Discrete varifolds and regularization of the generalized curvature

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Why varifolds ?

- Varifolds : a space containing both
  - regular objects (surfaces, sub-manifolds, rectifiables sets),
  - discrete objects (triangulations, volumetric approximations, point clouds).
- Define the generalized curvature of a varifold :
  - Classical mean curvature for regular objects,
  - Example of computation in the discrete case.
- Link between the control of the generalized curvature and rectifiability.
- Compactness properties.

Outline

Definition

A d–varifold in $\mathbb{R}^n$ *is a Radon measure* V *in* $\mathbb{R}^n \times G_{d,n}$ *with*

$$G_{d,n} = \{d\text{-vector planes of } \mathbb{R}^n\}$$

A varifold is a measure giving

- information of position : measure in $\mathbb{R}^n$.
- information of tangent plane : measure in the Grassmannian $G_{d,n}$

Example

Take a line $D \subset \mathbb{R}^n$ directed by $\overrightarrow{D} \in G_{1,n}$ and define the associated 1–varifold

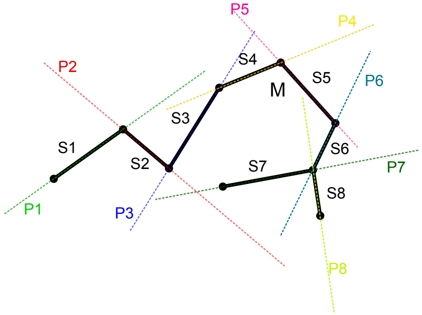
$$V = \mathcal{H}^1_{|D} \otimes \delta_{\overrightarrow{D}}$$

Varifold associated with a piecewise linear curve

Definition (mass of a varifold)

The masse of a d–varifold V is the positive Radon measure $\|V\|$ *defined by*

$$\|V\|(A) = V(A \times G_{d,n}) \text{ for every Borel set } A \subset \mathbb{R}^n.$$



Example

A 1–varifold canonically associated to this piecewise linear curve :

$$V = \sum_{i=1}^{8} \underbrace{\mathcal{H}^1_{|S_i}}_{\mathbb{R}^2} \otimes \underbrace{\delta_{P_i}}_{G_{1,2}} .$$

Measure in $\mathbb{R}^2 \times G_{1,2}$ with $G_{1,2} = \{1\text{–vector spaces of } \mathbb{R}^2\}$.
$\|V\| = \mathcal{H}^1_{|D}$.

Varifold associated with a surface

- $M \subset \mathbb{R}^3$ surface.
- A varifold canonically associated to M is the measure $v(M) = \mathcal{H}^2_{|M} \otimes \delta_{T_x M}$ i.e. for every Borel set $A \subset \mathbb{R}^2 \times G_{2,3}$,

$$v(M)(A) = \mathcal{H}^2(A \cap TM) \text{ où } TM = \{(x, T_x M) \,|\, x \in M\}.$$

or by duality for every $\varphi \in \mathrm{C_c}(\mathbb{R}^2 \times \mathrm{G}_{2,3})$,

$$\begin{aligned} \int \varphi(x,S)\, dv(M)(x,S) &= \int_M \int_{G_{2,3}} \varphi(x,S)\, d\delta_{T_x M}(S)\, d\mathcal{H}^2(x) \\ &= \int_M \varphi(x, T_x M)\, d\mathcal{H}^2(x)\,. \end{aligned}$$

$$\|V\| = \mathcal{H}^d_{|M}$$

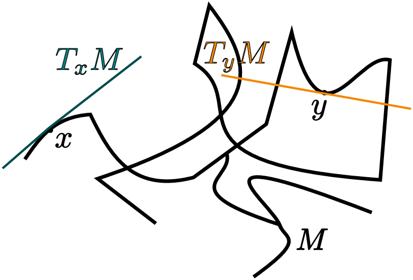
Rectifiable d–varifold : varifold associated with a d–rectifiable set

