A multimodal approach to surface representation

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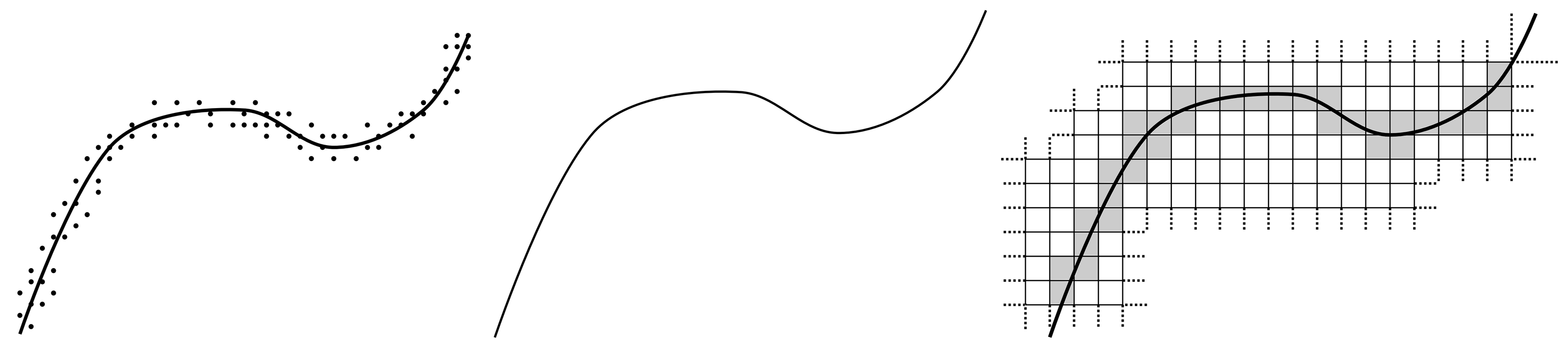
Motivations

Goal: A **unique** tool for describing **curves**, **surfaces** and their various **approximations** (point cloud, triangulations, volumetric approximations), with the purpose of:

- defining **(approximate) mean curvature** in any situation,
- providing **quantitative conditions** ensuring that a limit of given discretizations is (weakly) regular.



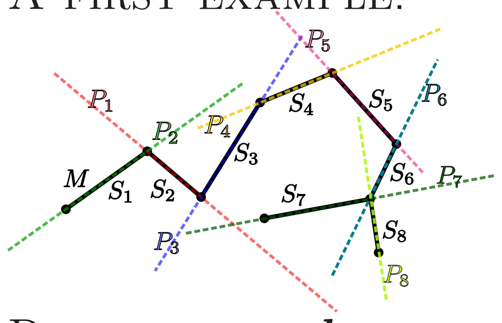
(a) Point cloud (b) Regular curve (c) Volumetric approximation

What is a varifold ?

A **varifold** is a measure containing both **spatial** information and **tangential** information. As a mathematical object, a d–varifold is a **Radon measure** in

$$\underbrace{\mathbb{R}^n}_{\substack{spatial\\information}} \times \underbrace{G_{d,n}\,\{d\text{-vector planes of } \mathbb{R}^n\}}_{\substack{tangential\\information}} .$$

- A first example:



A varifold canonically associated to M is the **measure**:

$$v_M(x,S) = \sum_{i=1}^{8} \mathcal{H}^1_{|S_i}(x) \otimes \delta_{P_i}(S) \text{ in } \mathbb{R}^2 \times G_{1,2}$$

- Rectifiable d–varifolds:

A **rectifiable** d**–varifold** is a measure of the form

$$v_{M,\theta}(x,S) = \underbrace{\theta(x)}_{\substack{>0\\ \text{multiplicity}}} \mathcal{H}^d_{|M}(x) \otimes \underbrace{\delta_{T_xM}(S)}_{\substack{\text{approximate}\\ \text{tangent plane}}} \text{ with } M \text{ rectifiable set.}$$

