LAURENT EXPANSIONS IN CLUSTER ALGEBRAS VIA QUIVER REPRESENTATIONS

PHILIPPE CALDERO AND ANDREI ZELEVINSKY

To Alexander Alexandrovich Kirillov on the occasion of his seventieth birthday

Abstract. We study Laurent expansions of cluster variables in a cluster algebra of rank 2 associated to a generalized Kronecker quiver. In the case of the ordinary Kronecker quiver, we obtain explicit expressions for Laurent expansions of the elements of the canonical basis for the corresponding cluster algebra.

1. Introduction

Cluster algebras introduced in [11] have found applications in a diverse variety of settings which include (in no particular order) total positivity, Lie theory, quiver representations, Teichmüller theory, Poisson geometry, discrete dynamical systems, tropical geometry, and algebraic combinatorics. See, e.g., [6, 9, 10, 14] and references therein.

Among these connections, the one with quiver representations has been developed especially actively. This development started with an observation made in [20] that the underlying combinatorial structure for a cluster algebra has a natural interpretation in terms of quiver representations. The subsequent work aimed to extend this interpretation from combinatorics to algebraic properties of cluster algebras. In the process, new concepts of cluster categories and cluster-tilted algebras have been introduced and studied in [4, 3] and many subsequent publications. These new concepts extend the classical theory of quiver representations and provide an interesting generalization of classical tilting theory.

In this paper, we focus on one important algebraic feature of cluster algebras: the *Laurent phenomenon* established in [11]. We will deal only with *coefficient-free* cluster algebras. In the nutshell, such an algebra is a commutative ring $\mathcal{A}$ (in fact, an integral domain) with a family of distinguished generators (*cluster variables*) grouped into (overlapping) *clusters* of the same finite cardinality n. Each cluster is algebraically independent and generates the field of fractions of $\mathcal{A}$. Thus, every cluster variable can be uniquely expressed as a rational function of the elements of every given cluster. The Laurent phenomenon asserts that these rational functions are in fact Laurent polynomials with integer coefficients.

We would like to know more about the coefficients of these Laurent polynomials. As conjectured by S. Fomin and A. Zelevinsky (see e.g. [14]), these coefficients are

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positive integers. The conjecture was recently confirmed in [7] for some special class of the Laurent polynomials in question. Unfortunately, this proof provides no explicit expression for the coefficients. We feel however that the main ingredient of the proof has a potential to provide more explicit information and deserves further study. This ingredient is a geometric interpretation (due to P. Caldero and F. Chapoton [5]) of a coefficient in question as the Euler-Poincaré characteristic of an appropriate Grassmannian of quiver representations. One of the main goals of this paper is to attract attention to the problem (that we find very interesting) of studying these Grassmannians and in particular, finding an explicit way to compute their Euler-Poincaré characteristics.

Our main new result is a complete solution of the latter problem for the cluster algebra associated with a root system of affine type $A_1^{(1)}$. For this algebra, the positivity of the Laurent polynomials in question was established by elementary means in [22], while a combinatorial expression for their coefficients was given in [21]. In this paper, we show that the Euler characteristic interpretation implies an unbeatably simple explicit expression for every coefficient in question as a product of two binomial coefficients. After such an expression is found, proving it is not hard; the point is however that this expression (left unnoticed in [22, 21]) follows naturally from the geometric study of the appropriate Grassmannians of quiver representations. In this case, the underlying quiver is the Kronecker quiver with two vertices and two arrows from one vertex to another.

We are happy to present these results in a paper dedicated to A. A. Kirillov. One of the authors (A.Z.) had been very fortunate to have A. A. Kirillov as one of his teachers at Moscow State University. One of the most impressive features of Kirillov's teaching style is his ability to explain mathematical ideas in the simplest possible terms, clearing them of unnecessary technical background so that they can be appreciated by inexperienced young researchers. We are trying to follow his example in our exposition. In particular, we never go beyond the ordinary quiver representations; and we make the paper self-contained by giving a new elementary proof of the Caldero-Chapoton result for the generalized Kronecker quiver.

In Kirillov's spirit, we now state our new formula for the cluster variables in type $A_1^{(1)}$ in a self-contained and elementary way. Let $x_1, x_2, x_3, \dots$ be a sequence of rational functions in two independent variables x_1, x_2 defined recursively by

$$x_{n+1} = \frac{x_n^2 + 1}{x_{n-1}} \quad (n \geq 2). \tag{1.1}$$

In Theorem 4.1 we show that, for every $n \geq 0$, the term x_{n+3} is given by

$$x_{n+3} = x_1^{-n-1} x_2^{-n} \left(x_2^{2(n+1)} + \sum_{q+r \leq n} \binom{n-r}{q} \binom{n+1-q}{r} x_1^{2q} x_2^{2r} \right); \tag{1.2}$$

in particular, all terms of the sequence are Laurent polynomials with positive coefficients in x_1 and x_2.

