From Classical to Quantum Field Theories: Perturbative and Nonperturbative Aspects

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Introduction

Typical (rigorous) approaches to Classical Field Theory mainly via geometric techniques ((multi)symplectic geometry (Kijowski, Marsden et alt.), algebraic geometry/topology (Vinogradov)) whereas physicists (B. de Witt) like to deal with (formal) functional methods, tailored to the needs of (path-integral-based) quantum field theory. In this last case we have:

- Heuristic infinite-dimensional generalisation of Lagrangian mechanics;
- Making it rigorous is possible – usually done in Banach spaces

However, one of our results entails that:

Classical field theory is not as "infinite dimensional" as it appears!

Aims/Bias

- Structural Foundations: We wish to give a fresh look, along the algebraic setting, of interacting classical field theories. From that a "new" quantization procedure for perturbation theory.
- pAQFT: Many structures suggested by perturbation theory in the algebraic fashion [Dütsch-Fredenhagen (CMP-2003), Brunetti-Dütsch-Fredenhagen (ATMP-2009) , Brunetti-Fredenhagen (LNP-2009), Keller (JMP-2009)]

Setting

- Model: Easiest example, real scalar field φ
- Geometry: The geometric arena is the following: $(\mathscr{M}, g)$ globally hyperbolic Lorentzian manifold (fixed, but otherwise generic dimension $d \geq 2$), with volume element $\mathrm{d}\mu_g = \sqrt{|\det g|}\mathrm{d}x$

States and observables of classical field theory

We mainly need to single out

STATES & OBSERVABLES

Reminder

In classical mechanics, states can be seen as points of a smooth finite dimensional manifold M (configuration space) and observables are taken to be the smooth functions over it $C^\infty(M)$. Moreover, we know that it has also a Poisson structure. This is the structure we would like to have;

- CONFIGURATION SPACE – OBSERVABLES $\longrightarrow$ Kinematics
- POISSON STRUCTURE $\longrightarrow$ Dynamics

Configuration Space

We start with the

CONFIGURATION SPACE

Also motivated by the finite-dimensional road map, we choose $\varphi \in C^\infty(\mathscr{M}, \mathbb{R})$ with the usual Fréchet topology (simplified notation $\mathscr{E} \equiv C^\infty(\mathscr{M}, \mathbb{R})$)

This choice corresponds to what physicists call

OFF-SHELL SETTING

namely, we do not consider *solutions* of equations of motion (which haven't yet been considered at all!)

Observables

As far as observables are concerned, we define them (step-by-step)

$$F : \mathscr{E} \longrightarrow \mathbb{R}$$

i.e. real-valued non-linear functionals.

The $\mathbb{R}$-linear space of all functionals is certainly an associative commutative algebra $\mathfrak{F}_{00}(\mathscr{M})$ under the pointwise product defined as

$$(F.G)(\varphi) = F(\varphi)G(\varphi)$$

However, in this generality not much can be said. We need to restrict the class of functionals to have good working properties:

Restrictions

- Support Properties
- Regularity Properties

Support

Definition: Support

We define the spacetime support of a functional F as

$$\mathrm{supp}F \doteq \mathscr{M} \setminus \{x \in \mathscr{M} : \exists U \ni x \text{ open s.t. } \forall \phi, \psi, \mathrm{supp}\phi \subset U, F(\phi+\psi) = F(\psi)\}$$

Lemma: Support properties

Usual properties for the support

- Sum: $\mathrm{supp}(F + G) \subseteq \mathrm{supp}(F) \cup \mathrm{supp}(G)$
- Product: $\mathrm{supp}(F.G) \subseteq \mathrm{supp}(F) \cap \mathrm{supp}(G)$

We require that all functionals have COMPACT support.

