Far field geometric structures of 2D flows with localised vorticity *

Lorenzo Brandolese†

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Abstract

We show that 2D Navier–Stokes and Euler flows with localised vorticity always feature regular structures in the far field. The level lines of each component of the velocity, at large distances, tend to have the symmetries of a regular polygon: a digon if the total circulation is non-zero; a square for flows with zero total circulation and non-integrable velocity; an hexagon for flows with integrable velocity and, exceptionally, a polygon with more than six sides.

1 Introduction

In this paper we establish a spatial asymptotic expansion for two-dimensional Biot–Savart integrals

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\omega(y)\,\mathrm{d}y,$$

where $x^\perp=(-x_2,x_1)^t$. We show that, under appropriate moment conditions on ω, the above integral can be written, for $|x|\to+\infty$, as

$$\sum_{n=1}^{m}\frac{A_n}{|x|^n}\begin{pmatrix}\cos(n\theta+\phi_n)\\ \sin(n\theta+\phi_n)\end{pmatrix}+o(|x|^{-m}),\tag{1.1}$$

for some amplitudes $A_1,\dots,A_m$ and phases $\phi_1,\dots,\phi_m$. Here θ is the argument of $x\in\mathbb{R}^2$, $x\neq0$. All the terms in (1.1) are divergence-free. Such expansion has a few remarkable geometric consequences for solutions of the two dimensional Euler and Navier–Stokes equations. Indeed, it allows us to give the precise geometric description of the polygonal symmetries of level lines of the solutions at the spatial infinity.

Consider, for example, the vorticity formulation of the 2D Navier–Stokes equations

$$\begin{cases}\partial_t\omega+(u\cdot\nabla)\omega=\Delta\omega,\\ \omega(x,0)=\omega_0(x).\end{cases}\qquad x\in\mathbb{R}^2,t>0,\tag{NS}$$

with the velocity field u given by the Biot–Savart law

$$u(x,t)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\omega(y,t)\,\mathrm{d}y,\qquad x\in\mathbb{R}^2.\tag{BS}$$

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†Institut Camille Jordan, Université de Lyon, Université Lyon 1, 43 bd. du 11 Novembre 69622 Villeurbanne Cedex, France

Let us recall that if $\omega_0 \in L^1(\mathbb{R}^2)$, then the Navier–Stokes equations (NS) possess a unique solution $\omega \in C^0([0,\infty), L^1(\mathbb{R}^2)) \cap C^0((0,\infty), L^\infty(\mathbb{R}^2))$, with velocity field given by (BS). See [1, 7, 11, 12]. In what follows we will consider this solution.

The application of our expansion gives our main result:

Theorem 1.1. *Let $m \in \mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Let $\omega_0 \in L^1(\mathbb{R}^2, (1+|x|)^m\,\mathrm{d}x)$ and ω be the unique solution of (NS) with velocity u given by (BS). Then, for any $t > 0$, there exist $A_n(t) \geq 0$ and $\phi_n(t) \in \mathbb{R}$ ($n = 1, \dots, m$) such that, for $x \neq 0$,*

$$u(x,t) = \sum_{n=1}^{m} \frac{A_n(t)}{|x|^n} \begin{pmatrix} \cos(n\theta + \phi_n(t)) \\ \sin(n\theta + \phi_n(t)) \end{pmatrix} + R(x,t), \tag{1.2}$$

for some remainder term $R(x,t) = o(|x|^{-m})$ as $|x| \to +\infty$, such that

$$|R(x,t)| \leq t^{-1/2}(1 + t^{m/2})|x|^{-m}\varepsilon(x),$$

where $\varepsilon(x)$ is a time-independent function with vanishing limit as $|x| \to \infty$.

The amplitudes A_1 and A_2 and the phases ϕ_1 and ϕ_2 are in fact independent on time and the following formulas hold:

$$A_1(t) = \tfrac{1}{2\pi}|\textstyle\int \omega_0|, \qquad \phi_1 = \tfrac{\pi}{2}$$
$$A_2(t) = \tfrac{1}{2\pi}|\textstyle\int y\omega_0|, \qquad \phi_2 = \tfrac{\pi}{2} - \gamma_0,$$

where γ_0 denotes the argument of $\int y\omega_0$, when this is not the zero vector. If $\int \omega_0 \neq 0$, i.e. if the flow has non-zero total circulation, then it is often convenient to choose the center of the vorticity as the origin of the coordinates. This choice ensures that $A_2 = 0$.

An attractive feature of Theorem 1.1 is the elementary character of the proof. This relies on two basic ingredients: the application of Carlen and Loss optimal size estimates of damped conservation laws and the analysis of the far-field behavior of convolution-type integrals in Lorentz spaces.

The conclusion of the Theorem 1.1 can be reached also when the conditions on ω_0 are replaced by

$$\omega_0 \in L^1(\mathbb{R}^2, (1+|x|)^{m-1}\,\mathrm{d}x) \qquad \text{and} \qquad |x|^m\omega_0 \in L^{2,1}(\mathbb{R}^2),$$

where $L^{2,1}$ denotes the Lorentz space. Replacing the condition $|x|^m\omega_0 \in L^1(\mathbb{R}^2)$ with the condition $|x|^m\omega_0 \in L^{2,1}(\mathbb{R}^2)$ requires slightly more regularity, but it is less demanding from the spatial localization point of view. If we additionally assume $\omega_0 \in L^{2,1}(\mathbb{R}^2)$, then in the estimate for $R(x,t)$ we can replace the factor $t^{-1/2}(1 + t^{m/2})$ by the factor $(1+t)^{(m-1)/2}$: in this case, our spatial decay estimates become uniform in time near $t = 0$.

The velocity u thus admits, for any fixed $t > 0$, an asymptotic expansion for $|x| \to +\infty$ of the form

$$u(x,t) = \sum_{n=1}^{m} \frac{V_n(x,t)}{|x|^n} + o(|x|^{-m}), \tag{1.3}$$

(with V_1 and V_2 time-independent) where $V_n(\cdot,t)$ is a bounded homogeneous vector fields of degree zero, of the form $V_n(x,t) = H_n(x,t)/|x|^n$, and where $H_n(\cdot,t)$ is a pair of homogeneous harmonic polynomials. Expansions in the spirit of (1.3) were obtained, e.g., in [5, 6, 13, 15, 16, 18]. But the fact that $V_n(x,t)$ satisfies the very simple formula

$$V_n(x,t) = A_n(t) \begin{pmatrix} \cos(n\theta + \phi_n(t)) \\ \sin(n\theta + \phi_n(t)) \end{pmatrix} \tag{1.4}$$

remained unnoticed in the previous works on the Navier–Stokes equations: the geometric consequences of (1.2)-(1.4) will be illustrated in Fig. 1.

