On the localization of the magnetic and the velocity fields in the equations of magnetohydrodynamics

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Abstract

We study the behavior at infinity, with respect to the space variable, of solutions to the magnetohydrodynamics equations in $\mathbb{R}^d$. We prove that if the initial magnetic field decays sufficiently fast, then the plasma flow behaves as a solution of the free nonstationnary Navier–Stokes equations when $|x| \to +\infty$, and that the magnetic field will govern the decay of the plasma, if it is poorly localized at the beginning of the evolution. Our main tools are boundedness criteria for convolution operators in weighted spaces.

Keywords: decay at infinity, instantaneous spreading, magnetohydrodynamics, MHD, spatial localisation, weighted spaces, convolution, asymptotic behavior.
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1 Introduction

The magnetohydrodynamics equations are a well-known model in plasma physics, describing the interactions between a magnetic field and a fluid made of moving electrically charged particles. A common example of an application of this model is the design of tokamaks: the purpose of these machines is to confine a plasma in a region, with a density and a temperature large enough to entertain thermonuclear fusion reactions. This can be achieved, at least during a small time interval, by applying strong magnetic fields. We refer to [12] for other applications of this model, in particular to the study of the dynamics of the solar corona.

In non-dimensional form, the magnetohydrodynamics equations can be written in the following way:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - S(B\cdot\nabla)B + \nabla\left(p + \dfrac{S}{2}|B|^2\right) = \dfrac{1}{R_e}\Delta u \\ \dfrac{\partial B}{\partial t} + (u\cdot\nabla)B - (B\cdot\nabla)u = \dfrac{1}{R_m}\Delta B \\ \operatorname{div} u = \operatorname{div} B = 0 \\ u(0) = u_0 \quad \text{and} \quad B(0) = B_0. \end{cases} \tag{MHD}$$

Here the unknowns are the velocity field u of the fluid, the pressure p and the magnetic field B, all defined in $\mathbb{R}^d$ ($d \geq 2$). The positive constants R_e and R_m are respectively the

Reynolds number and the magnetic Reynolds number; moreover $S = M^2/(R_e R_m)$, where M is the Hartman number. After rescaling u and B, we can assume that $S = R_e = 1$. With minor loss of generality, from now on we shall also assume that $R_m = 1$. All the results however remain valid in the general case with simple modifications in the constants.

In the particular case $B \equiv 0$, the system (MHD) reduces to the celebrated Navier–Stokes equations. Just as in this particular case, global weak solutions to (MHD) do exist, but their unicity, as well their smoothness in the case of smooth data, remains an open problem for $d \geq 3$. Partial regularity results, which provide bounds of the Hausdorff dimension of the possible singular set of weak solutions, have been obtained in [7]. Constantin and Fefferman's theory [5] relating the regularity of the flow to the directions of the vorticity has been extended to magnetohydrodynamics in [8]. A construction of forward selfsimilar solutions is given in [9], where the nonexistence of backward selfsimilar solutions is also discussed. Moreover, the asymptotic behavior of the solutions for $t \to +\infty$ is quite well understood: for example, [13] provides the optimal decay rates of the L^2 norm of u and B for a large class of flows.

On the other hand, nothing seems to have been done to study the decay of solutions of (MHD) with respect to the *space variable*. In this paper, motivated by recent results obtained by several authors for the Navier–Stokes equations (see, *e.g.*, [1], [2], [6], [11] and [14]), we would like to describe in which way the presence of the magnetic field affects the spatial localization of the velocity field.

Definitions and notations. We start by introducing the notion of decay rate at infinity in a *w*eak sense, which generalizes the usual notion of pointwise decay rate in the framework of locally square integrable functions. A simple motivation is that the $L^2_{\rm loc}$ regularity is the minimal one for which the system (MHD) makes sense.

1. Let $f \in L^2_{\rm loc}(\mathbb{R}^d)$. We define the L^2 *decay rate as* $|x| \to +\infty$ of f, as

$$\eta(f) = \sup\biggl\{ \eta \in \mathbb{R}\ ;\ \lim_{R\to+\infty} R^{2\eta} \int_{1\leq|x|\leq 2} |f(Rx)|^2\,dx = 0 \biggr\}. \tag{1.1}$$

   If $\eta = \eta(f)$ is finite then we will write $f \overset{L^2}{\sim} |x|^{-\eta}$ when $|x| \to +\infty$. On the other hand, when we write $f \overset{L^2}{=} \mathcal{O}(|x|^{-\eta})$ when $|x| \to +\infty$, we mean that $\eta(f) \geq \eta$. Of course, any measurable function such that $|f(x)| \leq C(1+|x|)^{-\eta}$ satisfies $f \overset{L^2}{=} \mathcal{O}(|x|^{-\eta})$ when $|x| \to +\infty$.

2. For $a \in [1, +\infty]$ and $\alpha \in \mathbb{R}$, the space $L^a_\alpha(\mathbb{R}^d)$ is the Banach space normed by

$$\|f\|_{L^a_\alpha} = \biggl(\int_{\mathbb{R}^d} |f(x)|^a (1+|x|)^{a\alpha}\,dx \biggr)^{1/a} \qquad \text{if } 1 \leq a < +\infty \tag{1.2a}$$

   and, if $a = +\infty$, by

$$\|f\|_{L^\infty_\alpha} = \operatorname*{ess\ sup}_{x\in\mathbb{R}^d} |f(x)|(1+|x|)^\alpha. \tag{1.2b}$$

   From the *localization point of view* the two spaces $L^a_\alpha(\mathbb{R}^d)$ and $L^b_\beta(\mathbb{R}^d)$ must be considered as equivalent, when

$$\alpha + \frac{d}{a} = \beta + \frac{d}{b}.$$

Indeed, if $f \in L^a_\alpha(\mathbb{R}^d)$ and $a \geq 2$, then $f \overset{L^2}{=} \mathcal{O}\big(|x|^{-(\alpha+d/a)}\big)$ when $|x| \to +\infty$. Hölder inequality implies that

$$L^a_\alpha \subset L^b_\beta \tag{1.3}$$

whenever $\alpha + d/a > \beta + d/b$ and $a \geq b$. It also implies that

$$\eta(f) = \sup\left\{\alpha + \frac{d}{a}\,;\, a \geq 2 \text{ and } f \in L^a_\alpha\right\} \tag{1.4}$$

for any $f \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$.

We shall use the following additional notations :

3. If A and B are two expressions containing a parameter α, then when we write

$$A \leq B - \varepsilon_\alpha,$$

we mean that $A \leq B$ if $\alpha = 0$ and $A < B$ if $\alpha \neq 0$. We shall also often write expressions of the form $A \leq B - \varepsilon_{1/a}$ meaning that the inequality must be strict for finite a and can be large when $a = +\infty$.

4. The positive part of a real number will be denoted by $(\cdot)^+ = \max\{\cdot, 0\}$.

Main results. We are concerned with the persistence problem of the spatial localization of the magnetic and the velocity fields. Our main results (Theorem 1.1 and 1.3 below) aim to answer the following questions. Consider a localization condition like

$$(u_0, B_0) \in L^{p_0}_{\vartheta_0}(\mathbb{R}^d) \times L^{p_1}_{\vartheta_1}(\mathbb{R}^d). \tag{1.5}$$

Will the unique solution of (MHD) preserve such a condition in some future time interval ? Depending on the parameters, the answer can be positive or negative. In case of a negative answer, can we still ensure that the spatial localization of the solution is conserved *in the weak sense* ? In other words, we would like to know whether

$$u(t) \overset{L^2}{=} \mathcal{O}\big(|x|^{-(\vartheta_0+d/p_0)}\big) \quad \text{and} \quad B(t) \overset{L^2}{=} \mathcal{O}\big(|x|^{-(\vartheta_1+d/p_1)}\big) \qquad \text{when } |x| \to +\infty.$$

Again, this condition may be conserved, or instantaneously break down.

We will prove the following:

Theorem 1.1 *Let $u_0 \in L^{p_0}_{\vartheta_0}(\mathbb{R}^d)$, $B_0 \in L^{p_1}_{\vartheta_1}(\mathbb{R}^d)$ be two divergence-free vector fields in $\mathbb{R}^d$ ($d \geq 2$). Assume that*

$$\begin{cases} \vartheta_0 \geq 0 \\ d < p_0 \leq +\infty \end{cases} \quad \text{and} \quad \begin{cases} \vartheta_1 \geq 0 \\ d < p_1 \leq +\infty. \end{cases} \tag{1.6a}$$

Let us also assume that

$$\delta + \varepsilon_\delta \leq \eta_0 \leq \min\big\{d+1\,;\, 2\eta_1 - \delta\big\}, \tag{1.6b}$$

with $\eta_0 = \vartheta_0 + d/p_0$, $\eta_1 = \vartheta_1 + d/p_1$ and $\delta = \left(\frac{2d}{p_1} - 1\right)^+$. Finally, define $p_0^* = \min\{p_0\,;\, \frac{d}{\delta} - \varepsilon_\delta\}$.