One further crucial requirement is

Additivity

If for all $\phi_1, \phi_2, \phi_3 \in \mathscr{E}$ such that supp$\phi_1 \cap$ supp$\phi_3 = \varnothing$, then

$$F(\phi_1 + \phi_2 + \phi_3) = F(\phi_1 + \phi_2) - F(\phi_2) + F(\phi_2 + \phi_3);$$

This replaces sheaf-like properties typical of distributions (in fact, it is a weak replacement of linearity) and that allows to decompose them into small pieces. Indeed,

Lemma

Any additive and compactly supported functional can be decomposed into finite sums of such functionals with arbitrarily small supports

Additivity goes back to Kantorovich (1938-1939)!

Regularity

We would like to choose a subspace of the space of our functionals which resembles that of the observables in classical mechanics, i.e. smooth observables. We consider $\mathscr{E}$ our manifold but is not even Banach, so one needs a careful definition of differentiability

[Michal (PNAS-USA-1938!), Bastiani (JAM-1964), popularized by Milnor (Les Houches-1984) and Hamilton (BAMS-1982)]

Definition

The derivative of a functional F at φ w.r.t. the direction ψ is defined as

$$\mathrm{d}F[\varphi](\psi) \doteq F^{(1)}[\varphi](\psi) \doteq \frac{d}{d\lambda}|_{\lambda=0} F(\varphi + \lambda\psi) \doteq \lim_{\lambda\to 0} \frac{F(\varphi+\lambda\psi) - F(\varphi)}{\lambda}$$

whenever it exists. The functional F is said to be

- differentiable at φ if $\mathrm{d}F[\varphi](\psi)$ exists for any ψ,
- continuosly differentiable if it is differentiable for all directions and at all evaluations points, and $\mathrm{d}F$ is a *jointly continuos* map from $\mathscr{E} \times \mathscr{E}$ to $\mathbb{R}$, then F is said to be in $C^1(\mathscr{E}, \mathbb{R})$.

$\mathrm{d}F[\varphi](\psi)$ as a map is typically non-linear at φ but certainly linear at ψ. Higher-order derivatives can be defined by iteration. What is important is that many of the typical important results of calculus are still valid: Leibniz rule, Chain rule, First Fundamental Theorem of Calculus, Schwarz lemma etc...

So, for our specific task we require

Definition: Smooth Observables

Our observables are all possible functionals $F \in \mathfrak{F}_{00}(\mathscr{M})$ such that

- they are smooth, i.e. $F \in C^\infty(\mathscr{E}, \mathbb{R})$,
- k-th order derivatives $\mathrm{d}^k F[\varphi]$ are distributions of compact support, i.e. $\mathrm{d}^k F[\varphi] \in \mathscr{E}'(\mathscr{M}^k)$.

Since we want an algebra that possesses, among other things, also a Poisson structure, the above definition is not enough...we need restrictions on wave front sets for every derivative!

Several possibilities, but let us point out the most relevant

Definition: Local Functionals

A smooth functional F is local whenever there hold

1. $\mathrm{supp}(F^{(n)}[\varphi]) \subset \Delta_n$, where Δ_n is the small diagonal,
2. $\mathrm{WF}(F^{(n)}[\varphi]) \perp T\Delta_n$.

Example: $F_f(\varphi) = \int \mathrm{d}\mu_g(x) f(x) P(j_x(\varphi))$, where $f \in C_0^\infty(\mathscr{M})$.
However, the space of local functionals $\mathfrak{F}_{loc}(\mathscr{M})$ is not an algebra! Hence, we need some enlargement...

Definition: Microlocal Functionals

A functional F is called microlocal if the following holds ($\mathscr{V}^n_\pm \equiv (\mathscr{M} \times J_\pm(0))^n$)

$$\mathrm{WF}(F^{(n)}[\varphi]) \cap (\overline{\mathscr{V}}^n_+ \cup \overline{\mathscr{V}}^n_-) = \emptyset$$

One checks that now the derivatives can be safely multiplied (Hörmander).
Let's call $\Gamma_n = T^*\mathscr{M}^n \setminus (\overline{\mathscr{V}}^n_+ \cup \overline{\mathscr{V}}^n_-)$ (warning!!! It is an open cone!), and the algebra of microlocal functionals as $\mathfrak{F}(\mathscr{M})$.

