Space-time decay of Navier–Stokes flows invariant under rotations

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Abstract

We show that the solutions to the non-stationary Navier–Stokes equations in $\mathbb{R}^d$ ($d = 2, 3$) which are left invariant under the action of discrete subgroups of the orthogonal group $O(d)$ decay much faster as $|x| \to \infty$ or $t \to \infty$ than in generic case and we compute, for each subgroup, the precise decay rates in space-time of the velocity field.

1 Introduction and main results

This paper is devoted to the study of the asymptotic behavior of viscous flows of incompressible fluids filling the whole space $\mathbb{R}^d$ ($d \geq 2$) and not submitted to the action of external forces. These flows are governed by the Navier–Stokes equations, which we may write in the following form

$$\begin{cases} \partial_t u - \Delta u + \mathbb{P}\nabla \cdot (u \otimes u) = 0 \\ u(x,0) = a(x) \\ \nabla \cdot u = 0. \end{cases} \tag{NS}$$

Here $u(\cdot, t) : \mathbb{R}^d \to \mathbb{R}^d$ ($d \geq 2$) denotes the velocity field and $\mathbb{P}$ is the Leray–Hopf projector onto the soleinodal vectors field, defined by $\mathbb{P}f = f - \nabla \Delta^{-1}(\nabla \cdot f)$, with $f = (f_1, \ldots, f_d)$.

It is now well known (see *e.g.* [10], [4], [30], [18]) that *generic* solutions u to (NS) decay at infinity at considerably slow rates in space-time. Indeed, even if the data have the form $a(x) = \epsilon\phi(x)$, where $\epsilon > 0$ is a small constant, the components of ϕ belong to the Schwartz class and have vanishing moments, then the corresponding strong solution

u to (NS) satisfies $|u(x,t)| \leq C(1+|x|)^{-(d+1)}$ and $|u(x,t)| \leq C(1+t)^{-(d+1)/2}$ for all $x \in \mathbb{R}^d$ and $t > 0$; but such decay rates, in general, are optimal (we refer to [25], [27] [1], [20] for a proof of these bounds under different assumptions). Furthermore, only few examples of solutions which decay faster are known so far: we should mention here the classical example of a two dimensional flow with *radial vorticity* ([30], [12], see also [31]) and the examples of flows in $\mathbb{R}^d$ ($d \geq 2$) constructed in [4], [5] by imposing some special symmetries on the initial data.

The purpose of this paper is to provide a systematic study of the connection between symmetry and space-time decay of viscous flows in dimension two and three. Our starting point is the observation that the Navier–Stokes equations are invariant under the transformations of the orthogonal group $O(d)$: if $u(x,t)$ is a solution to the Navier–Stokes equations in $\mathbb{R}^d$, and $P \in O(d)$ is an orthogonal matrix, then $\tilde{u}(x,t) = P^T u(Px,t)$ is a Navier–Stokes flow as well (here, P^T is the transposed matrix).

Roughly, it follows that if the initial datum commutes with $P \in O(d)$, then the velocity field will satisfy

$$P\,u(x,t) = u(P\,x,t), \tag{1}$$

whenever the unique strong solution u to (NS) is defined. In the case in which only a weak solution is known to exist (then it is not known if such weak solution is unique or not), then at least one of them satisfies (1), as it can be easily checked following step-by-step any of the constructions of weak solutions known so far.

If G is any subgroup of $O(d)$, then a natural problem is that of computing the space-time decay rates of solutions that are invariant under all the transformations of G. In this paper we will consider only *discrete* subgroups of the orthogonal group, the reason being the following: in the two dimensional case, solutions that are invariant under the continuous subgroup $SO(2)$ do exist, but boil down to flows with *radial vorticity*. These flows are "trivial" in the sense that the non-linear term $\mathbb{P}\nabla \cdot (u \otimes u)$ in (NS) identically vanishes. On the other hand, in the three dimensional case, one easily sees via the Fourier transform that flows $u(x,t) \not\equiv 0$ which are invariant under the whole group $SO(3)$ do not exist. Furthermore, solutions which are invariant under other continuous subgroups of $SO(3)$ (such as the complete direct symmetry group of a cylinder) do not decay faster than general solutions.

As an immediate consequence of our results in the $d = 2$ case we will prove in section 3 the following theorem.

Theorem 1.1 *Let $a(x)$ be a soleinoidal vector field with components in $\mathcal{S}(\mathbb{R}^2)$ (the Schwartz class). If a is invariant under the cyclic group C_n of order n, then the global strong solution $u(x,t)$ such that $u(0) = a$ satisfies $u(x,t) = O(|x|^{-(n+1)})$ for all $t \geq 0$. If, in addition, a is invariant under the dihedral group D_n of order $2n$, then this decay is uniform in time and, moreover, $||u(t)||_p \leq C(1+t)^{-(n+1)/2+1/p}$ (with $2 < p \leq \infty$).*

As we shall see later on, the flows obtained in Theorem 1.1 do not have radial vorticity and hence they do not boil down to "trivial solutions" (or to solutions of the homogeneous heat equation $\partial_t u = \Delta u$). At best of our knowledge, no other examples of highly localized flows in $\mathbb{R}^2$ were known so far.

In the three dimensional case, the problem of the existence of (trivial or non-trivial) rapidly decreasing solutions $u = (u_1, u_2, u_3) \not\equiv 0$ as $|x| \to \infty$ to Navier–Stokes equations was raised in [11] and it is still open.

However, non-trivial and localized divergence-free vector fields in $\mathbb{R}^3$ $a(x)$, which are invariant under discrete subgroups of $O(3)$ can be easily constructed, and we may expect that such fields should lead to solutions with fast decay at infinity. If G is one of these subgroups, then we know that G is either

- a subgroup of the complete symmetry group of a regular polyhedron, or
- a subgroup of the complete symmetry group of a prism (and hence isomorphic to a cyclic or a dihedral group).

As we will see in section 5, solutions which are invariant under the complete symmetry group of a prism (which is not a cube) in general do not decay faster than $|x|^{-4}$. On the other hand, flows with polyhedral symmetry decay much faster. As a byproduct of our constructions, we shall be able to provide examples of solutions $u(x,t)$ decaying at infinity as $|x|^{-8}$ and t^{-4}, thus improving the results of [4], [26] and [5]. The most interesting cases are described in the theorem below (see section 4 for the exaustive study of the asymptotic behavior for all the other finite groups of isometries in $\mathbb{R}^3$).

Let us denote by $L^\infty_\gamma(\mathbb{R}^d)$ ($\gamma \geq 0$) the space of all measurable functions (or vector fields) f, defined on $\mathbb{R}^d$, and such that $(1+|x|)^\gamma |f(x)| \in L^\infty(\mathbb{R}^d)$. For any positive T, $0 < T \leq \infty$, we denote by $C([0,T], L^\infty_\gamma(\mathbb{R}^d))$ the space of continuous and bounded $L^\infty_\gamma(\mathbb{R}^d)$-valued functions, the continuity at $t = 0$ being understood in the distributional sense. Then we have the following:

Theorem 1.2 *Let $a = (a_1, a_2, a_3)$ a divergence-free and rapidly decreasing vector field in $\mathbb{R}^3$: $a \in L^\infty_k(\mathbb{R}^3)$, for all $k = 0, 1, \ldots$. Then we know ([25], [7]) that there exists T ($0 < T \leq \infty$) and a unique strong solution u to the Navier-Stokes equations in $\mathbb{R}^3$, such that $u(0) = a$ and $u \in C([0,T], L^\infty_4(\mathbb{R}^3))$.*

1. *If $a(x)$ is invariant under the complete symmetry group of the tetrahedron, then $u \in C([0,T], L^\infty_5(\mathbb{R}^3))$.*
2. *If $a(x)$ is invariant under the complete symmetry group of the cube (or of the octahedron), then $u \in C([0,T], L^\infty_6(\mathbb{R}^3))$.*
3. *If $a(x)$ is invariant under the complete symmetry group of the dodecahedron (or of the icosahedron), then $u \in C([0,T], L^\infty_8(\mathbb{R}^3))$.*

Furthermore, if we know that $u(x,t)$ is global ($T = \infty$), then $||u(t)||_p$ decays, respectively, at least as fast as $t^{-5/2+3/(2p)}$, $t^{-3+3/(2p)}$ and $t^{-4+3/(2p)}$ as $t \to \infty$ ($\frac{3}{2} < p \leq \infty$).

