi'}}{\sqrt{\varphi'}} - \operatorname{arccoth}\sqrt{2} = \pm\sqrt{\frac{1}{6}\,b_0(\varphi,t)\varphi'}\;\frac{\tau}{2},$$

under assumption $a_0(\varphi(t),t) > 0$, $t \in [0;T]$.

Thus, taking into account equation (75) for the function $\varphi = \varphi(t)$, we can represent the function $v_0(t,\tau)$ as

$$v_0(t,\tau) = 6\,\frac{a_0(\varphi,t)}{b_0(\varphi,t)}\,\sinh^{-2}\left(\sqrt{a_0(\varphi,t)}\;\frac{|\tau|}{2} + \operatorname{arccoth}\sqrt{2}\right). \tag{78}$$

Since $v_0(t,\tau)$ is a rapidly decreasing function as $|\tau| \to \infty$, the main term $V_0(x,t,\tau)$ of the asymptotic peakon-like solution is written as $V_0(x,t,\tau) = v_0(t,\tau)$. So, it completes the search the main term of asymptotic peakon-like solution (66).

Remark 5. *Equation (75) is an ODE for the phase function $\varphi = \varphi(t)$, the initial condition for which can be taken as $\varphi(0) = 0$. It coincides with the ODE for the discontinuity curve (25) deduced for the asymptotic soliton-like solutions in Section 3.1.*

Remark 6. *The right-hand side of (78) is a peakon-like function. If $a(x,t,\varepsilon) = a_0(x,t) = 1$ and $b(x,t,\varepsilon) = b_0(x,t) = 3$, then we have $\varphi(t) = 2t$, $\tau = (x-2t)/\varepsilon$, and the obtained main term (78) of solution (66) completely coincides with the exact solution of the mCH equation (6) given by (11).*

4.2. The higher terms

The higher terms of the asymptotic peakon-like solution (66) are determined by PDE of the form (19). The procedure of solving is based on the idea of constructing the functions in the neighborhood of the discontinuity curve. In contrast to asymptotic soliton-like solutions, here we consider the two cases $\tau \geq 0$ and $\tau < 0$. The obtained functions are prolonged in such a way that they are continuous and belong to the space $G^{\pm}$. Thus, consider equations (19) for the functions $v_j \in \widetilde{G}^{\pm}$, $j = 1, \ldots, N$. Below we use notation

$$v_j(t,\tau) = \begin{cases} v_j^+(t,\tau), & \tau \geq 0, \\ v_j^-(t,\tau), & \tau < 0, \end{cases} \tag{79}$$

provided that

$$v_j^+(t,0) = \lim_{\tau \to 0-} v_j^-(t,\tau).$$

Taking into account the exact formula (78) for the main term analogously (29) we represent equations for the higher terms as

$$(\varphi' - v_0)v^+_{j\tau\tau} - v_{0\tau}v^+_{j\tau} + \left(b_0(\varphi,t)v_0^2 - a_0(\varphi,t)\varphi' - v_{0\tau\tau}\right) v_j^+ = \Phi_j^+(t,\tau), \quad \tau \geq 0, \tag{80}$$

$$(\varphi' - v_0)v^-_{j\tau\tau} - v_{0\tau}v^-_{j\tau} + \left(b_0(\varphi,t)v_0^2 - a_0(\varphi,t)\varphi' - v_{0\tau\tau}\right) v_j^- = \Phi_j^-(t,\tau), \quad \tau < 0, \tag{81}$$

where

$$\Phi_j^+(t,\tau) = \int_0^\tau \mathcal{F}_j(t,\tau)\, d\tau + E_j^+(t), \quad \tau \geq 0, \tag{82}$$

$$\Phi_j^-(t,\tau) = \int_0^\tau \mathcal{F}_j(t,\tau)\, d\tau + E_j^-(t), \quad \tau < 0, \tag{83}$$

and $\mathcal{F}_j(t,\tau)$, $j = 1, \ldots, N$, is the right-hand side function in equation (19).

Here, the values $E_j^+(t)$, $E_j^-(t)$ are constants of the integrations chosen in such a way that

$$\lim_{\tau \to +\infty} \Phi_j^+(t,\tau) = 0,$$

$$\lim_{\tau \to -\infty} \Phi_j^-(t,\tau) = 0.$$

In the particular case $j = 1$, we have

$$\Phi_1^+(t,\tau) = B_1[\coth\kappa_+ - 1] + [B_2\tau + B_3]\sinh^{-2}\kappa_+ + \left[B_4\cosh\kappa_+ + B_5\tau\sinh^{-1}\kappa_+\right]\sinh^{-3}\kappa_+ \\ + \left[(B_6\tau + B_7)\sinh^{-1}\kappa_+ + B_8\cosh\kappa_+\right]\sinh^{-5}\kappa_+, \tag{84}$$

and

$$\Phi_1^-(t,\tau) = -B_1[\coth\kappa_- - 1] + [B_2\tau + B_3]\sinh^{-2}\kappa_- + \left[-B_4\cosh\kappa_- + B_5\tau\sinh^{-1}\kappa_-\right]\sinh^{-3}\kappa_- \\ + \left[(B_6\tau + B_7)\sinh^{-1}\kappa_- - B_8\cosh\kappa_-\right]\sinh^{-5}\kappa_-, \tag{85}$$

where

$$\kappa_+ = \alpha\tau + \operatorname{arccoth}\sqrt{2}, \quad \kappa_- = -\alpha\tau + \operatorname{arccoth}\sqrt{2},$$