In [15, Chapt. 6], a development was just carried to the second order. In [5,6] a development of the form $u(x,t) \sim V_3(x,t)/|x|^3$ was obtained with a different method under decay assumptions on the initial velocity (such decay assumptions imply the vanishing of V_1 and V_2, and do not cover, e.g., the case of infinite energy solutions). In [13], Kukavica and Reis obtained an asymptotic profile of any order for u for $|x|$ or $t \to +\infty$. The terms appearing in Kukavica and Reis' development are written as linear combinations of $\partial^\beta R_i R_j G$, where $G(x) = (2\pi)^{-1} e^{-|x|^2/4}$ is the normalised Gaussian and R_i, R_j the Riesz transforms; these terms are not made explicit.

The papers [16–18] deal with inviscid flows. In [16], McOwen and Topalov obtain an asymptotic expansion for a class of solutions to the Euler equation in terms of inverse powers of $|x|$ and $\log |x|$. In [18, Corollary 1.2], Sultan and Topalov put in evidence a subclass of solutions inside which the logarithmic terms can be dropped: the resulting asymptotic expansion in [18] is essentially the same as ours (1.2)–(1.4), but their argument more involved. The recent preprint [17] covers the case of flows possibly growing at infinity in $\mathbb{R}^d$ ($d \geq 2$). The approach of [16–18] relies on well-posedness results of the Euler equations in suitable weighted Sobolev spaces and differs considerably in the technical aspects from the present paper. Its adaptation when the viscosity is taken into account is not straightforward.

Expanding the Biot–Savart integral as in (1.1) has immediate applications to solutions of the Euler equations. The following result is just another possible illustration of our analysis on the Biot-Savart law. Let us recall that, if $\omega_0 \in L^\infty_c(\mathbb{R}^2)$ (the space of L^∞ and compactly supported functions), then the Cauchy problem for Euler equation in $\mathbb{R}^2$,

$$\partial_t \omega + (u \cdot \nabla)\omega = 0,$$

with u given by (BS) and initial data ω_0, possess a unique weak solution $\omega \in L^\infty([0,\infty), L^\infty_c(\mathbb{R}^2))$. See, e.g., [2, Chapt. 8]. The simple statement below should be compared with the more technical result in [18].

Theorem 1.2. *Let $\omega_0 \in L^\infty_c(\mathbb{R}^2)$ and $\omega \in L^\infty([0,\infty), L^\infty_c(\mathbb{R}^2))$ be the weak solution of the Euler equation starting from ω_0, with velocity u given by (BS). Then, for all positive integer m, expansion* (1.2) *holds for u, with a remainder function R depending on m, such that*

$$|R(x,t)| \leq (1+t)^m |x|^{-m} \epsilon_m(x),$$

where $\epsilon_m(x) \to 0$ as $|x| \to \infty$ is independent on $t > 0$. Moreover, in the case $\omega_0 \geq 0$, the factor $(1+t)^m$ can be replaced by $((1+t)\log(e+t))^{m/4}$.

Geometric consequences of expansion (1.2)

We will discuss the consequences of our analysis in the case of the Navier–Stokes equations. The most obvious application of formula (1.2) is the following: the velocity has an algebraic decay and the speed $|u(\cdot,t)|$, at a fixed time $t > 0$, of fluid particles tends to be constant on circles $\{x \in \mathbb{R}^2 \colon |x| = R\}$ of large radii.

Corollary 1.3. *Let $m \in \mathbb{N}^*$ and $\omega_0 \in L^1(\mathbb{R}^2, (1+|x|)^m\,\mathrm{d}x)$. Let ω be the corresponding solution to (NS). Then, for any $t > 0$, the limits*

$$\ell_a(t) := \lim_{|x| \to +\infty} |x|^a \big| u(x,t) \big| \qquad (0 \leq a \leq m) \tag{1.5}$$

do exist, with $0 \leq \ell_a(t) \leq \infty$.

In particular, for $0 \le a \le m$ and all $\sigma \in \mathbb{S}^1$, the radial limits $\lim_{R\to+\infty} R^a\big|u(R\sigma,t)\big|$ are independent on σ. Notice that, if $0 < \ell_a(t) < \infty$, then a is necessarily an integer.

Let us assume, as before, that $\omega_0 \in L^1(\mathbb{R}^2,(1+|x|)^m\,\mathrm{d}x)$, for some $m \in \mathbb{N}^*$. For any $t > 0$, we define

$$\kappa(t) = \min\{k \in \mathbb{N}^* \colon 0 < \ell_k(t) < \infty\}, \tag{1.6}$$

where $\ell_k(t)$ is the limit given by (1.5). In the very particular case in which the moments of ω_0 are finite to *any order* and $\ell_k(t) = 0$ for all $k \in \mathbb{N}^*$, we set $\kappa(t) = +\infty$. The value of the parameter $\kappa(t)$ has deep geometrical implications. Let us first discuss some examples:

