Recovering measures from approximate values on balls

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Abstract

In a metric space (X, d) we show how to reconstruct an approximation of a Borel measure μ starting from a premeasure p defined on the collection of metric balls, and such that $p(B)$ approximates up to constant factors the value of $\mu(B)$ for any ball B. To this aim, we exploit a suitable packing-type construction of measures coupled with a key condition on the metric space (i.e., the existence of an asymptotically doubling measure). The problem has been originally motivated by the study of certain regularized mean curvature operators defined on varifolds.

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1 Introduction

Is a Borel measure μ on a metric space (X, d) fully determined by its values on balls? In the context of general measure theory, such a question appears to be of extremely basic nature. The answer (when it is known) strongly depends upon the interplay between the measure and the metric space. A clear overview on the subject is given in [8]. Let us mention some known facts about this issue. When $X = \mathbb{R}^n$ equipped with a norm, the answer to the above question is in the affirmative. The reason is the following: if two locally finite Borel measures μ and ν coincide on every ball $B_r(x) \subset \mathbb{R}^n$, then in particular they are mutually absolutely continuous, therefore by the Radon-Nikodym-Lebesgue Differentiation Theorem one has $\mu(A) = \int_A \eta \, d\nu = \nu(A)$ for any Borel set A, where

$$\eta(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{\nu(B_r(x))} = 1$$

is the Radon-Nikodym derivative of μ with respect to ν (defined for ν-almost all $x \in \mathbb{R}^n$). More generally, the same fact can be shown for any pair of Borel measures on a finite-dimensional Banach space X. Unfortunately, the Differentiation Theorem is valid on a Banach space X if and only if X is finite-dimensional. Of course, this does not prevent in general the possibility that Borel measures are uniquely determined by their values on balls. Indeed, Preiss and Tišer proved in [10] that in separable Banach spaces, two finite Borel measures coinciding on all balls also coincide on all Borel sets. Nevertheless, if we only take into account balls of radius less than 1, the question still stands. In the case of separable metric spaces, Federer introduced in [7] a geometrical condition on the distance (see Definition 2.5) implying a Besicovitch-type covering lemma that can be used to show the property above, i.e., that any finite Borel measure is uniquely identified by its values on closed balls. When this condition on the distance is dropped, some examples of measure spaces and of pairs of distinct Borel measures coinciding on balls of upper-bounded diameter are known (see [5]).

Here we consider the case of a separable metric space (X, d) where Besicovitch covering lemma (or at least some generalized version of it) holds, and we ask the following questions:

Question 1. *How can we reconstruct a Borel measure from its values on balls, and especially, what about the case of signed measures?*

A classical approach to construct a measure from a given *premeasure* p defined on a family $\mathcal{C}$ of subsets of X (here the premeasure p is defined on closed balls) is to apply Carathéodory constructions (Method I and Method II, see [1]) to obtain an outer measure. We recall that a premeasure p is a nonnegative function, defined on a given family $\mathcal{C}$ of subsets of X, such that $\emptyset \in \mathcal{C}$ and $p(\emptyset) = 0$. By Method I, an outer measure μ^* is defined starting from p as

$$\mu^*(A) = \inf \left\{ \sum_{k=1}^{\infty} p(B_k) \ : \ B_k \in \mathcal{C} \text{ and } A \subset \bigcup_{k=1}^{\infty} B_k \right\} ,$$

for any $A \subset X$. But, as it is explained in [1] (Section 3.2), Method I does not take into account that X is a metric space, thus the resulting outer measure can be incompatible with the metric on X (in the sense that open sets are not necessarily μ^*-measurable). On the other hand, Method II is used to define Hausdorff measures (see Theorem 2.4) and it always produces a metric outer measure μ^*, for which Borel sets are μ^*-measurable.

As for a signed measure $\mu = \mu^+ - \mu^-$, the main problem is that, given a closed ball B, it is impossible to directly reconstruct $\mu^+(B)$ and $\mu^-(B)$ from $\mu(B)$. The idea is, then, to apply Carathéodory's construction to the premeasure $p^+(B) = (\mu(B))_+$ (here a_+ denotes the positive part of $a \in \mathbb{R}$) and check that the resulting outer measure is actually μ^+. Then, by a similar argument we recover μ^-.

Question 2. *Given a positive Borel measure μ and a premeasure q defined on balls, such that for some $C \geq 1$ one has*

$$C^{-1}\mu(B_{r/C}(x)) \leq q(B_r(x)) \leq C\mu(B_r(x)) \tag{1}$$

for all $x \in X$ and $r > 0$, is it possible to reconstruct an approximation up to constants of μ from q? What about the case when μ is a signed measure?

Some minimal explanations about the assumption (1) on $q(B_r(x))$ are in order. Indeed, a possible choice of $q(B_r(x))$ satisfying (1) is

$$q(B_r(x)) = \frac{1}{r}\int_0^r \mu(B_s(x))\, ds\,. \tag{2}$$

This kind of premeasures arises from the problem of approximating the first variation δV of a (rectifiable) d–varifold V in $\mathbb{R}^n$, which is the weak–$*$ limit of a sequence of more general d–varifolds $(V_k)_k$, by means of suitably defined "regularized first variations" (see [2, 3, 4] for more details on this topic).

We point out that, in order to address Question 2, Carathéodory's Method II is not the right choice. Indeed, considering the simple example of μ given by a Dirac delta, the measure reconstructed from the premeasure q by means of Method II can be quite far from μ. More specifically, a loss of mass could happen in the recovery process. Indeed, if $\mu = \delta_y$, the closer to $\partial B_r(x)$ is the mass concentrated at y, the smaller is $q(B_r(x))$ (and indeed $q(B_r(x))$ vanishes when $y \in \partial B_r(x)$). Then for any $\varepsilon > 0$ one can consider x with $r(1-\varepsilon) < |x-y| < r$ and observe that $y \in B_r(x)$ and, at the same time, that $q(B_r(x))$ is small in terms of ε. This shows that the measure reconstructed by Method II is identically zero (see section 3.1 for more details).

In order to recover μ, or at least some measure equivalent or comparable to μ, the choice of centers of the balls in the collection used to cover the support of μ is crucial. Indeed they must be placed in some nearly-optimal positions, such that even the concentric balls with small radius have a significant overlapping with the support of μ. This has led us to considering a packing-type construction. Packing constructions are used to define the packing s-dimensional measure and its associated notion of packing dimension. They are in some sense dual to the constructions leading to Hausdorff measure and dimension, and were introduced by C. Tricot in [12]. Then Tricot and Taylor extended this notion to a general premeasure in [11].

The paper is organized as follows. In Section 2, we explain how reconstruct a positive measure and then a signed measure (Theorem 2.11) from their values on balls, thanks to Carathéodory's construction, answering Question 1. Section 3 deals with Question 2, that is, the reconstruction of a measure starting from a premeasure satisfying (1). After explaining the limitations of Carathéodory's construction for this problem, we prove our main result, Theorem 3.6, saying that by suitable packing constructions one can reconstruct a signed measure equivalent to the initial one in any metric space (X, d) which is directionally limited and endowed with an asymptotically doubling measure ν (see Hypothesis 1).

Some notations

Let (X, d) be a metric space.

- $\mathcal{B}(X)$ denotes the σ–algebra of Borel subsets of X.
- $B_r(x) = \{y \in X \mid d(y, x) \leq r\}$ is the closed ball of radius $r > 0$ and center $x \in X$.
- $U_r(x) = \{y \in X \mid d(y, x) < r\}$ is the open ball of radius $r > 0$ and center $x \in X$.
- $\mathcal{C}$ denotes the collection of closed balls of X and for $\delta > 0$, $\mathcal{C}_\delta$ denotes the collection of closed balls of diameter $\leq \delta$.
- $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$.
- $\mathcal{P}(X)$ is the set of all subsets of X.
- $\mathrm{card} A$ is the cardinality of the set A.

2 Carathéodory metric construction of outer measures

We recall here some standard definitions and well-known facts about general measures, focusing in particular on the construction of measures from premeasures, in the sense of Carathéodory Method II [1].

2.1 Outer measures and metric outer measures

Definition 2.1 (Outer measure)**.** *Let X be a set, and let $\mu^* : \mathcal{P}(X) \to [0; +\infty]$ satisfying*

(i) $\mu^*(\emptyset) = 0$.

(ii) *μ^* is monotone: if $A \subset B \subset X$, then $\mu^*(A) \le \mu^*(B)$.*

(iii) *μ^* is* countably subadditive*: if $(A_k)_{k\in\mathbb{N}}$ is a sequence of subsets of X, then*

$$\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \mu^*(A_k)\,.$$

Then μ^* is called an outer measure *on X.*

In order to obtain a measure from an outer measure, one defines the measurable sets with respect to μ^*.

Definition 2.2 (μ^*–measurable set)**.** *Let μ^* be an outer measure on X. A set $A \subset X$ is* μ^*–measurable *if for all sets $E \subset X$,*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)\,.$$

We can now define a measure associated with an outer measure. Thanks to the definition of μ^*–measurable sets, the additivity of μ^* among the measurable sets is straightforward, actually it happens that μ^* is σ–additive on μ^*–measurable sets.

Theorem 2.1 (Measure associated with an outer measure, see Theorem 2.32 in [1])**.** *Let X be a set, μ^* an outer measure on X, and $\mathcal{M}$ the class of μ^*–measurable sets. Then $\mathcal{M}$ is a σ–algebra and μ^* is countably additive on $\mathcal{M}$. Thus the set function μ defined on $\mathcal{M}$ by $\mu(A) = \mu^*(A)$ for all $A \in \mathcal{M}$ is a measure.*

We now introduce *metric outer measures.*

Definition 2.3. *Let (X, d) be a metric space and μ^* be an outer measure on X. μ^* is called a* metric outer measure *if*

$$\nu(E \cup F) = \nu(E) + \nu(F)$$

for any $E, F \subset X$ such that $d(E, F) > 0$.

