# ABOUT THE DYNAMICAL YANG-BAXTER EQUATION(S)

## AN INVITATION TO DYNAMICAL QUANTUM GROUPS

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ABSTRACT. These are the notes of a short talk given on some aspects of the dynamical Yang-Baxter equation during the meeting of the *GDR "Tresses"* in Clermont-Ferrand (September 3-6, 2006). It is largely inspired from the lecture notes of ICM talks by Felder [2] and Etingof [1].

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### 1. THE (QUANTUM) DYNAMICAL YANG-BAXTER EQUATION

Let $\mathfrak{h}$ be a finite dimensional abelian Lie algebra, $V$ a semi-simple $\mathfrak{h}$-module and $\hbar \in \mathbb{C}^\times$.

For any (meromorphic) function $R(\lambda, z) : \mathfrak{h}^* \times \mathbb{C} \to \mathrm{End}_{\mathfrak{h}}(V \otimes V)$, the *quantum dynamical Yang-Baxter equation* (QDYBE) with step $\hbar$ reads:

$$\begin{aligned} &R^{1,2}(\lambda - \hbar h^{(3)}, z_1 - z_2) R^{1,3}(\lambda, z_1 - z_3) R^{2,3}(\lambda - \hbar h^{(1)}, z_2 - z_3) \\ &\quad = R^{2,3}(\lambda, z_2 - z_3) R^{1,3}(\lambda - \hbar h^{(2)}, z_1 - z_3) R^{1,2}(\lambda, z_1 - z_2) \,. \end{aligned}$$

Here we adopt the *dynamical notation*: $R^{1,2}(\lambda - \hbar h^{(3)}, z)$ is defined by

$$R^{1,2}(\lambda - \hbar h^{(3)}, z)(v_1 \otimes v_2 \otimes v_3) := R(\lambda - \hbar\mu, z)(v_1 \otimes v_2) \otimes v_3 \,,$$

where $v_1, v_2 \in V$, $v_3 \in V[\mu]$, and $V[\mu]$ denotes the weight subspace of weight $\mu$.

1.1. **An example motivated by quantum integrable models.** In [2] Felder proved that any solution of the QDYBE produces solutions of the famous star-triangle relation that is usefull to construct solvable models of statistical mechanics. The following example produces the so-called $A^{(1)}_{n-1}$ face model: let $V$ be the vector representation of $\mathfrak{gl}_n$, $\mathfrak{h}$ the subalgera of diagonal matrices, and denote by $E_{ij}$ the elementary matrix defined by $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Then we write $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{h}^*$, with $\lambda_i = E^*_{ii}$, and define

$$R(\lambda, z) = \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{1 \le i \neq j \le n} \Big( \frac{\theta(\lambda_i - \lambda_j + \hbar)\theta(z)}{\theta(\lambda_i - \lambda_j)\theta(z - \hbar)} E_{ii} \otimes E_{jj} \frac{\theta(z - \lambda_j + \lambda_i)\theta(\hbar)}{\theta(z - \hbar)\theta(\lambda_j - \lambda_i)} E_{ij} \otimes E_{ji} \Big) \,,$$

where $\theta(z) := \theta(z|\tau)$ is the standard theta-function, with normalization $\partial_z \theta(0) = 1$. It is a solution of the QDYBE.

**Remark.** One can replace $\theta(z)$ by $\sin(z)$ or $z$. In these two last cases taking the limit $z \to \infty$ one obtains new solutions of the QDYBE, that are $z$-independant.

1.2. **An example coming from representation theory.** We follow [1]. Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. For any $\lambda \in \mathfrak{h}^*$ let us denote by $M_\lambda$ the corresponding Verma module with heighest weight $\lambda$, $v_\lambda$ its highest weight vector, and $v_\lambda^*$ the lowest weight vector of its dual module.

To any intertwining operator $\Phi : M_\lambda \to M_\mu \otimes V$, where $V$ is a finite dimensional $\mathfrak{g}$-module and $\lambda, \mu \in \mathfrak{h}^*$, we associate its *expectation value* $< \Phi >:= v_\mu^*(\Phi v_\lambda) \in V[\lambda-\mu]$. Assume that $M_\mu$ be irreducible (this holds for generic $\mu$). Then it is known that the expectation value defines an isomorphism $\mathrm{Hom}_{\mathfrak{g}}(M_\lambda, M_\mu \otimes V) \to V[\lambda-\mu]$ for any $\lambda \in \mathfrak{h}^*$ and any finite dimensional $\mathfrak{g}$-module $V$. Therefore one can define the intertwining operator $\Phi_\lambda^v$ such that $< \Phi_\lambda^v >= v$ ($v \in V$ of weight $|v| = \lambda - \mu$).