Results

Two most interesting results:

Lemma: Equivalence

In the smooth case, any local functional is equivalently an additive functional.

Theorem:

$\mathfrak{F}(\mathscr{M})$ is a (Haussdorf, locally convex) nuclear and sequentially complete topological algebra.

Sketch:

Initial topology: $F \longrightarrow F^{(n)}[\varphi]$, $n \in \mathbb{N} \cup \{0\}$. Then nuclear if all $\mathscr{E}'_{\Gamma_n}(\mathscr{M}^n)$ are such; easy if Γ_n were closed. Since it is not, one needs to work more.

It is here we see that since the algebra is nuclear then, roughly speaking, classical field theory is not terribly infinite dimensional!

Summary of Kinematical Structures

Summary

1. Configuration space $\mathscr{E} \equiv C^\infty(\mathscr{M})$ (off-shell formalism);
2. Observables as smooth non-linear functionals (with compact support) over $\mathscr{E}$, with appropriate restrictions on the wave front sets of their derivatives, i.e. $\mathrm{WF}(F^{(n)}[\varphi]) \subset \Gamma_n$;
3. Notions of smooth additive and local functionals, but actually equivalent;
4. The algebra $\mathfrak{F}(\mathscr{M})$ is nuclear and sequentially complete.

Lagrangians

We need to single out the most important object in our study, namely the local functional which generalize the notion of Lagrangian [Brunetti-Dütsch-Fredenhagen (ATMP-2009)]

Definition: Lagrangians

A generalized Lagrangian (or Action Functional) is a map

$$\mathscr{L} : \mathscr{D}(\mathscr{M}) \longrightarrow \mathfrak{F}_{loc}(\mathscr{M}) \ ,$$

such that the following hold;

1. $\mathrm{supp}(\mathscr{L}(f)) \subseteq \mathrm{supp}(f)$;
2. $\mathscr{L}(f+g+h) = \mathscr{L}(f+g) - \mathscr{L}(g) + \mathscr{L}(g+h)$, if $\mathrm{supp}(f) \cap \mathrm{supp}(h) = \emptyset$.

Example: Action (linear map)

$$\mathscr{L}(f)(\varphi) = \int \mathrm{d}\mu_g(x) f(x) \mathscr{L} \circ j_x(\varphi) \quad \text{with} \quad \mathscr{L} = \frac{1}{2} g(\mathrm{d}\varphi, \mathrm{d}\varphi) - V(\varphi)$$

Dynamics

Suppose $f \equiv 1$ on a relatively compact open subspacetime $\mathscr{N} \subset \mathscr{M}$,then

Euler-Lagrange equations

$$\mathscr{L}(f)^{(1)} \restriction_{\mathscr{N}} [\varphi] = \frac{\partial \mathscr{L}}{\partial \varphi} - \nabla_\mu \frac{\partial \mathscr{L}}{\partial \nabla_\mu \varphi} = -\Box \varphi - V'(\varphi) = 0$$

However, $\mathscr{N}$ is arbitrary, and the equation of motions hold everywhere in $\mathscr{M}$.

Linearization

We can linearize the field equations around any arbitrary field configuration φ. This means computing the second order derivative of the Lagrangian. We restrict again to a relatively compact open subspacetime $\mathscr{N}$ and determine $\mathscr{L}(f)^{(2)}$, $f \equiv 1$ on $\mathscr{N}$, in our example we get,

$$\mathscr{L}(f)^{(2)}[\varphi]\psi(x) = (-\Box - V''(\varphi))\psi(x)$$

So we may consider the second derivative as a differential operator, and in the general case, we require that it is a strictly hyperbolic operator, which possesses, as known, unique retarded and advanced Green functions $\Delta_{\mathscr{L}}^{ret,adv}$.