Then there exists $T > 0$ and a unique mild solution (u, B) of (MHD) in $\mathcal{C}([0,T]; L^{p_0^*} \times L^{p_1})$. This solution satisfies

$$u(t) \overset{L^2}{=} \mathcal{O}\big(|x|^{-\eta_0}\big) \quad \text{and} \quad B(t) \overset{L^2}{=} \mathcal{O}\big(|x|^{-\eta_1}\big) \qquad \text{when } |x| \to +\infty. \tag{1.7}$$

If $d = 2$, the time T can be arbitrarily large.

Moreover, if (u_0, B_0) also belongs to $L^{\widetilde{p_0}}_{\widetilde{\vartheta}_0} \times L^{\widetilde{p_1}}_{\widetilde{\vartheta}_1}$, with the corresponding indices satisfying assumptions (1.6), then the lifetimes in $L^{p_0^*} \times L^{p_1}$ and $L^{\widetilde{p_0}^*} \times L^{\widetilde{p_1}}$ agree and both maximal solutions are actually the same one.

Next we discuss the optimality of the above restrictions. Such restrictions are of two kinds: there are a few conditions related to the well-posedness of the system, and a condition (namely, the upper bound for η_0 in (1.6b)) which is related to the spatial localization of the solution. Here, we will only focus on this condition. The following theorem implies that the restriction $\eta_0 \le d+1$ is sharp. We expect that the other restriction is also sharp, or at least that $\eta_0 \le 2\eta_1$ for stable weak solutions. But we were not able to prove such a result.

Theorem 1.2 *Let $(u, B) \in \mathcal{C}([0,T]; L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ a solution to (MHD) such that*

$$\sup_{t\in[0,T]} |u(t,x)| \overset{L^2}{=} \mathcal{O}\big(|x|^{-(d+1+\varepsilon)}\big) \tag{1.8a}$$

$$\text{and} \qquad \sup_{t\in[0,T]} |B(t,x)| \overset{L^2}{=} \mathcal{O}\big(|x|^{-(d+1+\varepsilon)/2}\big) \tag{1.8b}$$

for some $\varepsilon > 0$. Then, for all $t \in [0,T]$, there exists a constant $C(t) \ge 0$ such that the components of $u(t)$ and $B(t)$ satisfy the following integral identity :

$$\int_{\mathbb{R}^d} (u^j u^k - B^j B^k)(t,x)\, dx = \delta_{j,k}\, C(t), \qquad (j,k = 1,\dots,d) \tag{1.9}$$

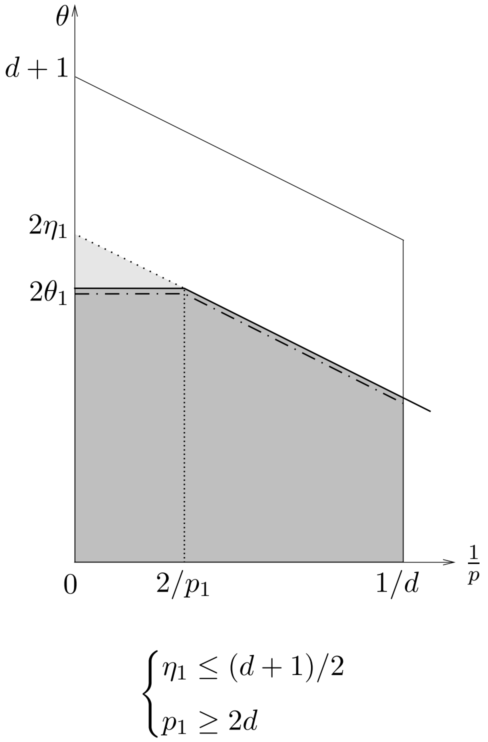
with $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ otherwise.

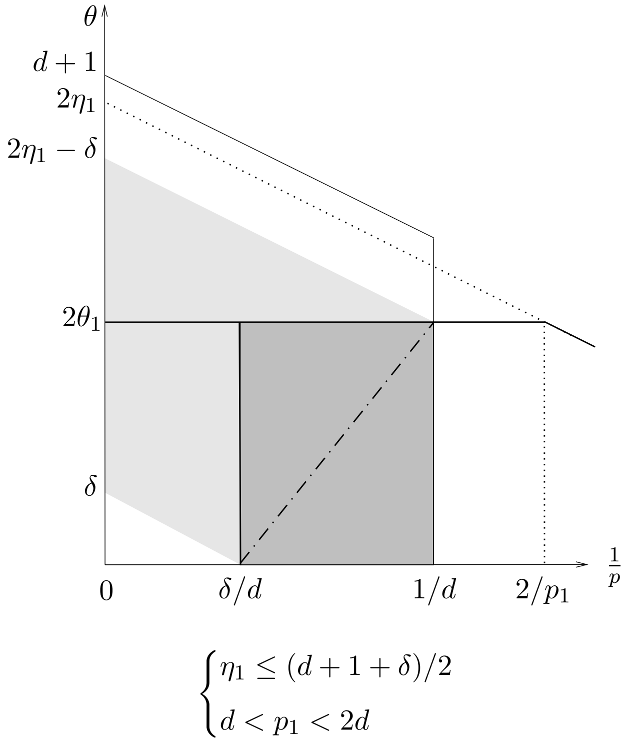
By Theorem 1.3 below, condition (1.8b) will be fulfilled as soon as u_0 and B_0 belong to $L^p_\vartheta(\mathbb{R}^d)$, with $p > d$ and $\vartheta + \frac{d}{p} = (d+1+\varepsilon)/2$, for some $\epsilon > 0$. This means that if we start with a well localized initial datum (u_0, B_0), but such that (1.9) does not hold for $t = 0$, then condition (1.8a) must brake down.

On the other hand, the integral identities (1.9) are obviously unstable. Neverthless, in section 5 we shall see that a class of exceptional solutions satisfying (1.9) does exist. Inside this class, one can exhibit solutions such that u decays much faster than in the generic case.

Physical interpretation of Theorem 1.1. This theorem reinforces mathematically some facts that can be observed in the applications. Three conclusions can be drawn:

1. Any spatial localization assumption on the magnetic field will be conserved by the flow. Indeed, the L^2 decay rate η_1 can be arbitrarily large. The spatial localization of the velocity field is also conserved, but there are some limitations to this property.





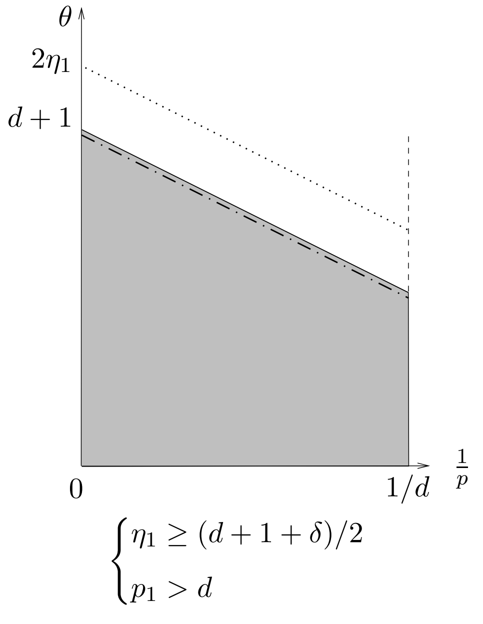


Fig.1 The figures show the admissible values for (p_0, ϑ_0) allowing (1.7) to hold, once (p_1, ϑ_1) is given (*all gray regions*).

ABOVE : Slowly decaying magnetic field. The results depends slightly on the regularity of B through $\delta = \left(\frac{2d}{p_1} - 1\right)^+$.

DOWN-LEFT : Fast decaying magnetic field. The velocity field behaves at infinity as the solution of Navier–Stokes equations with the same initial datum u_0 (see [14]).

The *dark gray* regions correspond to initial data for wich we will prove in addition that $u \in L^\infty([0,T]; L^{p_0}_{\vartheta_0})$. The *dash-dotted lines* illustrate the barriers used in the proof of §4.3.

2. For poorly localized magnetic fields (namely $\eta_1 \leq (d+1+\delta)/2$), the behavior of u when $|x| \to +\infty$ is governed by the decay of the magnetic field. As $0 \leq \delta < 1$ in (1.6b), the maximal L^2 decay rate of u that can be conserved by the flow exceeds $2\eta_1 - 1$. When $p_1 \geq 2d$, one has $\delta = 0$ and this rate is improved up to twice that of B_0. The pathological lower bound on η_0 disappears too. Roughly speaking, requiring p_1 to be larger (for a given L^2 decay rate $\eta_1 = \vartheta_1 + d/p_1$ of the magnetic field) means that the

behavior at infinity of B_0 is closer and closer to that of a function that decays as $|x|^{-\eta_1}$, in the usual pointwise sense.

3. For sufficiently fast decaying magnetic fields, the decay of u is not affected by B, but is provided by the fundamental laws of hydrodynamics. The reason is the following: for magnetic fields such that $\eta_1 \geq (d+1+\delta)/2$, our limitations on the L^2 decay rate at infinity of the velocity field (1.6b) boil down to the only restriction $\eta_0 \leq d+1$. This is exactly the same restriction that appears for the Navier–Stokes equations. Indeed, we know from F. Vigneron's result [14] that the mild solution of the Navier–Stokes equations remains in $L^{p_0}_{\vartheta_0}(\mathbb{R}^d)$ if the initial velocity belongs to such space and

$$\vartheta_0 + d/p_0 \leq d + 1 - \varepsilon_{1/p_0}.$$

This condition in known to be sharp. One may notice however that, thanks to (1.4), the equality case is possible even if $p_0 < +\infty$, provided that stability is asserted as in (1.7).