$$B_1 = a_0(\varphi,t)\frac{d}{dt}\left(\frac{B}{\alpha}\right) + \frac{B}{\alpha}\,\varphi'\,a_{0x}(\varphi,t) - \frac{8}{45}\frac{B^3}{\alpha}b_{0x}(\varphi,t),$$

$$B_2 = -a_0(\varphi,t)\frac{B}{\alpha}\,\alpha_t + 4B\alpha\alpha_t + \frac{1}{6}\frac{B}{b_0(\varphi,t)}\,a_{0x}(\varphi,t),$$

$$B_3 = 6\frac{a_0(\varphi,t)}{b_0(\varphi,t)}\,a_1(\varphi,t)\,B, \quad B_4 = -2\frac{d}{dt}(\alpha B) + \frac{4}{45}\frac{B^3}{\alpha}\,b_{0x}(\varphi,t),$$

$$B_5 = 6B\alpha\alpha_t, \quad B_6 = -\frac{1}{3}B^3 b_{0x}(\varphi,t), \quad B_7 = -\frac{1}{3}b_1(\varphi,t)B^3, \quad B_8 = -\frac{B^3}{\alpha}\,b_{0x}(\varphi,t),$$

and

$$\alpha=\alpha(t)=\frac{\sqrt{a_0(\varphi,t)}}{2},\quad B=B(t)=6\,\frac{a_0(\varphi,t)}{b_0(\varphi,t)},\quad \varphi=\varphi(t),\quad t\in[0;T].$$

The assumption $v_j\in\widetilde{G}^{\pm}$ implies inclusions $\Phi_j^+\in\widetilde{G}^+$, $\Phi_j^-\in\widetilde{G}^-$. In particular case $j=1$ it provides us with necessary condition of belonging the function v_1^+ to the space $\widetilde{G}^+$ and v_1^- to the space $\widetilde{G}^-$ as

$$a_0(\varphi,t)\frac{d}{dt}\left(\frac{B}{\alpha}\right)+\frac{B^2}{\alpha}\,a_{0x}(\varphi,t)-\frac{8}{45}\frac{B^3}{\alpha}b_{0x}(\varphi,t)=0 \tag{86}$$

that are similar to condition (52).

Let us move on to analysis of equations (80), (81). General solutions of these equations can be found by means of the method of variation of constants using a solution of the correspondent homogeneous equations. Because the function $w=w(t,\tau)=v_{0\tau}(t,\tau)$ is a solution of the homogeneous equation for both relations (80), (81), another solution of these homogeneous linear equations can be found by using Abel's formula

$$w_0(t,\tau)=v_{0\tau}(t,\tau)\int\limits_{\tau_0}^{\tau}\frac{d\xi}{\rho_1(t,\xi)v_{0\xi}^2(t,\xi)},\quad \tau_0\in[-\infty;\infty). \tag{87}$$

So, the solution can be taken as

$$w_0(t,\tau)=\frac{1}{9}\,\frac{b_0^2(\varphi,t)}{a_0^3(\varphi,t)}\,\Bigg[-\frac{35}{32}\sqrt{a_0(\varphi,t)}|\tau|\sinh^{-2}\kappa\coth\kappa$$
$$+\frac{5}{8}\coth^2\kappa-\frac{1}{4}\cosh^2\kappa+\frac{5}{3}\,\sinh^{-2}\kappa-\frac{1}{6}\,\sinh^{-4}\kappa\Bigg],$$

where

$$\kappa=\alpha|\tau|+\text{arccoth}\,\sqrt{2},\quad \tau\in\mathbb{R}.$$

Thus, general solutions of (80), (81) can be represented by formula

$$v_j^{\pm}(t,\tau)=-v_{0\tau}(t,\tau)\int\limits_0^{\tau}\Phi_j^{\pm}(t,\tau)w_0(t,\tau)d\tau+w_0(t,\tau)\int\limits_0^{\tau}\Phi_j^{\pm}(t,\tau)v_{0\tau}(t,\tau)d\tau \tag{88}$$
$$+c_{j1}^{\pm}v_{0\tau}(t,\tau)+c_{j2}^{\pm}w_0(t,\tau),$$

where c_{j1}^+, c_{j1}^-, c_{j2}^+, c_{j2}^- are constants taken in order to satisfy conjugation conditions at $\tau=0$, yielding:

$$c_{j1}^+v_{0\tau}(t,0)+c_{j2}^+w_0(t,0)=c_{j1}^-v_{0\tau}(t,0)+c_{j2}^-w_0(t,0).$$

According to the choice of values c_{j1}^+, c_{j1}^-, c_{j2}^+, c_{j2}^- formula (88) gives the function $v_j(t,\tau)$, $j=1,\dots,N$, which is continuous and belongs to the space $\widetilde{G}^{\pm}$.

Analogously to the prolongation procedure of the terms of asymptotic soliton-like solutions in the case $v_j(t,\tau)\in\widetilde{G}_0$, $j=1,\dots,N$ (see Subsection 3.5), we prolong the function $v_j(t,\tau)\in\widetilde{G}^{\pm}$, $j=1,\dots,N$, as $V_j(x,t,\tau)=v_j(t,\tau)$, $j=1,\dots,N$. It is clear that this function has a peak at $\tau=0$ and belongs to the space $G^{\pm}$.