First variation of a varifold: a generalized notion of curvature

- Let $M \subset \mathbb{R}^n$ be a C^2 d–sub-manifold. Then for every $X \in C^1_c(\mathbb{R}^n,\mathbb{R}^n)$,

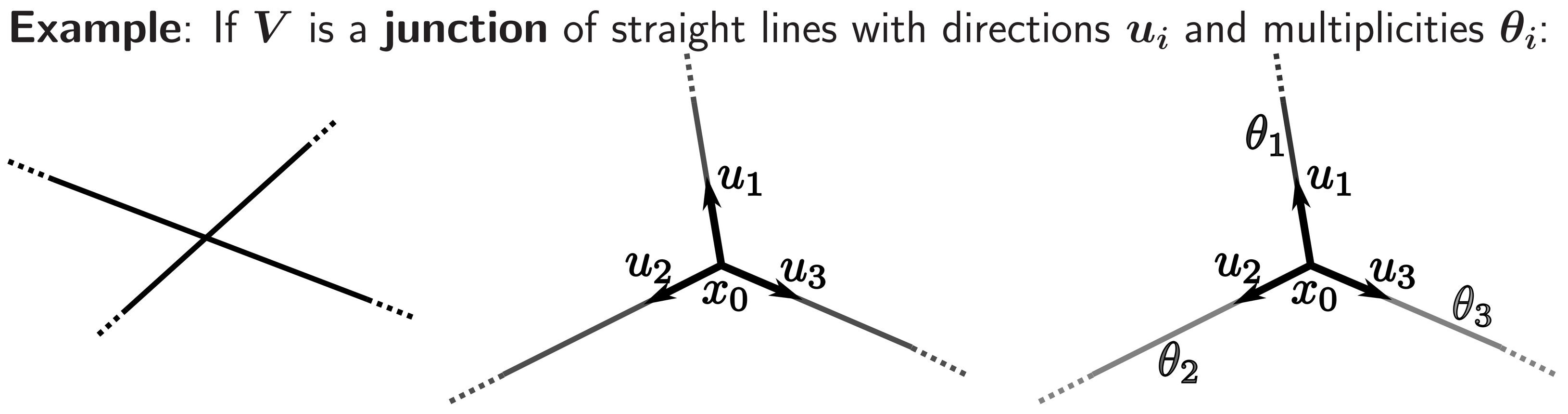
$$\int_M \operatorname{div}_M X \, d\mathcal{H}^d = -\int_M < X, H > \, d\mathcal{H}^d$$

This is actually a way to define the mean curvature vector H*:*

- The **first variation** of a d–varifold V is the **distribution of order** 1

$$\delta V : C^1_c(\mathbb{R}^n,\mathbb{R}^n) \longrightarrow \mathbb{R}$$
$$X \longmapsto \int_{\mathbb{R}^n \times G_{d,n}} \operatorname{div}_S X(x) \, dV(x,S) .$$

- When δV is actually a distribution of order 0, it can be represented as a vector Radon measure in $\mathbb{R}^n$ and V *has locally bounded first variation.*
- **Example**: If V is a **junction** of straight lines with directions u_i and multiplicities θ_i:



$$\delta V = 0, \quad \delta V = (u_1 + u_2 + u_3)\delta_{x_0}, \quad \delta V = (\theta_1 u_1 + \theta_2 u_2 + \theta_3 u_3)\delta_{x_0}$$

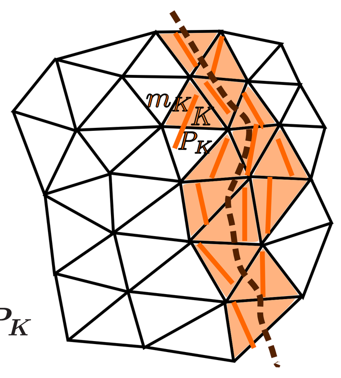
Variation of multiplicity affects generalized curvature.