A more physical explanation for the above conclusions is the following[1]. The induction equation means that the magnetic field lines are transported by the flow while simultaneously undergoing resistive diffusion. This transport-diffusion process guarantees that, where the velocity vanishes, the magnetic field will not spatially spread out during small time intervals, since the mechanism of diffusion is quite slow. As for the fluid flow, the magnetic field acts upon it only through the Lorentz force: whenever this disappears the velocity acts in a purely Navier–Stokes way; thus, the spatial spreading of the initial velocity is essentially governed by the competition between diffusion, whose effect is important only for large time, and incompressibility, that immediately prevents the flow from remaining too localized.

Stability in weighted spaces. Conclusion (1.7) does not mean that

$$(u, B) \in L^\infty\left([0,T]\,;\, L^{p_0}_{\vartheta_0} \times L^{p_1}_{\vartheta_1}\right).$$

Actually, we do not know if this property holds when $u_0 \in L^{p_0}_{\vartheta_0}$ and (p_0, ϑ_0) is in the light-gray regions of Fig.1. However, if (p_0, ϑ_0) is in a dark-gray region, then such property does hold. This is essentially the statement of our next theorem. It extends to the case of non-vanishing magnetic fields, the result established in [14] for the Navier–Stokes equations.

Theorem 1.3 *Let $u_0 \in L^{p_0}_{\vartheta_0}(\mathbb{R}^d)$, $B_0 \in L^{p_1}_{\vartheta_1}(\mathbb{R}^d)$ be two divergence-free vector fields in $\mathbb{R}^d$ $(d \geq 2)$. Assume that $\vartheta_0, \vartheta_1 \geq 0$, $d < p_0 \leq +\infty$ and*

$$\frac{2}{p_1} < \frac{1}{p_0} + \frac{1}{d}. \tag{1.10a}$$

Then there exist $T > 0$ (if $d = 2$, one may take $T = +\infty$) and a unique mild solution of (MHD)

$$(u, B) \in \mathcal{C}\left([0,T]; L^{p_0} \times L^{p_1}\right). \tag{1.10b}$$

If, in addition, the decay rates of u_0 and B_0 defined by $\eta_0 = \vartheta_0 + d/p_0$ and $\eta_1 = \vartheta_1 + d/p_1$ satisfy

$$\eta_0 \leq \min\left\{ d + 1 - \varepsilon_{1/p_0}\,;\, 2\eta_1 - \varepsilon_{2\vartheta_1 - \vartheta_0}\,;\, 2\eta_1 + \frac{d}{p_0} - \frac{2d}{p_1} \right\}, \tag{1.11a}$$

[1] This explanation was suggested to us by the Referee.

then we have more precisely

$$(u,B) \in \mathcal{C}\big([0,T]; L^{p_0}_{\vartheta_0} \times L^{p_1}_{\vartheta_1}\big). \tag{1.11b}$$

Moreover, if (u_0, B_0) *also belongs to* $L^{\widetilde{p}_0}_{\widetilde{\vartheta}_0} \times L^{\widetilde{p}_1}_{\widetilde{\vartheta}_1}$, *with new indices again satisfying* (1.10a) *and* (1.11a), *then the lifetimes in* $L^{p_0}_{\vartheta_0} \times L^{p_1}_{\vartheta_1}$ *and* $L^{\widetilde{p}_0}_{\widetilde{\vartheta}_0} \times L^{\widetilde{p}_1}_{\widetilde{\vartheta}_1}$ *are the same and both maximal solutions agree.*

The assumption (1.10a) is not really related to spatial localization problems, but rather to well-posedness issues of the equations, and in particular, to the invariance of the equation under the natural scaling

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x), \qquad B_\lambda(t,x) = \lambda B(\lambda^2 t, \lambda x) \qquad (\lambda > 0).$$

We expect that Theorem 1.3 remains true in limit cases $p = d$, or $\frac{2d}{p_1} = \frac{1}{p_1} + \frac{1}{d}$ (with several modifications in the proof). We did not treat these limit cases since they would require Kato's two-norm approach for proving the boundedness of the operators involved, as described in [3, chap. 3] or [4] for the Navier–Stokes equations. The proof would be more complicated, without providing any substantial clarification of the spatial localization problem.

Let us also observe that one could replace the weights $(1+|x|)^\vartheta$ with homogeneous weights. But in this case the conditions to be imposed on the parameters would be much more restrictive, *e.g.*

$$\vartheta + \frac{d}{p} < 1.$$

Again, this would not help to understand the spatial localization of the fields.

Main methods and organization of the paper. We shall first prove Theorem 1.3 and later deduce Theorem 1.1 as a corollary of the natural embedding (1.3) between weighted spaces. The idea consists in observing that the assumptions (1.6), together with the inclusion (1.3), ensure that the initial datum belongs to the product of two larger Lebesgue spaces, in which we can prove the existence and uniqueness of a mild solution.

Our proof of Theorem 1.3 consists in applying the contraction mapping principle to the integral form of (MHD), in a suitable ball of the space $\mathcal{C}([0,T], L^{p_0}_{\vartheta_0} \times L^{p_1}_{\vartheta_1})$. This is why we refer to (u,B) as a mild solution. The only difficulty is establishing the bicontinuity of the bilinear operator involved.

For small values of η_0, the bicontinuity would be a straightforward consequence of the well-known Young convolution inequality in weighted Lebesgue spaces (recalled in [14, §2.2]). But this argument does not go through when η_0 is close to the upper bound of (1.11a), since the kernel of the operator governing the evolution of the velocity field decays too slowly at infinity. In this case, the proof requires more careful estimates. The main one is given by Proposition 3.1 below.

Several generalizations of the weighted convolution inequalities are known (see, *e.g.*, the recent boundedness criterion for asymmetric kernel operators [14, §2.3], which applies to Navier–Stokes). However, we could not deduce the bicontinuity of the bilinear operator by applying directly any known inequality, unless we put additional artificial restrictions on the parameters.

The main issue with the spatial localization of magnetohydrodynamics fields is that the system cannot be treated as a scalar equation. When dealing with the Navier–Stokes system, one may often reduce the problem to a single equation, because all the components of the kernels of the Navier–Stokes operators satisfy the same estimates. This is no longer true for (MHD). In the following, we shall derive sharp bounds for the magnetohydrodynamics kernels and take advantage of the fact that a few components decay much faster than the others.

This paper is organized as follows. Section 2 contains some generalities on magnetohydrodynamics. In Section 3 we study the boundedness of convolution operators in weighted spaces. We use these results in Section 4, proving first the local existence of a unique solution in weighted spaces (1.11b), then the fact that lifetimes do not depend on the choice of the indices. Then we deduce Theorem 1.1 as a corollary.

Theorem 1.2 will be proved in Section 5, using a Fourier transform method developed in [2]. Section 5 also contains the description of a method for obtaining special solutions, such that the velocity field is more localized than in (1.6b). Those solutions are however unstable.

Remark 1.4 When we deal with the space $\mathcal{C}([0,T];L^{p_0}_{\vartheta_0}\times L^{p_1}_{\vartheta_1})$, with $p_0=+\infty$ or $p_1=+\infty$, the continuity at $t=0$ must be understood in the weak sense, as is usually done in nonseparable spaces.

2 The integral form of the equations

Let $\mathbb{P}$ be the Leray-Hopf projector onto the divergence-free vector field, defined by

$$\mathbb{P}f = f-\nabla\Delta^{-1}(\operatorname{div} f).$$

Applying $\mathbb{P}$ to the first equation of (MHD) and then the Duhamel formula, we obtain the integral equations

$$\begin{cases} u(t)=e^{t\Delta}u_0-\displaystyle\int_0^t e^{(t-s)\Delta}\,\mathbb{P}\operatorname{div}(u\otimes u-B\otimes B)(s)\,ds\\ B(t)=e^{t\Delta}B_0-\displaystyle\int_0^t e^{(t-s)\Delta}\,\operatorname{div}(u\otimes B-B\otimes u)(s)\,ds\\ \operatorname{div}u_0=\operatorname{div}B_0=0\end{cases}\tag{IE}$$

where $e^{t\Delta}$ is the heat semigroup (recall that the Reynolds numbers and the Hartman numbers have been set equal to 1). The semigroup method that we use in this paper to solve (IE) provides mild solutions of (MHD) that are in fact smooth for strictly positive t.