- If the total circulation is non-zero, i.e., if $\int \omega_0 \neq 0$, then $\kappa(t) \equiv 1$ for all $t > 0$. This corresponds to infinite energy flows, as $u(\cdot,t) \notin L^2(\mathbb{R}^2)$.
- If the total circulation is zero, but at least one of the two moments $\int x_j\omega_0(x)\,\mathrm{d}x$ $(j = 1,2)$ does not vanish, then $\kappa(t) \equiv 2$. Notice that in this case the flow has finite energy, but $u(\cdot,t) \notin L^1(\mathbb{R}^2)$.
- If the moments of ω_0 vanish up to the first order, then $\kappa(t) \ge 3$ for all $t > 0$. In this case, one generically expects $\kappa(t) \equiv 3$. In fact, one can put a "non-symmetry assumption" on ω_0 and deduce that $\kappa(t) \equiv 3$ at least in some nontrivial time interval $[0,T_0]$. However, it can also happen for some quite particular flows that $\kappa(t)$ does depend on time. Indeed, in [4], the author constructed special solutions featuring un arbitrarily large number of "concentration-diffusion" effects during the evolution: nontrivial flows were constructed that are "spatially concentrated" at some given times $t_1, t_2, \ldots$. For such flows $\kappa(t_j) \ge 4$ and $\kappa(t) = 3$ for $t \neq t_j$ $(j = 1,2\ldots)$. This phenomenon is of course related to the fact that the higher-order moments of the vorticity are non-constant in time.
- For any $k \in \mathbb{N}^*$, nontrivial examples of solutions such that $\sup_{t>0}|u(\cdot,t)| = O(|x|^{-k})$ as $|x| \to +\infty$ were constructed in [3] imposing discrete rotational symmetries on the flow (the invariance under a cyclic group of rotations of order $k-1$). For such flows, one generically has $\kappa(t) \equiv k$ (k can by abritrarily large).
- Inside the narrow class of planar flows with radial vorticity one easily constructs solutions such that $\kappa(t) \equiv +\infty$. Indeed, one just needs to start from a radial function $\omega_0 \in \mathcal{S}(\mathbb{R}^2)$, such that $0 \notin \operatorname{supp}(\widehat{\omega}_0)$ to obtain a solution of the Navier–Stokes equation invariant under rotations (with trivial nonlinearity: $\mathbb{P}\nabla\cdot(u\otimes u) \equiv 0$), that is a solution of the heat equation as well, such that the velocity belongs to the Schwartz class for all $t > 0$. This is essentially the only known example of flow such that $\kappa(t) = +\infty$.

Formula (1.4) reveals that each component of $V_n(\cdot,)$ vanishes exactly on n straight lines passing through the origin (unless $V_n(\cdot,t)$ identically vanishes). The intersections of these lines with the unit circle are the $2n$ complex roots of an equation of the form $z^{2n} = e^{i\theta_n(t)}$, for some (time-dependent, if $n \ge 3$, or constant-in-time, if $n = 1,2$) angle $\theta_n(t)$. This implies the following:

> *Each component v of the velocity field decays exactly like $|x|^{-\kappa}$ as $|x| \to +\infty$, excepted for 2κ exceptional directions, along which the decay is faster. Moreover, for all $t > 0$ the level lines of $|v|$ at the spatial infinity, tend to have the symmetry of a regular polygon with 2κ-sides.*

Here $\kappa = \kappa(t)$ is given by (1.6). The polygon thus degenerates to a digon when $\kappa = 1$.

Let u_0 be the velocity field associated with the initial vorticity ω_0, obtained applying (BS) with $t = 0$. Our last corollary shows that the components of $u(\cdot,t) - u_0$ naturally have an *hexagonal structure*.

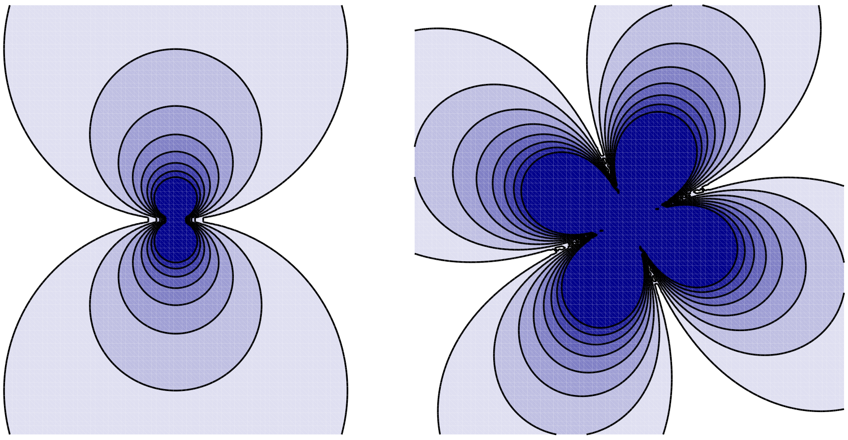

Figure 1: Typical shape of the level sets of $|u_1(\cdot,t)|$ (the horizontal speed of the fluid) for a fixed time $t>0$. The curves in the figure are obtained neglecting the lower-order terms at the spatial infinity. *Left:* the case of flows with non-zero total circulation ($A_1 \neq 0$), *i.e.*, infinite energy flows. *Middle:* the case of flows with zero total circulation but non-zero first moments ($A_1 = 0$, $A_2 \neq 0$). This is the case of flows with non-integrable velocity. *Right:* Typical level lines of flows with integrable velociy: $A_1 = A_2 = 0$. The right picture also illustrates the typical shape of the level sets of the difference $|u_1(\cdot,t) - u_{0,1}|$, for any flow (with zero or non-zero total circulation, and integrable or non-integrable velocity), provided the vorticity is well localised, and no special symmetry is initially prescribed on ω_0. The hexagonal structures (and higher-order structures not represented here) rotate during the evolution. The digonal and the quadrilobe structures, after a suitable time-dependent rescaling, remain in a fixed position.

Corollary 1.4. *Let $\omega_0 \in L^1(\mathbb{R}^2,(1+|x|)^2\,\mathrm{d}x)$ such that $|x|^3\omega_0 \in L^{2,1}(\mathbb{R}^2)$. Let ω the solution to (NS) and u the corresponding velocity. Let $(|x|,\theta)$ be the polar coordinates of $x \in \mathbb{R}^2$. Then, for any $t>0$, there exist $L(t) \geq 0$ and $\phi(t) \in \mathbb{R}$ (depending only on the second-order moments of ω) such that, as $|x| \to \infty$,*

$$u(x,t) - u_0(x) = \frac{L(t)}{|x|^3}\begin{pmatrix}\cos(3\theta+\phi(t))\\ \sin(3\theta+\phi(t))\end{pmatrix} + o(|x|^{-3}). \tag{1.7}$$

The formation of hexagonal structures for the component of $u(x,t) - u_0$ was first recently observed in a companion paper of the present work [5]. Therein, the focus was on Leray's solutions, with possibly non-localised vorticities. The result of Corollary 1.4 cannot be directly obtained from the main result of [5] (and conversely) even though the conclusions are comparable, because the assumptions are quite different. Working with Leray's solutions, allows to characterise $L(t)$ in terms of quadratic moments $\int_0^t\!\int u_j u_k$, that are well defined for L^2-velocities; this is useful to estimate the angular speed of the hexagonal structures during the evolution; another advantage is that one can encompass solutions with non-algebraically decaying velocity (e.g., $u(x,t) \simeq |x|^{-\alpha}$, with $1<\alpha<2$), which are (a fortiori) excluded by approach of the present paper. On the other hand, the vorticity approach developed in the present paper allows us to deal with infinite energy solutions and looks more adapted to compute higher-order asymptotics, as illustrated in Theorem 1.1.