Remark 1.3 The conclusion of Theorem 1.2 is sharp. Optimality of the above space decay rates should be understood in the following sense: if G is one of the previous three groups and $\gamma = 5$, 6 or 8 (respectively), then there exists a solution $u(x,t)$ to (NS) which is invariant under G and localized at $t = 0$, but which does not decay faster than $|x|^{-\gamma}$, uniformely in any time interval $[0, \epsilon]$ ($\epsilon > 0$). For each group G we shall provide examples of such flows.

We shall see in section 4 that, because of the symmetries imposed on the initial data, the velocity field has vanishing moments $\int x^\alpha u(x,t)\,dx$ up the the order 1, 2 and 4, for all $t \in [0,T]$, respectively in the case 1, 2 and 3 of Theorem 1.1. In particular, the fact that $a(x)$ has these cancelations allows us to see that the estimates in space-time obtained for $u(x,t)$ hold true also for the linear evolution $e^{t\Delta}a(x)$ (here $e^{t\Delta}$ denotes the heat semigroup).

Let us point out that the existence of a *global* strong solution is usually ensured by some smallness assumption on the initial data: common suitable assumptions are *e.g.* that $||a||_3$ is small enough (see [21]), or that a is small in some Besov norm (see [8]). However, for the flows treated in Theorem 1.2, the three equations contained in the first of (NS) reduce to a simpler *single scalar equation* on the first component $u_1(x,t)$. Thus, it would be an interesting problem to study the global solvability of those "symmetric" solutions in the case of "large" initial data.

There is an extensive literature on the asymptotic behavior of the Navier–Stokes equations (see *e.g.* [20], [17], [18], [25], [30], [32] and the references therein contained), but not so much has been written on symmetry of viscous flows. See, however, [22], [23] for applications of symmetries to the numerical simulation of turbulence and [14], [29] for the construction of ansatzes to (NS). The connection between symmetry and space-time decay has been first noticed in [4] and subsequentely studied in [5], [26]. The symmetries which are considered in these papers are only those corresponding to a subgroup of the group of the symmetries of the cube. Hence, a few results of [4], [26], [5] are contained in the present paper as a particular case.

It is worth observing that recently Th. Gallay and C. E. Wayne were able to prove the existence of flows with a fixed, but arbitrarily large, *time* decay rate (see [17] and [18], respectively for $d = 2, 3$). Indeed, using the vorticity formulation of the Navier–Stokes equations, they showed in [17] and [18] that the class of solutions which decay faster than a given rate as $t \to \infty$ lies on an invariant manifold of finite codimension, in a suitable functional space. Their method, which is a combination of the spectral decomposition of the Fokker–Planck operator and the theory of dynamical systems, would be effective in any space dimension. However, this approach yields no *explicit* examples of initial data leading to such solutions with fast decay.

The rest of this paper is organized as follows. In section 2 we briefly recall the vorticity formulation of (NS) and a general result of the author on the space decay of solutions to the Navier–Stokes equations that we will use throughout this paper. As an application of this result to the two-dimensional case, in section 3 we will prove Theorem 1.1 in a slightly more general form. In section 4 we start recalling the complete list of the discrete subgroups of $O(3)$ and we subsequently compute the space decay rates of flows invariant under the action of all these groups. There we will also discuss the closely related problem of the cancellations of the vorticity of such flows and the applications to the time decay. In section 5 we will show by means of some examples the optimality of the decay rates that we obtain.

2 Decay of the velocity field and the vorticity

Throughout this paper we shall assume that the initial datum a is a rapidly decreasing function in $\mathbb{R}^d$ ($d \geq 2$). This requirement is not essential (the optimal assumptions should be expressed in terms of Besov and weak-Hardy spaces, as in [25], [27] and [5]) but it considerably simplifies the presentations of our main results. Then we know that a *necessary condition* on the data, in order to avoid that the velocity field instantaneously "spreads out", is that the components of a are orthogonal with respect to the L^2 inner product (see [10]):

$$\int (a_h a_k)(x)\, dx = c\delta_{h,k}, \qquad (h, k = 1, \ldots, d) \tag{2}$$

($\delta_{h,k} = 1$ if $h = k$ and $\delta_{h,k} = 0$ if $h \neq k$).

Let us briefly recall how this condition can be obtained and generalized, since its generalization plays a central role in this paper. Let us first note that (2) can be conveniently restated by saying that the homogeneous polynomial

$$P_0(a)(\xi) \equiv \sum_{h,k=1}^{d} \left(\int (a_h a_k)(x)\, dx \right) \xi_h \xi_k, \qquad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \tag{3}$$

is divisible by $\xi_1^2 + \cdots + \xi_d^2$.

Such necessary condition can be very easily deduced under the (somewhat artificial) assumption that, at the beginning of the evolution, the Fourier transform $\widehat{p}(\xi, t)$ of the pressure is continuous at $\xi = 0$. Indeed, applying the Fourier transform in the classical relation $-\Delta p(x, t) = \sum_{h,k=1}^{d} \partial_h \partial_k (u_h u_k)(x, t)$, we get

$$-|\xi|^2 \widehat{p}(\xi, t) = \sum_{h,k=1}^{d} \xi_h \xi_k \widehat{u_h u_k}(\xi, t) \tag{4}$$

and our claim follows taking $t = 0$ and letting $\xi \to 0$ in (4). But it follows from the result of [7] that $P_0(a)(\xi)$ is divisible by $\xi_1^2 + \cdots + \xi_d^2$ even if we drop the assumption on the pressure, and if we assume, instead, that the velocity field is well-localized. We may ask, for example, that $u(x, t)$ decays at infinity faster than $|x|^{-(d+1)}$ uniformely in some time interval $[0, T]$, with $T > 0$ (roughly speaking, this is due to the fact that in this case the pressure is "localized in a weak sense" and $\widehat{p}(\xi, t)$ still has some kind of regularity).

This argument can be generalized in the following way (see [7]): if we put much more stringent *a priori* assumptions on the decay of the velocity field, then we get a better regularity for $\widehat{p}(\xi, t)$. In the particular case in which $u(x, t)$ is rapidly decreasing as $|x| \to \infty$ for all $t \in [0, T]$, we obtain $\widehat{p}(\xi, t) \in C^\infty(\mathbb{R}^3)$ in such time interval. In particular, we can apply the Taylor formula of any order in (4). We deduce that, for all $t \in [0, T]$, and $m = 0, 1, \ldots$ the homogeneous polynomial $P_m(u(t))$, defined by

$$P_m(u(t))(\xi) \equiv \sum_{h,k=1}^{d} \sum_{|\alpha|=m} \left(\frac{1}{\alpha!} \int x^\alpha (u_h u_k)(x, t)\, dx \right) \xi^\alpha \xi_h \xi_k, \tag{5}$$

(we adopted the usual notations for the multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$) must be divisible by $\xi_1^2 + \cdots + \xi_d^2$.

Conversely, if we assume that the initial datum a is rapidly decreasing as $|x| \to \infty$ and such that all the polynomials $P_m(u(t))$ (where u is the strong solution defined in some time interval $[0,T]$ such that $u(0) = a$) are divisible by $\xi_1^2+\cdots+\xi_d^2$ *for all* $t \in [0,T]$, then u will be rapidly decreasing as $|x| \to \infty$ for all t in such interval. More precisely, let us recall the following result from [7], which will be our main tool for our study of the spatial localization.