We denote respectively by $F^k_{j,h}(t,x)$ and $G^k_{j,h}(t,x)$ $(j,h,k=1,\dots,d)$ the components of the kernels of the matricial operators $e^{t\Delta}\mathbb{P}\nabla$ and $e^{t\Delta}\nabla$. Thus,

$$\widehat{F}^k_{j,h}(\xi,t)=e^{-t|\xi|^2}\xi_h(\delta_{j,k}-\xi_j\xi_k|\xi|^{-2}).\tag{2.1}$$

This expression of the symbol allows us to see that

$$\begin{aligned} &F(t,x)=t^{-(d+1)/2}\,\Phi(x/\sqrt{t}),\\ \text{with}\quad &|\Phi(x)|\le C(1+|x|)^{-(d+1)}.\end{aligned}\tag{2.2a}$$

This low decay rate of Φ is due to the fact that $F(t,\cdot) \notin L^1_1(\mathbb{R}^d)$; otherwise $\widehat{F}(t,\cdot)$ would be a $\mathcal{C}^1$ function on $\mathbb{R}^d$. On the other hand,

$$\begin{aligned} G(t,x) &= t^{-(d+1)/2}\,\Psi(x/\sqrt{t}), \\ \text{with} \quad \Psi &\in \mathcal{S}(\mathbb{R}^d) \quad \text{(the Schwartz class).} \end{aligned} \tag{2.2b}$$

Let us introduce the bilinear operators on $\mathbb{R}^d$-vector fields $\mathbb{U}$ and $\mathbb{B}$ whose k^{th} component is

$$\begin{aligned} \mathbb{U}^k(f,g)(t,x) &= \sum_{j,h} \int_0^t F^k_{j,h}(t-s) * \left(f^j \otimes g^h\right)(s)\,ds \\ \mathbb{B}^k(f,g)(t,x) &= \sum_{j,h} \int_0^t G^k_{j,h}(t-s) * \left(f^j \otimes g^h\right)(s)\,ds, \end{aligned}$$

and the bilinear operator $\mathbb{V} = (\mathbb{V}_1, \mathbb{V}_2)$ on $\mathbb{R}^{2d}$-vector fields $v = (v_1, v_2)$ defined by

$$\begin{aligned} \mathbb{V}_1(v,w) &= \mathbb{U}(v_1,w_1) - \mathbb{U}(v_2,w_2) \\ \mathbb{V}_2(v,w) &= \mathbb{B}(v_1,w_2) - \mathbb{B}(v_2,w_1). \end{aligned}$$

Here and below, for $v \in \mathbb{R}^{2d}$, we denote by v_1 the first d components and by v_2 the last d components.

With these notations and setting $v = (u, B)$, $v_0 = (u_0, B_0)$, the system (IE) can be rewritten as

$$v = e^{t\Delta} v_0 - \mathbb{V}(v,v). \tag{2.3}$$

As it is well known (we refer, *e.g.*, to [3, Lemma 1.2.6]), if X is a Banach space, then for solving an equation like (2.3) one just needs to check that

$$e^{t\Delta} v_0 \in \mathcal{C}([0,T]; X) \tag{2.4a}$$

and

$$\mathbb{V} \,:\, \mathcal{C}([0,T];X) \times \mathcal{C}([0,T];X) \to \mathcal{C}([0,T];X), \tag{2.4b}$$

with the operator norm of $\mathbb{V}$ tending to 0 as $T \to 0$. Then the existence of a solution $v \in \mathcal{C}([0,T];X)$ is ensured, at least for $T > 0$ small enough.

In order to prove Theorem 1.3 we shall take $X = L^{p_0}_{\vartheta_0} \times L^{p_1}_{\vartheta_1}$. In this setting, condition (2.4a), the unicity and the continuity of the solution with respect to the time variable are all straightforward. Therefore, our attention will now be exclusively devoted to the more subtle problem of the bicontinuity of $\mathbb{V}$ in $L^\infty([0,T]; L^{p_0}_{\vartheta_0} \times L^{p_1}_{\vartheta_1})$.

We need three estimates, namely

$$\|\mathbb{U}(u,u)(t)\|_{L^{p_0,\vartheta_0}} \le C_T \|u\|^2_{\mathcal{C}([0,T],L^{p_0,\vartheta_0})} \tag{2.5a}$$

$$\|\mathbb{U}(B,B)(t)\|_{L^{p_0,\vartheta_0}} \le C_T \|B\|^2_{\mathcal{C}([0,T],L^{p_1,\vartheta_1})} \tag{2.5b}$$

$$\|\mathbb{B}(u,B)(t)\|_{L^{p_1,\vartheta_1}} \le C_T \|u\|_{\mathcal{C}([0,T],L^{p_0,\vartheta_0})} \|B\|_{\mathcal{C}([0,T],L^{p_1,\vartheta_1})} \tag{2.5c}$$

for all $0 \leq t \leq T$ and some constant C_T such that $C_T \to 0$ as $T \to 0$. These bounds will not rely on the specific structure of the operators $\mathbb{U}$ and $\mathbb{B}$, but only on the decay properties of their respective kernels:

$$\begin{aligned} |F(t,x)| &\leq C(\sqrt{t}+|x|)^{-(d+1)} \\ |G(t,x)| &\leq C_N \sqrt{t}^{N-d-1}(\sqrt{t}+|x|)^{-N} \end{aligned} \tag{2.6}$$

for all $N \geq 0$.

We start by observing that by Hölder inequality,

$$\begin{aligned} \|u \otimes u\|_{L^{p_0/2}_{2\vartheta_0}} &\leq \|u\|^2_{L^{p_0}_{\vartheta_0}} \\ \|B \otimes B\|_{L^{p_1/2}_{2\vartheta_1}} &\leq \|B\|^2_{L^{p_1}_{\vartheta_1}} \\ \|u \otimes B\|_{L^{\mathbb{H}(p_0,p_1)}_{\vartheta_0+\vartheta_1}} &\leq \|u\|_{L^{p_0}_{\vartheta_0}} \|B\|_{L^{p_1}_{\vartheta_1}} \end{aligned}$$

where $\frac{1}{\mathbb{H}(p_0,p_1)} = \frac{1}{p_0} + \frac{1}{p_1}$ denotes the Hölder exponent (the assumptions of Theorem 1.3 imply that $p_0, p_1 \geq 2$). Set $\lambda = \sqrt{t}$ and

$$\Gamma^N_\lambda(x) = (\lambda + |x|)^{-N}. \tag{2.7}$$

Then the only thing that we have to do to obtain (2.5a)-(2.5c) is to establish that for all $0 < \lambda \leq 1$:

$$\|\Gamma^{d+1}_\lambda * f\|_{L^{p_0}_{\vartheta_0}} \leq C\lambda^{\sigma_0} \|f\|_{L^{p_0/2}_{2\vartheta_0}}, \tag{2.8a}$$

$$\|\Gamma^{d+1}_\lambda * f\|_{L^{p_0}_{\vartheta_0}} \leq C\lambda^{\sigma_0'} \|f\|_{L^{p_1/2}_{2\vartheta_1}} \tag{2.8b}$$

$$\text{and} \quad \|\Gamma^{N}_\lambda * f\|_{L^{p_1}_{\vartheta_1}} \leq C\lambda^{\sigma_1} \|f\|_{L^{\mathbb{H}(p_0,p_1)}_{\vartheta_0+\vartheta_1}} \tag{2.8c}$$

with an arbitrarily large $N \geq 0$ and exponent σ_0, σ_0', σ_1 such that

$$\sigma_0 > -2, \qquad \sigma_0' > -2, \qquad \sigma_1 > -N + d - 1. \tag{2.9}$$

The constant $C > 0$ has to be independent of λ. Assumption (2.9) ensures that the integrals

$$\int_0^T \|F(t-s) * (u \otimes u)(s)\|_{L^{p_0}_{\vartheta_0}} \, ds, \qquad \int_0^T \|F(t-s) * (B \otimes B)(s)\|_{L^{p_0}_{\vartheta_0}} \, ds$$

and

$$\int_0^T \|G(t-s) * (u \otimes B)(s)\|_{L^{p_1}_{\vartheta_1}} \, ds$$

converge.

3 Convolution estimates in weighted spaces

The fundamental estimates (2.8a)-(2.8c) will be a simple consequence of the following proposition.

Proposition 3.1 *Let $a, p \in [1; +\infty]$ and $\alpha, \vartheta \geq 0$. For any real numbers $\lambda > 0$ and $N \geq 1$ let us set*

$$\Gamma_\lambda^N(x) = (\lambda + |x|)^{-N}.$$

Let also $f \in L_\alpha^a(\mathbb{R}^d)$ and $N > d$.