When μ^* is a metric outer measure, every Borel set is μ^*–measurable and thus the measure μ associated with μ^* is a Borel measure.

Theorem 2.2 (Carathéodory's Criterion, see Theorem 3.8 in [1])**.** *Let μ^* be an outer measure on a metric space (X, d). Then every Borel set in X is μ^*-measurable if and only if μ^* is a metric outer measure. In particular, a metric outer measure is a Borel measure.*

We recall two approximation properties of Borel measures defined on metric spaces.

Theorem 2.3 (see Theorems 3.13 and 3.14 in [1])**.** *Let (X, d) be a metric space and μ be a Borel measure on X.*

- Approximation from inside*: Let F be a Borel set such that $\mu(F) < +\infty$, then for any $\varepsilon > 0$, there exists a closed set $F_\varepsilon \subset F$ such that $\mu(F \setminus F_\varepsilon) < \varepsilon$.*
- Approximation from outside*: Assume that μ is finite on bounded sets and let F be a Borel set, then*
$$\mu(F) = \inf\{\mu(U) \ : \ F \subset U, U \text{ open set}\}.$$

We can now introduce Carathéodory's construction of metric outer measures (Method II, see [1]).

Definition 2.4 (Premeasure)**.** *Let X be a set and $\mathcal{C}$ be a family of subsets of X such that $\emptyset \in \mathcal{C}$. A nonnegative function p defined on $\mathcal{C}$ and such that $p(\emptyset) = 0$ is called a* premeasure*.*

Theorem 2.4 (Carathéodory's construction, Method II)**.** *Suppose (X, d) is a metric space and $\mathcal{C}$ is a family of subsets of X which contains the empty set. Let p be a non-negative function on $\mathcal{C}$ which vanishes on the empty set. For each $\delta > 0$, let*

$$\mathcal{C}_\delta = \{A \in \mathcal{C} \mid \operatorname{diam}(A) \leq \delta\}$$

and for any $E \subset X$ define

$$\nu_\delta^p(E) = \inf\left\{\sum_{i=0}^{\infty} p(A_i) \,\middle|\, E \subset \bigcup_{i\in\mathbb{N}} A_i, \forall i,\ A_i \in \mathcal{C}_\delta\right\}.$$

As $\nu_\delta^p \geq \nu_{\delta'}^p$ when $\delta \leq \delta'$,

$$\nu^{p,*}(E) = \lim_{\delta\to 0} \nu_\delta^p(E)$$

exists (possibly infinite). Then $\nu^{p,*}$ is a metric outer measure on X.

2.2 Effects of Carathéodory's construction on positive Borel measures

Let (X, d) be an open set and μ be a positive Borel σ–finite measure on X. Let $\mathcal{C}$ be the set of closed balls and let p be the premeasure defined in $\mathcal{C}$ by,

$$\begin{array}{rcrcl} p & : & \mathcal{C} & \to & [0, +\infty] \\ & & B & \mapsto & \mu(B) \end{array} \tag{3}$$

Let $\mu^{p,*}$ be the metric outer measure obtained by Carathéodory's metric construction applied to $(\mathcal{C}, p)$ and then μ^p the Borel measure associated with $\mu^{p,*}$. Then, the following question arises.

Question 3. *Do we have $\mu^p = \mu$? In other terms, can we recover the initial measure by Carathéodory's Method II?*

The following lemma shows one of the two inequalities needed to positively answer Question 3.

Lemma 2.5. *Let (X, d) be an open set and μ be a positive Borel measure on X. Then, in the same notations as above, we have $\mu \leq \mu^p$.*

Proof. Let $A \subset X$ be a Borel set, we have to show that $\mu(A) \le \mu^p(A) = \mu^{p,*}(A)$. This inequality relies only on the definition of μ^p_δ as an infimum. Indeed, let $\delta > 0$, then for any $\eta > 0$ there exists a countable collection of closed balls $(B^\eta_j)_{j\in\mathbb{N}} \subset \mathcal{C}_\delta$ such that

$$A \subset \bigcup_j B^\eta_j \qquad \text{and} \qquad \mu^p_\delta(A) \ge \sum_{j=1}^{\infty} p(B^\eta_j) - \eta\,,$$

so that

$$\mu^p_\delta(A) + \eta \ge \sum_{j=1}^{\infty} p(B^\eta_j) = \sum_{j=1}^{\infty} \mu(B^\eta_j) \ge \mu\Big(\bigcup_j B^\eta_j\Big) \ge \mu(A)\,.$$

Letting $\eta \to 0$ and then $\delta \to 0$ leads to $\mu(A) \le \mu^p(A)$. □

The other inequality is not true in general. We need extra assumptions on (X, d) ensuring that open sets are "well approximated" by closed balls, that is, we need some specific covering property. In $\mathbb{R}^n$ with the euclidean norm, this approximation of open sets by disjoint unions of balls is provided by Besicovitch Theorem, which we recall here:

Theorem 2.6 (Besicovitch Theorem, see Corollary 1 p. 35 in [6])**.** *Let μ be a Borel measure on $\mathbb{R}^n$ and consider any collection $\mathcal{F}$ of non degenerated closed balls. Let A denote the set of centers of the balls in $\mathcal{F}$. Assume $\mu(A) < +\infty$ and that*

$$\inf\{r > 0 \mid B_r(a) \in \mathcal{F}\} = 0 \qquad \forall\, a \in A\,.$$

Then, for every open set $U \in \mathbb{R}^n$, there exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigsqcup_{B\in\mathcal{G}} B \subset U \quad \textit{and} \quad \mu\left((A \cap U) - \bigsqcup_{B\in\mathcal{G}} B\right) = 0\,.$$

A generalization of Besicovitch Theorem for metric measure spaces is due to Federer, under a geometric assumption involving the distance function.

Definition 2.5 (Directionally limited distance, see 2.8.9 in [7])**.** *Let (X, d) be a metric space, $A \subset X$ and $\xi > 0$, $0 < \eta \le \frac{1}{3}$, $\zeta \in \mathbb{N}^*$. The distance d is said to be* directionally (ξ, η, ζ)–limited at *A if the following holds. Take any $a \in A$ and $B \subset A \cap \Big(B^\circ_\xi(a) \setminus \{a\}\Big)$, such that*

$$\frac{d(x, c)}{d(a, c)} \ge \eta \tag{4}$$

for all $b, c \in B$ and all $x \in X$, such that $b \ne c$, $d(a, x) = d(a, c)$, $d(b, x) = d(a, b) - d(a, c)$ and $d(a, b) \ge d(a, c)$. Then $\mathrm{card}B \le \zeta$.

Let us say a few words about this definition. If $(X, |\cdot|)$ is a Banach space with strictly convex norm, then the above relations involving x imply that

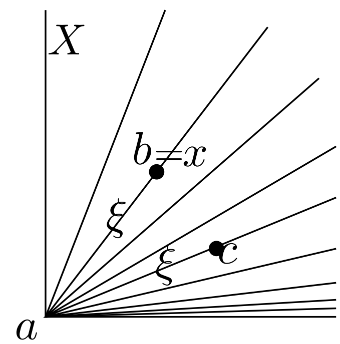
$$x = a + \frac{|a - c|}{|a - b|}(b - a)\,,$$

hence in this case (4) is equivalent to

$$\frac{d(x,c)}{d(a,c)} = \left| \frac{c-a}{|c-a|} - \frac{b-a}{|b-a|} \right| \geq \eta \, .$$

Consequently, if X is finite-dimensional, and thanks to the compactness of the unit sphere, for a given η there exists $\zeta \in \mathbb{N}$ such that $(X, |\cdot|)$ is directionally (ξ, η, ζ)-limited for all $\xi > 0$. Hereafter we provide two examples of metric spaces that are not directionally limited.

Example 2.7. Consider in $\mathbb{R}^2$ the union X of a countable number of half-lines, joining at the same point a. Then the geodesic metric d induced on X by the ambient metric is not directionally limited at $\{a\}$.



Indeed let $B = X \cap \{y \,:\, d(a,y) = \xi\}$, let b and $c \in B$ lying in two different lines, at the same distance $d(a,b) = d(a,c) = \xi$ of a. Then $x \in X$ such that $d(a,x) = d(a,c) = \xi$ and $d(b,x) = d(a,b) - d(a,c) = 0$ implies $x = b$ and thus

$$\frac{d(x,c)}{d(a,c)} = \frac{d(b,c)}{\xi} = \frac{2\xi}{\xi} = 2 \, .$$

but $\operatorname{card} B$ is not finite.

Example 2.8. If X is a separable Hilbert space and $B = (e_k)_{k\in\mathbb{N}}$ a Hilbert basis, $a \in H$ and $b = a + e_j$, $c = a + e_k \in a + B$, the Hilbert norm is strictly convex so that $d(a,x) = d(a,c)$, $d(b,x) = d(a,b) - d(a,c)$ uniquely define x as

$$x = a + \frac{|e_k|}{|e_j|} e_j = b \text{ and } \frac{d(x,c)}{d(a,c)} = |e_k - e_j| = 2 \geq \eta$$

for all $\eta \leq \frac{1}{3}$ and $\operatorname{card}(a+B)$ is infinite. Therefore H is not directionally limited (nowhere).

We can now state the generalized versions of Besicovitch Covering Lemma and Besicovitch Theorem for directionally limited metric spaces.