Proposition 2.1 *Let M be a fixed non-negative integer and $a(x)$ a divergence-free and rapidly decreasing vector field in $\mathbb{R}^d$. Let also $u(x,t)$ be the unique strong solution to (NS), defined in some time interval $[0,T]$ $(0 < T \le \infty)$ such that $u(0) = a$ and $u \in C([0,T], L^\infty_{d+1}(\mathbb{R}^d))$.*

If the polynomials $P_m(u(t))(\xi)$, defined by (5), are divisible by $\xi_1^2 + \cdots + \xi_d^2$ for all $t \in [0,T]$ and $m = 0,1,\dots,M$, then the spatial decay of u is improved by $u \in C([0,T'], L^\infty_{d+2+M}(\mathbb{R}^d))$, for any $T' \in \mathbb{R}^+$ $(0 \le T' \le T)$. Furthermore, if $T = +\infty$ and the moments of a vanish up to the order $1+M$, then

$$u \in C([0,+\infty[, L^\infty_{d+2+M}(\mathbb{R}^d)), \tag{6}$$

The assumpion on the moments of a ensures that, if the solution is globally defined, then the norm $||u(t)||_{L^\infty_{d+2+M}} \equiv \sup_x (1+|x|)^{d+2+M}|u(x,t)|$ does not blow up as $t \to \infty$. Condition (6) does not ensures that the solution decays fast as $t \to \infty$. It can be shown, however, that if the moments of a vanish up to the order $1+M$ *and if the following identities hold true*:

$$\sum_{h,k=1}^{d} \sum_{|\alpha|=m} \left(\frac{1}{\alpha!}\int x^\alpha (u_h u_k)(x,t)\,dx\right)\xi^\alpha \xi_j \xi_h \xi_k \equiv \tag{7}$$
$$\sum_{h=1}^{d} \sum_{|\alpha|=m} \left(\frac{1}{\alpha!}\int x^\alpha (u_h u_j)(x,t)\,dx\right)\xi^\alpha \xi_h (\xi_1^2+\cdots+\xi_d^2), \qquad (j=1,\dots,d)$$

for all $\xi \in \mathbb{R}^d$, $t \ge 0$ and $m = 0,1,\dots,M$, then

$$\sup_{x\in\mathbb{R}^d, t\ge 0} (1+|x|)^\gamma (1+t)^{(d+2+M-\gamma)/2}|u(x,t)| < \infty \qquad (0 \le \gamma \le d+2+M). \tag{8}$$

We refer to [5] for a proof of this claim and examples of flows satisfying (7) for $m = 0,1$. See also [12], [28], [18] for related results. Condition (7), however, is difficult to check for large m. To construct examples of flows with fast decay in space-time we shall rather make use of the vorticity formulation of the Navier–Stokes equations. This allows us to give a much more natural sufficient condition which ensures (8).

From now on we shall work only in two or three space dimension. We recall that the vorticity is defined by

$$\omega = \partial_1 u_2 - \partial_2 u_1 \qquad (\text{if } d = 2)$$

or,

$$\Omega = \nabla \times u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1) \qquad (\text{if } d = 3).$$

Note that the vorticity is a scalar function when $d = 2$ and a soleinoidal vector field if $d = 3$. Then the vorticity verifies the integro-differential equations

$$\partial_t \omega + (u \cdot \nabla)\omega = \Delta\omega \qquad (d = 2) \tag{9}$$

or,

$$\partial_t \Omega + (u \cdot \nabla)\Omega - (\Omega \cdot \nabla)u = \Delta\Omega, \qquad \nabla \cdot \Omega = 0 \qquad (d = 3). \tag{10}$$

Here, the velocity field u has to be expressed in terms of its vorticity via the Biot–Savart laws:

$$u(x,t) = \frac{1}{2\pi} \int \frac{(x-y)^{\perp}}{|x-y|^2} \omega(y,t)\, dy, \qquad (d = 2), \tag{11}$$

$$u(x,t) = -\frac{1}{4\pi} \int \frac{x-y}{|x-y|^3} \times \Omega(y,t)\, dy \qquad (d = 3), \tag{12}$$

where we denoted $(x_1, x_2)^{\perp} = (-x_2, x_1)$ in (11).

We collect in the following proposition several known facts on the vorticity equation:

Proposition 2.2 *1. Let n be a positive integer, ω_0 a rapidly decreasing function in $\mathbb{R}^2$ with vanishing moments up to the order $n-1$ and $\omega(x,t)$ the unique global strong solution of* (9), (11), *such that $\omega(0) = \omega_0$. If we know that the moments of $\omega(t)$ vanish up to the order $n-1$, for all $t \geq 0$, then we have*

$$\sup_{x,t} (1 + |x|)^{\gamma} (1+t)^{(2+n-\gamma)/2} |\omega(x,t)| < \infty \tag{13}$$

for all $\gamma \geq 0$ and

$$||\omega(t)||_p \leq C(1+t)^{-(2+n)/2+1/p} \qquad (1 \leq p \leq \infty). \tag{14}$$

2. Let Ω_0 be a rapidly decreasing and divergence-free vector field in $\mathbb{R}^3$, with vanishing moments up to the order $n-1$. If $\sup_x |x|^2 |\Omega_0(x)|$ is small, then there exists a unique strong solution $\Omega(x,t)$ of (10), (12), *such that $\Omega(0) = \Omega_0$ and $\Omega \in C([0,\infty[, L^{\infty}_2(\mathbb{R}^3))$. If we know that the moments of $\Omega(t)$ vanish up to the order $n-1$, for all $t \geq 0$, then we have*

$$\sup_{x,t} (1 + |x|)^{\gamma} (1+t)^{(3+n-\gamma)/2} |\Omega(x,t)| < \infty \tag{15}$$

for all $\gamma \geq 0$ and

$$||\Omega(t)||_p \leq C(1+t)^{-(3+n)/2+3/(2p)} \qquad (1 \leq p \leq \infty). \tag{16}$$

Remark 2.3 We refer to [2] and [15] for the study of the well-posedness of the Cauchy problem for equations (9) and (10). In particular, it is well known that (9), (11) can be uniquely solved *e.g.* in $C([0,\infty), L^1(\mathbb{R}^2)) \cap C(]0,\infty), L^\infty(\mathbb{R}^2))$. In the three dimensional case (and in the case of small initial data), the fact that (9), (11) can be uniquely solved in $C([0,\infty[, L^\infty_2(\mathbb{R}^3))$ is easily seen, see [6].

Note that the decay profiles of ω and Ω are the same which can be obtained for the solutions of the homogeneous heat equations $e^{t\Delta}\omega_0(x)$ and $e^{t\Delta}\Omega_0(x)$, respectively. We refer to [6] for a proof of (15) (the proof of (13) is identical). The decay of the vorticity in the L^p-norm are formally a consequence of (13) and (15), respectively if $d = 2$ or 3. Estimates (14) and (16), however, can be proved with straightforward adaptations of the arguments of [12], [6], [17], or [18] (see also [9]).

We would like to stress the fact that, since the moments of the vorticity are *not invariant* during the time evolution (excepted for the integral and the first order moments), profiles (13) and (15) will hold true only for $n = 0, 1, 2$ even if the vorticity has many cancelations at time $t = 0$. This reflects the fact that, in general, the velocity field does not decay faster than $|x|^{-(d+1)}$. We will show, however, that for the special flows described in Theorem 1.1 and Theorem 1.2 the vorticity has a large number of vanishing moments for all time (in particular, we shall be able to prove that the assumptions of the second part of Proposition 2.2 are non-vacuous for $n = 0, \dots, 6$).

We will finish this section stating another simple result which allows us to deduce time decay estimates for the velocity field from (13), (15).

Lemma 2.4 *Let $\omega(x,t)$ ($d = 2$), or $\Omega(x,t)$ ($d = 3$), as in Proposition 2.2, and let $u(x,t)$ be the corresponding velocity field obtained via the Biot–Savart law* (11) *(resp.* (12)*). Then we have,*

$$||u(t)||_p \le C(1+t)^{-(n+1)/2+1/p}, \qquad 2 < p \le \infty \qquad (d = 2), \tag{17}$$

or

$$||u(t)||_p \le C(1+t)^{-(n+2)/2+3/(2p)}, \qquad 3/2 < p \le \infty \qquad (d = 3) \tag{18}$$

and the spatial moments of $u(x,t)$ exist and vanish up to the order $n-2$.