1. *Then $\Gamma_\lambda^N * f \in L_\vartheta^p(\mathbb{R}^d)$, provided that*

$$\vartheta \leq \alpha \qquad \text{and} \qquad \vartheta + \frac{d}{p} \leq \min\Big\{N - \varepsilon_{1/p}\,; \alpha + \frac{d}{a} - \varepsilon_{\alpha-\vartheta}\Big\}. \tag{3.1}$$

Moreover, if $N \neq d(1 + \frac{1}{p} - \frac{1}{a})$, then there exists $C > 0$ such that

$$\|\Gamma_\lambda^N * f\|_{L_\vartheta^p} \leq C\lambda^{-N}(1+\lambda)^N \|f\|_{L_\alpha^a}. \tag{3.2}$$

2. *If one assumes in addition that*

$$\frac{1}{a} < \frac{1}{p} + \frac{1}{d}, \tag{3.3}$$

then there exists $\epsilon > 0$ and two constants $C, m > 0$ such that

$$\|\Gamma_\lambda^N * f\|_{L_\vartheta^p} \leq C\lambda^{-N+d-1+\epsilon}(1+\lambda)^m \|f\|_{L_\alpha^a}. \tag{3.4}$$

When $N = d(1+\frac{1}{p}-\frac{1}{a})$, the bounds (3.2) *and* (3.4) *hold with an additional factor* $(1+|\log\lambda|)$ *in the right-hand sides. In* (3.2) *and* (3.4) *the constant C may depend on ϑ, a, α, N and d, but it does not depend on λ or f.*

Remark 3.2 We shall see in the proof that we can take

$$\epsilon = \min\left\{\frac{d}{p} - \frac{d}{a} + 1\,;\ \frac{N-d+1}{2}\right\},$$
$$m = \max\left\{N - d + 1 - 2\epsilon\,;\ -N + d\left(\frac{1}{p} - \frac{1}{a} + 1\right)\right\}.$$

Proof. We start by observing that by Hölder's inequality,

$$\|f\|_{L^q} \leq C\|f\|_{L_\alpha^a} \qquad \text{if } \frac{1}{a} \leq \frac{1}{q} \leq \min\left\{1\,;\ \frac{1}{a} + \frac{\alpha}{d} - \varepsilon_\alpha\right\}. \tag{3.5}$$

Next we have

$$(1+|x|)^\vartheta\,|\Gamma_\lambda^N * f(x)| \leq \left[\int_{\mathbb{R}^d} \Gamma_\lambda^N(x-y)|f(y)|\,dy\right](1+|x|)^\vartheta = I_{\vartheta,\lambda}(x) + J_{\vartheta,\lambda}(x) + K_{\vartheta,\lambda}(x),$$

with the following definitions :

$$I_{\vartheta,\lambda}(x) = \left(\int_{|y|\geq|x|/2} \Gamma_\lambda^N(x-y)|f(y)|\,dy\right)(1+|x|)^\vartheta,$$
$$J_{\vartheta,\lambda}(x) = \left(\int_{|y|\leq|x|/2} \Gamma_\lambda^N(x-y)|f(y)|\,dy\right)(1+|x|)^\vartheta\,\mathbb{1}_{B(0,1)}(x),$$
$$K_{\vartheta,\lambda}(x) = \left(\int_{|y|\leq|x|/2} \Gamma_\lambda^N(x-y)|f(y)|\,dy\right)(1+|x|)^\vartheta\,\mathbb{1}_{B(0,1)^c}(x).$$

Here and below, $B(0,1)$ denotes the unit ball and $\mathbb{1}_E$ is the indicator function of a set $E \subset \mathbb{R}^d$.

The bound for $K_{\vartheta,\lambda}$. Since $|y| \le |x|/2$, we have

$$(\lambda + |x-y|)^{-N} \le 2^N (\lambda + |x|)^{-N}.$$

Hence, using (3.5) with $\frac{1}{q'} = 1 - \frac{1}{q} = \left(1 - \frac{\alpha}{d} - \frac{1}{a} + \varepsilon_\alpha\right)^+$,

$$\begin{aligned} 0 \le K_{\vartheta,\lambda}(x) &\le C\,(\lambda + |x|)^{-(N-\vartheta)} \int_{|y| \le \frac{|x|}{2}} |f(y)|\,dy \\ &\le C\,(\lambda + |x|)^{-(N-\vartheta)}\, \|f\|_{L^q}\, \|\mathbb{1}_{B(0,|x|/2)}\|_{L^{q'}} \\ &\le C\,(\lambda + |x|)^{-(N-\vartheta)}\, |x|^{[d-(\alpha+\frac{d}{a})+\varepsilon_\alpha]^+}\, \|f\|_{L^a_\alpha}. \end{aligned}$$

As $|x| \ge 1$, it follows that $\|K_{\vartheta,\lambda}\|_{L^p} \le C\,\|f\|_{L^a_\alpha}$, uniformly for $\lambda > 0$, provided that

$$\vartheta + \frac{d}{p} \le N - \left[d - \left(\alpha + \frac{d}{a}\right) + \varepsilon_\alpha\right]^+ - \varepsilon_{1/p}. \tag{3.6}$$

Since $N > d$, this condition is weaker than (3.1).

The bound for $J_{\vartheta,\lambda}$. Using (3.5) again, but with $q = a$, gives us

$$\begin{aligned} 0 \le J_{\vartheta,\lambda}(x) &\le C\,\mathbb{1}_{B(0,1)}(x)\,(\lambda + |x|)^{-N} \int_{|y| \le \frac{|x|}{2}} |f(y)|\,dy \\ &\le C\,\mathbb{1}_{B(0,1)}(x)\,(\lambda + |x|)^{-N} |x|^{d(1-1/a)}\, \|f\|_{L^a}, \end{aligned}$$

whence

$$\|J_{\vartheta,\lambda}\|_{L^p} \le C\left[\lambda^{-Np} \int_{|x| \le \lambda} |x|^{dp(1-1/a)}\,dx + \mathbb{1}_{\{\lambda<1\}} \int_{\lambda \le |x| \le 1} |x|^{-Np+dp(1-1/a)}\,dx\right]^{1/p} \|f\|_{L^a}.$$

Thus, for all $\vartheta \ge 0$ and $p \in [1, +\infty]$, we have

$$\|J_{\vartheta,\lambda}\|_{L^p} \le C\left(1 + \lambda^{-N+d+\frac{d}{p}-\frac{d}{a}}\right) \|f\|_{L^a} \qquad \text{if } N \ne d\left(1 + \frac{1}{p} - \frac{1}{a}\right), \tag{3.7a}$$

$$\text{and} \qquad \|J_{\vartheta,\lambda}\|_{L^p} \le C\,(1 + |\log\lambda|)\, \|f\|_{L^a} \qquad \text{if } N = d\left(1 + \frac{1}{p} - \frac{1}{a}\right). \tag{3.7b}$$

Note that $\|J_{\vartheta,\lambda}\|_{L^p}$ is bounded by the right-hand side of (3.2). Moreover, if $\frac{1}{a} < \frac{1}{p} + \frac{1}{d}$, then $\|J_{\vartheta,\lambda}\|_{L^p}$ is also bounded by the right-hand side of (3.4), provided that $0 < \epsilon \le d(\frac{1}{p} - \frac{1}{a} + \frac{1}{d})$.

The bound for $I_{\vartheta,\lambda}$. Set $F(x) = (1 + |x|)^\alpha\,|f(x)|$, so that $F \in L^a(\mathbb{R}^d)$ and

$$0 \le I_{\vartheta,\lambda}(x) \le C\,(1 + |x|)^{-(\alpha-\vartheta)} \int_{\mathbb{R}^d} \Gamma^N_\lambda(x-y) F(y)\,dy.$$

But $\Gamma^N_\lambda \in L^b_\beta(\mathbb{R}^d)$ for all $b \in [1; +\infty]$ and $\beta \ge 0$ such that $\beta + \frac{d}{b} \le N - \varepsilon_{1/b}$. Moreover, one has

$$\|\Gamma^N_\lambda\|_{L^b_\beta} \le C\lambda^{-N+\frac{d}{b}}(1+\lambda)^\beta. \tag{3.8}$$

The remaining part of the proof of Proposition 3.1 relies on the following lemma.

Lemma 3.3 *Let $a, b, p \in [1; +\infty]$ and $\alpha, \beta, \vartheta \geq 0$. For $f \in L^a_\alpha(\mathbb{R}^d)$, $g \in L^b_\beta(\mathbb{R}^d)$, define*

$$I_\vartheta(x) = (1 + |x|)^{-(\alpha-\vartheta)}\, F * g(x)$$

with $F(x) = (1 + |x|)^\alpha\, |f(x)|$. If there exists $s \in [1, +\infty]$ such that:

$$\begin{cases} \vartheta \leq \alpha \\ \dfrac{d}{s} \leq \min\left\{\dfrac{d}{a}\,;\ \left(\alpha + \dfrac{d}{a}\right) - \left(\vartheta + \dfrac{d}{p}\right) - \varepsilon_{\alpha-\vartheta}\,;\ d\left(1 - \dfrac{1}{b}\right)\right\} \\ \dfrac{d}{s} \geq \max\left\{\dfrac{d}{a} - \dfrac{d}{p}\,;\ \left[d - \left(\beta + \dfrac{d}{b}\right) + \varepsilon_\beta\right]^+\right\} \end{cases} \tag{3.9a}$$

then $I_\vartheta \in L^p(\mathbb{R}^d)$ and

$$\|I_\vartheta\|_{L^p} \leq C \|f\|_{L^a_\alpha} \|g\|_{L^b_\beta}. \tag{3.9b}$$

Proof. According to (3.5), we have $g \in L^{s'}(\mathbb{R}^d)$ for all $s' \in [1; +\infty]$ such that

$$\frac{1}{b} \leq \frac{1}{s'} \leq \min\left\{1\,;\ \frac{1}{b} + \frac{\beta}{d} - \varepsilon_\beta\right\}.$$

Let $\frac{1}{s} + \frac{1}{s'} = 1$. We now use that $\frac{1}{a} - \frac{1}{s} \geq 0$. The Young exponent $\mathbb{Y}(a, s')$ of a and s' is well defined by $\frac{1}{\mathbb{Y}(a,s')} = \frac{1}{a} - \frac{1}{s}$. Moreover, one has $F * g \in L^{\mathbb{Y}(a,s')}(\mathbb{R}^d)$, *i.e.*

$$I_\vartheta \in L^{\mathbb{Y}(a,s')}_{\alpha-\vartheta}.$$

Since $\vartheta \leq \alpha$, (3.5) implies that $I_\vartheta \in L^p(\mathbb{R}^d)$ for all p such that

$$\frac{1}{a} - \frac{1}{s} \leq \frac{1}{p} \leq \min\left\{1\,;\ \frac{1}{a} - \frac{1}{s} + \frac{\alpha - \vartheta}{d} - \varepsilon_{\alpha-\vartheta}\right\},$$

and (3.9b) is satisfied. □

Let us now come back to the proof of Proposition 3.1. We are going to apply the lemma with $g = \Gamma^N_\lambda$, $I_\vartheta = I_{\vartheta,\lambda}$, $b = +\infty$ and $\beta = N$.