Discretizations endowed with a varifold structure

- Volumetric approximation varifolds and Point cloud varifolds:

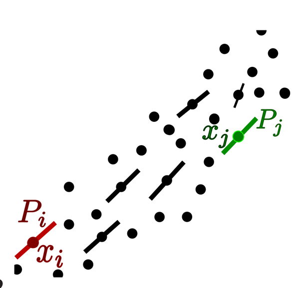
$\mathcal{K}$ mesh of $\mathbb{R}^n$. Given $\{m_K, P_K\}_K \subset \mathbb{R}_+ \times G_{d,n}$. We can associate the **volumetric** d**–varifold**:

$$V_{\mathcal{K}} = \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K}$$

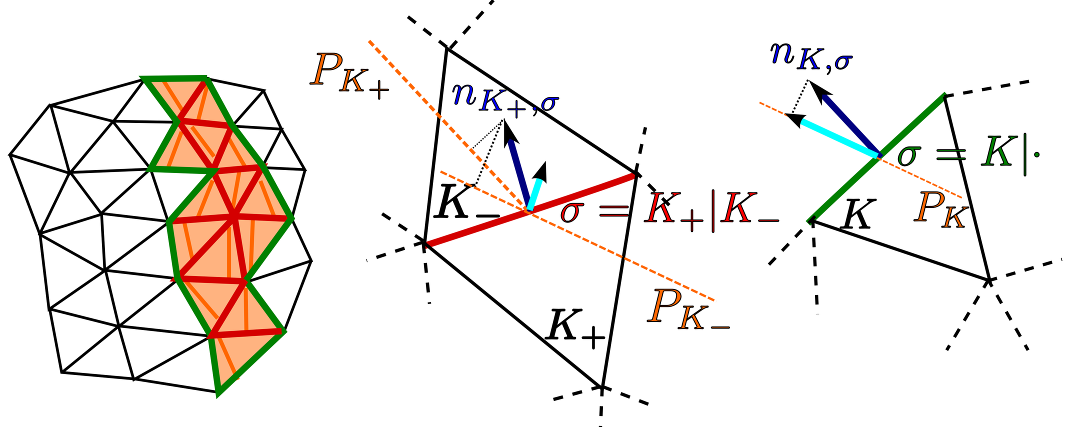


Given a **weighted** (weights $(m_i)_i$) and **oriented** (directions $(P_i)_i$) **point cloud** $(x_i)_{i=1}^N$,

$$V_N = \sum_i m_i \delta_{x_i} \otimes \delta_{P_i}$$



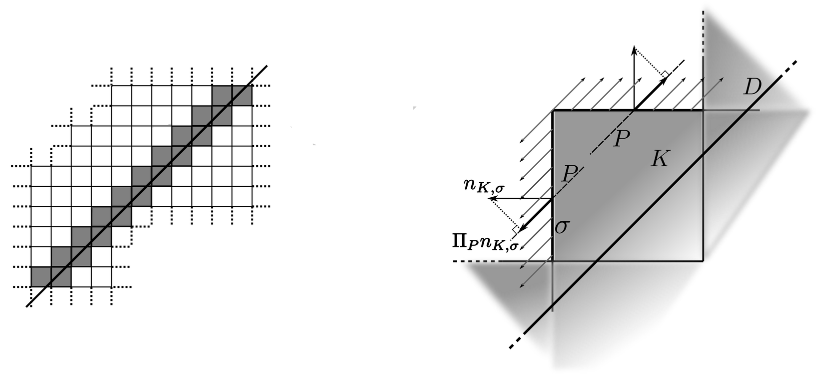
- First variation of a Volumetric varifold:
  - **curvature concentrated** on the faces $\mathcal{E}$,
  - contribution of boundary faces strongly **depends on the mesh**.



$$|\delta V_{\mathcal{K}}| = \sum_{\substack{\sigma \in \mathcal{E}_{int},\\ \sigma = K_-|K_+}} \left| \left[\frac{m_{K_+}}{|K_+|} \Pi_{P_{K_+}} - \frac{m_{K_-}}{|K_-|} \Pi_{P_{K_-}} \right] \cdot (n_{K_+,\sigma}) \right| \mathcal{H}^{n-1}_{|\sigma}$$
$$+ \sum_{\substack{\sigma \in \mathcal{E}_b,\\ \sigma = K|\cdot}} \frac{m_K}{|K|} |\Pi_{P_K} n_{K,\sigma}| \, \mathcal{H}^{n-1}_{|\sigma} .$$