Proof. Note that the Biot–Savart kernels $x^\perp/|x|^2$ ($d = 2$) and $x/|x|^3$ ($d = 3$) belong respectively to the the weak-Lebesgue spaces $L^{2,\infty}(\mathbb{R}^2)$ and $L^{3/2,\infty}(\mathbb{R}^3)$. Bounds (17) and (18) then immediately follow from the Biot–Savart laws (11) and (12), the corresponding bounds for the vorticity and elementary results on convolution and interpolation of Lorentz spaces (see [3]). The condition on the moments of u is easily seen by taking the Fourier transform in (11) and (12) and applying the Taylor formula (see *e.g.* [6] and [18] for this type of calculations). •

3 Space-time decay of two-dimensional flows

Let us recall that all finite subgroups of the orthogonal group $O(2)$, are of two types: cyclic groups (which are indeed subgroups of the special orthogonal group $SO(2)$ of proper rotations), and dihedral groups. We shall denote by C_n the cyclic group of order n and by D_n the dihedral group of order $2n$. This group contains C_n and its presentation is given by two generators R and τ, together with the relations $R^n = \mathbf{1}$, $\tau^2 = \mathbf{1}$ and $\tau R = R^{-1}\tau$ (R corresponds to a rotation of $2\pi/n$ around the origin and τ to a reflection with respect to a straight line passing through the origin).

Divergence-free vector fields, which are rapidly decreasing as $|x| \to \infty$ and which are left invariant under the actions of C_n or D_n are easily constructed by means of the vorticity. Indeed, in general, if $P \in O(2)$, $u(Px) = Pu(x)$ for all $x \in \mathbb{R}^2$ and $\omega = \partial_1 u_2 - \partial_2 u_1$, then $\omega(x) = \det(P)\omega(Px)$ (in the distributional sense). Conversely, if $\omega(x) = \det(P)\omega(Px)$ for all x and u is given by the Biot–Savart law, then $u(Px) = Pu(x)$, whenever the singular integral (11) makes sense.

Assume now that the initial datum $a = u(0)$ is rapidly decreasing as $|x| \to \infty$ and that it is *invariant under the cyclic group of order* n ($n \geq 3$). Since we know that strong solutions of the two dimensional Navier–Stokes equations are globally defined, we have $u(Rx, t) = Ru(x, t)$, for all $x \in \mathbb{R}^2$ and $t \geq 0$. We now apply to u Proposition 2.1: let us show that the polynomial

$$P_m(u)(\xi) \equiv \sum_{h,k=1}^{2} \sum_{|\alpha|=m} \left(\frac{1}{\alpha!} \int x^\alpha (u_h u_k)(x,t)\, dx \right) \xi^\alpha \xi_h \xi_k, \tag{19}$$

is divisible by $\xi_1^2 + \xi_2^2$, for all $t \geq 0$ and the first values of m.

But this is easily checked, since $P_m(u)(\xi) = P_m(u)(R\xi)$ for all $\xi \in \mathbb{R}^2$. Passing to polar cordinates, we write $P_m(u)(\xi_1, \xi_2) \equiv \tilde{P}_m(\rho, \theta)$, with $\xi_1 = \rho\cos\theta$ and $\xi_2 = \rho\sin\theta$ and observe that for each fixed $\rho > 0$, the trigonometric polynomial $\tilde{P}_m(\rho, \theta)$ has degree smaller or equal than $m + 2$ and period $2\pi/n$. If $m \leq n - 3$, then it follows that $\tilde{P}_m(\rho, \theta) \equiv \tilde{P}_m(\rho, 0)$ for all θ, *i.e.* $P_m(u)(\xi)$ is radial. This implies that $P_m(u)(\xi)$ identically vanishes for odd m and, for even m, that $P_m(u)(\xi)$ has the form $c_m(t)(\xi_1^2 + \xi_2^2)^{(m+2)/2}$ for some constant $c_m(t)$: in any case, $P_m(u)(\xi)$ is divisible by $\xi_1^2 + \xi_2^2$ for $m = 0, \ldots, n - 3$.

If $n \geq 3$, then Proposition 2.1 applies with $M = n - 3$ and we deduce that, for all $T \geq 0$, $u \in C([0, T], L^\infty_{n+1}(\mathbb{R}^2))$.

Solutions that are invariant just under the group C_n, in general, decay slowly as $t \to \infty$. To obtain solutions with large decay rates in space-time we will put more symmetries on the data and make use of the vorticity formulation. Assume now that *the flow is invariant under the dihedral group* D_n ($n \geq 3$) at the beginning of the evolution: $a(Rx) = Ra(x)$, $a(\tau x) = \tau a(x)$, and that $\omega_0 = \partial_1 a_2 - \partial_2 a_1$ is rapidly decreasing as $|x| \to \infty$. If $\omega(x, t)$ is the unique solution to the two-dimensional vorticity equation starting from ω_0, then $\omega(x, t) = \omega(Rx, t) = -\omega(\tau x, t)$, for all $x \in \mathbb{R}^2$ and $t \geq 0$ and, by Proposition 2.2, $\omega(x, t)$ is rapidly decreasing as $|x| \to \infty$ for all fixed $t \geq 0$. It now remains to compute the number of vanishing moments of $\omega(x, t)$:

Lemma 3.1 *If $\omega(x, t)$ is as above, then the moments $\int x^\alpha \omega(x, t)\, dx$ vanish for all $t \geq 0$, and all double-index $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha| = \alpha_1 + \alpha_2 \leq n - 1$. Moreover, $\int x^\alpha u(x, t)\, dx = 0$ for all α such that $|\alpha| \leq n - 2$ and all $t \geq 0$.*

Proof. We just have to establish the property for ω (see the last conclusion of Lemma 2.4). Taking the Fourier transform in the space variables, we see that $\widehat{\omega}(\xi, t) \in C^\infty(\mathbb{R}^2)$ for all $t \geq 0$ and $\widehat{\omega}(\xi, t) = \widehat{\omega}(R\xi, t)$, $\widehat{\omega}(\xi, t) = -\widehat{\omega}(\tau\xi, t)$. In particular, $\widehat{\omega}(\xi, t)$ identically vanishes on n different straight lines passing through $\xi = 0$. The Taylor formula then implies that $\partial_\xi^\alpha \widehat{\omega}(0, t) = 0$ for all α such that $|\alpha| \leq n - 1$ and Lemma 3.1 follows. •

We can summarize the results of this section in the following theorem, which sharpen the conclusion of Theorem 1.1.

Theorem 3.2 *Let $a = (a_1, a_2)$ be a rapidly decreasing and divergence-free vector field in $\mathbb{R}^2$. If a is invariant under the cyclic group C_n (n=3,4...) then the strong solution $u(x,t)$ to (NS) such that $u(0) = a$ satisfies $u(x,t) = O(|x|^{-(n+1)})$ as $|x| \to \infty$ for all $t \geq 0$.*

If, in addition, a is invariant under the dihedral group D_n and $\omega_0 = \partial_1 a_2 - \partial_2 a_1$ is also rapidly decreasing in $\mathbb{R}^2$, then the moments of the vorticity $\omega(x,t)$ of the flow vanish up to the order $n-1$ for all $t \geq 0$, $\sup_{t\geq 0} |u(x,t)| \leq C(1+|x|)^{-(n+1)}$ and (13)-(14) *and* (17) *hold true.*

In the second part of Theorem 3.2, the fact that the decay as $|x| \to \infty$ of $u(x,t)$ is uniform in $[0, \infty[$ follows from (6) and the cancelations $\int x^\alpha a(x)\,dx = 0$, (with $0 \leq |\alpha| \leq n-2$).

Remark 3.3 In the case $n = 4$, the symmetries described in the second part of Theorem 3.2, are the same as those studied in [4]: indeed the fact that the flow is invariant under the dihedral group D_4, can be written as follows: $u_1(x_1, x_2, t) = -u_1(-x_1, x_2, t)$, $u_1(x_1, x_2, t) = u_1(x_1, -x_2, t)$ and $u_1(x_1, x_2, t) = u_2(x_2, x_1, t)$, which are exactly the conditions of [4] in the two dimensional case.

4 The three dimensional case

We now study the class of flows which are invariant under finite subgroups of the group of all the isometries of the space. We shall identify two of such groups G and G' if they are conjugate in $O(3)$ (*i.e.* $G \sim G'$ if there exists an orthogonal matrix T such that $G' = TGT^{-1}$). Note that two flows that that are invariant under groups which are isomorphic, but not conjugate, may behave quite differently and this is why will not identify groups which are simply isomorphic in what follows.