Let $V, W$ be finite dimensional $\mathfrak{g}$-modules and consider $v \in V$, $w \in W$ homogeneous vectors. The expectation value $< \Phi_\lambda^{v,w} >$ of the composition of two intertwining operators

$$\Phi_\lambda^{v,w} := (\Phi_{\lambda-|w|}^v \otimes \mathrm{id}) \circ \Phi_\lambda^w : M_\lambda \to M_{\lambda-|v|-|w|} \otimes V \otimes W$$

is a bilinear function of $v$ and $w$. Therefore there exists $J_{V,W}(\lambda) \in \mathrm{End}_{\mathfrak{h}}(V \otimes V)$ such that $< \Phi_\lambda^{v,w} >= J_{V,W}(\lambda)(v \otimes w)$.

Then one can prove that $J_{V,W}(\lambda)$ is an invertible meromorphic function of $\lambda$, and satisfies the *dynamical twists equation* (DTE):

$$J_{V_1\otimes V_2,V_3}(\lambda) J_{V_1,V_2}(\lambda - h^{(3)}) = J_{V_1,V_2\otimes V_3}(\lambda) J_{V_2,V_3}(\lambda)\,.$$

The DTE implies that $R(\lambda) := J_{VV}(\lambda)^{-1} J_{V,V}^{2,1}(\lambda)$ is a ($z$-independant) solution of the QDYBE with step 1.

1.3. **Categorical interpretation of the DTE.** Let $\mathcal{C} = \mathrm{Rep}(U\mathfrak{g})$ and $\mathcal{M} = \mathrm{Rep}(Mer(\mathfrak{h}^*))$. For any finite dimensional $\mathfrak{g}$-module $V$ one has an algebra morphism $Mer(\mathfrak{h}^*) \to \mathrm{End}(V) \otimes Mer(\mathfrak{h}^*)\,;\, f(\lambda) \mapsto f(\lambda - h)$. Therefore one has a functor

$$\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}\,.$$

Let us now interprete $J_{V,W}(\lambda)$ as a natural "associativity isomorphism"

$$V \otimes (W \otimes M) \xrightarrow{\sim} (V \otimes W) \otimes M \quad (V, W \in \mathcal{C}\,,\, M \in \mathcal{M})\,.$$

Then the DTE implies that $J$ defines a structure of a $\mathcal{C}$-module category on $\mathcal{M}$ (in fact it is equivalent). Namely, the following diagram commutes:

$$\begin{tikzcd}
- \otimes (- \otimes (- \otimes \bullet)) \arrow[r, "1\otimes J_{V_2,V_3}(\lambda)"] \arrow[d, "J_{V_1,V_2}(\lambda-h^{(3)})"] & - \otimes ((- \otimes -) \otimes \bullet) \arrow[r, "J_{V_1,V_2\otimes V_3}(\lambda)"] & (- \otimes (- \otimes -)) \otimes \bullet \arrow[d, equal] \\
(- \otimes -) \otimes (- \otimes \bullet) \arrow[rr, "J_{V_1\otimes V_2,V_3}(\lambda)"] & & ((- \otimes -) \otimes -) \otimes \bullet
\end{tikzcd}$$

## 2. Classical limit

If $R(\lambda, z) = \mathrm{id}_{V\otimes V} - \hbar r(\lambda, z) + O(\hbar^2) \in Mer(\mathfrak{h}^* \times \mathbb{C}, \mathrm{End}_{\mathfrak{h}}(V \otimes V))$ is a solution of the QDYBE with step $\hbar$, then $r(\lambda, z)$ satisfies the *classical dynamical Yang-Baxter equation* (CDYBE):

$$[r^{1,2}(\lambda, z_1-z_2), r^{2,3}(\lambda, z_2-z_3)]+[r^{1,2}(\lambda, z_1-z_2), r^{1,3}(\lambda, z_1-z_3)]+[r^{1,3}(\lambda, z_1-z_3), r^{2,3}(\lambda, z_2-z_3)]$$

$$+\sum_\nu \big(h_\nu^{(1)} \frac{\partial r^{2,3}}{\partial \lambda^\nu}(\lambda, z_2 - z_3) - h_\nu^{(2)} \frac{\partial r^{1,3}}{\partial \lambda^\nu}(\lambda, z_1 - z_3) + h_\nu^{(3)} \frac{\partial r^{1,2}}{\partial \lambda^\nu}(\lambda, z_1 - z_2)\big) = 0$$

2.1. **Relation to integrable systems.** In [2] Felder proved that given a solution of the CDYBE one can define a compatible system of differential equations as follows:

$$\partial_{z_i} F(z_1, \ldots, z_n) = \sum_{j|j\neq i} r^{i,j}(\lambda, z_i - z_j) \cdot F - \sum_{\nu} h_{\nu}^{(i)} \cdot \frac{\partial F}{\partial \lambda^{\nu}} \quad (i = 1, \ldots, n),$$

where $F(z_1, \ldots, z_n) : \mathbb{C}^n \to V^{\otimes n}$.