- If $\frac{1}{a} \leq \frac{1}{p}$, then we further choose $s = +\infty$ and conditions (3.9a) boil down (recall that $N > d$) to the only restriction $\vartheta + \frac{d}{p} \leq \alpha + \frac{d}{a} - \varepsilon_{\alpha-\vartheta}$.
- If $\frac{1}{a} > \frac{1}{p}$, then we choose $\frac{1}{s} = \frac{1}{a} - \frac{1}{p}$. In this case conditions (3.9a) boil down to $\vartheta \leq \alpha$.

The first part of Proposition 3.1 now follows from the bounds obtained for $I_{\vartheta,\lambda}$, $J_{\vartheta,\lambda}$ and $K_{\vartheta,\lambda}$.

To prove (3.4), we fix ϵ such that $0 < \epsilon \leq \frac{N-d+1}{2}$. Then we apply Lemma 3.3 again with $g = \Gamma^N_\lambda$ and $I_\vartheta = I_{\vartheta,\lambda}$, but with b and β defined by

$$\frac{d}{b} = d - 1 + \epsilon, \qquad \text{and} \qquad \beta = N - d + 1 - 2\epsilon.$$

By (3.8), one has $\Gamma^N_\lambda \in L^b_\beta(\mathbb{R}^d)$ with $\|\Gamma^N_\lambda\|_{L^b_\beta} \leq \lambda^{-N+d-1+\epsilon}\phi(\lambda)$ and $\phi \in L^\infty_{loc}([0; +\infty))$.

As before,

– if $\frac{1}{a} \leq \frac{1}{p}$, then we choose $s = +\infty$ in (3.9a) and Lemma 3.3 implies that

$$\|I_{\vartheta,\lambda}\|_{L^b_\beta} \leq \lambda^{-N+d-1+\epsilon}\phi(\lambda)\|f\|_{L^a_\alpha}, \tag{3.10}$$

provided that $\vartheta + \frac{d}{p} \leq \alpha + \frac{d}{a} - \varepsilon_{\alpha-\vartheta}$.

– If $\frac{1}{a} > \frac{1}{p}$, then $\frac{1}{s} = \frac{1}{a} - \frac{1}{p}$ leads again to (3.10), provided that $\vartheta \leq \alpha$ and $\frac{1}{a} \leq \frac{1}{p} + \frac{1}{d} - \frac{\epsilon}{d}$.

The proof of Proposition 3.1 is now complete. □

4 End of the proof of Theorems 1.1 and 1.3

4.1 Existence of a unique mild solution in weighted spaces

We are now in a position to prove Theorem 1.3.

Under the assumptions of Theorem 1.3, one applies (3.4) with $N = d + 1$ and with $\epsilon = 1 - \frac{d}{p_0}$ or $\epsilon = 1 - \left(\frac{2d}{p_1} - \frac{d}{p_0}\right)^+$ respectively; assumption (3.3) is ensured by (1.10a). This proves (2.8a) and (2.8b) with

$$\sigma_0 = -1 - \frac{d}{p_0} \qquad \text{and} \qquad \sigma_0' = -1 - \left(\frac{2d}{p_1} - \frac{d}{p_0}\right)^+.$$

A new application of (3.4) with any N such that $N \geq \max\{d+1\,;\,\vartheta_1 + \frac{d}{p_1}\} + \varepsilon_{1/p_1}$ and $\epsilon = 1 - \frac{d}{p_0}$ yields (2.8c) with $\sigma_1 = -N + d - d/p_0$.

With the preceeding values of σ_0, σ_0' and σ_1, the assumption (1.10a) implies (2.9). As indicated in section 2, this yields (2.4b) and ensures that the operator norm of $\mathbb{V}$ tends to zero as a power of T, when $T \to 0$:

$$|||\mathbb{V}|||_{\mathcal{C}([0,T];X)} \leq C \max\left\{T^{1+\frac{\sigma_0}{2}}\,;\,T^{1+\frac{\sigma_0'}{2}}\,;\,T^{1+\frac{1}{2}(\sigma_1+N-d-1)}\right\}.$$

This ensures finally the conclusions (1.10b) and (1.11b) of Theorem 1.3.

More precisely, our argument proves that under the assumptions of Theorem 1.3, the maximal lifetime T^* of the mild solution in $X = L^{p_0}_{\vartheta_0} \times L^{p_1}_{\vartheta_1}$ satisfies

$$T^* \geq c \min\left\{1\,;\,\|(u_0,B_0)\|_X^{-2/(1-\frac{d}{p_0})}\,;\,\|(u_0,B_0)\|_X^{-2/\left(1-\left[\frac{2d}{p_1}-\frac{d}{p_0}\right]^+\right)}\right\}, \tag{4.1}$$

with a constant $c > 0$, depending on all the parameters, but not on u_0 or on B_0.

4.2 Comparison of lifetimes in Theorem 1.3

It only remains to establish that lifetimes are independent of the admissible pairs of indices chosen to construct the solution.

Proposition 4.1 *Let* $u_0 \in L^{p_0}_{\vartheta_0}(\mathbb{R}^d) \cap L^{\widetilde{p_0}}_{\widetilde{\vartheta}_0}(\mathbb{R}^d)$ *and* $B_0 \in L^{p_1}_{\vartheta_1}(\mathbb{R}^d)$. *Set* $\eta_0 = \vartheta_0 + d/p_0$, $\widetilde{\eta_0} = \widetilde{\vartheta}_0 + d/\widetilde{p_0}$ *and* $\eta_1 = \vartheta_1 + d/p_1$. *Assume that* $d \geq 2$ *and*

$$\begin{cases} d < p_0, \widetilde{p_0} \leq +\infty \\ \dfrac{2}{p_1} < \min\left\{\dfrac{1}{p_0} + \dfrac{1}{d}\,;\, \dfrac{1}{\widetilde{p_0}} + \dfrac{1}{d}\right\} \\ \eta_0 \leq \min\left\{d + 1 - \varepsilon_{1/p_0}\,;\, 2\eta_1 - \varepsilon_{2\vartheta_1 - \vartheta_0}\,;\, 2\eta_1 + \dfrac{d}{p_0} - \dfrac{2d}{p_1}\right\} \\ \widetilde{\eta_0} \leq \min\left\{d + 1 - \varepsilon_{1/\widetilde{p_0}}\,;\, 2\eta_1 - \varepsilon_{2\vartheta_1 - \widetilde{\vartheta}_0}\,;\, 2\eta_1 + \dfrac{d}{\widetilde{p_0}} - \dfrac{2d}{p_1}\right\}. \end{cases} \tag{4.2}$$

Let T^* *and* $\widetilde{T}$ *be the lifetimes of the solution* (u, B) *of (mhd) emanating from* (u_0, B_0) *in the respective weighted spaces, i.e.*

$$\begin{aligned} T^* &= \sup\left\{ T > 0 \ \text{s.t.}\ (u,B) \in \mathcal{C}([0,T]; L^{p_0}_{\vartheta_0} \times L^{p_1}_{\vartheta_1}) \right\}, \\ \widetilde{T} &= \sup\left\{ T > 0 \ \text{s.t.}\ (u,B) \in \mathcal{C}([0,T]; L^{\widetilde{p_0}}_{\widetilde{\vartheta}_0} \times L^{p_1}_{\vartheta_1}) \right\}. \end{aligned}$$

Then $\widetilde{T} = T^*$.

Proof. The structure of the proof is similar to that of [14]. Let us assume that we have, for example, $\widetilde{T} < T^*$. Unicity of mild solutions ensures that they agree on $[0, \widetilde{T}[$. We are going to prove that

$$\sup_{t \in [0, \widetilde{T}[} \left(\|u(t)\|_{L^{\widetilde{p_0}}_{\widetilde{\vartheta}_0}} + \|B(t)\|_{L^{p_1}_{\vartheta_1}} \right) < +\infty.$$

Then (4.1) would imply that the mild solution (u, B) in $L^{\widetilde{p_0}}_{\widetilde{\vartheta}_0} \times L^{p_1}_{\vartheta_1}$ could be extended beyond $\widetilde{T}$, and that would contradict the definition of $\widetilde{T}$.