4.1 Finite subgroups of $O(3)$

Finite groups of proper rotations The material of this section is very classical, but we present it to fix some notations. We start recalling the well known classification of all finite subgroups of the special orthogonal group $SO(3)$. We closely follow the presentation given in [24]. If $\mathcal{S}$ is any subset of $\mathbb{R}^3$, the group $G(\mathcal{S})$ of all $P \in SO(3)$ such that P leaves $\mathcal{S}$ globally invariant is called *the complete direct symmetry group of $\mathcal{S}$*. For different choices of $\mathcal{S}$ we obtain in this way only five different types of groups that are listed below: For each group we shall indicate a set of matrices generating $G(\mathcal{S})$ since we will need these generators in our subsequent calculations.

1. If $\mathcal{S}$ is a *n-pyramid*, $n = 1, 2, \ldots$ (*i.e.* a right piramid with base a n-sided regular polygon such that the distance from the vertex of the pyramid to a vertex of the base is not equal to one side of the polygon, with an obvious modification if $n = 1, 2$), then $G(\mathcal{S})$ is the cyclic group C_n of order n. A generator of this group is *e.g.*
$$R_n = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) & 0 \\ \sin(2\pi/n) & \cos(2\pi/n) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

2. If $\mathcal{S}$ is a *n-prism*, $n = 2, 3, \ldots$ (*i.e.* a right cylinder with base a n-sided regular polygon and height not equal to one side of the polygon, modification if $n = 2$), then $G(\mathcal{S})$ is the dihedral group D_n of order $2n$. This group is generated by R_n and by
$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{21}$$

3. If $\mathcal{S}$ is a *tetrahedron*, then $G(\mathcal{S}) = \mathbf{T}$ (the tetrahedral group). This group has order 12, is isomophic to the alterning group A_4 and it is generated by a rotation by $2\pi/3$ around an axis passing through a vertex and the center of $\mathcal{S}$ and by a rotation by π around an axis passing through the midpoints of two opposite edges. If $(-1, -1, -1)$, $(1, 1, -1)$, $(-1, 1, 1)$ and $(1, -1, -1)$ are the vertices of $\mathcal{S}$, then we see that two generators of $\mathbf{T}$ are U and
$$S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{22}$$

4. If $\mathcal{S}$ is a *cube (or an octahedron)* then $G(\mathcal{S}) = \mathbf{O}$ (the octahedral group). This group has order 24 and is isomorphic to the symmetric group S_4. If $(\epsilon_1, \epsilon_2, \epsilon_3)$, ($\epsilon_j = 1$ or -1, $j = 1, 2, 3$) are the vertices of the cube, then we see that $\mathbf{O}$ is generated by U, S and a rotation V by π around an axis passing through the midpoints of two opposite edges of the cube. We may choose
$$V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{23}$$
We finally observe that $\mathbf{T}$ is a subgroup of index 2 in $\mathbf{O}$.

5. If $\mathcal{S}$ is an *icosahedron (or a dodecahedron)* then $G(\mathcal{S}) = \mathbf{Y}$ (the icosahedral group). This group has order 60, is isomorphic to the alterning group A_5, it is generated by a rotation by $2\pi/5$ around an axis passing through two opposite vertices of the icosahedron and a rotation by $2\pi/3$ around an axis passing through the center of two opposite faces.

If $(\pm\phi, 0, \pm 1)$, $(0, \pm 1, \pm\phi)$ and $(\pm 1, \pm\phi, 0)$ are the 12 vertices of $\mathcal{S}$ (here $\phi = (\sqrt{5}-1)/2$ is the gold number) then we see that $\mathbf{Y}$ is generated by S and

$$J = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{\sqrt{5}-1}{4} & -\frac{1}{2} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} & \frac{\sqrt{5}+1}{4} \end{pmatrix}. \tag{24}$$

It also easily seen that $\mathbf{Y}$ contains $\mathbf{T}$. Indeed, the transfomation U corresponds now to a rotation by π around an axis passing through the midpoints of to opposite edges of the icosahedron.

Remark 4.1 A classical result states that if G is a finite subgroup of $SO(3)$ then G is conjugate to one the preceding five groups. For more details on those groups we refer *e.g.* to [24].

Other finite groups of isometries. It $\mathcal{S}$ is a subset of $\mathbb{R}^3$, then *the complete symmetry group of* $\mathcal{S}$ si defined as the group of all orthogonal transformations which leave $\mathcal{S}$ globally invariant. Let us recall that if G is a finite subgroup of $O(3)\backslash SO(3)$, such that the inversion I (the symmetry with respect to the origin) belongs to G, then $G = G_1 \cup IG_1$, where $G_1 = G \cap SO(3)$ is one of the five groups of proper rotation considered in the preceding paragraph. In this case G is obtained as direct product of G_1 and a cyclic group of order 2, and the generators of G are the same of G_1, together with

$$I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{25}$$

On the other hand, if G is a finite subgroup of $O(3)\backslash SO(3)$, but I does not belong to G, then $G^+ \equiv G_1 \cup \{Ig \,:\, g \in G\backslash G_1\}$ is a finite subgroup of $SO(3)$, containing G_1 as a subgroup of index 2. Further, G^+ is isomorphic (but not conjugate) to G. The group G is usually denoted by G^+G_1 in the literature of point groups. We thus can form four more types of group in this way, namely $C_{2n}C_n$, D_nC_n, $D_{2n}D_n$ and $\mathbf{OT}$.

We now follow the classical classification of Schöenflies (see also [24], [33]), starting with the complete symmetry groups of suitably modified prisms.

1. We lump togheter the groups $C_n \cup IC_n$ for odd n with the groups $C_{2n}C_n$ for even n, to form *the cyclic group* $\mathbf{S}_{2n}$ of order $2n$ (the complete symmetry group of an *alternating* $2q$*-prism*, see [33] for a plot). This group is generated by a rotation-inversion by π/n (a rotation of π/n around an axis followed by a reflection with respect to a plane perpendicular to the axis):

$$\tilde{R}_{n/2} = \begin{pmatrix} \cos(\pi/n) & -\sin(\pi/n) & 0 \\ \sin(\pi/n) & \cos(\pi/n) & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{26}$$

2. Lumping together the groups $C_n \cup IC_n$ for even n with the groups $C_{2n}C_n$ for odd n, we form an *abelian group* of order $2n$, denoted by $\mathbf{C}_{nh}$ (the complete symmetry

group of a *shaved q-prism*). This group is generated by a rotation-inversion $\tilde{R}_n$ by $2\pi/n$ and a rotation R_n by $2\pi/n$ around the same axis. Note that $\mathbf{C}_{nh}$ turns out to be a cyclic group if n is odd, but this group is not conjugate to $\mathbf{S}_{2n}$.

3. The group D_nC_n has order $2n$ and is usually denoted by $\mathbf{C}_{nv}$. This is the complete symmetry group of a n-pyramid and is formed by n rotations by multiples of $2\pi/n$ around the axis of the pyramid and n reflections in n vertical planes passing through this axis. A system of generators of $\mathbf{C}_{nv}$ is R_n and

$$W_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (27)$$

4. Combining $D_{2n}D_n$ for odd n with $D_n \cup ID_n$ for even n forms the complete group of symmetry of a n-prism, which is denoted by $\mathbf{D}_{nh}$. This group has order $4n$, and it is generated by R_n, W_2 and by

$$W_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \qquad (28)$$

5. Combining $D_{2n}D_n$ for even n with $D_n \cup ID_n$ for odd n forms the group $\mathbf{D}_{nd}$, which is the complete symmetry group of a *twisted n-prism* (the solid obtained pasting two n-prisms at their basis, in a such way that the prisms are rotated by π/n). This group has order $4n$ and is generated by $\tilde{R}_{n/2}$ and W_2.
6. The group $\mathbf{T} \cup I\mathbf{T}$ is denoted by $\mathbf{T}_h$. This group has order 24, is isomorphic to $A_4 \times \mathbb{Z}/2\mathbb{Z}$ and is generated by S, U and I (or simply by S and W_2).[1] The group $\mathbf{T}_h$ corresponds to the complete symmetry group of a solid obtained from a cube shaving off the eight vertices (this solid is often called *modified cube*, see [33]).
7. The group $\mathbf{OT}$ is denoted by $\mathbf{T}_d$. This is the complete symmetry group of a tetrahedron, it has order 24, is isomorphic to $\mathbf{O}$ (hence to S_4), but $\mathbf{T}_d$ and $\mathbf{O}$ are not conjugate. This group is generated by the two generators S and U of $\mathbf{T}$, together with a reflection Z with respect to a plane passing through the midpoint of an edge and containing the opposite edge of the tetrahedron:

$$Z = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (29)$$

8. The group $\mathbf{O} \cup I\mathbf{O}$, denoted by $\mathbf{O}_h$, is the complete groupe of symmetry of a cube (and of an octahedron).[2] This group is isomorphic to $S_4 \times \mathbb{Z}/2\mathbb{Z}$ and contains the 48 orthogonal matrices formed by 0, 1 and -1. A system of generators for $\mathbf{O}_h$ is *e.g.* S, V and I.