Møller Scattering

The off-shell dynamics is defined as a sort of scattering procedure, similar to Møller in quantum mechanics, i.e. consider $\mathscr{L}^{(1)}$ as a map from $\mathscr{E}$ to $\mathscr{E}'$, then

Retarded Møller operators

We look for a map $r_{\mathscr{L}_1,\mathscr{L}_2} \in \mathrm{End}(\mathscr{E})$, for which

$$\begin{aligned} \mathscr{L}_1^{(1)} \circ r_{\mathscr{L}_1,\mathscr{L}_2} &= \mathscr{L}_2^{(1)} && (*) \text{ intertwining} \\ r_{\mathscr{L}_1,\mathscr{L}_2}(\varphi(x)) &= \varphi(x) \quad \text{if } x \notin J_+(\mathrm{supp}(\mathscr{L}_1 - \mathscr{L}_2)) && (**) \text{ retardation} \end{aligned}$$

Our task will be the following:

Main Task

Consider $\mathscr{L}_2 = \mathscr{L}$ and $\mathscr{L}_1 = \mathscr{L} + \lambda\mathscr{I}(h)$, then prove existence and uniqueness of $r_{\mathscr{L}+\lambda\mathscr{I}(h),\mathscr{L}}$, around a general configuration φ

Main Result

Main Theorem

$r_{\mathscr{L}+\lambda\mathscr{I}(h),\mathscr{L}}$ exists and is unique in an open nbh of h.

- Write down a differential version of $(*)(**)$, i.e. a flow equation in λ $(\varphi_\lambda = r_{\mathscr{L}+\lambda\mathscr{I}(h),\mathscr{L}}(\varphi))$

$$\langle(\mathscr{L}+\lambda\mathscr{I}(h))^{(2)}[\varphi_\lambda], \frac{d}{d\lambda}\varphi_\lambda \otimes h\rangle + \langle\mathscr{I}(h)^{(1)}[\varphi_\lambda], h\rangle = 0$$

  By the $(**)$ property and strong hyperbolicity, we use the retarded propagator to write it in the form

$$\frac{d}{d\lambda}\varphi_\lambda = -\Delta^{ret}_{\mathscr{L}+\lambda\mathscr{I}(h)}[\varphi_\lambda] \circ \mathscr{I}(h)^{(1)}[\varphi_\lambda]$$

- Break-up the perturbation part into small pieces (put on $\mathbb{R}^d$) and use the composition property $r_{\mathscr{L}_2,\mathscr{L}_3} \circ r_{\mathscr{L}_1,\mathscr{L}_2} = r_{\mathscr{L}_1,\mathscr{L}_3}$ to go back to spacetime.
- Nash-Moser-Hörmander Implicit Function Theorem, tame estimates via (a priori) energy estimates for Δ^{ret}

Peierls Brackets

Peierls Brackets

For any pair $F, G \in \mathfrak{F}(\mathscr{M})$ we pose

$$\{F, G\}_{\mathscr{L}}(\varphi) = \langle F^{(1)}[\varphi], \Delta_{\mathscr{L}}[\varphi] G^{(1)}[\varphi] \rangle$$

This bracket satisfies all the axioms for being a Poisson bracket, especially Leibinz and Jacobi identities. This entails that

Poisson Structure

The triple $(\mathfrak{F}(\mathscr{M}), \mathscr{L}, \{.,.\}_{\mathscr{L}})$ defines a Poisson algebra, namely it has additionally (an infinite dimensional) Lie algebra structure given by the Peierls brackets.

Summary of Dynamical Structures

Summary

1. Generalized Lagrangians, hyperbolic equations (linear, semilinear, quasilinear...)
2. Off-shell dynamics, i.e. Møller intertwiners
3. Off-shell Peierls brackets and Poisson structure

Structural consequences

The existence and properties of $r_{\mathscr{L}+\lambda\mathscr{I}(h),\mathscr{L}}$ have fundamental implications for the underlying Poisson structure of any classical field theory determined by an action functional $\mathscr{L}$

Darboux

$r_{\mathscr{L}+\lambda\mathscr{I}(h),\mathscr{L}}$ is a canonical transformation, i.e. it intertwines the Poisson structures associated to $\mathscr{L}$ and $\mathscr{L}+\lambda\mathscr{I}(h)$:

$$\{.,.\}_{\mathscr{L}+\lambda\mathscr{I}(h)} \circ r_{\mathscr{L}+\lambda\mathscr{I}(h),\mathscr{L}} = \{.,.\}_{\mathscr{L}}.$$

In particular, even off-shell does it allow one to put $\{.,.\}_{\mathscr{L}+\lambda\mathscr{I}(h)}$ in normal form, i.e. to make it locally background-independent (Functional Darboux Theorem).