Theorem 2.9 (Generalized Besicovitch Covering Lemma, see 2.8.14 in [7])**.** *Let (X,d) be a separable metric space directionally (ξ, η, ζ)–limited at $A \subset X$. Let $0 < \delta < \frac{\xi}{2}$ and $\mathcal{F}$ be a family of closed balls of radii less than δ such that each point of A is the center of some ball of $\mathcal{F}$. Then, there exists $2\zeta + 1$ countable subfamilies of $\mathcal{F}$ of disjoint closed balls, $\mathcal{G}_1, \ldots, \mathcal{G}_{2\zeta+1}$ such that*

$$A \subset \bigcup_{j=1}^{2\zeta+1} \bigsqcup_{B \in \mathcal{G}_j} B \, .$$

Remark 2.1. In $\mathbb{R}^n$ endowed with the Euclidean norm it is possible to take $\xi = +\infty$ and ζ only dependent on η and n. If we fix $\eta = \frac{1}{3}$, then $\zeta = \zeta_n$ only depends on the dimension n.

Theorem 2.10 (Generalized Besicovitch Theorem, see 2.8.15 in [7])**.** *Let (X,d) be a separable metric space directionally (ξ, η, ζ)–limited at $A \subset X$. Let $\mathcal{F}$ be a family of closed balls of X satisfying*

$$\inf \{ r > 0 \mid B_r(a) \in \mathcal{F} \} = 0, \qquad \forall \, a \in A \, , \tag{5}$$

and let μ be a positive Borel measure on X, finite on bounded sets. Then, for any open set $U \subset X$ there exists a countable disjoint family $\mathcal{G}$ of $\mathcal{F}$ such that

$$\bigsqcup_{B\in\mathcal{G}} B \subset U \quad \textit{and} \quad \mu\left((A\cap U) - \bigsqcup_{B\in\mathcal{G}} B\right) = 0\,.$$

We can now prove the coincidence of the initial measure and the reconstructed measure under assumptions of Theorem 2.10.

Proposition 2.11. *Let (X,d) be a separable metric space directionally (ξ,η,ζ)–limited at X. Let μ be a positive Borel measure on X, finite on bounded sets. Let $\mathcal{C}$ be the family of closed balls in X and let p be the premeasure defined in $\mathcal{C}$ by (3). Denote by $\mu^{p,*}$ the metric outer measure obtained by Carathéodory's metric construction applied to $(\mathcal{C},p)$ and by μ^p the Borel measure associated with $\mu^{p,*}$. Then μ^p is finite on bounded sets and $\mu^p = \mu$.*

Proof. **Step one.** We prove that $\mu^{p,*}$ is finite on bounded sets. First we recall that by Theorem 2.4 $\mu^{p,*}$ is a metric outer measure, then thanks to Theorem 2.1 μ^p is a Borel measure. Moreover, μ is finite on bounded sets, let us prove that μ^p is finite on bounded sets. Let $A \subset X$ be a bounded Borel set and apply Besicovitch Covering Lemma (Theorem 2.9) with the family

$$\mathcal{F}_\delta = \{B = B_r(x) \text{ closed ball } : \, x\in A \text{ and } \operatorname{diam} B \le \delta\}\ ,$$

to get $2\zeta+1$ countable subfamilies $\mathcal{G}_1^\delta \ldots, \mathcal{G}_{2\zeta+1}^\delta$ of disjoint balls in $\mathcal{F}$ such that

$$A \subset \bigcup_{j=1}^{2\zeta+1} \bigsqcup_{B\in\mathcal{G}_j^\delta} B\,.$$

Therefore,

$$\mu_\delta^p(A) \le \sum_{j=1}^{2\zeta+1}\sum_{B\in\mathcal{G}_j^\delta} p(B) \le \sum_{j=1}^{2\zeta+1}\mu\left(\bigsqcup_{B\in\mathcal{G}_j^\delta} B\right) \le (2\zeta+1)\mu(A+B_\delta(0)) \le (2\zeta+1)\mu(A+B_1(0))\,,$$

where $A+B_1(0) = \bigcup_{x\in A} B_1(x)$ is bounded, thus $\mu(A+B_1(0)) < +\infty$ and hence for all $0<\delta<1$

$$\mu_\delta^p(A) \le (2\zeta+1)\mu(A+B_1(0)) < +\infty\,,$$

whence $\mu^{p,*}(A) < +\infty$.

Step two. We now prove that for any open set $U \subset X$ it holds $\mu^p(U) \le \mu(U)$. Let $U \subset X$ be an open set and let $\delta > 0$ be fixed. Consider the collection of closed balls

$$\mathcal{C}_\delta = \{B_r(x) \mid x\in U,\ 0 < 2r \le \delta\}\ .$$

The family $\mathcal{C}_\delta$ satisfies the assumption (5), thus we can apply Theorem 2.10 to μ^p and get a countable collection $\mathcal{G}^\delta$ of disjoint balls in $\mathcal{C}_\delta$ such that

$$\bigsqcup_{B\in\mathcal{G}^\delta} B \subset U \quad \text{and} \quad \mu^p(U) = \mu^p\left(\bigsqcup_{B\in\mathcal{G}^\delta} B\right)\,.$$

However we also have

$$\mu_\delta^p\left(\bigsqcup_{B\in\mathcal{G}^\delta} B\right) \le \sum_{B\in\mathcal{G}^\delta} p(B) = \sum_{B\in\mathcal{G}^\delta} \mu(B) = \mu\left(\bigsqcup_{B\in\mathcal{G}^\delta} B\right) \le \mu(U)\,. \tag{6}$$

By taking $A = \bigcap_{\substack{\text{countable}\\ \delta\downarrow 0}} \left(\bigsqcup_{B\in\mathcal{G}^\delta} B\right)$ we obtain $\mu^p(U) = \mu^p(A)$ and for any $\delta > 0$, $A \subset \bigsqcup_{B\in\mathcal{G}^\delta} B$. Thus, thanks to (6),

$$\mu_\delta^p(A) \le \mu_\delta^p\left(\bigsqcup_{B\in\mathcal{G}^\delta} B\right) \le \mu(U) \quad \Rightarrow \quad \mu^p(U) = \mu^p(A) \le \mu(U)\,.$$

This shows that $\mu^p(U) \le \mu(U)$, as wanted.

Step three. Since μ and μ^p are Borel measures, finite on bounded sets, they are also outer regular (see Theorem 2.3), then for any Borel set $B \subset X$, and owing to Step two, it holds

$$\begin{aligned}\mu^p(B) &= \inf\{\mu^p(U)\,|\,U \text{ open, } B\subset U\}\\ &\le \inf\{\mu(U)\,|\,U \text{ open, } B\subset U\} = \mu(B)\,.\end{aligned}$$

Coupling this last inequality with Lemma 2.5 we obtain $\mu^p = \mu$. □

2.3 Carathéodory's construction for a signed measure

We recall that a Borel signed measure μ on (X, d) is an extended real-valued set function $\mu : \mathcal{B}(X) \to [-\infty, +\infty]$ such that $\mu(\emptyset) = 0$ and, for any sequence of disjoint Borel sets $(A_k)_k$, one has

$$\sum_{k=1}^{\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)\,. \tag{7}$$

Remark 2.2. Notice that when $\mu\left(\bigcup_{k=1}^{\infty} A_k\right)$ is finite, its value does not depend on the arrangement of the A_k, therefore the series on the right hand side of (7) is commutatively convergent, thus absolutely convergent. In particular, if we write the Hahn decomposition $\mu = \mu^+ - \mu^-$, with μ^+ and μ^- being two non-negative and mutually orthogonal measures, then $\mu^+(X)$ and $\mu^-(X)$ cannot be both $+\infty$.

The question is now the following:

Question 4. *Let (X, d) be a metric space, separable and directionally (ξ, η, ζ)–limited at X, and let μ be a Borel signed measure, finite on bounded sets. Is it possible to recover μ from its values on closed balls by some Carathéodory-type construction?*

The main difference with the case of a positive measure is that μ is not monotone and thus the previous construction is not directly applicable. A simple idea could be to rely on the Hahn decomposition of μ: indeed, μ^+ and μ^- are positive Borel measures, and since one of them is finite, both are finite on bounded sets (recall that μ is finite on bounded sets by assumption). Once again, we cannot directly apply Carathéodory's construction to μ^+ or μ^- since we cannot directly reconstruct $\mu^+(B)$ and $\mu^-(B)$ simply knowing $\mu(B)$ for any closed balls B. We thus try to apply Carathéodory's construction not with $\mu^+(B)$, but with $(\mu(B))_+$, where a_+ (resp. a_-) denote the positive part $\max(a,0)$ (resp. the negative part $\max(-a,0)$) for any $a \in \mathbb{R}$. To be more precise, we state the following definition.

Definition 2.6. *Let μ be a Borel signed Radon measure in X. We define*

$$\begin{array}{rccc} p_+ & : & \mathcal{C} & \longrightarrow & \mathbb{R}_+ \\ & & B & \longmapsto & (\mu(B))_+ \end{array} \quad \text{and} \quad \begin{array}{rccc} p_- & : & \mathcal{C} & \longrightarrow & \mathbb{R}_+ \\ & & B & \longmapsto & (\mu(B))_- \end{array} .$$

Then according to Carathéodory's construction, we define the metric outer measure $\mu^{p_+,*}$ such that for any $A \subset X$,

$$\mu^{p_+,*}(A) = \lim_{\delta \to 0} \mu_\delta^{p_+,*}(A) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=0}^{\infty} p_+(A_i) \,\middle|\, A \subset \bigcup_{i \in \mathbb{N}} A_i, \forall i,\ A_i \in \mathcal{C}_\delta \right\} .$$

Similarly we define $\mu^{p_-,*}$ and then call μ^{p_+} and μ^{p_-} the Borel measures associated with $\mu^{p_+,*}$ and $\mu^{p_-,*}$. Finally, we set $\mu^p = \mu^{p_+} - \mu^{p_-}$.