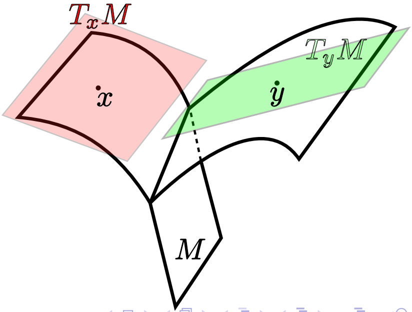
- $\mathcal{M} \subset \mathbb{R}^n$ countably d–rectifiable set

$$\mathcal{M} = \mathcal{M}_0 \cup \bigcup_{i \in \mathbb{N}} f_i(\mathbb{R}^d)$$

  with $f_i : \mathbb{R}^d \to \mathbb{R}^n$ of class $\mathcal{C}^1$ and $\mathcal{H}^d(\mathcal{M}_0) = 0$.

- $$\begin{array}{rcl} M & \to & G_{d,n} = \{d\text{-vector plane } \mathbb{R}^n\} \\ x & \mapsto & T_x M \text{ approximate tangent plane to } \mathcal{M} \text{ at } x \end{array}.$$





Definition (Rectifiable d–varifold)

- $\mathcal{M} \subset \mathbb{R}^n$ *d-rectifiable set,*
- $\theta : \mathcal{M} \to \mathbb{R}_+ \in \mathrm{L}^1_{loc}(M)$ *multiplicity.*

The varifold $v(M,\theta)$ is the Radon measure associated to the continuous linear form

$$\begin{array}{rcl} \mathcal{C}^0_c\left(\mathbb{R}^n \times G_{d,n}\right) & \longrightarrow & \mathbb{R} \\ \varphi & \longmapsto & \displaystyle\int_M \int_{G_{d,n}} \varphi(x,S)\, d\delta_{T_xM}(S)\, \theta(x)\, d\mathcal{H}^d(x) \\ & = & \displaystyle\int_M \varphi(x, T_xM)\, \theta(x)\, d\mathcal{H}^d(x)\,. \end{array}$$

$$v(M,\theta) = \theta \mathcal{H}^d_{|M} \otimes \delta_{T_xM} \text{ and } \|V\| = \theta\, \mathcal{H}^d_{|M}\,.$$

Examples of discrete varifolds : Point cloud varifolds

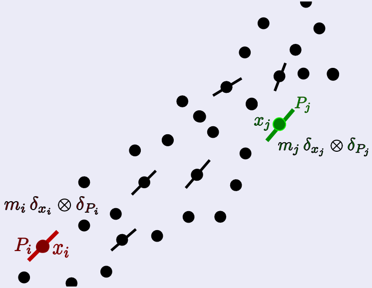
Definition (Point cloud varifolds)

Let $\{x_i\}_{i=1\dots N} \subset \mathbb{R}^n$ *be a point cloud, weighted by the masses* $\{m_i\}_{i=1\dots N}$ *and provided with directions* $\{P_i\}_{i=1\dots N} \subset G_{d,n}$. *We can thus associate a* $d-$*varifolds on* $\mathbb{R}^n \times G_{d,n}$ *with this point cloud :*

$$V = \sum_{i=1}^{N} m_i \, \delta_{x_i} \otimes \delta_{P_i} \, ,$$

so that for $\varphi \in \mathrm{C_c}(\Omega \times \mathrm{G_{d,n}})$,

$$\int \varphi \, dV = \sum_{i=1}^{N} \varphi(x_i, P_i) \, .$$



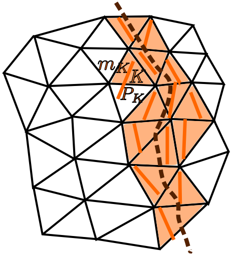
Discrete volumetric varifolds

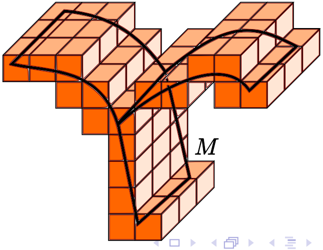
Definition (Discrete volumetric varifolds)

Consider a mesh $\mathcal{K}$ in $\mathbb{R}^n$ and a family $\{m_K, P_K\}_{K\in\mathcal{K}} \subset \mathbb{R}_+ \times G_{d,n}$. We can associate the d–varifold :

$$V_{\mathcal{K}} = \sum_{K\in\mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K} \text{ with } |K| = \mathcal{L}^n(K)\,.$$

This d–varifold is not rectifiable since its support is n–rectifiable but not d–rectifiable.