[1] The "symmetric solutions" $u(x,t)$ introduced in [4] are precisely the flows which are invariant under the group $\mathbf{T}_h$. These solutions have been later considered in [13], [18], [26], [27] and [5], but the connection with the group $\mathbf{T}_h$ does not seem to have been noticed.

[2] It was pointed out by Kida [22] that it is possible to construct solutions that are both invariant under $\mathbf{O}_h$ and 2π-periodic in any direction.

9. The group $\mathbf{Y}\cup I\mathbf{Y}$ is denoted by $\mathbf{Y}_h$, and it is the complete groupe of symmetry of an icosahedron (and of a dodecahedron). This group has order 120, is isomorphic to $A_5 \times \mathbb{Z}/2\mathbb{Z}$ and is generated by S, J and I (or by S, J and W_2).

4.2 Application to the Navier–Stokes equations

Space decay This paragraph is devoted to the computation of the space decay rates of flows $u(x,t)$ which are invariant under a discrete subgroup of $O(3)$ and such that $u(x,0)$ is localized. We will not consider here all the possible groups G listed in the preceding section, but we will just treat the case in which G is either $\mathbf{T}$, $\mathbf{T}_h$, $\mathbf{O}$ or $\mathbf{Y}$. Combining the results of this paragraph with the examples of section 5, however, will immediately give the optimal space decay rates for *all* groups.

We need the following lemma.

Lemma 4.2 *Let $P_m(\xi)$, where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, be a homogenous polynomial of degree $m+2$ $(m = 0, 1, \ldots)$.*

1. *If P_0 is invariant under the transformations of the tetrahedral group $\mathbf{T}$, then $P_0(\xi) \equiv c_0(\xi_1^2 + \xi_2^2 + \xi_3^2)$.*
2. *If P_1 is invariant under the transformations of either $\mathbf{O}$, $\mathbf{T}_h$ or $\mathbf{Y}$, then $P_1(\xi) \equiv 0$.*
3. *If P_2 and P_3 are invariant under $\mathbf{Y}$, then $P_2(\xi) \equiv c_2(\xi_1^2+\xi_2^2+\xi_3^2)^2$ and $P_3(\xi) \equiv 0$.*

Proof. The proof follows by imposing $P_m(\xi) \equiv P_m(Q\xi)$ where, in the first case $Q = S, U$; in the second case we take, respectively, $Q = S, U, V$, $Q = S, W_2$, or $Q = S, J$; in the third case we choose $Q = S, J$. We thus obtain linear systems where the unknowns are the coefficients of $P_m(\xi)$. Conclusion of Lemma 4.2 then immediately follows from lengthy but elementary calculations. •

Applying Proposition 2.1 we immediately get the following

Corollary 4.3 *Let $a = (a_1, a_2, a_3)$ be a soleinoidal and rapidly decreasing vector field in $\mathbb{R}^3$ and $u(x,t)$ the strong solution to (NS), which is defined in some time interval $[0,T]$ $(T > 0)$, such that $u(0) = a$. If a is invariant under the transformations of $\mathbf{T}$, then $u(x,t) = O(|x|^{-5})$ as $x \to \infty$ uniformely in $t \in [0,T]$. Such decay rate is improved up to $u(x,t) = O(|x|^{-6})$, if a is invariant under either $\mathbf{O}$ or $\mathbf{T}_h$, and up to $u(x,t) = O(|x|^{-8})$ if a is invariant under $\mathbf{Y}$.*

Remark 4.4 Note that the homogeneous polynomial $P_1(\xi) \equiv \xi_1\xi_2\xi_3$ satisfies $P_1(\xi) = P_1(S\xi) = P_1(U\xi)$ for all ξ. This polynomial is then invariant under $\mathbf{T}$, but it is not divisible by $\xi_1^2 + \xi_2^2 + \xi_3^2$. On the other hand, the polynomial $P_2(\xi) \equiv \xi_1^4 + \xi_2^4 + \xi_3^4$ is invariant under both $\mathbf{O}$ and $\mathbf{T}_h$, but it is not divisible by $\xi_1^2+\xi_2^2+\xi_3^2$. At the same way, it is not difficult to construct a homogeneous polynomial of degree 6, which is invariant under $\mathbf{Y}$ and is not divisible by $\xi_1^2+\xi_2^2+\xi_3^2$ (see the polynomial $Q(\xi)$ in the last section).

These considerations show that the decay rates computed in Corollary 4.3 seem to be optimal for generic flows invariant under one of the four preceding groups. We will see by means of the examples of section 5 that (a) the decay $|x|^{-5}$ is indeed optimal, in general, for flows which are invariant under the group $\mathbf{T}_d$, which contains $\mathbf{T}$, (b) the decay $|x|^{-6}$ is optimal inside the group $\mathbf{O}_h$ which contains both $\mathbf{O}$ and $\mathbf{T}_h$, (c) the decay $|x|^{-8}$ is optimal inside the group $\mathbf{Y}_h$ which contains $\mathbf{Y}$. Finally, we will see that flows which are invariant under the complete group of symmetries D_{2nh} of a $2n$-prism (which contains all the other groups C_n, S_{2n}, D_n, C_{nh}, C_{nv}, D_{nd} and D_{nh}), in general, do not decay faster than $|x|^{-4}$. This provides a complete answer to the space decay problem of flows with this kind of symmetries.

Time decay We now compute the time decay rate of flows invariant under the complete symmetry group of the solids listed in the preceding section. We will give detailed arguments only for the group of the icosahedron $\mathbf{Y}_h$, since this group provides the largest decay rates. Of course similar (but simpler!) considerations can be repeated for the other groups.

As in the two-dimensional case, we shall make use of the vorticity formulation. We start observing that requiring the condition $P\,u(x,t) \equiv u(P\,x,t)$, for a given $P \in O(3)$, is equivalent, at least when the singular integral (12) makes sense, to requiring that

$$P\,\Omega(x,t) = \det(P)\,\Omega(P\,x,t) \qquad (30)$$

for all $x \in \mathbb{R}^n$ and $t \geq 0$. A simple way to prove the equivalence between (1) and (30) is to use the Fourier transform and the identity $(Pv) \times (Pw) = \det(P)P(v \times w)$, which holds for all $v, w \in \mathbb{R}^3$ and $P \in O(3)$.

From now on we shall assume that $a(x)$ is invariant under the group $\mathbf{Y}_h$ and that the initial vorticity $\Omega_0 = \nabla \times a$ is a rapidly decreasing vector field as $|x| \to \infty$. Then we have the following.

Lemma 4.5 *Let Ω be a rapidly decreasing vector field in $\mathbb{R}^3$, such that $P\Omega(x) \equiv \det(P)\Omega(Px)$ for all transformations P belonging to the complete symmetry group of the icosahedron. Then the moments of Ω vanish up to the order* 5.