Poisson Ideals

The subspace of functionals $J_{\mathscr{L}}(\mathscr{M}) = \{F \in \mathfrak{F}(\mathscr{M}) \mid F(\varphi) = 0 \text{ if } \mathscr{L}^{(1)}[\varphi] = 0\}$ is a Poisson ideal.

Idea: Let φ be a solution of $\mathscr{L}^{(1)}[\varphi](x) = 0$, $x \in \mathscr{M}$. Consider the one-parameter family of functions $t \mapsto \varphi_t$ such that $\varphi_0 = \varphi$ and satisfy

$$\frac{d}{dt}\varphi_t = \Delta_{\mathscr{L}}[\varphi_t] \circ G^{(1)}[\varphi] \qquad \text{any } G \qquad (*).$$

Provided a solution exists it is a solution for $\mathscr{L}^{(1)}$. Indeed, $\mathscr{L}^{(1)}[\varphi_t] = \mathscr{L}^{(1)}[\varphi_0] = 0$ and

$$\frac{d}{dt}\mathscr{L}^{(1)}[\varphi_t] = \mathscr{L}^{(2)}[\varphi_t]\frac{d}{dt}\varphi_t$$

Hence by the above this is zero. This means $F(\varphi_t) = 0$ any t, hence taking derivative we get $\{F, G\}(\varphi) = 0$.
To prove that $(*)$ is a solution one applies the same reasoning for the construction of the Møller maps.

Symplectic-Poisson Structure

We may characterize as well the Casimir functionals $\mathrm{Cas}(\mathfrak{F}(\mathscr{M}))$, namely the elements of the center w.r.t. the Poisson structure, i.e. those F such that $\{F, G\}_{\mathscr{L}}(\varphi) = 0$ for any G, φ.
They are generated by the elements $\mathscr{L}^{(1)}[\varphi]h(x)$ and constant functionals.
So we may quotient the algebra by the Poisson ideal and/or the Casimir ideal (which is just an ideal for the Lie structure).
The quotient represent the (Symplectic-)Poisson algebras for the on-shell theory, namely

$$\mathfrak{F}(\mathscr{M})/J_{\mathscr{L}}(\mathscr{M}) \,, \qquad \text{or} \qquad \mathfrak{F}(\mathscr{M})/\mathrm{Cas}(\mathscr{M})$$

Since all the ideals are linear subspaces they are nuclear, and since they are sequentially closed, we get that the quotient remains nuclear as well.
By restrictions one may get the net structure, instead we shall present it in the locally covariant form.

Local Covariance

Let us generalize the previous discussion for the sake of local covariance. We have a functor $\mathfrak{F}$ from Loc to Obs where the elements of the second category are the algebras of observables $\mathfrak{F}(\mathscr{M})$ we defined before. One can use another category by the use of the Peierls-Poisson brackets.To do it we need to enlarge the meaning of the generalized Lagrangians, namely

Natural Lagrangians

A natural Lagrangian $\mathscr{L}$ is a natural transformation from $\mathscr{D}$ to $\mathfrak{F}$, i.e. a family of maps $\mathscr{L}_{\mathscr{M}} : \mathscr{D}(\mathscr{M}) \to \mathfrak{F}(\mathscr{M})$ (Lagrangians) such that if $\chi : \mathscr{M} \to \mathscr{N}$ is an embedding we have

$$\mathscr{L}_{\mathscr{M}}(f)(\varphi \circ \chi) = \mathscr{L}_{\mathscr{N}}(\chi_* f)(\varphi)$$

A crucial point is that

Theorem

$\mathscr{L}_{\mathscr{M}}$ is an additive functional, i.e. local by the equivalence Theorem.