First of all, let us recall (see, *e.g.*, [14, §2.2]) that there exists a constant $C_0 > 0$ depending only on d and ϑ, such that

$$\sup_{\tau \in [0, \widetilde{T}]} \|e^{\tau\Delta} v\|_{L^{p_1}_{\vartheta_1}} \leq C_0\,(1 + \widetilde{T})^{\vartheta_1/2}\, \|v\|_{L^{p_1}_{\vartheta_1}}. \tag{4.3}$$

In the following, we set $A = C_0\,(1 + \widetilde{T})^{\vartheta_1/2}$.

Note also that we can obviously assume that $u \not\equiv 0$ in $[0, \widetilde{T}]$.

The bound for B. By the second of the integral equations (IE), one has for $0 \leq s \leq t < \widetilde{T}$:

$$B(t) = e^{(t-s)\Delta} B(s) - \int_s^t G(t-\tau) * (u \otimes B - B \otimes u)(\tau)\, d\tau.$$

Proposition 3.1 applied to the upper bound of G given by (2.6), with $\epsilon = 1 - \frac{d}{p_0}$ in (3.4), yields

$$\forall\, \tau \leq t \leq \widetilde{T}, \qquad \|G(t-\tau) * (u \otimes B)(\tau)\|_{L^{p_1}_{\vartheta_1}} \leq K(t-\tau)^{-\sigma} \|\,(u \otimes B)(\tau)\|_{L^{\mathbb{H}(p_0,p_1)}_{\vartheta_0+\vartheta_1}}$$

where $\sigma = \frac{1}{2}(1+\frac{d}{p_0})$ and K is a constant, possibly depending on T^* and all the parameters contained in (4.2), but not on $\widetilde{T}$. Note that $\sigma < 1$. Thus, for all $t \in [0;\widetilde{T}]$,

$$\|B(t)\|_{L^{p_1}_{\vartheta_1}} \le A\,\|B(s)\|_{L^{p_1}_{\vartheta_1}} + K\,\frac{(t-s)^{1-\sigma}}{1-\sigma} \sup_{\tau\in[s,t]} \|u(\tau)\|_{L^{p_0}_{\vartheta_0}} \cdot \sup_{\tau\in[s,t]} \|B(\tau)\|_{L^{p_1}_{\vartheta_1}}\,. \tag{4.4}$$

Now let $(T_n)_{n\ge 0}$ be the increasing sequence defined by

$$T_n = n\Delta \qquad \text{with} \qquad \Delta = \left(\frac{2K}{1-\sigma} \sup_{\tau\in[0,\widetilde{T}]} \|u(\tau)\|_{L^{p_0}_{\vartheta_0}}\right)^{-1/(1-\sigma)}$$

and $N\in\mathbb{N}$ such that $T_N \le \widetilde{T} < T_{N+1}$. For $0\le n\le N$, let I_n be the interval $[T_n, T_{n+1}]\cap[0,\widetilde{T}[$ and

$$M_n = \sup_{\tau\in I_n} \|B(\tau)\|_{L^{p_1}_{\vartheta_1}}.$$

Applying (4.4) with $s = T_n$ and $t\in I_n$ for $n = 0,\ldots,N$, we get

$$M_0 \le 2A\|B_0\|_{L^{p_1}_{\vartheta_1}} \qquad \text{and} \qquad M_n \le 2AM_{n-1} \quad (1\le n\le N),$$

whence

$$\sup_{t\in[0,\widetilde{T}[} \|B(t)\|_{L^{p_1}_{\vartheta_1}} = \max_{0\le n\le N} M_n \le (2A)^{N+1}\|B_0\|_{L^{p_1}_{\vartheta_1}}.$$

Finally, this leads to :

$$\sup_{t\in[0,\widetilde{T}[} \|B(t)\|_{L^{p_1}_{\vartheta_1}} \le C\,\|B_0\|_{L^{p_1}_{\vartheta_1}} \exp\left(\left(1+\widetilde{T}\sup_{s\in[0,\widetilde{T}]} \|u(s)\|_{L^{p_0}_{\vartheta_0}}^{2/(1-\frac{d}{p_0})}\right)\left(1+\vartheta_1\log(1+\widetilde{T})\right)\right). \tag{4.5}$$

The right-hand side is finite because we assumed $\widetilde{T} < T^*$.

The bound for u. For $0\le s\le t<\widetilde{T}$, one has

$$u(t) = e^{(t-s)\Delta}u(s) - \int_s^t F(t-\tau)*(u\otimes u)(\tau)\,d\tau + \int_s^t F(t-\tau)*(B\otimes B)(\tau)\,d\tau.$$

Proposition 3.1, applied this time to the upper bound of F given by (2.6), yields

$$\begin{aligned}\|u(t)\|_{L^{\widetilde{p}_0}_{\widetilde{\vartheta}_0}} \le A\|u(s)\|_{L^{\widetilde{p}_0}_{\widetilde{\vartheta}_0}} &+ K\,\frac{(t-s)^{1-\sigma}}{1-\sigma}\sup_{\tau\in[s,t]}\|u(\tau)\|_{L^{p_0}_{\vartheta_0}}\cdot\sup_{\tau\in[s,t]}\|u(\tau)\|_{L^{\widetilde{p}_0}_{\widetilde{\vartheta}_0}}\\ &+ K\,\frac{(t-s)^{1-\widetilde{\sigma}}}{1-\widetilde{\sigma}}\left(\sup_{\tau\in[s,t]}\|B(\tau)\|_{L^{p_1}_{\vartheta_1}}\right)^2\end{aligned}$$

with $\sigma = \frac{1}{2}(1+\frac{d}{p_0})$ and $\widetilde{\sigma} = \frac{1}{2}(1+(\frac{2d}{p_1}-\frac{d}{p_0})^+)$. Note that σ is the same as before and that $\widetilde{\sigma} < 1$; K depends on T^* and all the parameters, except $\widetilde{T}$. The last term is uniformly bounded by

$$L = \frac{K\,\widetilde{T}^{1-\widetilde{\sigma}}}{1-\widetilde{\sigma}}\left(\sup_{\tau\in[0,\widetilde{T}[}\|B(\tau)\|_{L^{p_1}_{\vartheta_1}}\right)^2$$

which is a finite constant because (4.5) holds. Define $(T_n)_{n\geq 0}$ and I_n as before. Let also

$$\widetilde{M}_n = \sup_{\tau\in I_n} \|u(\tau)\|_{L^{\widetilde{p}_0}_{\widetilde{\vartheta}_0}}.$$

Recall that N is the integer part of $\widetilde{T}/\Delta$. Then, for $1 \leq i \leq N$, one has

$$\widetilde{M}_0 \leq 2A\,\|u_0\|_{L^{\widetilde{p}_0}_{\widetilde{\vartheta}_0}} + 2L \qquad \text{and} \qquad \widetilde{M}_n \leq 2A\,\widetilde{M}_{n-1} + 2L,$$

hence

$$\sup_{t\in[0,\widetilde{T}[} \|u(t)\|_{L^{\widetilde{p}_0}_{\widetilde{\vartheta}_0}} = \max_{0\leq n\leq N} \widetilde{M}_n \leq (2A)^{N+1}\,\|u_0\|_{L^{\widetilde{p}_0}_{\widetilde{\vartheta}_0}} + 2L\left[1+\ldots+(2A)^{N-1}+(2A)^N\right] < +\infty.$$

Combined with (4.1) and (4.5), this estimate ensures that $\widetilde{T} \geq T^*$. Exchanging the roles of $\widetilde{T}$ and T^*, one finally obtains that $\widetilde{T} = T^*$. □

An analogous result holds if we assume instead $u_0 \in L^{p_0}_{\vartheta_0}(\mathbb{R}^d)$ and $B_0 \in L^{p_1}_{\vartheta_1}(\mathbb{R}^d)\cap L^{\widetilde{p}_1}_{\widetilde{\vartheta}_1}(\mathbb{R}^d)$, with obvious modifications in (4.2) :

$$\begin{cases} d < p_0 \leq +\infty \\ \max\left\{\dfrac{2}{p_1}\,;\,\dfrac{2}{\widetilde{p}_1}\right\} < \dfrac{1}{p_0} + \dfrac{1}{d} \\ \eta_0 \leq \min\left\{d+1-\varepsilon_{1/p_0}\,;\,2\eta_1 - \varepsilon_{2\vartheta_1-\vartheta_0}\,;\,2\widetilde{\eta}_1 - \varepsilon_{2\vartheta_1-\vartheta_0}\right\}. \\ \eta_0 \leq \min\left\{2\eta_1 + \dfrac{d}{p_0} - \dfrac{2d}{p_1}\,;\,2\widetilde{\eta}_1 + \dfrac{d}{p_0} - \dfrac{2d}{\widetilde{p}_1}\right\}. \end{cases} \tag{4.2'}$$

Theorem 1.3 is now established.

4.3 The proof of Theorem 1.1

Let p_0, p_1 and ϑ_0, ϑ_1 such that (1.6a) and (1.6b) hold.

If $\vartheta_0 \leq 2\vartheta_1$, $p_0 \leq d/\delta - \varepsilon_\delta$ and $\eta_0 \leq d+1-\varepsilon_{1/p_0}$, then (1.10a) and (1.11a) hold, and there is nothing more to prove since Theorem 1.3 already gives a stronger conclusion.