Theorem 5. *Let the following conditions be assumed:*

1. *Functions $a_j(x,t)$, $b_j(x,t)\in C^\infty(\mathbb{R}\times[0;T])$, $j=0,1,\dots,N$;*
2. *Inequalities $a_0(\varphi(t),t)>0$, $b_0(\varphi(t),t)>0$, $t\in[0;T]$, hold, where the phase function $\varphi(t)$, $t\in[0;T]$, is a solution of equation (75);*
3. *The functions $v_j(t,\tau)$, $j=1,\dots,N$, defined by formulas (79), belong to the space $\widetilde{G}^{\pm}$.*

Then the function

$$u_N(x,t,\varepsilon)=\sum_{j=0}^{N}\varepsilon^j V_j(x,t,\tau),\quad \tau=\frac{x-\varphi(t)}{\varepsilon},\tag{89}$$

is the N–th asymptotic approximation of the peakon-like solution of equation (9) and satisfies the equation on the set

$$\{(x,t)\in\mathbb{R}\times[0;T]\colon x-\varphi(t)>0\}\cup\{(x,t)\in\mathbb{R}\times[0;T]\colon x-\varphi(t)<0\}$$

with an asymptotic accuracy $O(\varepsilon^N)$.

Proof of Theorem 5 is done analogously to the case of the asymptotic one-phase soliton-like solution of the vcmCH equation with a singular perturbation of form (9) for the case when the all functions of the constructed asymptotics belong to the space G_0. It should be also noted that, despite the discontinuity in the variable τ of the derivatives of the singular terms of the asymptotic peakon-like solution, this solution satisfies this equation with the accuracy declared in Theorem 5.

Remark 7. *The residual value for function (89) and the vcmCH equation (9) tends to zero as $\tau\to\pm\infty$ for any nonnegative integer N.*

4.3. Example 2

Let us consider the vcmCH equation with a singular perturbation of the form:

$$\left[1+\varepsilon\left(\frac{1}{36}x^2+1\right)\right]u_t-\varepsilon^2u_{xxt}+\left[1+\varepsilon(t^2+1)\right]u^2u_x-2\varepsilon^2u_xu_{xx}-\varepsilon^2uu_{xxx}=0.\tag{90}$$

The coefficients of the equation are given as

$$a_0(x,t)=b_0(x,t)=1,\quad a_1(x,t)=\frac{1}{36}x^2+1,\quad b_1(x,t)=t^2+1,$$

$$a_2(x,t)=b_2(x,t)=a_3(x,t)=b_3(x,t)=\ldots=0.$$

The phase function $\varphi=\varphi(t)$ for the discontinuity curve of the peakon-like solution can be found from the first-order ODE

$$\frac{d\varphi}{dt}=6,$$

that has global solution $\varphi(t)=6t$ satisfied initial condition $\varphi(0)=0$.

Formula (78) with $a_0(x,t)=b_0(x,t)=1$, yields the main term of the asymptotic peakon-like solution as

$$V_0(x,t,\tau)=v_0(t,\tau)=6\,\sinh^{-2}\kappa,\tag{91}$$

where

$$\kappa=\frac{|\tau|}{2}+\text{arccoth}\,\sqrt{2},\quad \tau=\frac{x-6t}{\varepsilon}.$$

Let us move on to definition of the first term $v_1(t,\tau)$ of the asymptotic solution. In this case the functions $\Phi_1^+(t,\tau)$, $\Phi_1^-(t,\tau)$ are given as

$$\Phi_1^{\pm}(t,\tau)=36(t^2+1)\sinh^{-2}\kappa_\pm-72(t^2+1)\sinh^{-6}\kappa_\pm,$$

where the values k_+, k_- are as

$$\kappa_\pm=\pm\frac{\tau}{2}+\text{arccoth}\,\sqrt{2}.$$

According to formula (87) the function $w_0(t,\tau)$ is written as

$$w_0(t,\tau)=\frac{1}{9}\left[\frac{5}{8}\coth^2\kappa-\frac{1}{4}\cosh^2\kappa-\frac{35}{32}|\tau|\coth\kappa\sinh^{-2}\kappa+\frac{5}{3}\sinh^{-2}\kappa-\frac{1}{6}\sinh^{-4}\kappa\right],$$

and the functions $v_1^+(t,\tau)$, $v_1^-(t,\tau)$ for the first term $v_1(t,\tau)$ in (79) are correspondingly represented as

$$v_1^\pm(t,\tau) = v_1^{\pm,1}(t,\tau) + v_1^{\pm,2}(t,\tau) + c_{11}^\pm v_{0\tau}(t,\tau) + c_{12}^\pm w_0(t,\tau), \tag{92}$$

where

$$v_1^{\pm,1}(t,\tau) = -v_{0\tau}(t,\tau)\int\limits_0^\tau \Phi_1^\pm(t,\tau)w_0(t,\tau)d\tau, \tag{93}$$

$$v_1^{\pm,2}(t,\tau) = w_0(t,\tau)\int\limits_0^\tau \Phi_1^\pm(t,\tau)v_{0\tau}(t,\tau)d\tau. \tag{94}$$

The values $c_{11}^+ = c_{11}^+(t)$, $c_{11}^- = c_{11}^-(t)$ have to obey relation $c_{11}^+(t) = c_{11}^-(t)$, $t \in \mathbb{R}$, following from the continuity condition of the function $v_1(t,\tau)$ at point $\tau = 0$. It allows us to put $c_{11}^+(t) = c_{11}^-(t) = 0$, $t \in \mathbb{R}$.