Classic first variation is not well adapted to discrete varifolds

- Volumetric approximation of a straight line in a cartesian grid:



$$\|\delta V_{\mathcal{K}}\| = |\delta V_{\mathcal{K}}|(\Omega) \geq \frac{\sqrt{2}}{2h_{\mathcal{K}}} \text{length}(V)$$

tends to $+\infty$ when the size of the mesh $h_{\mathcal{K}}$ tends to 0.

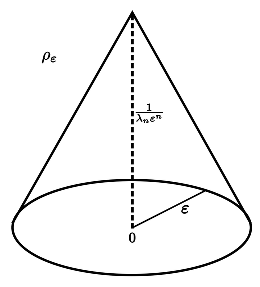
- First variation of Point cloud varifolds:

The first variation of a Point cloud varifold $V = \sum_i m_i \delta_{x_i} \otimes \delta_{P_i}$ is not even a Radon measure.

We propose a notion of approximate curvature for such varifolds, based on a regularization of the first variation.

Regularization of the first variation

- Expression of the regularized first variation:



Let V be a d–varifold $\mathbb{R}^n$ and denote Π_S the orthogonal projection on S,

$$\delta V * \rho_\epsilon(x) = \frac{-1}{\lambda_n \epsilon^{n+1}} \int_{B_\epsilon(x) \times G_{d,n}} \frac{\Pi_S(y-x)}{|y-x|} dV(y,S)$$

well-defined even if V *does not have bounded variation !*

- Quantitative conditions of rectifiability:

For any d–varifold V, we define

$$\mathcal{W}^1_\epsilon(V) = \|\delta V * \rho_\epsilon\|_{L^1} .$$

Let $V_i \xrightarrow{*} V$, then

1. For any $V_\epsilon \xrightarrow{*} V$, $\|\delta V\| \leq \liminf_\epsilon \mathcal{W}^1_\epsilon(V_\epsilon)$. *So that if there exists* ϵ_i *such that* $\sup_i \int_{\mathbb{R}^n} |\delta V_i * \rho_{\epsilon_i}(x)| \, dx < +\infty$ *then* V *has bounded first variation.*
   Moreover,
2. For any d–varifold of bounded first variation V, $\mathcal{W}_\epsilon(V) \to \|\delta V\|$.
3. For any rectifiable d–varifold with bounded first variation, there exists a sequence of volumetric varifolds $(V_i)_i$ such that

$$\mathcal{W}^1_{\epsilon_i}(V_i) \to \|\delta V\| .$$

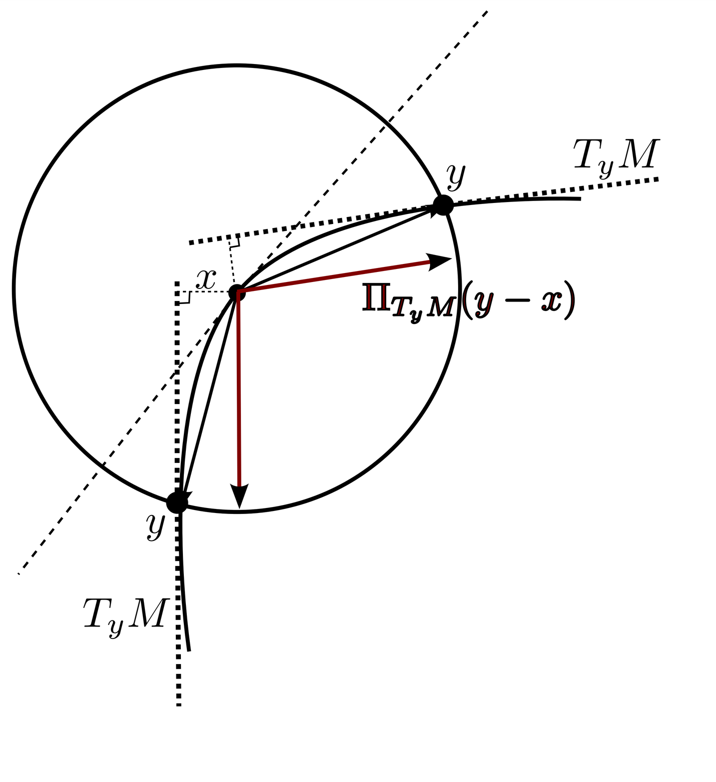
So that W^1_ϵ Γ*–converges to* $\|\delta V\|$ *(the total variation of* δV*) in the space of volumetric varifolds* .