Approximation of rectifiable varifolds by discrete varifolds

Question

Are rectifiable varifolds well-approximated by discrete varifolds ? in which sense ? is it possible to quantify it ?

- which sense ? : natural convergence in varifolds space : weak $*$–convergence, a sequence of varifolds $V_i \xrightarrow[i\to\infty]{*} V$ if for all $\varphi \in \mathrm{C_c}(\mathbb{R}^n \times \mathrm{G_{d,n}})$,

$$\int_{\mathbb{R}^n \times G_{d,n}} \varphi \, dV_i \to \int_{\mathbb{R}^n \times G_{d,n}} \varphi \, dV \, .$$

- quantify ? : measure the error of approximation, we need a distance between varifolds.

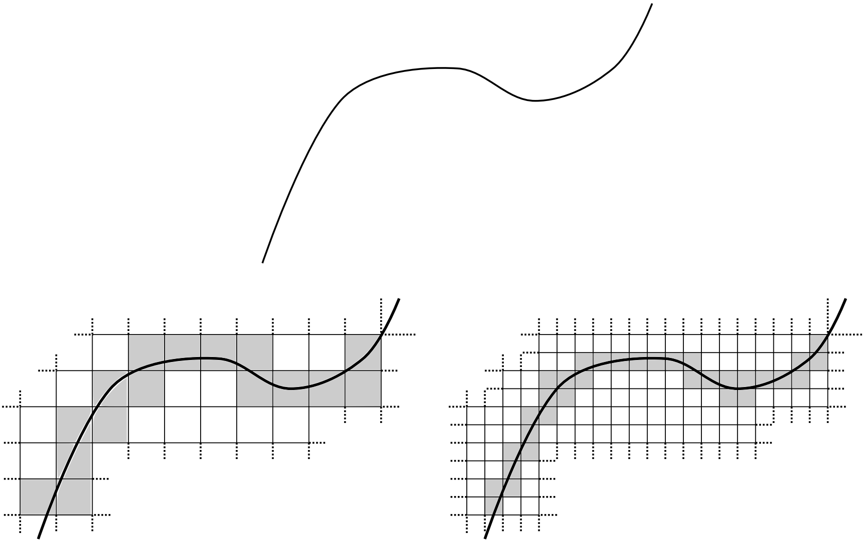
Approximation with discrete volumetric varifolds

Question

Considering a sequence of meshes $(\mathcal{K}_i)_i$ whose size

$$\sup_{K \in \mathcal{K}_i} \mathrm{diam} K = \delta_i \xrightarrow[i \to \infty]{} 0$$

is it possible to approximate rectifiable varifolds by discrete volumetric varifolds $(V_i)_i$ associated with these prescribed successive meshes ?



Definition of a discrete volumetric varifold by projection onto a mesh $\mathcal{K}$

Let $V = v(M,\theta) = \theta \mathcal{H}^d_{|M} \otimes \delta_{T_x M}$ be a rectifiable d–varifold in $\mathbb{R}^n$ and $\mathcal{K}$ be a mesh of $\mathbb{R}^n$. Define

$$V_{\mathcal{K}} = \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n \otimes \delta_{P_K} \, ,$$

with

$$m_K = \int_K \theta \, d\mathcal{H}^d \text{ and } P_K \in \operatorname*{arg\,min}_{P \in G_{d,n}} \int_{K \times G_{d,n}} \|P - S\| \, dV(x,S) \, .$$