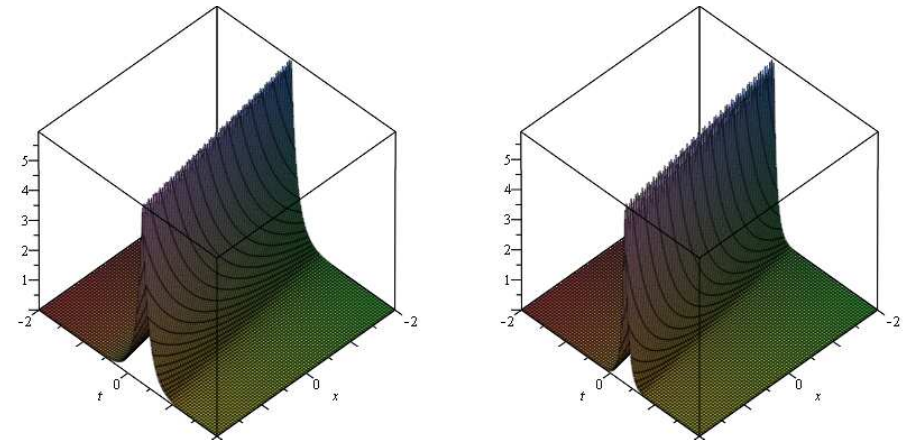

Figure 4: The main term $V_0(x,t,\tau)$ as $\varepsilon = 1$ (at the left) and $\varepsilon = 0.5$ (at the right).

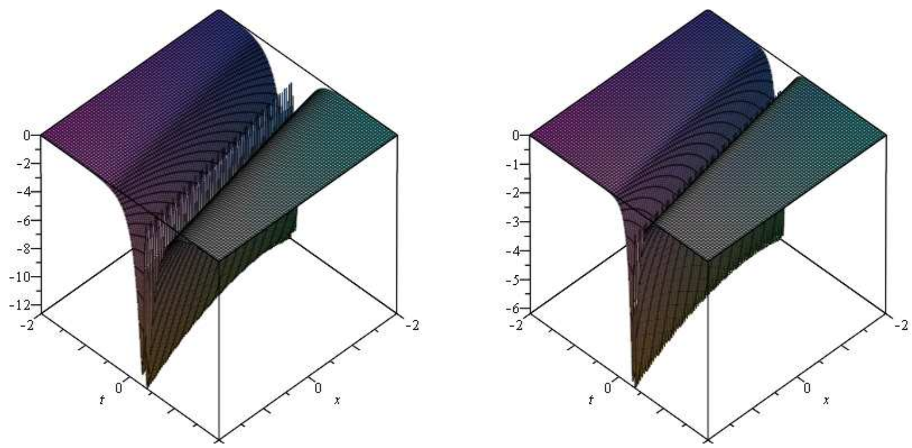

Figure 5: The term $\varepsilon V_1(x,t,\tau)$ as $\varepsilon = 1$ (at the left) and $\varepsilon = 0.5$ (at the right).

Taking into account the form of the functions $w_0(t,\tau)$ and $\Phi_1^\pm(t,\tau)$, we have that the functions $\Phi_1^\pm(t,\tau)w_0(t,\tau)$ are bounded in τ for any $t \in [0;T]$. Thus, $v_1^{\pm,1}(t,\tau)$ are rapidly decreasing functions as $|\tau| \to \pm\infty$.

For the functions $v_1^{\pm,2}(t,\tau)$ we can calculate their exact values as

$$v_1^{\pm,2}(t,\tau) = 108\,(t^2+1)\,w_0(t,\tau)\,\left[\sinh^{-4}\kappa_\pm - \sinh^{-8}\kappa_\pm - C_1\right]$$

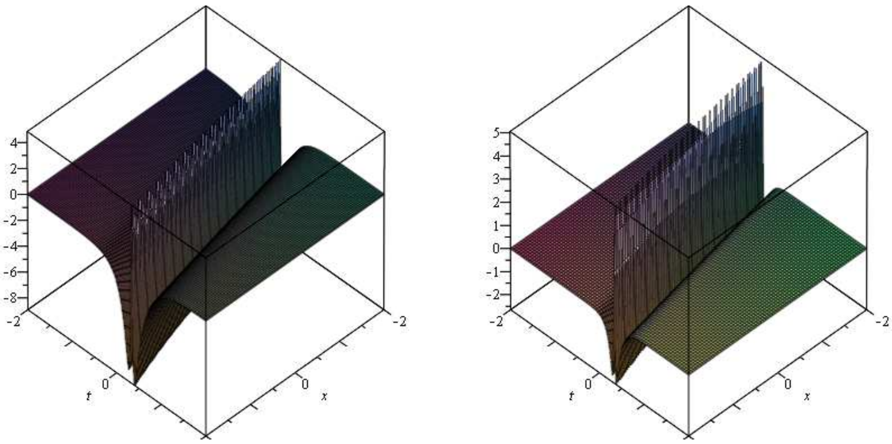

Figure 6: The first approximation of the peakon-like solution $u_1(x,t,\varepsilon)$ as $\varepsilon = 1$ (at the left) and $\varepsilon = 0.5$ (at the right).

with constant

$$C_1 = \sinh^{-4}\Big(\operatorname{arccoth}\sqrt{2}\Big) - \sinh^{-8}\Big(\operatorname{arccoth}\sqrt{2}\Big).$$

It is clear, that the function $v_1(t,\tau)$ constructed via formulae (79), (92)–(94) belongs to the space $G^{\pm}$ if the condition $c_{12}^{+} = c_{12}^{-} = 108\,(t^2+1)C_1$ holds.