Proof. Let us denote by $P_j \in \mathbf{Y}_h \subset O(3)$ the reflection with respect to a plane π_j of symmetry of the icosahedron ($j = 1, \ldots, 15$). Note that each of the six axes passing through two opposite vertices of the icosahedron belongs exactly to five distinct planes. Let $\mathbf{e}_1, \ldots, \mathbf{e}_6$ be six unit vectors corresponding to these axes and such that $\frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2) = (1,0,0)$, $\frac{1}{\sqrt{2}}(\mathbf{e}_3 + \mathbf{e}_4) = (0,1,0)$ and $\frac{1}{\sqrt{2}}(\mathbf{e}_5 + \mathbf{e}_6) = (0,0,1)$ (this is possible if we choose the vertices of the icosahedron as in the preceding section). If we show that $\int x^\alpha \langle \Omega(x), \mathbf{e}_k \rangle = 0$ for $k = 1, \ldots, 6$, and some $\alpha \in \mathbb{N}^3$ (where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^3$), then it will follow that $\int x^\alpha \Omega(x)\,dx = 0$.

Condition $P_j\Omega(x) \equiv -\Omega(P_jx)$ implies that $\Omega(x)$ is orthogonal to π_j, for all $x \in \pi_j$. In the same way, passing to the Fourier transform we see that $\widehat{\Omega}(\xi)$ is orthogonal to π_j for all $\xi \in \pi_j$. In particular, for each $k = 1, \ldots, 6$ there exist five planes containing the axis generated by $\mathbf{e}_k$, on which the function $f_k(\xi) \equiv \langle \widehat{\Omega}(\xi), \mathbf{e}_k \rangle$ identically vanish.

Now we use the general fact that if $g(\xi) \in C^\infty(\mathbb{R}^3)$ identically vanishes on n distinct planes passing through a given axis, then g has vanishing derivatives on this axis up to the order $n-1$ (this simple fact can be seen using the same argument as in Lemma 3.1).

Since f_k is a smooth function, it follows that the derivatives of f_k identically vanish up to the order 4 on the k-th axis. This shows that the moments of Ω vanish up to the order 4.

But the group $\mathbf{Y}_h$ contains the three reflections with respect to the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ (assuming that the icosahedron is orientated as above). Therefore $\Omega_i(x_1, x_2, x_3)$ is an even function with respect to x_i and an odd function with respect to x_h ($i, h = 1, 2, 3$ and $i \neq h$). It then follows that for any $\alpha \in \mathbb{N}^3$, such that $\alpha_1 + \alpha_2 + \alpha_3$ is an odd integer, $\int x^\alpha \Omega_i(x)\, dx = 0$ ($i = 1, 2, 3$). Lemma 4.5 is thus proved. •

Combining this result with Proposition 2.1 and Lemma 2.4 implies the following:

Corollary 4.6 *Let Ω_0 be a divergence-free and rapidly decreasing vector field, such that $P\Omega_0(x) = \det(P)\Omega_0(Px)$ for all $P \in \mathbf{Y}_h$. If $\sup_x |x|^2|\Omega_0(x)|$ is small, then the solution $\Omega(x,t)$ of Proposition 2.2 satisfies* (15)-(16) *with $n = 6$. Furthermore, the corresponding velocity field belongs to $C([0, +\infty[, L^\infty_8(\mathbb{R}^3))$ and satisfies* (18) *(with $n = 6$).*

These arguments apply also to the simpler case of flows invariant to complete symmetry group of the tetrahedron and the complete symmetry group of the cube (and, with slight modification, to their subgroups). This yields *e.g.* that (18) holds true with $n = 3$ in the case of flows invariant under $\mathbf{T}_d$, and with $n = 4$ in the case of flows invariant under $\mathbf{O}_h$. We leave the corresponding computations to the reader. Theorem 1.2 then follows.

5 Examples of localized flows

Here we provide explicit examples of initial data leading to flows invariant under the groups considered in the preceding section. These examples also show that the space decay rates previously computed are sharp. The proof of the optimality is based on the following fact (see [7]). Let $a(x)$ be a rapidly decreasing divergence-free vector field in $\mathbb{R}^d$ ($d \geq 2$) and $u(x,t)$ the strong solution to (NS) starting from a defined in a time interval $[0, T]$ ($T > 0$). If the homogeneous polynomial

$$P_m(a)(\xi) \equiv \sum_{h,k=1}^{d} \sum_{|\alpha|=m} \left(\frac{1}{\alpha!} \int x^\alpha (a_h a_k)(x)\, dx \right) \xi^\alpha \xi_h \xi_k, \qquad (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \tag{31}$$

is not divisible by $\xi_1^2 + \ldots + \xi_d^2$, then there exists a decreasing sequence $t_k \to 0$ such that

$$\limsup_{R\to\infty} \left(R^{1+m} \int_{R \leq |x| \leq 2R} |u(x, t_k)|\, dx \right) > 0 \qquad \text{for all } k = 1, 2, \ldots \tag{32}$$

Condition (32) implies that u cannot decay faster than $|x|^{-(d+1+m)}$ uniformely in $[0, T]$.

From now on we shall assume $d = 3$. To give examples of soleinodal vector fields a which are invariant under a given orthogonal transformation P, it will be often convenient first to construct a *potential vector* field b, such that $b(Px) = \det(P)Pb(x)$ and then to set $a = \nabla \times b$.

Let G be a finite subgroup of $O(3)$ which is not of polyhedral type (*i.e.* G does not contain $\mathbf{T}$). If G has order n then it is contained in either D_{nd} or in D_{nh}. These two groups are in turn both contained in the complete symmetry group of a $2n$-prism D_{2nh}, which has order $8n$. Let us show that generic flows which are invariant under D_{2nh} do not decay faster than $|x|^{-4}$. This is immediate: we can take *e.g.* a vector field of the form

$$a(x) = (-\partial_2\mu(x), \partial_1\mu(x), 0), \tag{33}$$

where $\mu \in \mathcal{S}(\mathbb{R}^3)$ is a non-trivial function such that $\mu(x_1, x_2, x_3) = -\mu(x_1, -x_2, x_3) = -\mu(x_1, x_2, -x_3)$ and μ is invariant under a rotation of π/n around the vertical axis. Then a is invariant under D_{2nh}, but $\int a_1^2(x)\,dx \neq \int a_3^2(x)\,dx$, hence (32) holds true with $m = 0$. A slight modification of the choice of μ would show, in the same way, that flows invariant under the complete group of direct symmetry of the cylinder do not decay faster than $|x|^{-4}$, in general.

A very simple example of a vector field which is invariant under the complete symmetry group $\mathbf{T}_d$ of a tetrahedron is obtained choosing *e.g.* the potential vector $\bar{b}_1(x_1, x_2, x_3) = x_1(x_2^2 - x_3^2)e^{-|x|^2}$, $\bar{b}_2(x_1, x_2, x_3) = \bar{b}_1(x_2, x_3, x_1)$ and $\bar{b}_3(x_1, x_2, x_3) = \bar{b}_1(x_3, x_1, x_2)$. Another possible simple choice for the first component of the potential vector would be $\tilde{b}_1(x_1, x_2, x_3) = x_2x_3(x_2^2 - x_3^2)e^{-|x|^2}$ These two choices give, respectively,

$$\bar{a}(x) = \begin{pmatrix} -2x_2x_3(2 + 2x_1^2 - x_2^2 - x_3^2)e^{-|x|^2} \\ -2x_3x_1(2 + 2x_2^2 - x_3^2 - x_1^2)e^{-|x|^2} \\ -2x_1x_2(2 + 2x_3^2 - x_1^2 - x_2^2)e^{-|x|^2} \end{pmatrix} \tag{34}$$

and

$$\tilde{a}(x) = \begin{pmatrix} x_1(2x_1^2 - 3x_2^2 - 2x_1^2x_2^2 + 2x_2^4 - 3x_3^2 + 2x_3^4 - 2x_1^2x_3^2)e^{-|x|^2} \\ x_2(2x_2^2 - 3x_3^2 - 2x_2^2x_3^2 + 2x_3^4 - 3x_1^2 + 2x_1^4 - 2x_2^2x_1^2)e^{-|x|^2} \\ x_3(2x_3^2 - 3x_1^2 - 2x_3^2x_1^2 + 2x_1^4 - 3x_2^2 + 2x_2^4 - 2x_3^2x_2^2)e^{-|x|^2} \end{pmatrix} \tag{35}$$

Accordingly with Lemma 3.1, the two polynomials $P_0(\bar{a})(\xi)$ and $P_0(\tilde{a})(\xi)$ are divisible by $\xi_1^2 + \xi_2^2 + \xi_3^3$. Hence the solutions $\bar{u}$ and $\tilde{u}$ starting from $\bar{a}$ and $\tilde{a}$ decay at least as fast as $|x|^{-5}$, at the beginnning of their evolution. However, a direct calculation shows that both $P_1(\bar{a})(\xi)$ and $P_1(\tilde{a})(\xi)$ identically vanish. This means that $\bar{u}$ and $\tilde{u}$ may decay faster than expected for generic flows invariant under the group $\mathbf{T}_d$. However, one easily checks that $P_1(\bar{a} + \tilde{a})(\xi) \equiv c\xi_1\xi_2\xi_3$ for some constant $c \neq 0$. Hence, the flow starting from $(\bar{a} + \tilde{a})(x)$, which is also invariant under $\mathbf{T}_d$ (and, in particular, under $\mathbf{T}$), cannot decay faster than $|x|^{-5}$. This decay rate is thus sharp, in general, for the groups $\mathbf{T}$ and $\mathbf{T}_d$.