Theorem 2.12. *Let (X,d) be a metric space, separable and directionally (ξ,η,ζ)–limited at X and let $\mu = \mu^+ - \mu^-$ be a Borel signed measure on X, finite on bounded sets. Let $\mu^p = \mu^{p_+} - \mu^{p_-}$ be as in Definition 2.6. Then $\mu^p = \mu$.*

Proof. We observe that μ^{p_+} and μ^{p_-} are Borel measures: indeed, by construction they are metric outer measures and Carathéodory criterion implies then that these are Borel. Furthermore, for any closed ball $B \in \mathcal{C}$, if we set $p(\mu^+)(B) = \mu^+(B)$, then

$$p_+(B) = (\mu(B))_+ \leq \mu^+(B) = p(\mu^+)(B) \quad \text{and} \quad p_-(B) = (\mu(B))_- \leq \mu^-(B) ,$$

thus by construction, $\mu^{p_+,*} \leq \mu^{p(\mu^+),*}$ and then

$$\mu^{p_+} \leq \mu^{p(\mu^+)} = \mu^+ \text{ thanks to Proposition 2.11} .$$

In the same way we can show that $\mu^{p_-} \leq \mu^-$. In particular, μ^{p_+} and μ^{p_-} are finite on bounded sets, as it happens for μ^+ and μ^-.

Let now $A \subset X$ be a Borel set. It remains to prove that $\mu^{p_+}(A) = \mu^{p_+,*}(A) \geq \mu^+(A)$ (and the same for μ^{p_-}). We argue exactly as in the proof of Lemma 2.5. Let $\delta > 0$, then for any $\eta > 0$ there exists a countable collection of closed balls $(B_j^\eta)_{j \in \mathbb{N}} \subset \mathcal{C}_\delta$ such that $A \subset \bigcup_j B_j^\eta$ and $\mu_\delta^{p_+}(A) \geq \sum_{j=1}^{\infty} p_+(B_j^\eta) - \eta$ so that

$$\mu_\delta^{p_+,*}(A) + \eta \geq \sum_{j=1}^{\infty} p_+(B_j^\eta) = \sum_{j=1}^{\infty} \left(\mu(B_j^\eta)\right)_+ \geq \sum_{j=1}^{\infty} \mu(B_j^\eta) \geq \mu\left(\bigcup_j B_j^\eta\right) \geq \mu(A) .$$

Letting $\eta \to 0$ and then $\delta \to 0$ gives

$$\mu(A) \leq \mu^{p_+,*}(A) = \mu^{p_+}(A) . \tag{8}$$

Recall that in Hahn decomposition, μ^+ and μ^- are mutually singular so that there exists a Borel set $P \subset X$ such that, for any Borel set A,

$$\mu^+(A) = \mu(P \cap A) \quad \text{and} \quad \mu^-(A) = \mu(A \cap (X - P)) \, .$$

Thanks to (8) we already know that $\mu \leq \mu^{p+}$, therefore we get $\mu^+(A) = \mu(P \cap A) \leq \mu^{p+}(P \cap A) \leq \mu^{p+}(A)$ for any Borel set A. We finally infer that $\mu^{p+} = \mu^+$, $\mu^{p-} = \mu^-$, i.e., that $\mu^p = \mu$. $\square$

Remark 2.3. If μ is a vector-valued measure on X, with values in a finite vector space E, we can apply the same construction componentwise.

3 Recovering measures from approximate values on balls

We now want to reconstruct a measure μ (or an approximation of μ) starting from approximate values on closed balls, given by a premeasure q satisfying (1). More precisely, we can now reformulate Question 2 in the context of directionally limited metric spaces.

Question 5. *Let (X, d) be a separable metric space, directionally (ξ, η, ζ)–limited at X and let μ be a positive Borel measure on X. Is it possible to reconstruct μ from q up to multiplicative constants? and what can be done when μ is a signed measure?*

In section 3.1 below we explain with a simple example involving a Dirac mass why Carathéodory's construction does not allow to recover μ from the premeasure q defined in (2). Then we define a *packing construction* of a measure, that is in some sense dual to Carathéodory Method II, and we show that in a directionally limited and separable metric space (X, d), endowed with an asymptotically doubling measure ν (see (21)) it produces a measure equivalent to the initial one.

3.1 Why Carathéodory's construction is not well-suited

Let us consider a Dirac mass $\mu = \delta_x$ in $\mathbb{R}^n$ and define

$$q(B_r(y)) = \frac{1}{r} \int_{s=0}^{r} \mu(B_s(y)) \, ds \, .$$

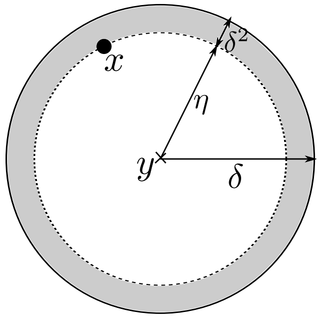
It is easy to check that this particular choice of premeasure q satisfies (1). First of all, for any $r > 0$,

$$q(B_r(x)) = \frac{1}{r} \int_{s=0}^{r} \delta_x(B_s(x)) \, ds = \frac{1}{r} \int_{s=0}^{r} 1 \, ds = 1 \, .$$

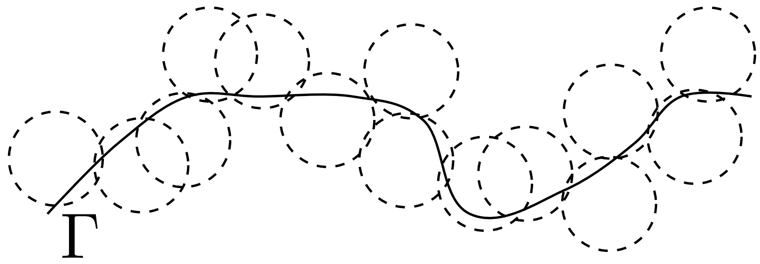
If now y is at distance η from x for some $0 < \eta < r$, we have

$$q(B_r(y)) = \frac{1}{r} \int_{s=0}^{r} \delta_x(B_s(y)) \, ds = \frac{1}{r} \int_{s=\eta}^{r} 1 \, ds = \frac{r - \eta}{r} \, .$$

Therefore, $q(B_r(y)) \to 0$ as $d(x, y) \to r$. We can thus find a covering made by a single ball of radius less than r for which $\mu_r^q(\{x\})$ is as small as we wish. This shows that Carathéodory's construction associated with this premeasure produces the zero measure.



(a) Bad covering of a Dirac mass



(b) Bad covering of a curve

More generally, as soon as it is possible to cover with small balls such that the mass of the measure inside each ball is close to the boundary, one sees that Carathéodory's construction "looses mass". For instance, take $\mu = \mathcal{H}^1_{|\Gamma}$, where $\Gamma \subset \mathbb{R}^n$ is a curve of length L_Γ and $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure in $\mathbb{R}^n$, then cover Γ with a family of closed balls $\mathcal{B}_\delta$ of radii δ with centers at distance η from Γ. Assuming that no portion of the curve is covered more than twice, then

$$\begin{aligned}
\sum_{B \in \mathcal{B}_\delta} q(B) = \sum_k \frac{1}{\delta} \int_{s=0}^{\delta} \mu(B_s(x_k)) &= \sum_k \frac{1}{\delta} \int_{s=\eta}^{\delta} \mu(B_s(x_k))\, ds \\
&\leq \frac{\delta - \eta}{\delta} \sum_k \mu(B_\delta(x_k)) \\
&\leq 2 L_\Gamma \frac{\delta - \eta}{\delta} \xrightarrow[\delta \to 0]{} 0\,,
\end{aligned}$$

with $\eta = \delta - \delta^2$ for instance.

The same phenomenon cannot be excluded by blindly centering balls on the support of the measure μ. Indeed, take a line D with a Dirac mass on it at a point x in $\mathbb{R}^2$, so that $\mu = \mathcal{H}^1_{|D} + \delta_x$. Then, by centering the balls on the support of μ, we may recover the line, but not the Dirac mass, for the same reason as before. We thus understand that the position of the balls should be optimized in order to avoid the problem. For this reason we consider an alternative method, based on a packing-type construction.

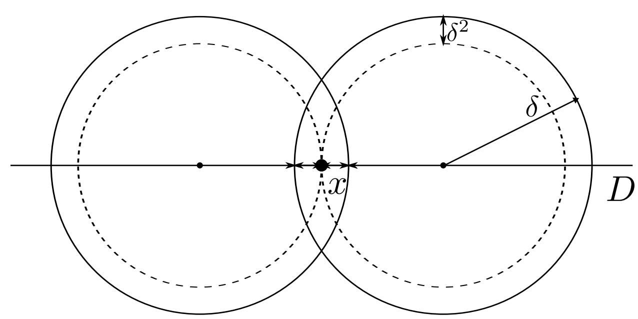


Figure 1: Bad covering with balls centered on the support of the measure

3.2 A packing-type construction

Taking into account the problems described in the examples of the previous section, one realizes the need to optimize the position of the centers of the balls in order to properly reconstruct the original measure μ. The idea is to consider a kind of dual construction, that is, take a supremum over packings rather than an infimum over coverings. To this aim we define the notion of *packing* of balls.