In all the other cases and for any $\epsilon > 0$, our assumptions yield an embedding $L^{p_0}_{\vartheta_0} \subset L^q_\mu$ such that Theorem 1.3 may be applied to

$$(u_0, B_0) \in L^q_\mu \times L^{p_1}_{\vartheta_1}$$

and with

$$\mu + \frac{d}{q} = \eta_0 - \epsilon.$$

It follows that $u \overset{L^2}{=} \mathcal{O}(|x|^{-(\eta_0-\epsilon)})$ and $B \overset{L^2}{=} \mathcal{O}(|x|^{-\eta_1})$ when $|x| \to +\infty$. Letting $\epsilon \to 0$, this will conclude the proof of Theorem 1.1.

Let us be more precise about the embedding $L^{p_0}_{\vartheta_0} \subset L^q_\mu$. Actually, various choices are possible for (q,μ). We have chosen the indices that are represented on the interpolation diagram (see Fig. 1 p. 5) by a *dash-dotted* line.

If the magnetic field decays sufficiently fast, namely if $\eta_1 \geq (d+1+\delta)/2$, the only case not included in Theorem 1.3 is that of $\eta_0 = d+1$ with p_0 finite. In this case, one may take

$$(q,\mu) = (p_0, \vartheta_0 - \epsilon).$$

Let us now assume that $\eta_1 \leq (d+1+\delta)/2$ and, for the moment, that $p_1 \geq 2d$. Then the cases to be dealt with correspond either to $\vartheta_0 > 2\vartheta_1$ or to $\eta_0 = 2\eta_1$, or to both.

– If $\vartheta_0 > 2\vartheta_1$, then

$$\frac{d}{q} = \vartheta_0 - 2\vartheta_1 + \frac{d}{p_0} - \epsilon \qquad \text{and} \qquad \mu = 2\vartheta_1$$

are suitable, even if $\eta_0 = 2\eta_1$.

– If $\vartheta_0 \leq 2\vartheta_1$ and $\eta_0 = 2\eta_1$, one may again choose $(q,\mu) = (p_0, \vartheta_0 - \epsilon)$.

Finally, if $d < p_1 < 2d$ and $\eta_1 \leq (d+1+\delta)/2$, one may use the following barrier :

$$\frac{d}{q} = 1 - (1-\delta)\kappa \qquad \mu = 2\vartheta_1(1-\kappa) \quad \text{and} \quad \kappa = 1 - \frac{\eta_0 - \delta - \epsilon}{2(\eta_1 - \delta)}.$$

The proof of Theorem 1.1 is now complete. □

5 Instantaneous spreading of rapidly decreasing fields

This section is included for completeness and contains the proof of theorem 1.2, and some remarks about exceptional solutions to (MHD) that decay extremely fast.

5.1 Proof of theorem 1.2

Following [2], we define E as the space of all functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\|f\|_E \underset{\mathrm{def}}{=} \int_{|x|\leq 1} |f(x)|\,dx + \sup_{R\geq 1} R \int_{|x|\geq R} |f(x)|\,dx \tag{5.1}$$

is finite, and

$$\lim_{R\to+\infty} R \int_{|x|\geq R} |f(x)|\,dx = 0.$$

Hölder inequality implies that :

$$L^{p_0}_{\vartheta_0}(\mathbb{R}^d) \subset E \qquad \text{whenever} \qquad \begin{cases} \vartheta_0 + \dfrac{d}{p_0} \geq d+1 \quad (p_0 < +\infty) \quad \text{or} \\ \vartheta_0 > d+1 \quad (p_0 = +\infty). \end{cases}$$

Let us prove that $\|u\|_E$ cannot remain uniformly bounded during a positive time interval, unless the orthogonality relations (1.9) are satisfied.

Proposition 5.1 *Let $(u,B) \in \mathcal{C}([0,T]; L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ a solution to (MHD) such that $u_0 \in E$. Assume that*

$$u \in L^\infty([0,T]; E) \tag{5.2a}$$

$$|u|^2 + |B|^2 \in L^\infty([0,T]; E). \tag{5.2b}$$

Then there exists a constant $c \geq 0$ such that the components of the initial data satisfy

$$\forall j,k \in \{1,\dots,d\}, \qquad \int_{\mathbb{R}^d} u_0^j u_0^k - B_0^j B_0^k = c\,\delta_{j,k}, \tag{5.3}$$

where $\delta_{j,k} = 1$ if $j = k$ and 0 otherwise.

Proof. The proof will only be sketched briefly since it is a straightforward adaptation of [2]. Let us write the first equation of (MHD) in the following form (recall that S and R_e can be set equal to 1):

$$u(t) - e^{t\Delta}u_0 + \sum_{j=1}^{d} \int_0^t e^{(t-s)\Delta}\,\partial_j(u^j u - B^j B)\,ds = -\int_0^t e^{(t-s)\Delta}\,\nabla P(s)\,ds, \tag{5.4}$$

where $P = p + \frac{|B|^2}{2}$ is the total pressure. Arguing as in [2], we see that (5.2) imply that all the terms in the left-hand side of (5.4) belong to $L^\infty([0,T];E)$. Thus, we have

$$\nabla\widetilde{P} \in L^\infty([0,T];E) \qquad \text{with} \qquad \widetilde{P}(t) = \int_0^t e^{(t-s)\Delta}\,P(s)\,ds.$$

Let

$$\widetilde{u}^{j,k}(t) = \int_0^t e^{(t-s)\Delta}\,u^j u^k(s)\,ds$$

and

$$\widetilde{B}^{j,k}(t) = \int_0^t e^{(t-s)\Delta}\,B^j B^k(s)\,ds.$$

Taking the divergence in (5.4) yields

$$-\Delta\widetilde{P} = \sum_{j,k=1}^{d} \partial_j\partial_k(\widetilde{u}^{j,k} - \widetilde{B}^{j,k}).$$

One now deduces (5.3), applying Lemma 2.3 and Proposition 2.4 of [2]. □

The proof of Theorem 1.2 is now very easy. Thanks to (1.3) and (1.4), assumptions (1.8a) and (1.8b) imply the existence of $\varepsilon' > \varepsilon'' > 0$ such that

$$\sup_{t\in[0,T]} |u(t,\cdot)| \in L^2_{\frac{d}{2}+1+\varepsilon'} \subset L^1_{1+\varepsilon''} \subset E.$$

Moreover, the definition of the L^2 *decay rate at infinity* (1.1) implies that

$$\lim_{R\to\infty} R^{d+2+2\varepsilon'} \int_{R\leq|x|\leq 2R} |u(t,x)|^2\,dx = 0$$

and

$$\lim_{R\to\infty} R^{1+\varepsilon'} \int_{R\leq|x|\leq 2R} |B(t,x)|^2\,dx = 0,$$

uniformly for $t \in [0,T]$. Therefore

$$\sup_{t\in[0,T]} \big(|u(t,\cdot)|^2 + |B(t,\cdot)|^2\big) \in L^1_{1+\varepsilon''} \subset E.$$

Conclusion (1.9) now follows from proposition 5.1.

5.2 Solutions of (MHD) with an exceptional spatial behavior

We finally observe that solutions that decay faster than predicted by Theorem 1.3 do exist.

Such solutions can be constructed starting with properly symmetric initial data. Assume, *e.g.*, that u_0 and B_0 are rapidly decreasing in the usual pointwise sense when $|x| \to +\infty$ (faster than any inverse polynomial) and that $Au_0(x) = u_0(Ax)$, $AB_0(x) = B_0(Ax)$ for all $x \in \mathbb{R}^d$ and all matrix $A \in G$, where G is a subgroup of the orthogonal group $O(d)$. Then the solution of (MHD) will inherit this property as far as it exists, the system being invariant under rotations. If the group G is rich enough, then these symmetry relations ensure the validity of conditions (1.9). Moreover the decay rate of the velocity field of the corresponding solution will depend on the symmetry group to which (u_0, B_0) belongs.

In dimension $d = 2, 3$ and for the Navier–Stokes equations, the optimal decay rates of the solution have been computed in [1] for each symmetry group. With simple modifications in the proofs, one could show that *the same* decay rates hold for the solution of (MHD). This is not surprising: indeed, since the magnetic field decays fast when $|x| \to +\infty$, the decay of the velocity field is governed only by the decay rate of the kernels $F^k_{j,h}$, defined by (2.1), and by the possible corresponding cancellations. These kernels are the same ones that appear in the Navier–Stokes system as well.

Thus, for example, in dimension $d = 2$ and when G is the cyclic group of order n, one has

$$\forall t \in [0, T^*), \qquad u(t,x) = \mathcal{O}(|x|^{-(n+1)})$$

in the usual pointwise sense, when $|x| \to +\infty$. In particular, the property of being simultaneously completely invariant under rotations (*i.e.* $G = SO(2)$) and rapidly decreasing at infinity will be conserved by (u, B) during the evolution, if such property already holds for (u_0, B_0).

In dimension three, the largest decay rates of the velocity field (*i.e.* like $|x|^{-8}$ as $x \to +\infty$) are obtained with the symmetry groups of the icosahedron. Those symmetric solutions are however unstable: in general, the velocity field of an infinitesimal perturbation of a highly symmetric flow will decay much more slowly at infinity.

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