Perspectives

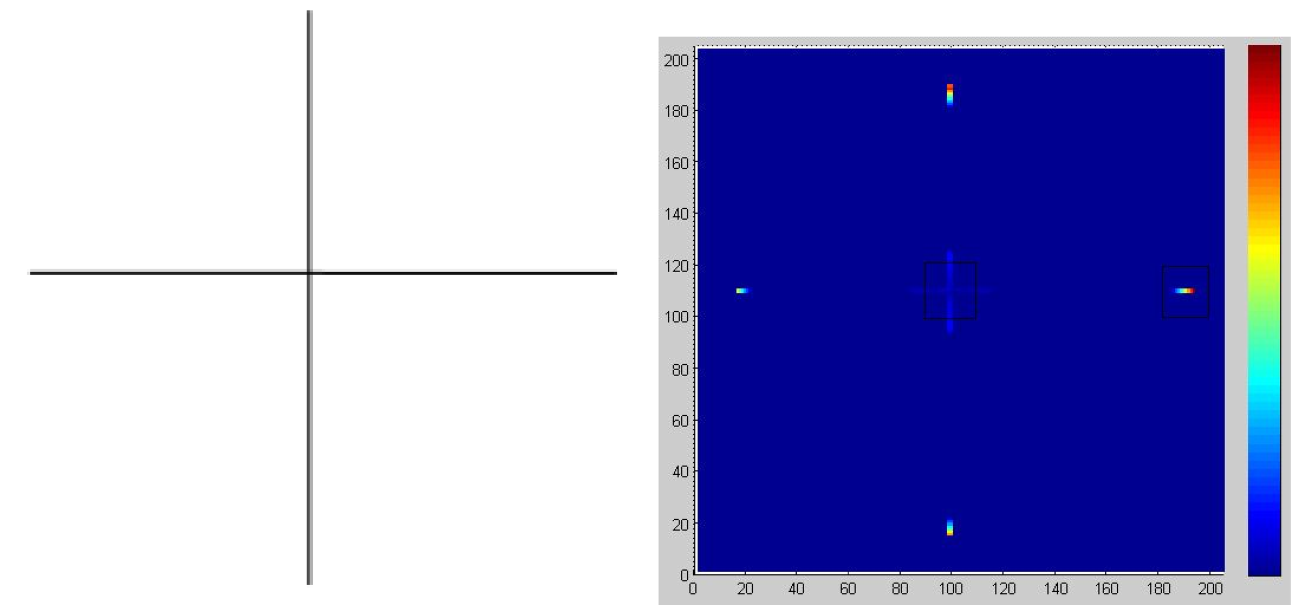
- Approximate Willmore energies:

$$\mathcal{W}^p_\epsilon(V) = \int_x \left| \frac{\delta V * \rho_\epsilon(x)}{\|V\| * \rho_\epsilon(x)} \right|^p \|V\| * \rho_\epsilon(x) \, d\mathcal{L}^n(x) = \int_x |H_\epsilon(x)|^p \, d\mu_\epsilon(x) .$$