Definition 3.1 (Packings). *Let (X, d) be a separable metric space and $U \subset X$ be an open set. We say that $\mathcal{F}$ is a packing of U of order δ if $\mathcal{F}$ is a countable family of disjoint closed balls whose radius is less than δ and such that*

$$\bigsqcup_{B \in \mathcal{F}} B \subset U\,.$$

Definition 3.2 (Packing construction of measures). *Let (X, d) be a separable metric space and let q be a non-negative set function defined on closed balls, such that $q(\emptyset) = 0$. Let $U \subset X$ be an open set and fix $\delta > 0$. We set*

$$\hat{\mu}^q_\delta(U) := \sup\left\{\sum_{B \in \mathcal{F}} q(B)\ :\ \mathcal{F} \textit{ is a packing of order } \delta \textit{ of } U\right\}$$

and, in a similar way as in Carathéodory construction, define

$$\hat{\mu}^q(U) = \lim_{\delta \to 0} \hat{\mu}^q_\delta(U) = \inf_{\delta > 0} \hat{\mu}^q_\delta(U)$$

and note that $\delta' \leq \delta$ implies $\hat{\mu}^q_{\delta'}(U) \leq \hat{\mu}^q_\delta(U)$. Then, $\hat{\mu}^q$ can be extended to all $A \subset X$ by setting

$$\hat{\mu}^q(A) = \inf\left\{\hat{\mu}^q(U)\ :\ A \subset U,\, U \textit{ open set}\right\}\,.$$

The main difference between Definition 3.2 and Carathéodory's construction is that the set function $\hat{\mu}^q$ is not automatically an outer measure: it is monotone but not sub-additive in general. In order to fix this problem we may apply the construction of outer measures, known as Munroe Method I, to the set function $\hat{\mu}^q$ restricted to the class of open sets. This amounts to setting, for any $A \subset X$,

$$\tilde{\mu}^q(A) = \inf\left\{\sum_{n \in \mathbb{N}} \hat{\mu}^q(U_n)\ :\ A \subset \bigcup_{n \in \mathbb{N}} U_n,\, U_n \text{ open set}\right\}\,. \tag{9}$$

One can check that $\tilde{\mu}^q$ is an outer measure.

Remark 3.1. The construction above is very similar to the one introduced in [11] for measures in $\mathbb{R}^n$. In that paper, starting from a given premeasure q, a so-called *packing premeasure* is defined for any $E \subset \mathbb{R}^n$ as

$$(q - P)(E) = \limsup_{\delta \to 0}\left\{\sum_{B \in \mathcal{B}} q(B)\ :\ \mathcal{B} \text{ is a packing of order } \delta \text{ of } E,\, \mathcal{B} \subset \{B^\circ_r(x)\ :\ x \in E,\, r > 0\}\right\}\,.$$

$(q - P)(E)$ coincides with $\hat{\mu}^q$ on open sets (actually, open balls are considered in the definition, which is not important in the Euclidean $\mathbb{R}^n$ if one deals with Radon measures). Then, from this packing premeasure, they define a packing measure μ^{q-P}, applying Carathéodory's construction, Method I, to $q-P$ on Borel sets. To be precise, for any $A \subset \mathbb{R}^n$,

$$\mu^{q-P}(A) = \inf \left\{ \sum_{k=1}^{\infty} (q-P)(A_k) \ : \ A_k \in \mathcal{B}(\mathbb{R}^n),\ A \subset \bigcup_k A_k \right\} .$$

The outer measure $\tilde{\mu}^q$ is constructed in a very similar way to μ^{q-P}.

We will prove in Proposition 3.1 that, for the class of set functions q we are focusing on, $\hat{\mu}^q$ is already a Borel outer measure, that is, $\hat{\mu}^q = \tilde{\mu}^q$.

Remark 3.2. In order to show that $\hat{\mu}^q = \tilde{\mu}^q$, it is enough to prove the sub-additivity of $\hat{\mu}^q$ in the class of open sets. Indeed, the inequality $\tilde{\mu}^q(A) \leq \hat{\mu}^q(A)$ comes directly from the fact that minimizing $\hat{\mu}^q(U)$ over U open such that $A \subset U$ is a special case of minimizing $\sum_k \hat{\mu}^q(U_k)$ among countable families of open sets U_k such that $A \subset \bigcup_k U_k$. Assuming in addition that $\hat{\mu}^q$ is sub-additivite on open sets implies that for any countable family of open sets $(U_k)_k$ such that $A \subset \bigcup_k U_k$,

$$\hat{\mu}^q(A) \leq \hat{\mu}^q(\bigcup_k U_k) \leq \sum_k \hat{\mu}^q(U_k) .$$

By definition of $\tilde{\mu}^q$, taking the infimum over such families leads to $\hat{\mu}^q(A) \leq \tilde{\mu}^q(A)$.

Proposition 3.1. *Let (X,d) be a separable metric space and let μ be a positive and locally finite Borel measure on X. Let q be a premeasure defined on the class $\mathcal{C}$ of closed balls contained in X, such that* (1) *holds. Then, for any countable family $(A_k)_k \subset X$ satisfying $\hat{\mu}^q\left(\bigcup_k A_k\right) < +\infty$, one has*

$$\hat{\mu}^q \left(\bigcup_{k\in\mathbb{N}} A_k \right) \leq \sum_{k\in\mathbb{N}} \hat{\mu}^q(A_k) . \tag{10}$$

In particular, if μ is finite, then $\hat{\mu}^q$ is an outer measure.

Proof. **Step 1.** We first prove (10) for open sets. Given a countable family $(U_k)_k$ of open subsets of X, such that $\sum_k \mu(U_k) < +\infty$, we show that

$$\hat{\mu}^q \left(\bigcup_{k\in\mathbb{N}} U_k \right) \leq \sum_{k\in\mathbb{N}} \hat{\mu}^q(U_k) . \tag{11}$$

Let $\varepsilon > 0$, then for all $k \in \mathbb{N}$ we define

$$U_k^\varepsilon = \{ x \in U_k \ : \ d(x, X - U_k) > \varepsilon \} .$$

Let $0 < \delta < \frac{\varepsilon}{2}$ be fixed. If B is a closed ball such that $\operatorname{diam} B \leq 2\delta$ and $B \subset \bigcup_k U_k^\varepsilon$, then there exists k_0 such that $B \subset U_{k_0}$. Indeed, $B = B_\delta(x)$ and there exists k_0 such that $x \in U_{k_0}^\varepsilon$ and thus

$$B_\delta(x) \subset U_{k_0}^{\varepsilon-\delta} \subset U_{k_0}^{\frac{\varepsilon}{2}} \subset U_{k_0} .$$

Of course the inclusion $B \subset U_{k_0}$ remains true for any closed ball B with $\operatorname{diam} B \leq 2\delta$. Therefore

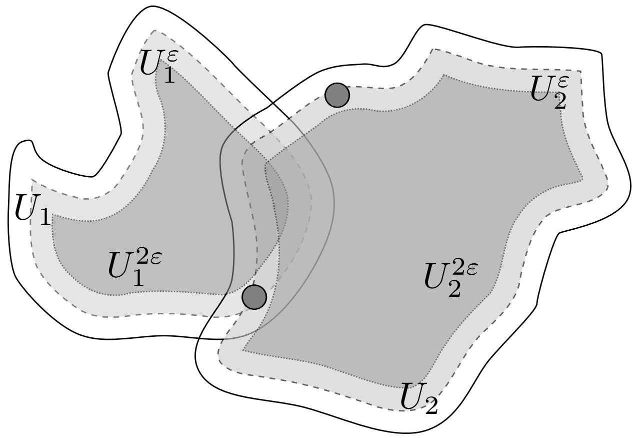


Figure 2: Sub-additivity for packing construction

any packing $\mathcal{B}$ of $\bigcup_k U_k^\varepsilon$ of order $\delta \leq \frac{\varepsilon}{2}$ can be decomposed as the union of a countable family of packings $\mathcal{B} = \bigsqcup_k \mathcal{B}_k$, where $\mathcal{B}_k$ is a packing of U_k of order δ. Thus for any $\delta < \frac{\varepsilon}{2}$,

$$\sum_{B \in \mathcal{B}} q(B) = \sum_k \sum_{B \in \mathcal{B}_k} q(B)$$

and therefore, taking the supremum over all such packings $\mathcal{B}$ of $\bigcup_k U_k^\varepsilon$, we get

$$\hat{\mu}_\delta^q \left(\bigcup_k U_k^\varepsilon \right) \leq \sum_k \hat{\mu}_\delta^q (U_k) \,.$$

Then, taking the infimum over $\delta > 0$ and then the supremum over $\varepsilon > 0$ gives

$$\sup_{\varepsilon > 0} \hat{\mu}^q \left(\bigcup_k U_k^\varepsilon \right) \leq \inf_{\delta > 0} \sum_{k \in \mathbb{N}} \hat{\mu}_\delta^q (U_k) \,. \tag{12}$$

We now want to prove that

$$\sup_{\varepsilon > 0} \hat{\mu}^q \left(\bigcup_{k \in \mathbb{N}} U_k^\varepsilon \right) = \hat{\mu}^q \left(\bigcup_{k \in \mathbb{N}} U_k \right) \,. \tag{13}$$

Let $\mathcal{B}$ be a packing of $\bigcup_k U_k$ of order $\delta < \frac{\varepsilon}{2}$. We have

$$\sum_{B \in \mathcal{B}} q(B) = \sum_{\substack{B \in \mathcal{B} \\ B \subset \bigcup_k U_k^\varepsilon}} q(B) \; + \sum_{\substack{B \in \mathcal{B} \\ B \not\subset \bigcup_k U_k^\varepsilon}} q(B) \,. \tag{14}$$