Theorem (Approximation by discrete volumetric varifolds)

If V is a d–rectifiable varifold and $(\mathcal{K}_i)_i$ is a sequence of meshes whose size tends to 0 then

$$V_{\mathcal{K}_i} \xrightarrow[i \to +\infty]{*} V \text{ in } \Omega \, .$$

Quantitative version

Assume moreover that the rectifiable varifold $V = \theta \mathcal{H}^d_{|M} \otimes \delta_{T_x M}$ satisfies in addition : there exist $0 < \beta < 1$ and $C > 0$ such that for $\mathcal{H}^d$–almost x, $y \in M$,

$$\| T_x M - T_y M \| \leq C |x - y|^\beta ,$$

then :

Theorem (Convergence with respect to the flat distance)

for $(\mathcal{K}_i)_i$ sequence of meshes with size δ_i and $V_{\mathcal{K}_i}$ successive projections of V onto $\mathcal{K}_i$,

$$\Delta^{1,1}(V, V_{\mathcal{K}_i}) \leq \left(\delta_i + 2C\delta_i^\beta \right) \|V\|(\mathbb{R}^n) .$$

Where $\Delta^{1,1}$ is the flat distance or bounded Lipschitz distance :

$$\Delta^{1,1}(V, W) = \sup \left\{ \left| \int \varphi \, dV - \int \varphi \, dW \right| \; : \; \varphi \in \mathrm{Lip}_1, \; \|\varphi\|_\infty \leq 1 \right\} .$$

Outline

Divergence Theorem

Theorem

Let $\mathcal{M} \subset \mathbb{R}^n$ be a $\mathcal{C}^2$ sub-manifold of dimension d and $\Omega \subset \mathbb{R}^n$ be some open set. Then for every $X \in C_c^1(\Omega, \mathbb{R}^n)$,

$$\int_{\mathcal{M}\cap\Omega} div_{\mathcal{M}} X \, d\mathcal{H}^d = - \int_{\mathcal{M}\cap\Omega} < X, \vec{H} > \, d\mathcal{H}^d$$

This is actually a way of defining the mean curvature vector $\vec{H}$ in a more general class : in the space of varifolds.

Curvature of a varifold

Definition (First variation)

The first variation of a d−varifold V is the linear form

$$\begin{array}{ccc} \mathrm{C}^1_\mathrm{c}(\mathbb{R}^\mathrm{n}, \mathbb{R}^\mathrm{n}) & \longrightarrow & \mathbb{R} \\ X & \longmapsto & \displaystyle\int_{\mathbb{R}^n \times G_{d,n}} div_S X(x) \, dV(x, S) \, . \end{array}$$

If $V = V(\text{sub-manifold } M)$, $\delta V = -H \, \mathcal{H}^d_{|M}$ is the classical mean curvature. But in general, we only know that it is a distribution of order 1.

Bounded first variation

Definition

If there exists $C > 0$ *such that for every* $X \in \mathrm{C}^1_c(\mathbb{R}^n, \mathbb{R}^n)$,

$$|\delta V(X)| \leq C\|X\|_\infty \, ,$$

then δV *extends into a continuous linear form in* $\mathrm{C}^0_c(\mathbb{R}^n, \mathbb{R}^n)$ *and we say that* V *has bounded first variation.*

EXAMPLES of varifolds whose first variation is not bounded :

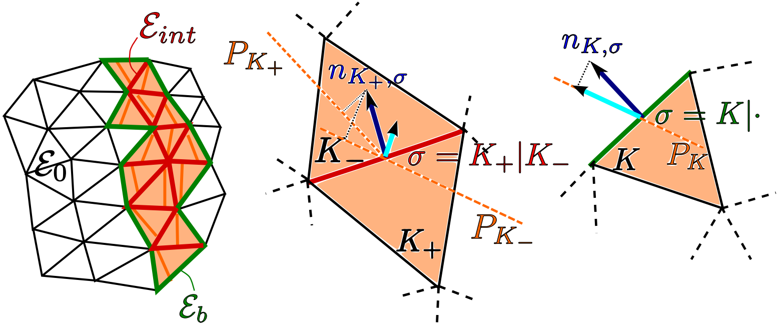
- Point clouds $\sum_i m_i \delta_{x_i} \otimes \delta_{P_i}$,
- A varifold associated to the line D, $V = \mathcal{H}^1_{|D} \otimes \delta_{D'}$ where the direction D' is constant and is not parallel to D.