The method that we use for the 2D Biot–Savart law goes through for studying the asymptotic expansion, as $|x| \to \infty$, of more general singular integrals. In particular, Proposition 2.3 below can be directly applied to the 3D Biot–Savart integral. This fact could be used to give an alternative proof of the asymptotic expansions for solutions of the 3D Navier–Stokes flows in the form (1.3), that were already obtained in [5,6,13,15]. However, in higher dimension the geometric implications of such expansions are less striking, because formula (1.4) is no longer valid.

2 Proof of the results

Let us recall that, for $1 < p < \infty$ and $1 \leq q \leq \infty$, a measurable function f belongs to the Lorentz space $L^{p,q}(\mathbb{R}^d)$ if and only if the map $t \mapsto t\big|\{|f| \geq t\}\big|^{1/p}$ belongs to $L^q(\mathbb{R}^+, \frac{\mathrm{d}t}{t})$. Or, equivalently, if and only if $2^j|E_j|^{1/p} \in \ell^q(\mathbb{Z})$, where $E_j = \{2^j \leq |f(x)| < 2^{j+1}\}$ and $|\cdot|$ denotes the Lebesgue measure. The quasi-norm $\|(2^j|E_j|^{1/p})_j\|_{\ell^q(\mathbb{Z})}$ is equivalent to a norm that makes $L^{p,q}(\mathbb{R}^d)$ a Banach space, with the same rescaling properties of the usual $L^p(\mathbb{R}^d)$-space. We refer to [14, Chapter 2] for the statement and a proof of the generalized Hölder and Young's inequality in Lorentz spaces.

Our results rely on the following proposition:

Proposition 2.1. *Let d be a positive integer, $0 < \vartheta < d$, and $g \in C^1(\mathbb{R}^d)\backslash\{0\})$ such that*

$$|g(x)| \leq C|x|^{-\vartheta} \qquad \text{and} \qquad |\nabla g(x)| \leq C|x|^{-\vartheta-1},$$

for some constant $C > 0$ independent on $x \neq 0$. Let $f \in L^1(\mathbb{R}^d)$ be such that $|x|^\vartheta f \in L^{d/(d-\vartheta),1}(\mathbb{R}^d)$. Then

$$(f * g)(x) = \Big(\int f\Big) g(x) + o(|x|^{-\vartheta}) \quad \text{as } |x| \to +\infty. \tag{2.1}$$

Proof. Let us write

$$f * g - \Big(\int f\Big) g = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},$$

Here,

$$\begin{aligned}
\mathrm{I}(x) &= \int_{|y|\leq|x|/2} \Big(g(x-y) - g(x)\Big) f(y)\,\mathrm{d}y,\\
\mathrm{II}(x) &= -\Big(\int_{|y|\geq|x|/2} f(y)\,\mathrm{d}y\Big) g(x),\\
\mathrm{III}(x) &= \int_{|y|\geq|x|/2,\ |x-y|\geq|x|/2} g(x-y) f(y)\,\mathrm{d}y,\\
\mathrm{IV}(x) &= \int_{|x-y|\leq|x|/2} g(x-y) f(y)\,\mathrm{d}y.
\end{aligned}$$

Applying the Taylor formula to $g(x-y) - g(x)$ and the decay condition on $|\nabla g|$, we can estimate

$$|I(x)| \leq C|x|^{-\vartheta-1} \int_{|y|\leq|x|/2} |y|\,|f(y)|\,\mathrm{d}x.$$

But, if we fix $0 < \epsilon < 1/2$, then

$$\begin{aligned}
|x|^{-1} \int_{|y|\leq|x|/2} |y|\,|f(y)|\,\mathrm{d}x &\leq \epsilon \int_{|y|\leq\epsilon|x|} |f(y)|\,\mathrm{d}x + \int_{\epsilon|x|\leq|y|} |f(y)|\,\mathrm{d}x\\
&\leq \epsilon\Big(\int |f|\Big) + o(1) \quad \text{as } |x| \to +\infty.
\end{aligned}$$

This implies $|\mathrm{I}|(x) = o(|x|^{-\vartheta})$ as $|x| \to +\infty$.

The fact that $|\mathrm{II}|(x) = o(|x|^{-\vartheta})$ as $|x| \to +\infty$ just follows from the dominated convergence theorem. The third term is also easy to treat, because we can use that $|g(x-y)| \le C|x|^{-\vartheta}$ when $|x-y| \ge |x|/2$ and conclude again by the the dominated convergence theorem that $|\mathrm{III}|(x) = o(|x|^{-\vartheta})$ as $|x| \to +\infty$.

Let us treat the fourth integral. For $x \neq 0$, we denote $B_x \subset \mathbb{R}^2$ the ball centered at x with radius $|x|/2$. Let

$$\phi_x(y) = |y|^{\vartheta}\, |f(y)|\, \mathbf{1}_{B_x}(y),$$

where $\mathbf{1}_E$ denotes the indicator function of the set E. Then, applying Young's inequality in Lorentz spaces, we get

$$\begin{aligned}|\mathrm{IV}(x)| &\le \int_{B_x} |g(x-y)|\, |f(y)|\, \mathrm{d}y \\ &\le C|x|^{-\vartheta} \int |g(x-y)|\, |y|^{\vartheta}\, |f(y)| \mathbf{1}_{B_x}(y)\, \mathrm{d}y \\ &\le C|x|^{-\vartheta} \|g\|_{L^{d/\vartheta,\infty}}\, \|\phi_x\|_{L^{d/(d-\vartheta),1}}.\end{aligned}$$