Notice that since $2\delta < \varepsilon$, for any $B \in \mathcal{B}$, if $B \not\subset \bigcup_k U_k^\varepsilon$ then $B \subset \bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon}$. Since $q(B) \leq C\mu(B)$ according to (1), we get

$$\sum_{\substack{B\in\mathcal{B}\\ B\not\subset\bigcup_k U_k^\varepsilon}} q(B) \leq C \sum_{\substack{B\in\mathcal{B}\\ B\not\subset\bigcup_k U_k^\varepsilon}} \mu(B) = C\mu\left(\bigsqcup_{\substack{B\in\mathcal{B}\\ B\not\subset\bigcup_k U_k^\varepsilon}} B\right) \leq C\mu\left(\bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon}\right). \tag{15}$$

Owing to the fact that $\bigcup_k U_k = \bigcup_{\substack{\text{countable}\\ \varepsilon>0}} \bigcup_k U_k^{2\varepsilon}$ is decreasing in ε, we have that

$$\mu\left(\bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon}\right) \xrightarrow[\varepsilon\to 0]{} 0 \tag{16}$$

as soon as $\mu(\bigcup_k U_k) < +\infty$, which is true under the assumption $\sum_k \mu(U_k) < +\infty$. Therefore, by(14), (15) and (16) we infer that

$$\sum_{B\in\mathcal{B}} q(B) \leq \sum_{\substack{B\in\mathcal{B}\\ B\subset\bigcup_k U_k^\varepsilon}} q(B) + C\mu\left(\bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon}\right). \tag{17}$$

Taking the supremum in (17) over all packings $\mathcal{B}$ of order δ of $\bigcup_k U_k$, we get

$$\begin{aligned}\hat{\mu}^q_\delta\left(\bigcup_k U_k\right) &\leq \sup\left\{\sum_{\substack{B\in\mathcal{B}\\ B\subset\bigcup_k U_k^\varepsilon}} q(B) \ :\ \mathcal{B} \text{ is a packing of } \bigcup_k U_k \text{ order } \delta\right\} + C\mu\left(\bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon}\right)\\ &\leq \hat{\mu}^q_\delta\left(\bigcup_k U_k^\varepsilon\right) + C\mu\left(\bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon}\right),\end{aligned}$$

Then taking the limit as $\delta \to 0$ we obtain

$$\hat{\mu}^q\left(\bigcup_k U_k\right) \leq \hat{\mu}^q\left(\bigcup_k U_k^\varepsilon\right) + C\mu\left(\bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon}\right)$$

and finally, letting $\varepsilon \to 0$, we prove that

$$\hat{\mu}^q\left(\bigcup_k U_k\right) = \lim_{\varepsilon\to 0} \hat{\mu}^q\left(\bigcup_k U_k^\varepsilon\right), \tag{18}$$

that is, (13).

We now turn to the right hand side of (12). For fixed k, $\hat{\mu}^q_\delta(U_k)$ is decreasing when $\delta \downarrow 0$, therefore

$$\lim_{\delta\downarrow 0} \sum_k \hat{\mu}^q_\delta(U_k) = \sum_k \lim_{\delta\downarrow 0} \hat{\mu}^q_\delta(U_k) = \sum_k \hat{\mu}^q(U_k) \tag{19}$$

provided that $\sum_k \hat{\mu}^q_\delta(U_k)$ is finite for some $\delta > 0$. But, since $q(B) \leq C\mu(B)$, $\hat{\mu}^q_\delta(U_k) \leq C\mu(U_k)$ for all k so that $\sum_k \hat{\mu}^q_\delta(U_k) \leq C \sum_k \mu(U_k) < +\infty$. Finally, thanks to (12) , (18) and (19) we obtain the countable sub-additivity for open sets (11).

Step 2. Let $(A_k)_k$ be a countable family of disjoint sets such that $\mu\left(\bigsqcup_k A_k\right) < +\infty$. We shall prove that

$$\hat{\mu}^q\left(\bigsqcup_k A_k\right) \leq \sum_k \hat{\mu}^q(A_k)\,. \tag{20}$$

Being μ a Borel measure, finite on bounded sets, let $(U_k)_k$ be a family of open sets such that, by outer regularity (Theorem 2.3) and for any k,

$$A_k \subset U_k \quad \text{and} \quad \mu(U_k) \leq \mu(A_k) + \frac{1}{2^k}\,,$$

so that $\sum \mu(U_k) \leq \sum \mu(A_k) + 2 < +\infty$. By (11) we thus find

$$\hat{\mu}^q\left(\bigsqcup_k A_k\right) \leq \hat{\mu}^q\left(\bigcup_k U_k\right) \leq \sum_k \hat{\mu}^q(U_k)\,.$$

Taking the infimum over such families of open sets $(U_k)_k$ leads to the required inequality (20).

Step 3. The case of a countable family $(A_k)_k$ such that $\mu\left(\bigcup_k A_k\right) < +\infty$ is obtained from Step 2, by the classical process to make the family disjoint, defining for all k, $B_k \subset A_k$ by

$$B_k = A_k - \bigcup_{i=1}^{k-1} A_i\,.$$

The family $(B_k)_k$ is disjoint and $\displaystyle\bigsqcup_{k\in\mathbb{N}} B_k = \bigcup_{k\in\mathbb{N}} A_k$ so that thanks to Step 2 (20),

$$\hat{\mu}^q\left(\bigcup_k A_k\right) = \hat{\mu}^q\left(\bigsqcup_k B_k\right) \leq \sum_k \hat{\mu}^q(B_k) \leq \sum_k \hat{\mu}^q(A_k)\,.$$

□

Remark 3.3. Notice that the fact that μ is finite on bounded sets is not used in Step 1, to get the sub-additivity on open sets, provided that $\displaystyle\sum_k \mu(U_k) < +\infty$.

In order to have the countable sub-additivity of $\hat{\mu}^q$ (in the case where μ is not assumed to be finite), we want to show that $\sum_k \mu(U_k) = +\infty$ implies $\sum_k \hat{\mu}^q(U_k) = +\infty$. If so, either $\sum_k \mu(U_k) < +\infty$ and the sub-additivity is given by Proposition 3.1, either $\sum_k \hat{\mu}^q(U_k) = +\infty$ and the sub-additivity is clear. Thus we try to estimate $\hat{\mu}^q$ from above, comparing it to μ. The main problem is that the stronger lower bound

$$q(B) \geq C^{-1}\mu(B)$$

cannot be deduced from the weaker lower bound in (1). Moreover, unless we know that the measure μ is doubling, the inequality $\mu(2B) \leq c\mu(B)$ for some $c \geq 1$ (where by $2B$ we denote the ball concentric to B with double radius) may not hold for any ball B. Nevertheless, by assuming that (X,d) is directionally bounded and by comparing μ with an asymptotically doubling measure ν on X, we are able to prove that a doubling property for μ actually holds for enough balls, so that we can choose packings among these balls. Before showing the result, we need to introduce the notion of *asymptotically doubling measure*: we say that ν is asymptotically doubling on X if $0<\nu(B)<+\infty$ for any ball B and there exists a constant $c\geq 1$ such that for all $x\in X$ it holds

$$\limsup_{r\to 0^+}\frac{\nu(B_{2r}(x))}{\nu(B_r(x))}\leq c\,. \tag{21}$$

Of course, this implies that for any fixed constant $C\geq 1$, taking $Q\in\mathbb{N}\cup\{0\}$ as the unique integer for which $2^Q\leq C\leq 2^{Q+1}$, the following holds: for all $x\in X$ there exists $r_0=r_0(x)>0$ such that

$$\nu(B_r(x))\leq (c+1)^{Q+1}\nu(B_{r/C}(x))\,,\quad \forall\, r\in(0,r_0)\,. \tag{22}$$

We conveniently state some key properties on the metric space (X,d) and on the measure μ on X, that will be constantly assumed in the rest of this section.

Hypothesis 1. *(X,d) is a directionally limited metric space endowed with an asymptotically doubling measure ν satisfying (22) for some constants $c,C\geq 1$, and μ is a positive Borel measure on X.*

Proposition 3.2. *Assume that (X,d) and μ satisfy Hypothesis (1). Let*

$$A_0=\left\{x\in X\ :\ \liminf_{r\to 0^+}\frac{\mu(B_r(x))}{\nu(B_r(x))}=0\right\}\quad \textit{and}\quad A_+=\left\{x\in X\ :\ 0<\liminf_{r\to 0^+}\frac{\mu(B_r(x))}{\nu(B_r(x))}\leq+\infty\right\}.$$

Then the following holds.

(i) For all $x\in A_+$, either $\mu(B_r(x))=+\infty$ for all $r>0$, or

$$\limsup_{r\to 0^+}\frac{\mu(B_{r/C}(x))}{\mu(B_r(x))}\geq\frac{1}{(c+1)^{Q+1}}\,.$$

(ii) $\mu(A_0)=0$.