What can be said when the first variation is bounded ?

- Thanks to Riesz Theorem, δV is a Radon measure in $\mathbb{R}^n$.
- And Radon-Nikodym decomposition with respect to the mass $\|V\|$ gives an absolutely continuous curvature H with respect to the mass and a singular curvature :

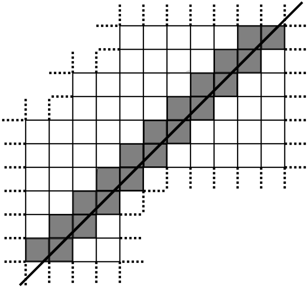
$$\delta V = -H \|V\| + (\delta V)_s \ .$$

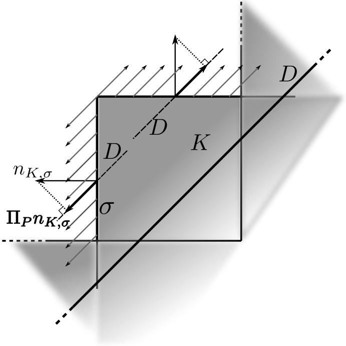
And what about the First variation of a discrete varifold



Curvature is concentrated on faces

$$\delta V = - \sum_{T \text{ edge of the mesh}} \left[\frac{\mu_{K_+}}{|K_+|} \Pi_{P_{K_+}} - \frac{\mu_{K_-}}{|K_-|} \Pi_{P_{K_-}} \right] (n_{ext}) \, d\mathcal{H}^{n-1}_{|T} \, .$$





So that if we now consider successive volumetric approximations $V_{\mathcal{K}_i}$ of $V = \mathcal{H}^1_{|D} \otimes \delta_D$ associated with successive meshes $\mathcal{K}_i$ whose size δ_i tends to 0,

$$|\delta V_{\mathcal{K}_i}|(\Omega) \geq \frac{C}{\delta_i} \|V\|(\mathbb{R}^n) \xrightarrow[i\to\infty]{} +\infty \,.$$

Outline

Idea :

Theorem (G.P. Leonardi-S. Masnou)

If $V = v(M,\theta)$ is a rectifiable d–varifold rectifiable with bounded first variation then for $x \in M$,

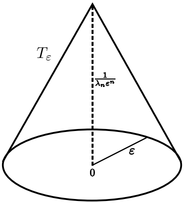
$$\delta V(B_r(x)) = \int_{\partial B_r(x) \cap M} \eta(y)\theta(y)\, d\mathcal{H}^{d-1}(y) \ \textit{ for almost every } r \, .$$

In an integrated form :

$$\frac{1}{\varepsilon}\int_{r=0}^{\varepsilon} \delta V(B_r(x))\, dr = \underbrace{\frac{1}{\varepsilon}\int_{B_\varepsilon(x)\times G_{d,n}} \frac{\Pi_S(y-x)}{|y-x|}\, dV(y,S)}_{\text{makes sense for any varifold}} \, .$$

Regularization of the first variation : initial idea

$$\text{Let } T(z) = \begin{cases} \frac{1}{\lambda_n}(1 - |z|) & \text{if } |z| \leq 1 \\ 0 & \text{otherwise} \end{cases}, \qquad (1)$$



where λ_n such that $\int_{\mathbb{R}^n} T = 1$, we can define the associated approximate identity $T_\varepsilon(z) = \frac{1}{\varepsilon^n} T\left(\frac{z}{\varepsilon}\right)$.