Thus, the first term $V_1(x,t,\tau)$ is written as

$$V_1(x,t,\tau) = v_1(t,\tau) = (t^2+1)\,\sinh^{-2}\kappa\,\bigg[-\frac{6071}{315} - 6\,|\tau|\coth\kappa - \frac{4519}{1260}\,\sinh^{-2}\kappa + \frac{105}{8}\,|\tau|\,\sinh^{-3}\kappa + \frac{1787}{126}\,\sinh^{-8}\kappa - \frac{7381}{6300}\,\sinh^{-4}\kappa + \frac{105}{16}\,|\tau|\sinh^{-5}\kappa + \frac{4231}{1260}\,\sinh^{-6}\kappa - \frac{105}{16}\,|\tau|\cosh\kappa\sinh^{-9}\kappa - \frac{7}{9}\,\sinh^{-10}\kappa\bigg], \quad \kappa = \frac{|\tau|}{2} + \operatorname{arccoth}\sqrt{2}. \tag{95}$$

Finally, the first asymptotic approximation for peakon-like solution of equation (90) is global and it is given as

$$u_1(x,t,\varepsilon) = V_0(x,t,\tau) + \varepsilon V_1(x,t,\tau), \quad \tau = \frac{x-6t}{\varepsilon}, \tag{96}$$

where the functions $V_0(x,t,\tau)$, $V_1(x,t,\tau)$ are defined by (91), (95).

According to Theorem 5 function (96) satisfies equation (90) with an asymptotic accuracy $O(\varepsilon)$.

Graphs of the main and first terms of the asymptotic peakon-like solution as well as of the first approximation are presented on Fig. 4–6 for a small parameter $\varepsilon = 1$ and $\varepsilon = 0.5$.

5. Conclusions and discussions

Research into various types of integrable models in modern mathematical and theoretical physics is currently attracting significant attention [69, 70, 71, 72]. This paper focuses on the variable-coefficient modified Camassa–Holm (vcmCH) equation, a direct generalization of the well-known modified Camassa–Holm equation (6). This integrable system supports both soliton solutions and peakon solutions.

Our primary objective is to construct soliton- and peakon-like solutions for the vcmCH equation in the regime of small dispersion. The approach employed is similar to those used for the variable-coefficient Korteweg–de Vries (vcKdV) equation and the variable-coefficient Benjamin–Bona–Mahony (vcBBM) equation, both of which describe wave propagation in media with heterogeneous characteristics. Consequently, it is natural to explore solutions to the vcmCH equation that resemble solitary wave solutions, with a particular focus on solitons and peakons.

In this extended context, exact solutions are unavailable, as traditional analytical methods prove ineffective due to the variable coefficients. Therefore, it is appropriate to seek approximate solutions that resemble the exact solutions of the corresponding equations with constant coefficients. For cases of small dispersion in the medium, the powerful tools of asymptotic analysis, particularly the WKB method, can be effectively applied to solve these problems.

This study constructs asymptotic peakon-like solutions for a partial differential equation (PDE) with variable coefficients for the first time. We emphasize that the soliton-like and peakon-like solutions do not coincide, but share the same discontinuity curve. This curve is determined by a first-order ordinary differential equation, unlike the second-order equations that govern the KdV and BBM equations [48, 54].

The novelty of this paper lies in developing a general methodology for constructing these solutions, supported by a thorough and rigorous justification. This methodology builds upon the results of [48, 54, 59], which address the construction of asymptotic soliton-like solutions for the vcKdV and vcBBM equations, as well as asymptotic step-like solutions for the variable-coefficient Burgers equation. The asymptotic soliton- and peakon-like solutions of the vcmCH equation reduce to the solitons and peakons of the original modified Camassa–Holm (mCH) equation when the variable coefficients are taken as constants. Thus, these solutions can be viewed as deformations of traveling-wave solutions induced by the variable coefficients.

It is important to note that these variable coefficients introduce significant challenges in analyzing the equations, even when seeking specific solutions. Although the proposed method offers an approximate description of soliton modulation, the solutions it produces are generally not global. This limitation arises from the phase function governing the solutions, which is determined by a nonlinear equation. As is well known, such equations often do not admit global solutions. The governing equation incorporates only the leading terms in the asymptotic expansion of the variable coefficients in the vcmCH equation. However, with an appropriate choice of these coefficients, the general theory of ordinary differential equations ensures that the phase function is globally well-defined in time. This holds for a sufficiently broad range of variable coefficients. Consequently, the existence of a global asymptotic solution for such systems requires careful analysis, with special attention given to the selection of coefficients.

Despite these limitations, the results presented here are significant. They greatly expand the potential applications of hydrodynamic-type equations with variable parameters and open new avenues for exploring wave propagation phenomena in inhomogeneous media. The findings are further enriched by nontrivial examples of asymptotic soliton- and peakon-like solutions with a global phase function. Specifically, both the main and first terms of these solutions are derived. Additionally, for various values of a small parameter, graphs are provided to illustrate these solutions. These examples confirm that for an adequate description of wave processes, it suffices to determine the main and first terms of the corresponding asymptotic solutions.

CRediT authorship contribution statement

Yuliia Samoilenko: Conceptualisation, Methodology, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Visualization. **Lorenzo Brandolese:** Conceptualisation, Validation, Writing – original draft, Writing – review & editing, Supervision. **Valerii Samoilenko:** Conceptualisation, Methodology, Validation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] F. Calogero, A. Degasperis, Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations, North–Holland, Amsterdam, 1982.

[2] D. J. Korteweg, G. de Vries, Xli. on the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, Philosophical Magazine Series 1 39 (1895) 422–443. doi:10.1080/14786449508620739.