Note that the field $\tilde{a}(x)$ turns out to be invariant under the transformations of the larger group $\mathbf{O}_h$. Now, it is not difficult to check that the homogeneous polynomial $P_2(\tilde{a})(\xi)$ is not divisible by $\xi_1^2 + \xi_2^2 + \xi_3^2$. Indeed, a necessary condition on the coefficients of $P_2(\tilde{a})(\xi)$, to obtain a polynomial which is divisible by $\xi_1^2 + \xi_2^2 + \xi_3^2$, would be $4\int x_1x_2\tilde{a}_1\tilde{a}_2(x)\,dx = \int (x_1^2 - x_2^2)(\tilde{a}_1^2 - \tilde{a}_2^2)(x)\,dx$. But the left hand side equals $\frac{57}{512}\pi^{3/2}\sqrt{2}$ and the right-hand side equals $\frac{15}{64}\pi^{3/2}\sqrt{2}$. Then, the decay rate $|x|^{-6}$ is optimal for generic flows invariant under the complete symmetry group of a cube (hence, also for the groups $\mathbf{O}$ and $\mathbf{T}_h$ which are both contained in $\mathbf{O}_h$).

The icosahedron groups. Examples of localized and soleinoidal vector fields which are invariant under the groups $\mathbf{Y}$ and $\mathbf{Y}_h$ are slightly more difficult to obtain. We propose here two different methods for their construction.

The first method is based on some simple geometric considerations: the fiveteen planes of symmetry of the icosahedron divide $\mathbb{R}^3$ into 120 congruent pyramidal regions (each of them is the convex hull of three half straight lines arising from the origin). If Γ is one of these regions, it is then sufficient to construct a vector field a which is localized and divergence-free in Γ, such that $a(x) = 0$ for all x belonging to the three half-straight lines and $\langle a(x), \mathbf{n}\rangle = 0$ on the boundary of Γ ($\mathbf{n}$ denotes here the exterior normal). The extension of such field by subsequent reflections with respect to the fiveteen planes is then invariant under the transformations of $\mathbf{Y}_h$ (we use here the well-known fact that every orthogonal transformation in $\mathbb{R}^3$ can be obtained by subsequent plane reflections). Note that, because of (1), the condition on the boundary of Γ is conserved by the Navier–Stokes evolution: this means that the fluid particles will remain in the same region Γ for all time.

The second method is based on elementary linear algebra: we start with the construction of a potential vector field $b = (b_1, b_2, b_3) \not\equiv 0$ such that $Sb(x) = b(Sx)$, $b(W_2 x) = -W_2 b(x)$ and $Jb(x) = b(Jx)$ (with the notations of section 4.1). This is possible, as it can be checked *e.g.* by imposing that each components of b is a homogeneous polynomial of degree larger or equal than six and then solving the corresponding linear system on the coefficients. Next we modify the definition of b multiplying its components by a fixed radial function in the Schwartz class. Such operation, of course, does not affect the three previous identities.

We would like to give an explicit example, since highly symmetric flows are useful in numerical simulations of turbulence (see Kida's papers [22], [23] for some results in this direction, at least in the periodic case). Let us take

$$\begin{aligned} b_1(x_1,x_2,x_3) = x_2 x_3 e^{-|x|^2}\big(&-x_1^4(5+\sqrt{5}) + x_2^4(3+\sqrt{5}) + 2x_3^4 \\ &-4\sqrt{5}x_1^2x_2^2 - (10+2\sqrt{5})x_2^2x_3^2 + (10+6\sqrt{5})x_1^2x_3^2\big). \end{aligned}$$

The two other components of b are then uniquely defined. The corresponding velocity field is the vector field $a = (a_1, a_2, a_3)$ such that

$$\begin{aligned} a_1(x) = x_1 e^{-|x|^2}\Big(& 6\,{x_3}^4\sqrt{5} - 15\,{x_2}^4 + 20\,{x_3}^4 - 6\,{x_3}^6 - {x_1}^4 + 4\,{x_2}^6 \\ &+30\,{x_2}^4{x_3}^2 - 10\,{x_2}^2{x_3}^4 - 2\,{x_3}^6\sqrt{5} - 20\,{x_1}^2{x_2}^4 + 6\,{x_1}^4{x_2}^2 \\ &+20\,{x_3}^4{x_1}^2 - 4\,{x_3}^2{x_1}^4 - 20\,{x_1}^2{x_3}^2{x_2}^2\sqrt{5} + 4\,{x_1}^2{x_3}^4\sqrt{5} \\ &-4\,{x_1}^2{x_2}^4\sqrt{5} + 2\,{x_1}^4\sqrt{5}{x_2}^2 + 6\,{x_2}^2{x_3}^4\sqrt{5} + 14\,{x_2}^4{x_3}^2\sqrt{5} \\ &-20\,{x_3}^2{x_1}^2{x_2}^2 + 40\,{x_2}^2{x_1}^2 - 30\,{x_3}^2{x_2}^2 - 30\,{x_3}^2{x_1}^2 - {x_1}^4\sqrt{5} \\ &-{x_2}^4\sqrt{5} + 12\,\sqrt{5}{x_1}^2{x_2}^2 - 30\,{x_3}^2{x_2}^2\sqrt{5} - 2\,{x_3}^2{x_1}^2\sqrt{5}\Big), \end{aligned}$$

and the other two components of a are given by the identity $a(Sx) = Sa(x)$. By construction, a is divergence-free and invariant under the transformations of $\mathbf{Y}_h$. Note that if $\epsilon > 0$ is small enough, then the initial vorticity $\Omega_0 = \epsilon(\nabla \times a)$ satisfies all the assumptions of Corollary 4.6. Then the solution such that $u(x,0) = \epsilon a(x)$ is globally defined and $|u(x,t)|$ is bounded at infinity by $|x|^{-8}$ and t^{-4}. No example of solution to the Navier–Stokes equation in $\mathbb{R}^3$ with a better localization in space-time seems to be known so far.

It now remains to prove that the decay $|x|^{-8}$ is optimal, for general solutions $\mathbf{Y}_h$-invariant (hence also for solutions $\mathbf{Y}$-invariant). Let us show that condition (32) holds true with $m = 4$, if u is the solution starting from the vector field a that we have just defined. Indeed, a long but elementary computation allows us to write explicitly the homogeneous polynomial $P_4(a)(\xi)$. This polynomial equals $\frac{45}{8192}\pi^{3/2}Q(\xi)$, where

$$\begin{aligned} Q(\xi) = 124(3+\sqrt{5})(\xi_1^6+\xi_2^6+\xi_3^6) + 6(216+77\sqrt{5})(\xi_1^4\xi_2^2+\xi_2^4\xi_3^2+\xi_3^4\xi_1^2) \\ +3(357+109\sqrt{5})(\xi_1^4\xi_3^2+\xi_2^4\xi_1^2+\xi_3^4\xi_2^2) + 564(3+\sqrt{5})\xi_1^2\xi_2^2\xi_3^2. \end{aligned}$$

Since $Q(\xi)$ is not divisible by $\xi_1^2+\xi_2^2+\xi_3^2$, the remark at the beginning of this section applies and $u(x,t)$ cannot decay faster than $|x|^{-8}$ uniformely in any positive time interval.

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