It then remains to prove that $\|\phi_x\|_{L^{d/(d-\vartheta),1}} \to 0$ as $|x| \to +\infty$. For this, we observe that

$$\|\phi_x\|_{L^{d/(d-\vartheta),1}} \simeq \sum_{j\in\mathbb{Z}} 2^j \Big|\{y \in B_x \colon 2^j \le |y|^{\vartheta}|f(y)| \le 2^{j+1}\}\Big|^{(d-\vartheta)/d}. \tag{2.2}$$

Let $|x_n| \to +\infty$ and denote

$$A_{n,j} = \cup_{k\ge n}\{y \in B_{x_k} \colon 2^j \le |y|^{\vartheta}|f(y)| \le 2^{j+1}\}$$

For all j, the sets $A_{n,j}$ decrease with $n \in \mathbb{N}$ and $\cap_{n\in\mathbb{N}} A_{n,j} = \emptyset$, because $B_{x_n} \subset \{x \colon |x| \ge |x_n|/2\}$ and $|x_n| \to +\infty$. Moreover,

$$A_{1,j} \subset \{y \in \mathbb{R}^d \colon 2^j \le |y|^{\vartheta}|f(y)| \le 2^{j+1}\}$$

and the latter, for all $j \in \mathbb{Z}$, is a set of finite Lebesgue measure, because $|y|^{\vartheta} f \in L^{d/(d-\vartheta),1}(\mathbb{R}^d)$. So, $|A_{n,j}| \to 0$ as $n \to +\infty$. This in turn implies that, for all $j \in \mathbb{Z}$,

$$Q_{n,j} := 2^j \Big|\{y \in B_{x_n} \colon 2^j \le |y|^{\vartheta}|f(y)| \le 2^{j+1}\}\Big|^{(d-\vartheta)/d} \to 0 \quad \text{as } n \to +\infty.$$

All the above terms $Q_{n,j}$ can be dominated by

$$2^j \Big|\{y \in \mathbb{R}^d \colon 2^j \le |y|^{\vartheta}|f(y)| \le 2^{j+1}\}\Big|^{(d-\vartheta)/d}$$

that is summable sequence with respect to j, because $|y|^{\vartheta} f \in L^{d/(d-\vartheta),1}$. By the dominated convergence theorem, we deduce that $\sum_{j\in\mathbb{Z}} Q_{n,j} \to 0$ as $n \to +\infty$. This proves that $\|\phi_{x_n}\|_{L^{d/(d-\vartheta),1}} \to 0$ as $n \to \infty$ and so $\|\phi_x\|_{L^{d/(d-\vartheta),1}} \to 0$ as $|x| \to +\infty$. This concludes our proof. □

Notice that the only condition $f \in L^1(\mathbb{R}^d)$ would not be enough to ensure the validity of (2.1), even though both the left and the right hand sides of (2.1) make sense. A counterexample is constructed by choosing $g(x) = |x|^{-\vartheta}$ and $f = \sum_{k=0}^{\infty} k 2^{-k} I_k$, where I_k is the indicator function of the ball of radius one, centered at $x_k = 2^{k/\vartheta}(1, 0, \ldots, 0)$. Indeed, $g * f(x_k) \ge (\int_{|y|\le 1} |y|^{-\vartheta}) k 2^{-k}$, and so $\lim_{k\to+\infty} |x_k|^{\vartheta}(f * g)(x_k) = \infty$, which contredicts (2.1).

The above proof goes through in the case $\vartheta = 0$ (and is in fact simpler, as no Lorentz space assumption on f is needed in this limit case). We state this in the following remark:

Remark 2.2. Let $g \in \mathrm{Lip}(\mathbb{R}^d)$ be the set of bounded Lipschitz real functions on $\mathbb{R}^d$, such that $\sup_{R>0} R\|g\|_{\mathrm{Lip}(\{|x|>R\})} < \infty$, and $f \in L^1(\mathbb{R}^d)$. Then,

$$(f * g)(x) = \Big(\int f\Big) g + o(1), \qquad \text{as } |x| \to +\infty.$$

Proposition 2.1 admits an obvious generalization. To state it in a more general form, we introduce the space $\dot{E}^m_\vartheta$ of all functions $g \in C^m(\mathbb{R}^d \backslash \{0\})$ such that, for all $x \neq 0$,

$$\forall \alpha \in \mathbb{N}^d,\ 0 \le |\alpha| \le m, \quad |\partial^\alpha g(x)| \le C_\alpha |x|^{-|\alpha|-\vartheta},$$

for some constant C_α independent on x.

Proposition 2.3. *Let $m \in \mathbb{N}$, $g \in \dot{E}^{m+1}_\vartheta$, with $0 < \vartheta < d$,*

$$f \in L^1(\mathbb{R}^d, (1+|x|)^m \,\mathrm{d}x) \qquad \text{and} \qquad |x|^{\vartheta+m} f \in L^{d/(d-\vartheta),1}(\mathbb{R}^d).$$

Then, as $|x| \to \infty$,

$$(f * g)(x) = \sum_{\substack{\gamma \in \mathbb{N}^d \\ 0 \le |\gamma| \le m}} \frac{(-1)^{|\gamma|}}{\gamma!} \Big(\int y^\gamma f(y)\, dy\Big) \partial^\gamma g(x) \;+\; o(|x|^{-m-\vartheta}). \tag{2.3}$$

Proof. The proof is very similar to the previous one: one writes the difference between $(f * g)(x)$ and the first term in the right-hand side, next splits the obtained expression in four integral terms, just as before. The first term is

$$I(x) = \int_{|y| \le |x|/2} \Big(g(x-y) - \sum_{0 \le |\gamma| \le m} \frac{(-1)^{|\gamma|}}{\gamma!} \partial^\gamma g(x) y^\gamma \Big) f(y) \,\mathrm{d}y$$

and this is treated with the Taylor formula to the order $m+1$: we get in this way, arguing as in the previous proposition, $|I(x)| \le C|x|^{-m-1-\vartheta} \int_{|y| \le |x|/2} |y|^{m+1} |f(y)| \,\mathrm{d}y = o(|x|^{-m-\vartheta})$ as $|x| \to \infty$. Next term is

$$\mathrm{II}(x) = -\sum_{0 \le |\gamma| \le m} \frac{(-1)^{|\gamma|}}{\gamma!} \Big(\int_{|y| \ge |x|/2} y^\gamma f(y) \,\mathrm{d}y \Big) \partial^\gamma g(x).$$