[3] N. J. Zabusky, M. Kruskal, Interaction of "solitons" in a collisionless plasma and the recurrence of initial states, Physical Review Letters 15 (6) (1965) 240–243. doi:10.1103/PhysRevLett.15.240.

[4] R. Bullough, P. J. Caudrey, Solitons, Springer-Verlag Berlin Heidelberg, 1980.

[5] M. J. Ablowitz, Nonlinear Dispersive Waves. Asymptotic Analysis and Solitons, North – Holland, Amsterdam, 2011. doi:10.1017 / CBO9780511998324.

[6] A. K. Prykarpatsky, I. V. Mykytiuk, Algebraic Integrability of Nonlinear Dynamical Systems on Manifolds. Classical and Quantum Aspects, Dordrekht, Kluwer Academic Publishers, 1998. doi:10.1007/978-94-011-4994-5.

[7] D. Blacmore, A. K. Prykarpatsky, V. H. Samoylenko, Nonlinear Dynamical Systems of Mathematical Physics. Spectral and Integrability Analysis, World Scientific, Singapore, 2011. doi:10.1142/7960.

[8] R. K. Dodd, J. C. Eilback, J. D. Gibbon, H. C. Morris, Solitons and Nonlinear Wave Equations, Academic Press, New York, 1982.

[9] A. C. Newell, Solitons in Mathematics and Physics, Society for Industrial and Applied Mathematics, 1985.

[10] T. B. Benjamin, J. L. Bona, J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences 272 (1972) 47–78. doi:10.1098/rsta.1972.0032.

[11] D. H. Peregrin, Calculations of the development of an undular bore, Journal of Fluid Mechanics 25 (2) (1966) 321–330. doi:10.1017/S0022112066001678.

[12] A. Cheviakov, D. Dutykh, Galilei-invariant and energy-preserving extension of Benjamin–Bona–Mahony–type equations, Partial Differential Equations in Applied Mathematics 7 (2023) 100519. doi:10.1016/j.padiff.2023.100519.

[13] R. I. Joseph, R. G. Egri, Another possible model equation for long waves in nonlinear dispersive systems, Physics Letters A 61 (7) (1977) 429–430. doi:10.1016/0375-9601(77)90739-3.

[14] F. Fan, Y. Zhou, Q. Liu, Bifurcation of traveling wave solutions for the Joseph–Egri equation, Reports on Mathematical Physics 83 (2) (2019) 175–190. doi:10.1016/S0034-4877(19)30038-2.

[15] P. Chiu, L. Lee, T. W. Sheu, A dispersion-relation-preserving algorithm for a nonlinear shallow–water wave equation, Journal of Computational Physics 228 (21) (2009) 8034–8052. doi:10.1016/j.jcp.2009.07.030.

[16] R. M. Chen, T. Hu, Y. Liu, The shallow-water models with cubic nonlinearity, Journal of Mathematical Fluid Mechanics 24 (2) (2022) 49. doi:10.1007/s00021-022-00685-4.

[17] F. Fedele, D. Dutykh, Vortexons in axisymmetric poiseuille pipe flows, Europhysics Letters 101 (3) (2013) 34003. doi:10.1209/0295-5075/101/34003.

[18] B. Fuchssteiner, A. S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, Physica D: Nonlinear Phenomena 4 (1) (1981) 47–66. doi:10.1016/0167-2789(81)90004-X.

[19] R. Camassa, D. Holm, An integrable shallow water equation with peaked soliton, Physical Review Letters 71 (11) (1993) 1661–1664. doi:10.1103/PhysRevLett.71.1661.

[20] S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, S. Wynne, The Camassa–Holm equations and turbulence, Physica D: Nonlinear Phenomena 133 (1) (1999) 49–65. doi:10.1016/S0167-2789(99)00098-6.

[21] F. Fedele, D. Dutykh, Camassa–Holm Type Equations for axisymmetric poiseuille pipe flows, Procedia IUTAM 9 (2013) 16–24. doi:https://doi.org/10.1016/j.piutam.2013.09.003.

[22] F. Fedele, D. Dutykh, Camassa–Holm equations and vortexons for axisymmetric pipe flows, Fluid Dynamics Research 46 (1) (2013) 015503. doi:10.1088/0169-5983/46/1/015503.

[23] A. Boutet de Monvel, I. Karpenko, D. Shepelsky, Riemann-Hilbert approach for the Camassa–Holm equation with nonzero boundary conditions, Journal of Mathematical Physics 61 (3) (2020) 031504. doi:10.1063/1.5139519.

[24] A. Constantin, V. S. Gerdjikov, R. I. Ivanov, Inverse scattering transform for the Camassa–Holm equation, Inverse Problems 22 (6) (2006) 2197–2207. doi:10.1088/0266-5611/22/6/017.

[25] A. Parker, On the Camassa–Holm equation and a direct method of solution I. Bilinear form and solitary waves, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences 460 (2050) (2004) 2929–2957. doi:10.1098/rspa.2004.1301.

[26] A. G. Rasin, J. Schiff, Bäcklund transformations for the Camassa–Holm equation, Journal of Nonlinear Science 27 (2017) 45–69. doi:10.1007/s00332-016-9325-6.

[27] T. W. Sheu, P. Chiu, C. Yu, On the development of a high-order compact scheme for exhibiting the switching and dissipative solution natures in the Camassa–Holm equation, Journal of Computational Physics 230 (2011) 5399–5416. doi:10.1016/j.jcp.2011.03.043.