(iii) For μ–almost any $x\in\Omega$, there exists a decreasing infinitesimal sequence $(r_n)_n$ of radii, such that

$$\mu(B_{r_n}(x))\leq (c+2)^{Q+1}\,\mu(B_{r_n/C}(x))\,,\quad \forall\, n\in\mathbb{N}\,.$$

Proof. **Proof of (i).** Let $x\in A_+$. By monotonicity, either $\mu(B_r(x))=+\infty$ for all $r>0$ (and then of course (iii) is also trivially satisfied) or there exists some R such that, for all $r\leq R$, $\mu(B_r(x))<+\infty$. In this case the function defined by

$$f(r)=\frac{\mu(B_r(x))}{\nu(B_r(x))}$$

is non-negative and finite for r small enough. Moreover, since $x\in A_+$, $\liminf_{r\to 0}f(r)>0$. Let us prove that

$$\limsup_{r\to 0}\frac{f(r/C)}{f(r)}\geq 1\,. \tag{23}$$

Assume by contradiction that $\limsup_{r\to 0} \dfrac{f(r/C)}{f(r)} < 1$, then necessarily $C > 1$ and there exists $r_0 > 0$ and $0 < \alpha < 1$ such that for all $r \leq r_0$, $f(r/C) \leq \alpha f(r)$. Consider now the sequence $(r_k)_k$ defined by $r_k = C^{-k} r_0$. Then $r_k \to 0$ and

$$f(r_k) \leq \alpha f(C r_k) = \alpha f(r_{k-1}) \leq \alpha^k f(r_0) \xrightarrow[k\to\infty]{} 0$$

which contradicts $\liminf_{r\to 0} f(r) > 0$ and thus proves (23). Let us then decompose

$$\frac{\mu(B_{r/C}(x))}{\mu(B_r(x))} = \frac{f(r/C)}{f(r)} \cdot \frac{\nu(B_{r/C}(x))}{\nu(B_r(x))} \,.$$

Since ν is asymptotically doubling, by (22) we get

$$\limsup_{r\to 0^+} \frac{\mu(B_{r/C}(x))}{\mu(B_r(x))} \geq \frac{1}{(c+1)^{Q+1}} \limsup_{r\to 0^+} \frac{f(r/C)}{f(r)} \geq \frac{1}{(c+1)^{Q+1}} \,.$$

Proof of (ii). Let us show that $\mu(A_0) = 0$. Assume that $\nu(\Omega) < +\infty$ and let $\varepsilon > 0$. Consider

$$\mathcal{F}_\varepsilon = \{B \subset \Omega \mid B = B_r(a), a \in A_0 \text{ and } \mu(B) \leq \varepsilon \nu(B)\} \,.$$

Let $a \in A_0$ be fixed. Since $\liminf_{r\to 0^+} \dfrac{\mu(B_r(a))}{\nu(B_r(a))} = 0$, there exists $r > 0$ such that $B_r(a) \in \mathcal{F}_\varepsilon$. Every point in A_0 is the center of some ball in $\mathcal{F}_\varepsilon$, so that we can apply Theorem 2.9 and obtain $2\zeta + 1$ countable families $\mathcal{G}_1, \dots \mathcal{G}_{2\zeta+1}$ of disjoint balls in $\mathcal{F}_\varepsilon$, such that

$$A_0 \subset \bigcup_{j=1}^{2\zeta+1} \bigsqcup_{B\in\mathcal{G}_j} B \,.$$

Therefore

$$\mu(A_0) \leq \sum_{j=1}^{2\zeta+1} \sum_{B\in\mathcal{G}_j} \underbrace{\mu(B)}_{\leq \varepsilon\nu(B)} \leq \varepsilon \sum_{j=1}^{2\zeta+1} \nu\left(\bigsqcup_{B\in\mathcal{G}_j} B\right) \leq \varepsilon(2\zeta+1)\nu(\Omega) \,.$$

Hence $\mu(A_0) = 0$ if $\nu(\Omega) < +\infty$. Otherwise, replace Ω by $\Omega \cap U_k(0)$ to obtain that for any $k \in \mathbb{N}$, $\mu(A_0 \cap U_k(0)) = 0$, then let $k \to \infty$ to conclude that $\mu(A_0) = 0$.

Proof of (iii). It is an immediate consequence of the fact that $X = A_0 \cup A_+$ coupled with (i) and (ii). $\square$

Corollary 3.3 (Besicovitch with doubling balls)**.** *Assume Hypothesis* (1) *and, for $\delta > 0$, define*

$$\mathcal{F}_\delta = \left\{B \text{ closed ball } \subset X \,:\, \mu(B) \leq (c+1)^{Q+1}\mu(C^{-1}B) \text{ and } \operatorname{diam} B \leq 2\delta\right\}$$

and, for any $A \subset X$,

$$\mathcal{F}_\delta^A = \{B \in \mathcal{F}_\delta \,:\, B = B_r(a),\ a \in A\} \,.$$

Then there exist $A_0 \subset X$ and $2\zeta+1$ countable subfamilies of $\mathcal{F}_\delta^A$ of disjoint closed balls, $\mathcal{G}_1, \dots \mathcal{G}_{2\zeta+1}$ such that

$$A \subset A_0 \cup \bigcup_{j=1}^{2\zeta+1} \bigsqcup_{B\in\mathcal{G}_j} B \quad \text{and} \quad \mu(A_0) = 0 \,.$$

Moreover, if $\mu(A) < +\infty$, then for any open set $U \subset X$, there exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}_\delta^A$ such that

$$\bigsqcup_{B\in\mathcal{G}} B \subset U \quad \textit{and} \quad \mu\left((A\cap U)\setminus \bigsqcup_{B\in\mathcal{G}} B\right) = 0\,.$$

Proof. Thanks to Proposition 3.2 (iii) we know that for μ-almost every $x \in X$ there exists a decreasing infinitesimal sequence $(r_n)_n$ such that $B_{r_n}(x) \in \mathcal{F}_\delta$ for all $n \in \mathbb{N}$. Hence for μ-almost any $x \in X$ we have

$$\inf\{r \mid B_r(x) \in \mathcal{F}_\delta\} = 0\,.$$

Then, the conclusion follows from Theorems 2.9 and 2.10. □

We can now prove that $\hat{\mu}^q$ and μ are equivalent on Borel sets.

Proposition 3.4. *Assume Hypothesis 1 and let q be a premeasure satisfying* (1)*. Let $\hat{\mu}^q$ be as in Definition 3.2. Then there exists a constant $K \geq 1$ only depending on the doubling constant c and on the constant C appearing in* (1)*, such that for any open set $U \subset X$ we have*

$$\frac{1}{K}\mu(U) \leq \hat{\mu}^q(U) \leq K\mu(U)\,.$$

Consequently, for any Borel set $A \subset X$ we have

$$\frac{1}{K}\mu(A) \leq \hat{\mu}^q(A) \leq K\inf\{\mu(U) \mid A \subset U \textit{ open set }\}$$

and if μ is outer regular, then

$$\frac{1}{K}\mu(A) \leq \hat{\mu}^q(A) \leq K\mu(A)\,.$$

Proof. Let $U \subset X$ be an open set, then the inequality $\hat{\mu}^q(U) \leq C\mu(U)$ is just a consequence of the definition of $\hat{\mu}^q$ and of the second inequality in (1), i.e., of the fact that, for any closed ball B, $q(B) \leq C\mu(B)$. Now we prove the other inequality by splitting the problem in two cases.

(i) Case $\mu(U) < +\infty$. Let $\delta > 0$, then we can apply Corollary 3.3 (Besicovitch with doubling balls) to get a countable family $\mathcal{G}_\delta$ of disjoint balls of

$$\mathcal{F}_\delta^U = \left\{B = B_r(x) \subset U \,:\, \mu(C^{-1}B) \geq \frac{1}{(c+2)^{Q+1}}\mu(B) \text{ and } \operatorname{diam} B \leq 2\delta\right\}$$

such that

$$\mu(U) = \mu\left(\bigsqcup_{B\in\mathcal{G}_\delta} B\right) \quad \text{and} \quad \bigsqcup_{B\in\mathcal{G}_\delta} B \subset U\,.$$

Therefore by (1)

$$\begin{aligned}\hat{\mu}^q_\delta(U) &\geq \sum_{B\in\mathcal{G}_\delta} q(B) \geq \sum_j \frac{1}{C}\mu(B_{r_j/C}(x_j)) \\ &\geq \frac{1}{C(c+2)^{Q+1}}\sum_j \mu(B_{r_j}(x_j)) = \frac{1}{K}\mu\left(\bigsqcup_{B\in\mathcal{G}_\delta} B\right) = \frac{1}{K}\mu(U)\,,\end{aligned}$$

with $K = C(c+2)^{Q+1} > C$. Letting $\delta \to 0$ gives $\hat{\mu}^q(U) \geq \dfrac{1}{K}\mu(U)$.

(ii) Case $\mu(U) = +\infty$. As before, let $\delta > 0$ and apply Corollary 3.3 (Besicovitch with doubling balls) with

$$\mathcal{F}_\delta^U \cap \{B \mid B \subset U\} \ ,$$

which gives $2\zeta + 1$ countable families $\mathcal{G}_\delta^1, \dots, \mathcal{G}_\delta^{2\zeta+1}$ of balls in $\mathcal{F}_\delta \cap \{B \mid B \subset U\}$ such that

$$U \subset U_0 \cup \bigcup_{j=1}^{2\zeta+1} \bigsqcup_{B \in \mathcal{G}_\delta^j} B \text{ with } \mu(U_0) = 0 \, .$$

Then we get

$$\sum_{j=1}^{2\zeta+1} \mu \left(\bigsqcup_{B \in \mathcal{G}_\delta^j} B \right) \geq \mu(U) = +\infty \, .$$

Consequently there exists $j_0 \in \{1, \dots, 2\zeta + 1\}$ such that $\mu \left(\bigsqcup_{B \in \mathcal{G}_\delta^{j_0}} B \right) = +\infty$. Therefore we have the same estimate as in the case $\mu(U) < +\infty$:

$$\begin{aligned} \hat{\mu}_\delta^q(U) \geq \sum_{B \in \mathcal{G}_\delta^{j_0}} q(B) \geq \sum_l \frac{1}{C} \mu(B_{r_l/C}(x_l)) \\ \geq \frac{1}{K} \sum_l \mu(B_{r_l}(x_l)) = \frac{1}{K} \mu \left(\bigsqcup_{B \in \mathcal{G}_\delta^{j_0}} B \right) = +\infty \, , \end{aligned}$$

and we conclude $\hat{\mu}^q(U) = +\infty$.