Note that if the exponent 2 in (1.1) gets replaced by a positive integer $b > 2$, then each x_n is still an integer Laurent polynomial in x_1 and x_2 by the Laurent phenomenon proved in [11]. However in this case no explicit combinatorial expression or closed formula is known for the coefficients; even positivity of these coefficients is still open. Each of the coefficients in question is the Euler-Poincaré characteristic of an appropriate Grassmannian of quiver representations for the generalized Kronecker quiver with two vertices and b arrows from one vertex to another. We find it a very interesting challenge to use this interpretation for finding an explicit expression for the coefficients.

The paper is organized as follows. In Section 2 we provide some general background on the Laurent phenomenon in cluster algebras and its geometric interpretation. In Section 3 we provide a self-contained treatment of the Laurent expansions of cluster variables in a cluster algebra of rank 2 associated with a generalized Kronecker quiver Q_b. Our main technical tool that allows us to give a new proof for the geometric interpretation of Laurent expansions (Theorem 3.2) are the functors T^+ and T^- on the category of Q_b-representations (see Definition 3.3) obtained by a slight modification of reflection functors in [2]. In Section 4 we work with the indecomposable preprojective and preinjective representations of the classical Kronecker quiver and prove equality (1.2) (Theorem 4.1). Finally, in Section 5 we use the regular indecomposable representations of the Kronecker quiver to obtain explicit Laurent expansions of the elements of the canonical basis (constructed in [22]) in the corresponding cluster algebra.

2. Some background

In this section we recall some background and results from [11, 7]. The definitions below are not the most general ones: we will deal only with coefficient-free cluster algebras having skew-symmetric exchange matrices (instead of more general skew-symmetrizable ones).

Let $\mathcal{F} = \mathbb{Q}(x_1, \dots, x_n)$ be the field of rational functions in n independent variables. Let B be a skew-symmetric integer $n \times n$ matrix. We will associate to B a commutative subring $\mathcal{A}(B) \subset \mathcal{F}$ called the (coefficient-free) *cluster algebra*. Let $\mathbb{T}_n$ be the *n-regular tree* whose edges are labeled by the numbers $1, \dots, n$, so that the n edges emanating from each vertex receive different labels. We write $t \overset{k}{\text{———}} t'$ to indicate that vertices $t, t' \in \mathbb{T}_n$ are joined by an edge labeled by k. We also fix some vertex $t_0 \in \mathbb{T}_n$ and refer to t_0 as the *initial vertex*. We associate to B and to every $t \in \mathbb{T}_n$ a skew-symmetric integer $n \times n$ matrix B_t, and an n-tuple $(x_{1;t}, \dots, x_{n;t})$ of elements of $\mathcal{F}$. They are uniquely determined by the *initial conditions*

$$B_{t_0} = B, \quad x_{j;t_0} = x_j,$$

and the *mutation relations* given as follows. Whenever $t \overset{k}{\text{———}} t'$, the matrices $B_t = (b_{ij})$ and $B_{t'} = (b'_{ij})$ are related by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \operatorname{sgn}(b_{ik})\ [b_{ik}b_{kj}]_+ & \text{otherwise,} \end{cases} \tag{2.1}$$

where we use the notation

$$[b]_+ = \max(b, 0);$$
$$\mathrm{sgn}(b) = \begin{cases} -1 & \text{if } b < 0; \\ 0 & \text{if } b = 0; \\ 1 & \text{if } b > 0; \end{cases}$$

and we have $x_{j;t'} = x_{j;t}$ for $j \neq k$, while $x_{k;t'}$ and $x_{k;t}$ satisfy the *exchange relation*

$$x_{k;t'} x_{k;t} = \prod_i x_{i;t}^{[b_{ik}]_+} + \prod_i x_{i;t}^{[-b_{ik}]_+} . \tag{2.2}$$

We refer to B_t as the *exchange matrix* at t, and to $(x_{1;t}, \ldots, x_{n;t})$ as the *cluster* at t. The elements $x_{j;t}$ are called *cluster variables* (note that there may be some equalities among them). The cluster algebra $\mathcal{A}(B)$ is defined as the subring of $\mathcal{F}$ generated by all cluster variables.

Being an element of $\mathcal{F}$, every cluster variable is a rational function in $x_1, \ldots, x_n$. The following result sharpens this considerably.