Proposition

Let V be a d–varifold in $\Omega \subset \mathbb{R}^n$. Then $\delta V * T_\varepsilon(x)$ is well defined for $\mathcal{L}^n$–almost every x and

$$\delta V * T_\varepsilon(x) = \frac{-1}{\lambda_n \varepsilon^n} \frac{1}{\varepsilon} \int_{B_\varepsilon(x) \times G_{d,n}} \frac{\Pi_S(y - x)}{|y - x|} \, dV(y, S)$$

Approximate first variation and curvature

We fix a symmetric positive function $\rho \in \mathrm{W}^{1,\infty}$ such that

$$\int \rho = 1 \text{ and } \mathrm{supp}\rho \subset B_1(0)\,, \tag{2}$$

and we also fix the associated family $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$.
If V has bounded first variation, then

- $$\delta V * \rho_\varepsilon \xrightarrow[\varepsilon\to 0]{*} \delta V$$
- If moreover V is rectifiable and ρ is radial, then, for $\|V\|$–almost any x,

$$H_\varepsilon(x) = \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \xrightarrow[\varepsilon\to 0]{} -H(x) \text{ where } \delta V = -H\|V\| + \delta V_s\ .$$

Approximate Willmore energies

Definition (Approximate Willmore energies)

Let $p \geq 1$ and $\varepsilon > 0$. For any d–varifold V in $\mathbb{R}^n$, we define

$$\mathcal{W}_\varepsilon^p(V) = \int_{x \in \mathbb{R}^n} \left| \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \right|^p \|V\| * \rho_\varepsilon(x) \, d\mathcal{L}^n(x) \, .$$

$$\mathcal{W}_\varepsilon^p \xrightarrow[\varepsilon \to 0]{\Gamma} \mathcal{W}^p \quad \text{for } 1 < p < +\infty$$

$$\mathcal{W}_\varepsilon^1 \xrightarrow[\varepsilon \to 0]{\Gamma} \text{the total variation of the first variation} \neq \mathcal{W}^1 \, .$$

$$\mathcal{W}^p(V) = \int_\Omega |H|^p \, d\|V\| \text{ if } \delta V = -H \, \|V\| \text{ and } + \infty \text{ otherwise} \, ,$$

Recall that for an explicit kernel ρ, the expression of $\delta V * \rho_\varepsilon$ is explicit .

The case of discrete varifolds

Theorem

- $V = v(M, \theta)$ *rectifiable d–varifold in $\mathbb{R}^n$ with finite mass* $\|V\|(\mathbb{R}^n) < +\infty$.
- $(\mathcal{K}_i)_i$ *a sequence of meshes satisfying* $\sup_{K \in \mathcal{K}_i} \mathrm{diam}(K) \leq \delta_i \xrightarrow[i \to +\infty]{} 0$.
- $(V_{\mathcal{K}_i})_i$ *the sequence of discrete volumetric varifolds obtained by projection on the mesh* $\mathcal{K}_i$.
- $\rho \in \mathrm{W}^{2,\infty}$.
- *Assume that there exist* $0 < \beta < 1$ *and* C *such that for* $\|V\|$*–almost every* x, y,
$$\| T_x M - T_y M \| \leq C |x - y|^\beta \, .$$

Then, for any sequence of infinitesimals $\varepsilon_i \downarrow 0$

$$\mathcal{W}^1_{\varepsilon_i}(V_{\mathcal{K}_i}) \xrightarrow[i \to +\infty]{} |\delta V|(\mathbb{R}^n) \text{ as soon as } \frac{\delta_i^\beta}{\varepsilon_i^2} \xrightarrow[i \to +\infty]{} 0 \, .$$