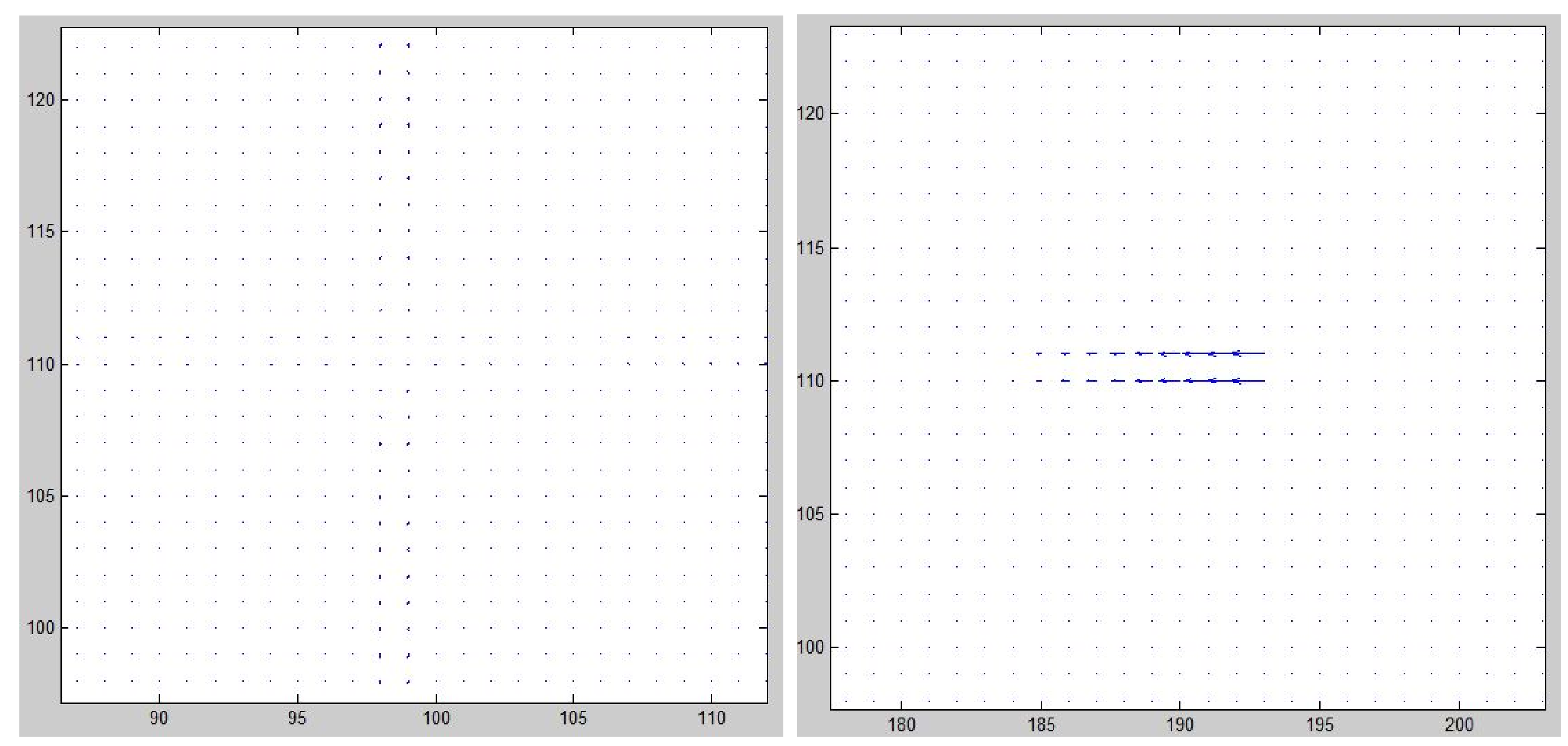
$\mathcal{W}^p_\epsilon$ Γ*–converges to the classical Willmore energy* $\mathcal{W}^p$ *in the space of varifolds but what about the* Γ*–*$\limsup$ *in the set of volumetric varifolds?*



- Numerical experiments:



(a) Pixellized cross (b) Curvature modulus



(a) Curvature vector at the center (b) Curvature vector at cross end points