[28] A. Constantin, D. Lannes, The hydrodynamical relevance of the Camassa–Holm and Degasperis–Processi equations, Archive for Rational Mechanics and Analysis 192 (2009) 165–186. doi:10.1007/s00205-008-0128-2.

[29] A. Constantin, J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, Acta Mathematica 181 (1998) 229–243. doi:10.1007/BF02392586.

[30] L. Brandolese, Local-in-space criteria for blowup in shallow water and dispersive rod equations, Communications in Mathematical Physics 330 (1) (2014) 401–414. doi:10.1007/s00220-014-1958-4.

[31] D. Dutykh, D. Ionescu-Kruse, Travelling wave solutions for some two-component shallow water models, Journal of Differential Equations 261 (2) (2016) 1099–1114. doi:https://doi.org/10.1016/j.jde.2016.03.035.

[32] D. Dutykh, D. Ionescu-Kruse, Effects of vorticity on the travelling waves of some shallow water two-component systems, Discrete and Continuous Dynamical Systems 39 (9) (2019) 5521–5541. doi:10.3934/dcds.2019225.

[33] I. Karpenko, D. Shepelsky, G. Teschl, A Riemann–Hilbert approach to the modified Camassa–Holm equation with step-like boundary conditions, Monatshefte für Mathematik 201 (2023) 127–172. doi:10.1007/s00605-022-01786-y.

[34] Y. Gao, On conservative sticky peakons to the modified Camassa–Holm equation, Journal of Differential Equations 365 (2023) 486–520. doi:10.1016/j.jde.2023.04.027.

[35] Z. Guo, X. Liu, C. Qu, Stability of peakons for the generalized modified Camassa–Holm equation, Journal of Differential Equations 266 (12) (2019) 7749–7779. doi:10.1016/j.jde.2018.12.014.

[36] R. McLachlan, X. Zhang, Well-posedness of modified Camassa–Holm equations, Journal of Differential Equations 246 (8) (2009) 3241–3259. doi:10.1016/j.jde.2009.01.039.

[37] R. M. Chen, Y. Liu, C. Qu, S. Zhang, Oscillation-induced blow-up to the modified Camassa–Holm equation with linear dispersion, Advances in Mathematics 272 (2015) 225–251. doi:10.1016/j.aim.2014.12.003.

[38] A. Yokus, H. Durur, K. Abro, Role of shallow water waves generated by modified Camassa–Holm equation: A comparative analysis for traveling wave solutions, Nonlinear Engineering 10 (1) (2021) 385–394. doi:10.1515/nleng-2021-0030.

[39] A. Degasperis, G. Gaeta, Symmetry and perturbation theory, World Scientific, Singapore, 1999, Ch. Asymptotic integrability, p. 23–37. doi:10.1142/9789812833037.

[40] L. Brandolese, A Liouville theorem for the Degasperis–Procesi equation, Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie V 16 (3) (2016) 759–765. doi:10.2422/2036-2145.201410_014.

[41] A. Parker, Wave dynamics for peaked solitons of the Camassa–Holm equation, Chaos, Solitons & Fractals 35 (2) (2008) 220–237. doi:https://doi.org/10.1016/j.chaos.2007.07.049.

[42] S. Abenda, T. Grava, C. Klein, Numerical solution of the small dispersion limit of the Camassa–Holm and Whitham equations and multiscale expansions, SIAM Journal on Applied Mathematics 70 (8) (2010) 2797–2821. doi:10.1137/090770278.

[43] M. V. Flamarion, E. Pelinovsky, Interactions of solitons with an external force field: Exploring the Schamel equation framework, Chaos, Solitons & Fractals 174 (2023) 113799. doi:10.1016/j.chaos.2023.113799.

[44] P. D. Lax, C. D. Levermore, The small dispersion limit of the Korteweg-de Vries equation I, Communications on Pure and Applied Mathematics 36 (3) (1983) 253–290. doi:10.1002/cpa.3160360302.

[45] P. D. Lax, C. D. Levermore, The small dispersion limit of the Korteweg-de Vries equation II, Communications on Pure and Applied Mathematics 36 (5) (1983) 571–593. doi:10.1002/cpa.3160360503.

[46] P. D. Lax, C. D. Levermore, The small dispersion limit of the Korteweg-de Vries equation III, Communications on Pure and Applied Mathematics 36 (6) (1983) 809–829. doi:10.1002/cpa.3160360606.

[47] R. M. Miura, M. D. Kruskal, Application of nonlinear WKB–method to the Korteweg–de Vries equation, SIAM Journal on Applied Mathematics 26 (2) (1974) 376–395. doi:10.1137/0126036.

[48] V. H. Samoylenko, Y. I. Samoylenko, Asymptotic expansions for one–phase soliton–type solutions of the Korteweg–de Vries equation with variable coefficients, Ukrainian Mathematical Journal 57 (1) (2005) 132–148. doi:10.1007/s11253-005-0176-9.

[49] V. H. Samoilenko, Y. I. Samoilenko, Asymptotic m–phase soliton–type solutions of a singularly perturbed Korteweg–de Vries equation with variable coefficients, Ukrainian Mathematical Journal 64 (7) (2012) 1109–1127. doi:10.1007/s11253-012-0702-5.

[50] V. H. Samoilenko, Y. I. Samoilenko, Asymptotic m–phase soliton–type solutions of a singularly perturbed Korteweg–de Vries equation with variable coefficients. II, Ukrainian Mathematical Journal 64 (8) (2013) 1241–1259. doi:10.1007/s11253-013-0713-x.