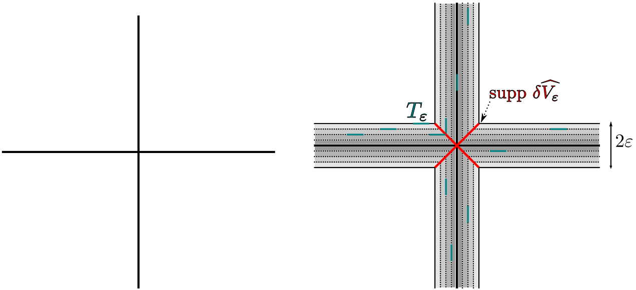
What is $\delta V * \rho_\varepsilon$

for $p > 1$,

$$\mathcal{W}^p_{\varepsilon_i}(V_{\mathcal{K}_i}) \xrightarrow[i \to +\infty]{?} \mathcal{W}^p(V)$$

Question

- Given a d–varifold V, is the regularization $\delta V * \rho_\varepsilon$ of the first variation δV, the first variation $\delta\left(\widehat{V_\varepsilon}\right)$ of some varifold $\widehat{V_\varepsilon}$?
- And if so, is $\widehat{V_\varepsilon}$ the regularization (in a sense to be defined) of V ?

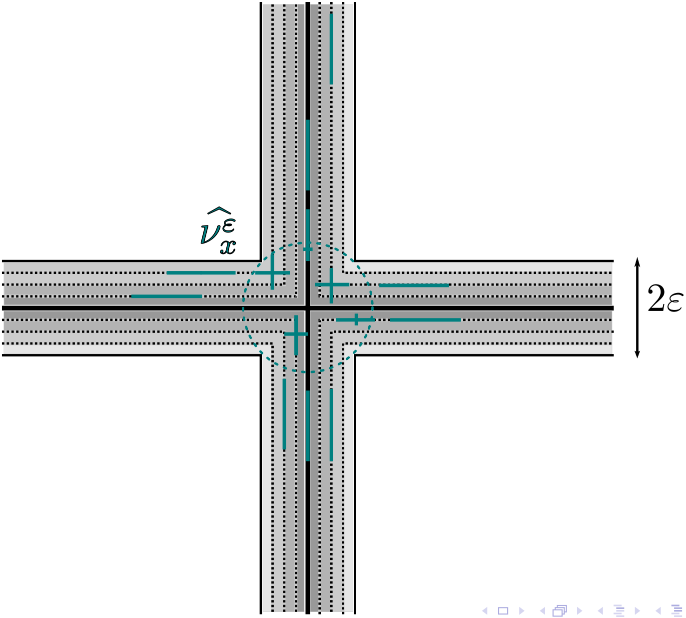


(a) (b)

$\widehat{V_\varepsilon} = (\|V\| * \rho_\varepsilon(x)) \otimes \widehat{\nu_x^\varepsilon}$ and $\widehat{\nu_x^\varepsilon}$

- is not the tangent to the level-line at x,
- but it is a convex combination of δ_{T_1} and δ_{T_2} where T_1 and T_2 are the directions of the lines.

Example of a cross



Let

$$\left\langle \widehat{V_\varepsilon}, \psi \right\rangle = \langle V, (y,S) \mapsto \psi(\cdot, S) * \rho_\varepsilon(y) \rangle \quad \text{for every } \psi \in \mathrm{C_c^0}(\Omega \times \mathrm{G_{d,n}})\,;$$

Then,

1. $\|\widehat{V_\varepsilon}\| = \|V\| * \rho_\varepsilon$,
2. $\delta\left(\widehat{V_\varepsilon}\right) = \delta V * \rho_\varepsilon$.