Theorem 2.1 (Laurent phenomenon [11, Theorem 3.1])**.** *The cluster algebra $\mathcal{A}(B)$ is contained in the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$; equivalently, every cluster variable is an integer Laurent polynomial in $x_1, \ldots, x_n$.*

We now turn to a geometric interpretation of the Laurent polynomials in Theorem 2.1. First of all, we represent a skew-symmetric integer $n \times n$ matrix $B = (b_{ij})$ by means of the quiver $Q = Q(B)$ with vertices $[1, n] = \{1, \ldots, n\}$, and b_{ij} arrows from i to j whenever $b_{ij} > 0$ (thus, Q is allowed to have multiple edges). Following [1], we say that B is *acyclic* if $Q(B)$ has no oriented cycles. (Equivalently, B is acyclic if and only if, by a simultaneous permutation of rows and columns, we can make $b_{ij} \geq 0$ for all $i > j$.) In the rest of the section we assume that the initial exchange matrix B is acyclic.

Let us recall some basics on quiver representations (we do not attempt to give a self-contained introduction to the subject, just fix some terminology and notation). Recall that a *representation* M (over some field K) of a quiver $Q = Q(B)$ is given by assigning a finite-dimensional K-vector space M_i to every vertex i of Q, and a b-tuple $(\varphi_{ji}^{(1)}, \ldots, \varphi_{ji}^{(b)})$ of linear maps $M_i \to M_j$ to every arrow $i \to j$ of multiplicity $b = b_{ij}$ in Q. For our current purposes it is sufficient to work over $K = \mathbb{C}$. The *morphisms* between quiver representations are defined in a natural way, giving rise to the category of quiver representations. This category is abelian, hence there is a well defined notion of *indecomposable* representations.

The *dimension* of a representation M is an integer vector $\mathbf{d} = (d_1, \ldots, d_n)$ given by $d_i = \dim M_i$. A quiver representation M of dimension $\mathbf{d}$ is called *rigid* if a generic representation of dimension $\mathbf{d}$ is isomorphic to M; equivalently, M has no nontrivial self-extensions.

A *subrepresentation* N of a representation M is specified by a collection of subspaces $N_i \subset M_i$ such that $\varphi_{ji}^{(k)}(N_i) \subset N_j$ for all i, j and k. For a representation M of dimension $\mathbf{d}$, and any nonnegative integer vector $\mathbf{e} = (e_1, \ldots, e_n)$ such that $e_i \leq d_i$ for all i, let $\mathrm{Gr}_{\mathbf{e}}(M)$ denote the variety of all subrepresentations of dimension $\mathbf{e}$ in M.

By the definition, $\mathrm{Gr}_{\mathbf{e}}(M)$ is a closed subvariety in the product of Grassmannians $\prod_i \mathrm{Gr}_{e_i}(M_i)$.

Let $\chi_{\mathbf{e}}(M)$ denote the Euler-Poincaré characteristic of $\mathrm{Gr}_{\mathbf{e}}(M)$ (see e.g., [16, Section 4.5]). We associate to any representation M of Q with dimension vector $\mathbf{d}$ a Laurent polynomial $X_M(x_1,\dots,x_n)$ given by

$$X_M(x_1,\dots,x_n) = x_1^{-d_1}\cdots x_n^{-d_n}\sum_{\mathbf{e}}\chi_{\mathbf{e}}(M)\prod_{i,j}(x_i^{d_j-e_j}x_j^{e_i})^{[b_{ij}]_+}; \tag{2.3}$$

this is easily seen to be equivalent to the definition in [5].

Now we are ready to state the following result obtained in [7, Theorem 3]; it generalizes [5, Theorem 3.4].

Theorem 2.2. *Let $\mathcal{A}(B)$ be the (coefficient-free) cluster algebra with an acyclic skew-symmetric initial exchange matrix B. The correspondence $M \mapsto X_M(x_1,\dots,x_n)$ is a bijection between the set of isomorphism classes of indecomposable rigid representations of the quiver $Q(B)$, and the set of all cluster variables in $\mathcal{A}(B)$ not belonging to the initial cluster $\{x_1,\dots,x_n\}$.*