[51] V. H. Samoilenko, Y. I. Samoilenko, V. O. Limarchenko, V. S. Vovk, K. S. Zaitseva, Asymptotic solutions of soliton type of the Korteweg–de Vries equation with variable coefficients and singular perturbation, Mathematical Modeling and Computing 6 (2) (2019) 374–385. doi:10.23939/mmc2019.02.374.

[52] S. I. Lyashko, V. H. Samoilenko, Y. I. Samoilenko, I. V. Gapyak, N. I. Lyashko, M. S. Orlova, Global asymptotic step–type solutions to singularly perturbed Korteweg–de Vries equation with variable coefficients, Journal of Automation and Information Sciences 52 (9) (2020) 27–38. doi:10.1615/JAutomatInfScien.v52.i9.30.

[53] S. I. Lyashko, V. H. Samoilenko, Y. I. Samoyienko, I. V. Gapyak, M. S. Orlova, Asymptotic step–wise solutions of the Korteweg–de Vries equation with a singular perturbation and their accuracy, Mathematical Modeling and Computing 8 (3) (2021) 410–421. doi:10.23939 / mmc2021.03.410.

[54] V. H. Samoilenko, Y. I. Samoilenko, Asymptotic soliton–like solutions to the singularly perturbed Benjamin–Bona–Mahony equation with variable coefficients, Journal of Mathematical Physics 60 (1) (2019) 011501–0115013. doi:10.1063/1.5085291.

[55] V. H. Samoilenko, Y. I. Samoilenko, Asymptotic multiphase Σ–solutions to the singularly perturbed Korteweg–de Vries equation with variable coefficients, Journal of Mathematical Sciences 200 (3) (2014) 358–373. doi:10.1007/s10958-014-1918-8.

[56] A. Nayfeh, Perturbation Methods, John Wiley & Sons, Ltd., 2000. doi:10.1002/9783527617609.

[57] X. Lu, L. Lu, A. Chen, New peakons and periodic peakons of the modified Camassa–Holm equation, Journal of Nonlinear Modeling and Analysis 2 (3) (2021) 345–353. doi:10.12150/jnma.2020.345.

[58] R. S. Johnson, On solutions of the Camassa–Holm equation, Proceedings of the Royal Society of London. Series A: Mathematical, Physical

and Engineering Sciences 459 (2035) (2003) 1687–1708. doi:10.1098/rspa.2002.1078.

[59] V. H. Samoilenko, Y. I. Samoilenko, E. Zappale, Asymptotic step–like solutions of the singularly perturbed Burgers equation, Physics of Fluids 35 (6) (2023) 067106. doi:10.1063/5.0150685.

[60] F. John, Partial Differential Equations. 4th ed, Vol. 1 of Appl. Math. Sci., Springer, Cham, 1982.

[61] L. C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, 2nd Edition, Vol. 19 of Grad. Stud. Math., Providence, RI: American Mathematical Society (AMS), 2010.

[62] V. P. Maslov, G. A. Omel'yanov, Geometric Asymptotics for PDE, I, Providence, American Mathematical Society, 2001.

[63] L. Hörmander, The Analysis of Linear Partial Differential Operators III. Pseudodifferential Operators, reprint of the 1994 ed. Edition, Class. Math., Springer, Berlin, 2007. doi:10.1007/978-3-540-49938-1.

[64] V. V. Grushin, Pseudodifferential operators on R^n with bounded symbols, Functional Analysis and Its Applications 4 (1970) 202–212. doi:10.1007/BF01075240.

[65] V. V. Grushin, On a class of elliptic pseudodifferential operators degenerating on a submanifold, Mathematics of the USSR-Sbornik 13 (2) (1971) 155–185. doi:10.1070/SM1971v013n02ABEH001033.

[66] V. H. Samoilenko, Samoilenko, Existence of a solution to the inhomogeneous equation with the one–dimensional Schrödinger operator in the space of quickly decreasing functions, Journal of Mathematical Sciences 187 (1) (2012) 70–76. doi:10.1007/s10958-012-1050-6.

[67] V. H. Samoilenko, Samoilenko, Existence of solutions to inhomogeneous higher order differential equation in the Schwartz space, Journal of Mathematical Physics, Analysis, Geometry 16 (4) (2020) 454–459. doi:10.15407/mag16.04.454.

[68] D. W. Jordan, P. Smith, Nonlinear Ordinary Differential Equations, 4th edition, Oxford University Press, 2007.

[69] H.-Y. Guan, J.-G. Liu, Variable–coefficient polynomial function method for finding the lump-type solutions of integrable system with variable coefficients, Modern Physics Letters B 38 (14) (2024). doi:10.1142/S0217984924501148.

[70] S. Opanasenko, A. Bihlo, R. O. Popovych, Group analysis of general Burgers–Korteweg–de Vries equations, Journal of Mathematical Physics 58 (8) (2017) 081511. doi:10.1063/1.4997574.

[71] P. K. Das, Three interaction scenarios of two orthogonally polarised optical pulses modelled by the variable coefficient coupled nonlinear Schrödinger equations, Nonlinear Dynamics 112 (24) (2024) 22355–22378. doi:10.1007/s11071-024-10217-6.

[72] Z.-Z. Lan, N-soliton solutions, Bäcklund transformation and Lax pair for a generalized variable–coefficient cylindrical Kadomtsev–Petviashvili equation, Applied Mathematics Letters 158 (2024). doi:10.1016/j.aml.2024.109239.