Outline

Approximation of the mean curvature vector of point clouds

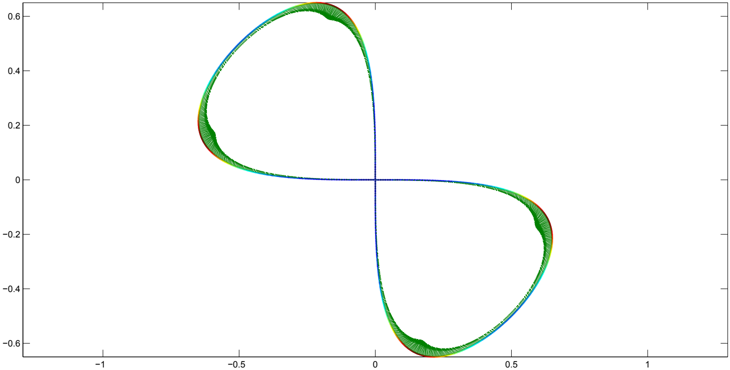
For a d–varifold V_N associated with a point cloud and for a radial kernel $\rho(y) = \zeta(|y|)$,

$$V_N = \sum_{j=1}^{N} m_j \delta_{x_j} \otimes \delta_{P_j} \ ,$$

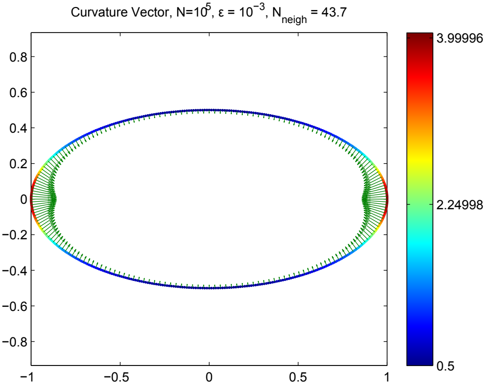
$$H_\varepsilon^N(x) = \frac{\delta V_N * \rho_\varepsilon(x)}{\| V_N \| * \rho_\varepsilon(x)} = \frac{\displaystyle\sum_{j=1}^{N} m_j \zeta' \left(\frac{|x_j - x|}{\varepsilon} \right) \frac{\Pi_{P_j}(x_j - x)}{|x_j - x|}}{\displaystyle\sum_{j=1}^{N} m_j \varepsilon \zeta \left(\frac{|x_j - x|}{\varepsilon} \right)} \ .$$

Convergence under same kind of assumptions as for the first variation, involving $\frac{\delta}{\varepsilon^2}$.

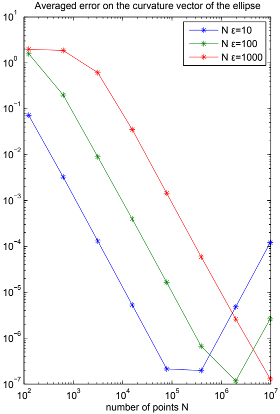
Recovering the 0–singular curvature



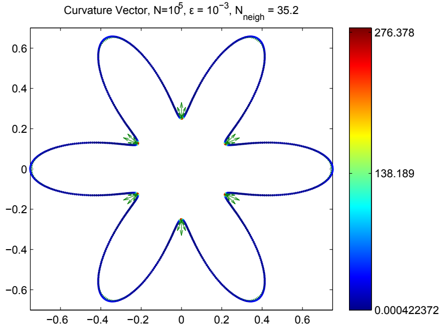
Recovering the classical curvature with projecting onto the normal vector

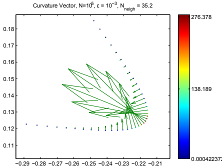


(a)



(b)


Curvature Vector, N=10^5, ε = 10^−3, N_neigh = 35.2
276.378
138.189
0.000422372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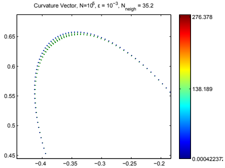
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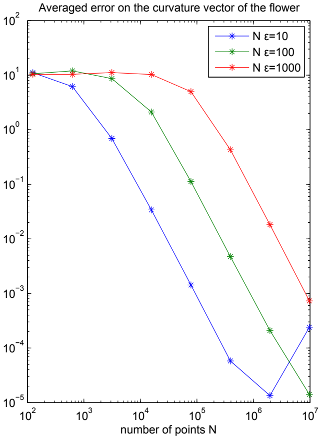
Averaged error on the curvature vector of the flower
N ε=10
N ε=100
N ε=1000
number of points N