□

Corollary 3.5. *Under Hypothesis 1, $\hat{\mu}^q$ is countably sub-additive.*

Proof. Let $(A_n)_n$ be a countable collection of subsets of Ω. If $\sum_n \mu(A_n) = +\infty$, by Proposition 3.4 we get $\mu(A_n) \leq K \hat{\mu}^q(A_n)$ for all n, therefore

$$\sum_n \hat{\mu}^q(A_n) \geq \frac{1}{K} \sum_n \mu(A_n) = +\infty \, ,$$

whence the countable sub-additivity follows. Recall that if $\sum_n \mu(A_n) < +\infty$ and A_n are open sets, then countable sub-additivity was proved in Proposition 3.1, without the assumption of finiteness on bounded sets. It remains to check the case $\sum_n \mu(A_n) < +\infty$, for any sequence of Borel sets A_n. For any sequence $(U_n)_n$ of open sets such that $A_n \subset U_n$ for all n, by sub-additivity on open sets we have

$$\hat{\mu}^q(\bigcup_n A_n) \leq \hat{\mu}^q(\bigcup_n U_n) \leq \sum_n \hat{\mu}^q(U_n) \, .$$

Taking the infimum over such families of open sets gives, by definition of $\hat{\mu}^q$,

$$\hat{\mu}^q(\bigcup_n A_n) \le \inf \left\{ \sum_n \hat{\mu}^q(U_n) \ : \ A_n \subset U_n \text{ open set } \right\} = \sum_n \hat{\mu}^q(A_n) \, .$$

□

Let us summarize the results contained in Proposition 3.1, Remark 3.2, Proposition 3.4 and Corollary 3.5:

Theorem 3.6. *Assume Hypothesis 1 and let q be a premeasure satisfying* (1)*. Let $\hat{\mu}^q$ be as in Definition 3.2. Then, the following holds:*

1. *$\hat{\mu}^q$ is a metric outer measure, coinciding with the measure $\tilde{\mu}^q$ defined in* (9)*.*
2. *there exists a constant $K \ge 1$ depending only on the constants c, C (respectively, the asymptotic doubling constant and the constant appearing in* (1)*), such that for any Borel set $A \subset \Omega$,*
$$\frac{1}{K}\mu(A) \le \hat{\mu}^q(A) \le K \inf\{\mu(U) \mid A \subset U \ \textit{open set} \, \} \, .$$
3. *if moreover μ is outer regular, then μ and the positive Borel measure associated with the outer measure $\hat{\mu}^q$ (still denoted as $\hat{\mu}^q$) are equivalent, that is, $\frac{1}{K}\mu \le \hat{\mu}^q \le K\mu$.*

3.3 The case of a signed measure

Our aim is to prove that the packing-type reconstruction applied to a signed measure μ, with premeasures $q_\pm$ satisfying

$$\frac{1}{C}\mu^+(B_{r/C}(x)) - \mu^-(B_r(x)) \le q_+(B_r(x)) \le C\mu^+(B_r(x)) \tag{24}$$

and

$$\frac{1}{C}\mu^-(B_{r/C}(x)) - \mu^+(B_r(x)) \le q_-(B_r(x)) \le C\mu^-(B_r(x)) \tag{25}$$

for some $C \ge 1$, produces a signed measure $\hat{\mu}^p$ whose positive and negative parts are comparable with those of μ. Notice that an example of such a pair of premeasures is given by

$$q_\pm(B_r(x)) = \left(\frac{1}{r} \int_{s=0}^r \mu(B_s(x)) \, ds \right)_\pm \, .$$

Theorem 3.7. *Let (X, d) be a directionally limited metric space endowed with an asymptotic doubling measure ν, and let $\mu = \mu^+ - \mu^-$ be a locally finite, Borel-regular signed measure on X. Take $\hat{\mu}^{q_\pm}$ as in Definition 3.2, corresponding to premeasures $q_\pm$ satisfying* (24) *and* (25)*. Then the following holds.*

(i) $\hat{\mu}^{q_+}$, $\hat{\mu}^{q_-}$ are locally finite, metric outer measures.

(ii) The measure $\hat{\mu}^q = \hat{\mu}^{q+} - \hat{\mu}^{q-}$ *is a signed measure and there exists a constant* $K \geq 1$ *such that, for any Borel set* $A \subset \Omega$,

$$\frac{1}{K}\mu^+(A) \leq \hat{\mu}^{q+}(A) \leq K\mu^+(A) \quad \text{and} \quad \frac{1}{K}\mu^-(A) \leq \hat{\mu}^{q-}(A) \leq K\mu^-(A)\,,$$

whence in particular

$$\frac{1}{K}|\mu|(A) \leq |\hat{\mu}^q|(A) \leq K|\mu|(A)\,.$$

Proof. The countable sub-additivity of $\hat{\mu}^{q+}$ and $\hat{\mu}^{q-}$ under the assumption $\sum_k \mu(A_k) < +\infty$ follows from the second inequalities in (24) and (25) (see Proposition 3.1). Similarly one can conclude that, for any open set $U \subset X$,

$$\hat{\mu}^{q+}(U) \leq C\mu^+(U)\,.$$

Arguing in the same way for q_-, one obtains the proof of (i).

Let now $A \subset X$ be a Borel set. We first derive an estimate concerning $\hat{\mu}^{q+}$ (the estimate for $\hat{\mu}^{q-}$ can be obtained in the same way). If $\mu^+(A) < +\infty$, we take an open set U containing A, such that $\mu^+(U) < +\infty$. Let $\delta > 0$, then apply Corollary 3.3 to μ_+ and get a family $\mathcal{G}_\delta$ of disjoint closed balls $B \subset U$ of radius $\leq \delta$, with $\mu^+(C^{-1}B) \geq \frac{1}{(c+2)^{Q+1}}\mu^+(B)$, such that

$$\mu^+(A) = \mu^+\left(\bigsqcup_{B\in\mathcal{G}_\delta} B\right)\,.$$

We have

$$\begin{aligned}\hat{\mu}^{q+}_\delta(U) &\geq \sum_{B\in\mathcal{G}_\delta} q_+(B) \geq \sum_{B\in\mathcal{G}_\delta} \frac{1}{C}\mu^+(C^{-1}B) - \mu^-(B)\\ &\geq \sum_{B\in\mathcal{G}_\delta} \frac{1}{K}\mu^+(B) - \mu^-(B) \geq \frac{1}{K}\mu^+(A) - \mu^-(U)\,.\end{aligned}$$

Letting $\delta \to 0$ we find

$$\hat{\mu}^{q+}(U) \geq \frac{1}{K}\mu^+(A) - \mu^-(U)\,. \tag{26}$$

By definition of $\hat{\mu}^{q+}(A)$, there exists a sequence of open sets $(U_k^1)_k$ such that, for all k, it holds $A \subset U_k^1$ and

$$\hat{\mu}^{q+}(U_k^1) \xrightarrow[k\to\infty]{} \hat{\mu}^{q+}(A)\,.$$

By outer regularity of μ^- (which is Borel and finite on bounded sets) there exists a sequence of open sets $(U_k^2)_k$ such that, for all k, we get $A \subset U_k^2$ and

$$\mu^-(U_k^2) \xrightarrow[k\to\infty]{} \mu^-(A)\,.$$

For all k, let $U_k = U_k^1 \cap U_k^2$, then U_k is an open set, $A \subset U_k$ and, by monotonicity,

$$\begin{aligned}\hat{\mu}^{q+}(A) &\leq \hat{\mu}^{q+}(U_k) \leq \hat{\mu}^{q+}(U_k^1)\,,\\ \mu^-(A) &\leq \mu^-(U_k) \leq \mu^-(U_k^2)\,,\end{aligned}$$

therefore

$$\hat{\mu}^{q+}(U_k) \xrightarrow[k\to\infty]{} \hat{\mu}^{q+}(A) \text{ and } \mu^-(U_k) \xrightarrow[k\to\infty]{} \mu^-(A)\,.$$

Evaluating (26) on the sequence $(U_k)_k$ and letting k go to $+\infty$, we eventually get

$$\hat{\mu}^{q+}(A) \geq C_n \mu^+(A) - \mu^-(A)\,. \tag{27}$$

Owing to Hahn decomposition of signed measures, there exists a Borel set P such that for all Borel A it holds

$$\mu^+(A) = \mu^+(A \cap P) = \mu(A \cap P) \text{ and } \mu^-(A) = \mu(A \setminus P))\,.$$

Finally, let A be a Borel set, then by (27) applied to $A \cap P$ we finally get

$$\begin{aligned}\hat{\mu}^{q+}(A) \geq \hat{\mu}^{q+}(A \cap P) &\geq \frac{1}{K}\mu^+(A \cap P) - \mu^-(A \cap P) \\ &= \frac{1}{K}\mu^+(A)\,.\end{aligned}$$

It remains to show that if $\mu^+(A) = +\infty$, then $\hat{\mu}^{q+}(A) = +\infty$. This can be easily obtained[1] by taking a sequence of open balls U_n with fixed center and radius $n \in \mathbb{N}$, and by considering the sequence $A_n = A \cap U_n$ for which $\mu^+(A_n) < +\infty$ and $\lim_n \mu^+(A_n) = \mu^+(A) = +\infty$. By applying the same argument as before, we get

$$\hat{\mu}^{q+}(A) \geq \hat{\mu}^{q+}(A_n) \geq \frac{1}{K}\mu^+(A_n)\,,$$

thus the conclusion follows by taking the limit as $n \to +\infty$. This completes the proof of (ii) and thus of the theorem. □

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