But $\int_{|y| \ge |x|/2} |y|^\gamma |f(y)| \,\mathrm{d}y \le (|x|/2)^{-m+|\gamma|} \int_{|y| \ge |x|/2} |y|^m |f(y)| \,\mathrm{d}y$. Hence, each term in the summation of $\mathrm{II}(x)$ decays faster than $|x|^{-m-\vartheta}$ as $|x| \to \infty$, by the dominated convergence theorem. For the last two terms there is nearly no change with respect to the proof of Proposition 2.1, as the arguments of the previous proof can be applied to $|x|^m f$. □

Remark 2.4. Formula (2.3) is closely related to the classical development of functions in the Dirac basis, as in [9]. Therein, a function $f \in L^1(\mathbb{R}^d, (1+|x|)^m \,\mathrm{d}x)$ is written as

$$f(x) = \sum_{\substack{\gamma \in \mathbb{N}^d \\ 0 \le |\gamma| \le m}} \frac{(-1)^{|\gamma|}}{\gamma!} \Big(\int y^\gamma f(y)\, dy \Big) \partial^\gamma \delta + \mathcal{R}(x)$$

where δ is the Dirac mass, with a remainder term $\mathcal{R}(x) = \sum_{|\gamma|=m} \partial^\gamma F_\gamma$ and where $F_\gamma \in L^1(\mathbb{R}^d)$ are suitable functions depending on f. The new feature of Proposition 2.3, with respect to the result of [9], is to make clear what additional conditions on f and g ensure the pointwise decay $\mathcal{R} * g(x) = o(|x|^{-m-\vartheta})$ as $|x| \to +\infty$.

Let us now apply our expansion to the 2D Biot–Savart integral.

Proposition 2.5. *Assume that, for some positive interger m, we have $\omega \in L^1(\mathbb{R}^2, (1+|x|)^{m-1}\,\mathrm{d}x)$ and $|x|^m\omega \in L^{2,1}(\mathbb{R}^2)$. Then, for some $A_n \geq 0$ and $\phi_n \in \mathbb{R}$ ($n = 1, \dots, m$), we have, as $|x| \to \infty$,*

$$\frac{1}{2\pi}\int \frac{(x-y)^\perp}{|x-y|^2}\omega(y)\,\mathrm{d}y = \sum_{n=1}^{m} \frac{A_n}{|x|^n}\begin{pmatrix}\cos(n\theta+\phi_n)\\ \sin(n\theta+\phi_n)\end{pmatrix} + o(|x|^{-m}). \tag{2.4}$$

Proof. Consider $K(x) = \frac{1}{2\pi}x^\perp/|x|^2 = (-\partial_2 E, \partial_1 E)^t$ with $E(x) = \frac{1}{2\pi}\log|x|$ the fundamental solution of the Laplacian. The Biot–Savart kernel K belongs to $\dot{E}^k_1$, for all $k \in \mathbb{N}$. For any $m \in \mathbb{N}^*$, applying Proposition 2.3 with $f = \omega$, $g = K$ and $\vartheta = 1$ (and with $m-1$ instead of m) gives

$$K * \omega(x) = \sum_{n=0}^{m-1}\sum_{|\gamma|=n} \frac{(-1)^\gamma}{\gamma!}\alpha_\gamma\,\partial^\gamma K(x) \;+\; o(|x|^{-m}), \tag{2.5}$$

as $|x| \to +\infty$, with

$$\alpha_\gamma = \int y^\gamma \omega(y)\,\mathrm{d}y.$$

The property that we will need is the following:

$$\forall\,\beta = (\beta_1,\beta_2) \in \mathbb{N}^2,\ |\beta| = n \geq 1: \quad \partial^\beta E(x) = \begin{cases} \dfrac{(-1)^{\beta_1-1}(-1)^{\beta_2/2}\,(n-1)!}{2\pi\,\rho^n}\cos(n\theta) & \text{if } \beta_2 \text{ is even}\\[2ex] \dfrac{(-1)^{\beta_1}(-1)^{(\beta_2-1)/2}\,(n-1)!}{2\pi\,\rho^n}\sin(n\theta) & \text{if } \beta_2 \text{ is odd,}\end{cases} \tag{2.6}$$

where (ρ,θ) are the polar coordinates of $x \in \mathbb{R}^2$. Property (2.6) can be proved by induction on β, by applying the chain rule and the usual addition formulae for sine and cosine functions. From this we readily get, using $K(x) = (-\partial_2, \partial_1)E(x)$, that for any $\gamma = (\gamma_1,\gamma_2) \in \mathbb{N}^2$, with $|\gamma| = n$,

$$\partial^\gamma K(x) = \begin{cases} \dfrac{(-1)^{\gamma_1}(-1)^{\gamma_2/2}\,n!}{2\pi\,\rho^{n+1}}\Big(-\sin((n+1)\theta), \cos((n+1)\theta)\Big)^t & \text{if } \gamma_2 \text{ is even}\\[2ex] \dfrac{(-1)^{\gamma_1-1}(-1)^{(\gamma_2-1)/2}\,n!}{2\pi\,\rho^{n+1}}\Big(\cos((n+1)\theta), \sin((n+1)\theta)\Big)^t & \text{if } \gamma_2 \text{ is odd.}\end{cases} \tag{2.7}$$

Let us set, for $n = 0, \dots, m-1$,

$$\begin{aligned} a_{n+1} &= \sum_{|\gamma|=n,\ \gamma_2 \text{ even}} \frac{(-1)^\gamma}{2\pi\,\gamma!}\alpha_\gamma(-1)^{\gamma_1}(-1)^{\gamma_2/2}\,n!\\ &= \sum_{|\gamma|=n,\ \gamma_2 \text{ even}} \frac{(-1)^{\gamma_2/2}}{2\pi\,\gamma!}\alpha_\gamma\,n! \end{aligned}$$

and

$$\begin{aligned} b_{n+1} &= \sum_{|\gamma|=n,\ \gamma_2 \text{ odd}} \frac{(-1)^\gamma}{2\pi\,\gamma!}\alpha_\gamma(-1)^{\gamma_1-1}(-1)^{(\gamma_2-1)/2}\,n!\\ &= \sum_{|\gamma|=n,\ \gamma_2 \text{ odd}} \frac{(-1)^{(\gamma_2-1)/2}}{2\pi\,\gamma!}\alpha_\gamma(t)\,n! \end{aligned}$$

Let us point out, in particular, that $a_1 = \frac{1}{2\pi}\int \omega$, $b_1 = 0$, $a_2 = \frac{1}{2\pi}\int y_1\omega$ and $b_2 = \frac{1}{2\pi}\int y_2\omega$.