As shown in [7, Corollary 1], for every representation M of an acyclic quiver Q, the dimension vector $\mathbf{d}=(d_1,\dots,d_n)$ of M is the *denominator vector* of the Laurent polynomial $X_M(x_1,\dots,x_n)$; that is, for every $j\in[1,n]$, the minimum of exponents of x_j in all the monomials of X_M is equal to $-d_j$. The dimension vectors of indecomposable rigid representations are called *real Schur roots*. This terminology comes from the well-known results due to V. Kac [18]. Namely, let $A=(a_{ij})$ be the *Cartan counterpart* of B, that is, the symmetric integer $n\times n$ matrix with all diagonal entries equal to 2, and off-diagonal entries given by $a_{ij}=-|b_{ij}|$. Then the dimension vectors of indecomposable representations of $Q(B)$ are precisely the positive roots of the root system associated to A, expanded in the basis $\{\alpha_1,\dots,\alpha_n\}$ of simple roots. Furthermore, a positive root $\alpha=\sum d_i\alpha_i$ is *real* if and only if there is a unique isomorphism class of indecomposable representations with dimension vector $(d_1,\dots,d_n)$. We see in particular, that every real Schur root is a positive real root. Note that the set of real Schur roots depends on the orientation of $Q(B)$, in contrast with positive roots and with real positive roots. There seems to be no easy way to distinguish real Schur roots among all positive real roots, see [8].

Returning to Theorem 2.2, we have the following corollary.

Corollary 2.3. *In the situation of Theorem 2.2, a cluster variable in $\mathcal{A}(B)$ is uniquely determined by the denominator vector in its Laurent expansion with respect to the initial cluster. Furthermore, the denominator vectors of the cluster variables not belonging to the initial cluster are precisely the real Schur roots of $Q(B)$.*

3. Rank 2 cluster algebras and generalized Kronecker quiver

In this section we discuss Theorem 2.2 and Corollary 2.3 (and give their independent proofs) for the cluster algebra $\mathcal{A}(B)$ associated with the matrix

$$B=\begin{pmatrix}0 & b\\ -b & 0\end{pmatrix},$$

where b is a positive integer. Unraveling the definitions in Section 2, we see that $\mathcal{A}(B)$ is the subring of the ambient field $\mathbb{Q}(x_1, x_2)$ generated by the cluster variables x_m ($m \in \mathbb{Z}$) defined recursively by the relations

$$x_{m-1}x_{m+1} = x_m^b + 1 \quad (m \in \mathbb{Z}) \ . \tag{3.1}$$

The clusters are the pairs $\{x_m, x_{m+1}\}$ for all $m \in \mathbb{Z}$, and we choose $\{x_1, x_2\}$ as the initial cluster. Theorem 2.1 asserts that each x_m is an integer Laurent polynomial in x_1 and x_2. For $m \in \mathbb{Z} - \{1, 2\}$, let $\alpha(m) \in \mathbb{Z}^2$ denote the denominator vector of the cluster variable x_m.

If $b = 1$, an easy calculation (done in many places before) gives

$$x_3 = \frac{x_2 + 1}{x_1}, \ x_4 = \frac{x_1 + x_2 + 1}{x_1 x_2}, \ x_5 = \frac{x_1 + 1}{x_2}, \ x_{m+5} = x_m \quad (m \in \mathbb{Z}) \ .$$

(As an easy exercise, one can check that these expressions agree with Theorem 2.2.) For the rest of this section we fix b and assume that $b \geq 2$.

Let $S_{-1}(x), S_0(x), S_1(x), \ldots$ be *normalized Chebyshev polynomials of the second kind* defined recursively by

$$S_{-1}(x) = 0, \quad S_0(x) = 1, \quad S_{n+1}(x) = xS_n(x) - S_{n-1}(x) \quad (n \geq 0). \tag{3.2}$$

As an easy consequence of (3.1), the denominator vectors $\alpha(m)$ of the cluster variables are given as follows.

Proposition 3.1. *For each $n \geq 0$, we have*

$$\alpha(n+3) = (S_n(b), S_{n-1}(b)), \quad \alpha(-n) = (S_{n-1}(b), S_n(b)) \ . \tag{3.3}$$

We identify the lattice $\mathbb{Z}^2$ with the root lattice for the Cartan matrix

$$A = \begin{pmatrix} 2 & -b \\ -b & 2 \end{pmatrix},$$

by identifying the standard basis vectors $(1, 0)$ and $(0, 1)$ with the simple roots α_1 and α_2 (for the properties of rank 2 root systems see, e.g., [22, Section 3.1]). Under this identification, the denominator vectors in (3.3) are precisely the real positive roots. This follows easily from the relations (for all $n \geq 0$)

$$s_1\alpha(-n) = \alpha(n+4) = \sigma\alpha(-n-1), \ s_2\alpha(n+3) = \alpha(-n-1) = \sigma\alpha(n+4), \tag{3.4}$$

where s_1 and s_2 are simple reflections acting on $\mathbb{Z}^2$ by matrices

$$s_1 = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 \\ b & -1 \end{pmatrix},$$

and $\sigma : \mathbb{Z}^2 \to \mathbb{Z}^2$ acts by interchanging the two components of a vector (cf. [22, (3.1)]).