2.2. **The universal classical dynamical Yang-Baxter equation and infinitesimal braids on the torus.**

**Definition.** The Lie algebra of *infinitesimal (pure) braids on the torus* is the graded Lie algebra generated by $x_i$'s and $y_i$'s ($1 \leq i \leq n$) in degree 1 and $t_{ij}$'s ($1 \leq i \neq j \leq n$) in degree 2, with relations

$$(1) \qquad [x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 0 \text{ and } [x_i, y_j] = t_{ij} = [x_j, y_i] \quad (i \neq j);$$

$$(2) \qquad [x_i, y_i] = -\sum_{j:j\neq i} t_{ij} \quad (\forall i); \quad [x_i, t_{jk}] = [y_i, t_{jk}] = 0 \quad (\#\{i,j,k\} = 3).$$

One can easily check that the usual infinitesimal pure braid relations on the plane ($t_{ij} = t_{ji}$, $[t_{ij}, t_{ik} + t_{jk}] = 0$, $[t_{ij}, t_{kl}] = 0$) are consequences of (1-2).

Let $i \neq j \in \{1, \ldots, n\}$. There is a Lie algebra morphism $\mathfrak{t}_{1,2} \to \mathfrak{t}_{1,n}; \alpha \mapsto \alpha^{i,j}$ defined by $x_k^{i,j} = \delta_{1k} x_i + \delta_{2k} x_j$ and $y_k^{i,j} = \delta_{1k} y_i + \delta_{2k} y_j$ ($k = 1, 2$).

For a (meromorphic) function $r(z) : \mathbb{C} \to \widehat{\mathfrak{t}_{1,2}}$, the *universal CDYBE* reads

$$[r(z_1 - z_2)^{1,2}, r(z_2 - z_3)^{2,3}] + [r(z_1 - z_2)^{1,2}, r(z_1 - z_3)^{1,3}] + [r(z_1 - z_3)^{1,3}, r(z_2 - z_3)^{2,3}]$$

$$= [y_1, r(z_2 - z_3)^{2,3}] + [y_2, r(z_1 - z_3)^{1,3}] + [y_3, r(z_1 - z_1)^{1,2}].$$

Here $\widehat{\ }$ means "the degree completion of" and the universal CDYBE takes place in $\widehat{\mathfrak{t}_{1,3}}$. It implies that the following system of differential equations is compatible:

$$\partial_{z_i} F(z_1, \ldots, z_n) = \sum_{j|j\neq i} r(z_i - z_j)^{i,j} \cdot F - y_i \cdot F \quad (i = 1, \ldots, n),$$

where $F(z_1, \ldots, z_n) : \mathbb{C}^n \to \widehat{\mathfrak{t}_{1,n}}$.

**Example ([3]).** Let $\theta(z) = \theta(z|\tau)$ be, as before, the standard theta-function. Then

$$r(z) = \Big(\frac{\theta(x_1 + z)}{\theta(x_1)\theta(z)} - \frac{1}{x_1}\Big)(t_{12})$$

is a solution of the universal CDYBE. The system of differential equations that we obtain actually defines a holomorphic flat connection on a principal $\exp(\widehat{\mathfrak{t}_{1,n}})$-bundle over the configuration space of $n$ points on the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Then one has a group homomorphism $PB_{1,n} \to \exp(\widehat{\mathfrak{t}_{1,n}})$, where $PB_{1,n}$ denotes the pure braid group of the torus. If we denote by $\mathfrak{pb}_{1,n}$ be the Malcev Lie algebra of $PB_{1,n}$ (i.e. the Lie algebra of its prounipotent completion), then it induces a Lie algebra morphism $\mathfrak{pb}_{1,n} \to \widehat{\mathfrak{t}_{1,n}}$.

**Proposition ([3]).** It is an isomorphism.

## References


[1] P. Etingof, *On the dynamical Yang-Baxter equation*, Proceedings of the ICM, Beijing (2002), `arXiv:math.QA/0207008`.

[2] G. Felder, *Conformal field theory and integrable systems associated to elliptic curves*, Proceedings of the ICM, Zürich (1994), `arXiv:hep-th/9407154`.

[3] D. Calaque, B. Enriquez, and P. Etingof, *Universal KZB equations*, in preparation.