Using that $\mathscr{L}^{(1)}_{\mathscr{M}}$ defines equation of motions ad $\mathscr{L}^{(2)}_{\mathscr{M}}$ is a strictly hyperbolic operator, we endow $\mathfrak{F}(\mathscr{M})$ with the Peierls structure of before and we have:

Locally Covariant Classical Field Theory

The functor $\mathfrak{F}_{\mathscr{L}}$ from Loc to the category of (Nuclear) Poisson algebras Poi satisfies the axioms of local covariance

Remarks

1. Actually, one can extend to the case of tensor categories, since nuclearity works well under tensor products.
2. If one takes the quotient w.r.t. the Poisson ideal, i.e. we pass to the on-shell theory, then the ideals transform as ($\chi : \mathscr{M} \to \mathscr{N}$)

$$\mathfrak{F}_{\mathscr{L}}\chi J_{\mathscr{L}}(\mathscr{M}) \subset J_{\mathscr{L}}(\mathscr{N})$$

   The quotient is again a good functor but due to the above (cp. blow-up) the morphisms of the Poisson category are not anymore injective homomorphisms! It would be interesting to see if the time-slice axiom is satisfied replacing injectivity by surjectivity.

Deformation Quantization

We restrict our attention to Minkowski spacetime.
To quantize one deforms the pointwise product to two different products:
Firstly a star product by posing:

- Wick's Theorem

$$(F \star G)(\varphi) = e^{\langle \Delta_+, \frac{\delta^2}{\delta\varphi\delta\varphi'} \rangle}(F(\varphi)G(\varphi')) \restriction_{\varphi=\varphi'}$$

  No problem if F and G have *smooth* functional derivatives.

- Vacuum state

$$\omega_0(F) = F(0)$$

- Time Ordering Operator

$$(TF)(\varphi) = e^{\langle \Delta_F, \frac{\delta^2}{\delta\varphi^2} \rangle} F(\varphi) \equiv \int d\mu_{\Delta_F}(\psi) F(\varphi - \psi)$$

where $d\mu_{\Delta_F}$ is a gaussian measure with covariance Δ_F.

- Time Ordered Product

$$F \cdot_T G = T(T^{-1}F . T^{-1}G)$$

which is not everywhere well defined..., and

$$(F \cdot_T G)(\varphi) = e^{\langle \Delta_F, \frac{\delta^2}{\delta\varphi\delta\varphi'} \rangle} (F(\varphi)G(\varphi')) \restriction_{\varphi=\varphi'}$$

with

$$F \star G = F \cdot_T G$$

whenever supp(F) is later than supp(G).

- Formal S-matrix

$$S = T \circ \exp \circ T^{-1}$$

- More informally (path integrals):

$$\omega_0(S(V)) = S(V)(\varphi = 0) = \int d\mu_{\Delta_F} e^{:V:}$$

- Retarded Interacting Fields and Møller Operators $R_V : \mathfrak{F} \to \mathfrak{F}$

$$S(V) \star R_V(F) = S(V) \cdot_T F$$

Usual Gell-Mann-Low formula in the adiabatic limit (+ unique vacuum)

$$\omega_0(R_V(F)) = \frac{\int d\mu_{\Delta_F} e^{:V:} :F:}{\int d\mu_{\Delta_F} e^{:V:}}$$

Up to now we did everything on the case of functionals with smooth derivatives. We wish to extend the construction to the case of local functionals, the result of which is the following

Reformulation of Theorem 0 of Epstein-Glaser

- $\star$-products for local observables exist and generate an associative *-algebra
- Time ordered products of local observables exist under conditions of supports and generate a partial algebra (Keller-loc. cit.)

What is left is the lift of this result to the case where the restrictions on supports are not there anymore, which is the essence of renormalization. In other words we wish to extend the local S-matrix to a map between local functionals (observables). The core of the technique is a careful extension procedure for distributions. You will hear a lot more on this...(Dütsch, Fredenhagen, Keller)