Now, for $n = 1, \dots, m$, we set

$$A_n = \sqrt{a_n^2 + b_n^2},$$

and we introduce a phase ϕ_n such that

$$\begin{cases} \cos(\phi_n) = b_n/A_n, \\ \sin(\phi_n) = a_n/A_n. \end{cases}$$

In this way,

$$\begin{pmatrix} -a_n \sin(n\theta) + b_n \cos(n\theta) \\ a_n \cos(n\theta) + b_n \sin(n\theta) \end{pmatrix} = A_n \begin{pmatrix} \cos(n\theta + \phi_n) \\ \sin(n\theta + \phi_n) \end{pmatrix}$$

and combining this with expansion (2.5) and formula (2.7) leads to the result. □

The pair $(\cos(n\theta), \sin(n\theta))$ forms an orthogonal basis of the vector space $\mathcal{H}_n$ of circular harmonics of degree n (see [19]). Thus, the terms in the summation in the right-hand side of (2.4) are of the form $|x|^{-n}V_n(x)$, where $V_{n,1}$ and $V_{n,2}$ are harmonic homogeneous polynomials of degree n (unless $A_n = 0$). See [19] for a detailed study of harmonic polynomials. We also observe that each term $V_n(x)/|x|^n$ in expansion (1.3) is divergence-free (in the classical sense, for $x \neq 0$), as one easily checks applying the chain rule and the elementary formula $\nabla\theta(x) = x^\perp/|x|^2$.

We are now in the position of proving Theorem 1.1.

Proof of Theorem 1.1. Let us start recalling two sharp bounds due to E. Carlen and M. Loss: if $\omega_0 \in L^1(\mathbb{R}^2)$, and ω and u are the solutions of (NS)-(BS), then, for all $t > 0$,

$$\|u(t)\|_\infty \le Bt^{-1/2},$$

with

$$B := \tfrac{\sqrt{2}}{2\pi}\|\omega_0\|_1$$

and, for any $0 < \beta < 1$,

$$|\omega(x,t)| \le \beta^{-1} e^{B^2\beta/(1-\beta)} \int (\beta/4\pi t) e^{-\beta|x-y|^2/(4t)} |\omega_0(y)| \, \mathrm{d}y.$$

(See [8, Theorem 3]). In particular, choosing $\beta = 1/2$ and denoting $G_t(x) = (4\pi t)^{-1}\exp(-|x|^2/(4t))$ the standard 2D heat kernel, we get

$$|\omega(x,t)| \le 2e^{B^2} \int G_{2t}(x-y)|\omega_0(y)| \, \mathrm{d}y. \tag{2.8}$$

The above inequality allows us to obtain for ω the same bounds as for the heat equation, in weighted-L^p for all $1 \le p \le \infty$ and in weighted-$L^{p,q}$ norms, for all $1 < p < \infty$ and $1 \le q \le \infty$.

Under the assumption $(1+|\cdot|)^m\omega_0 \in L^1(\mathbb{R}^2)$, applying the submultiplicativity of $x \mapsto (1+|x|)^k$ for $k = 0, \dots, m$, and the classical L^1-L^1 Young inequality we get

$$\begin{aligned} \int (1+|x|)^k |\omega(x,t)| \, \mathrm{d}x &\le \|(1+|\cdot|)^k G_{2t}\|_1 \int (1+|x|)^k |\omega_0(x)| \, \mathrm{d}x \\ &\le C(1+t)^{k/2}. \end{aligned} \tag{2.9}$$

On the other hand, using that $|x| \le |x-y| + |y|$ and applying twice the L^1-$L^{2,1}$-Young inequality we get, for all $t > 0$,

$$\begin{aligned} \|\,|\cdot|^m\,\omega(\cdot,t)\|_{L^{2,1}} &\le C\,e^{B^2}\left\|\,|\cdot|^m G_{2t}\right\|_{L^{2,1}} \int |\omega_0| \, + C\|G_{2t}\|_{L^{2,1}} \int |x|^m|\omega_0(x)| \, \mathrm{d}x \\ &\le Ct^{-1/2}(1+t^{m/2}). \end{aligned} \tag{2.10}$$

Here C depends only on m and the $L^1(\mathbb{R}^2,(1+|x|)^m\,\mathrm{d}x)$ norm of ω_0. We conclude that under the assumption

$$\omega_0 \in L^1(\mathbb{R}^2,(1+|x|)^m\,\mathrm{d}x)$$

the solution is such that

$$\omega \in L^\infty\big([0,T],L^1(\mathbb{R}^2,(1+|x|)^m\,\mathrm{d}x)\big), \quad \sqrt{t}\,\omega \in L^\infty\big((0,T],L^{2,1}(\mathbb{R}^2,|x|^m\,\mathrm{d}x)\big)$$

for all $T>0$, with norm bounded by $C(1+T)^{m/2}$. Combining now the asymptotic formula (2.4) (with $\omega(x,t)$ instead of $\omega(x)$) with (BS) proves expansion (1.2).

□

Remark 2.6. Notice that the moments of the vorticity $\alpha_\gamma=\int y^\gamma\omega$, if they are finite, are independent on time for $\gamma=0$ and $|\gamma|=1$. Therefore, for all $t\geq 0$, we have $a_1(t)=\frac{1}{2\pi}\int\omega_0$, $b_1(t)=0$, $a_2(t)=\frac{1}{2\pi}\int y_1\omega_0$ and $b_2(t)=\frac{1}{2\pi}\int y_2\omega_0$. This in turn implies that the amplitudes $A_n(t)$ and the phases $\phi_n(t)$ are independent on time when $n=1,2$. We can make more explicit the two first terms in expansion (1.3) (if $m\geq 1$):

$$u(x,t)=\frac{\alpha_0}{2\pi}\,\frac{x^\perp}{|x|^2}-\frac{1}{2\pi|x|^4}\left[\alpha_{(1,0)}\begin{pmatrix}2x_1x_2\\ x_2^2-x_1^2\end{pmatrix}+\alpha_{(0,1)}\begin{pmatrix}x_2^2-x_1^2\\ -2x_1x_2\end{pmatrix}\right]\;+\;o(|x|^{-2}).$$

If the assumptions of our theorem are satisfied with $m\geq 3$, then we can compute the term $V_3(x,t)/|x|^3$ as follows:

$$V_3(x,t)/|x|^3=\frac{1}{2}\Big[A\,\partial_1^2K(x)+B\,\partial_1\partial_2K(x)+C\,\partial_2^2K(x)\Big]$$

where

$$A(t)=\int y_1^2\omega(y,t)\,\mathrm{d}y, \qquad B(t)=\int 2y_1y_2\omega(y,t)\,\mathrm{d}y \quad\text{and}\quad C(t)=\int y_2^2\omega(y,t)\,\mathrm{d}y.$$

After some elementary calculations, we obtain that the euclidean norm of V_3 is given by

$$\ell_3(t)=|V_3|(x,t)=\frac{\sqrt{(A-C)^2(t)+B^2(t)}}{2\pi}.$$

Remark 2.7. The conclusion of the Theorem 1.1 can be reached also under the assumption

$$\omega_0\in L^1(\mathbb{R}^2,(1+|x|)^{m-1}\,\mathrm{d}x) \qquad\text{and}\qquad |x|^{m-1}\omega_0\in L^{2,1}(\mathbb{R}^2). \tag{2.11}$$

Indeed, the stronger moment condition $\omega_0\in L^1(\mathbb{R}^2,(1+|x|)^m\,\mathrm{d}x)$ was used in the proof only to ensure that, for all $t>0$, $\omega(t)\in L^{2,1}(\mathbb{R}^2,|x|^m\,\mathrm{d}x)$ by parabolic regularization. But if one puts the additional assumption $|x|^m\omega_0\in L^{2,1}(\mathbb{R}^2)$, then the condition $|x|^m\omega_0\in L^1(\mathbb{R}^2)$ can be dropped. The persistence of the condition (2.11) is a consequence of Carlen and Loss estimate (2.8).

Proof of Theorem 1.2. Let $\omega\in L^\infty([0,\infty),L_c^\infty(\mathbb{R}^2))$ be the weak solution of the Euler equation. Such solution is such that $\|\omega(t)\|_1+\|\omega(t)\|_\infty\leq c(\|\omega_0\|_1+\|\omega_0\|_\infty)$ for some absolute constant $c>0$ and all $t\geq 0$. See [2, Chapter 8]. We also know that, for all $t\geq 0$, the support of $\omega(t)$ is contained in a ball of radius $C(1+t)$, for some constant $C>0$ depending only on ω_0. See [10]. Then, for some other constant $C>0$ depending only on ω_0, we get, for $m\geq 1$, $\|(1+|x|)^{m-1}\omega(t)\|_{L^1}\leq C(1+t)^{m-1}$ and $\|\,|x|^m\omega(t)\|_{L^{2,1}}\leq C(1+t)^m\|\omega(t)\|_{L^{2,1}}\leq C(1+t)^m$, where the last inequality follows by interpolation. Applying Proposition 2.5 implies that u can be expanded, for all positive integer m, as in (1.2), with a remainder depending on m and satisfying

$|R(x,t)| \leq (1+t)^m |x|^{-m} \epsilon_m(x)$, with $\epsilon_m(x) \to 0$ independent on $t > 0$. In the case of a positive initial vorticity, the better control on the large time growth of the remainder term follows from the result by P. Gamblin, D. Iftimie and T. Sideris [10], asserting that the diameter of the support of $\omega(t)$ is, in this case, $\mathcal{O}((t \log t)^{1/4})$ as $t \to \infty$. □

Proof of Corollary 1.3. The assertion follows immediately from expansion (1.3), and formula (1.4), with $\ell_n(t) = A_n(t)$. □

Proof of Corollary 1.4. We can apply twice Proposition 2.5 with $m = 2$, first with $\omega = \omega(t)$, then with $\omega = \omega_0$. Taking the difference of the two expansions and recalling the invariance of the moments of the vorticity of order zero and one, we get, as $|x| \to \infty$.

$$u(x,t) - u_0(x) = \frac{1}{|x|^3} \left(A_3(t) \begin{pmatrix} \cos(3\theta + \phi_3(t)) \\ \sin(3\theta + \phi_3(t)) \end{pmatrix} - A_3(0) \begin{pmatrix} \cos(3\theta + \phi_3(0)) \\ \sin(3\theta + \phi_3(0)) \end{pmatrix} \right) + |x|^{-3}.$$

Here $u_0 = K * \omega_0$ and $A_3(t)$, $A_3(0)$ and $\phi_3(t)$, $\phi_3(0)$ are defined in terms of the second-order moments of the vorticity, as in the proof of Proposition 2.5. Applying elementary trigonometric formulas, we easily find an amplitude $L(t) \geq 0$ and a phase $\phi(t) \in \mathbb{R}$ such that the latter expression can be rewritten as (1.7). □

The formation of hexagonal structures for $u(\cdot,t) - u_0$ was pointed out also in companion paper [5], in the setting of 2D Leray's solutions with possibly non-integrable vorticity. The approaches of [5] and that of the present paper are equivalent only under the more stringent situation that both the velocity and the vorticity are sufficiently localised: in such situation one can deduce the result obtained with one approach from the other using classical integral formulae between ω and u. See [13] for a general version and a proof of such formulae. For example, the two relations

$$\int x_1 x_2 (\omega(x,t) - \omega_0(x))\,\mathrm{d}x = \int_0^t \int (u_1^2 - u_2^2),$$

and

$$\int x_1^2 (\omega(x,t) - \omega_0(x))\,\mathrm{d}x = -\int_0^t \int 2u_1 u_2 = -\int x_2^2 (\omega(x,t) - \omega_0(x))\,\mathrm{d}x$$

can be used to express $L(t)$ in terms of quadratic integrals of the velocity, as we did in [5]. As soon as one of these integrals is nonzero, one has $L(t) \neq 0$.

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