Turning to Theorem 2.2, we note that the quiver $Q_b = Q(B)$ associated to B has two vertices 1 and 2, and b arrows from 1 to 2. When $b = 2$ (resp. $b > 2$), this quiver is called the *Kronecker quiver* (resp. *generalized Kronecker quiver*).

A Q_b-representation M over a field K consists of a pair of finite dimensional K-vector spaces (M_1, M_2) and a b-tuple of linear maps $(\varphi_1, \ldots, \varphi_b)$ from M_1 to M_2. We will use the following two interpretations of the tuple $(\varphi_1, \ldots, \varphi_b)$: the *column interpretation* identifying it with a linear map $\varphi^c : M_1 \to M_2^b$, and the *row interpretation* identifying it with a linear map $\varphi^r : M_1^b \to M_2$.

The dimension vector of M is an integer vector $\mathbf{dim}\, M = (\dim M_1, \dim M_2) \in \mathbb{Z}^2$. A subrepresentation N of M is a pair of subspaces (N_1, N_2) such that $N_i \subset M_i$ for $i = 1, 2$, and $\varphi_k(N_1) \subset N_2$ for $k \in [1, b]$. In accordance with (2.3), we associate with every Q_b-representation M of dimension $\mathbf{d} = (d_1, d_2)$ a Laurent polynomial

$$X_M(x_1, x_2) = x_1^{-d_1} x_2^{-d_2} \sum_{\mathbf{e}=(e_1,e_2)} \chi_{\mathbf{e}}(M) x_1^{b(d_2-e_2)} x_2^{be_1}. \tag{3.5}$$

Since the vectors $\alpha(m)$ for $m \in \mathbb{Z} - \{1, 2\}$ are real positive roots, by Kac's theorem [18], each of them is the dimension vector of a unique (up to an isomorphism) indecomposable Q_b-representation $M(m)$. In particular, $M(3) = S_1$ is a simple representation of dimension α_1, and $M(0) = S_2$ is a simple representation of dimension α_2. It is well known (see e.g., [8]) that each $\alpha(m)$ is a real Schur root, i.e., all $M(m)$ are rigid, but we will not use this fact. The representations $M(-n)$ (resp. $M(n+3)$) for $n \geq 0$ are also known as *preprojective* (resp. *preinjective*) indecomposable Q_b-representations. The rest of indecomposable Q_b-representations are of dimension (n, n) for some $n \geq 1$. They are called *regular* and will be considered in the next section.

Theorem 3.2. *For every $m \in \mathbb{Z} - \{1, 2\}$, the cluster variable x_m is equal to $X_{M(m)}(x_1, x_2)$.*

Our main tool in proving Theorem 3.2 will be the following functors on the category of Q_b-representations.

Definition 3.3. Let $M = (M_1, M_2; \varphi_1, \ldots, \varphi_b)$ be a Q_b-representation.

- The duality functor D sends M to $M^* = (M_2^*, M_1^*; \varphi_1^*, \ldots, \varphi_b^*)$.
- The functor T^+ sends M to $M^+ = (M_1^+, M_2^+; \varphi_1^+, \ldots, \varphi_b^+)$ given by
$$M_1^+ = M_2, \ M_2^+ = \operatorname{coker}(\varphi^c : M_1 \to M_2^b),$$
and $(\varphi^+)^r : (M_1^+)^b = M_2^b \to M_2^+$ being the natural projection.
- The functor T^- sends M to $M^- = (M_1^-, M_2^-; \varphi_1^-, \ldots, \varphi_b^-)$ given by
$$M_2^- = M_1, \ M_1^- = \ker(\varphi^r : M_1^b \to M_2),$$
and $(\varphi^-)^c : M_1^- \to M_1^b = (M_2^-)^b$ being the natural embedding.

All these functors are additive and send direct sums to direct sums. The functors T^+ and T^- are slight modifications of *reflection functors* from [2]. The following properties are immediate from the definition.

Proposition 3.4. (1) $D^2 = \mathrm{Id}$, $\quad T^- = DT^+D$, $\quad T^+ = DT^-D$.

(2) *The composition T^+T^- sends a Q_b-representation $M = (M_1, M_2; \varphi_1, \ldots, \varphi_b)$ to $(M_1, M_1^b/\ker(\varphi^r); \psi_1, \ldots, \psi_b)$, with ψ^r being the natural projection $M_1^b \to M_1^b/\ker(\varphi^r)$.*

(3) *The composition T^-T^+ sends a Q_b-representation $M = (M_1, M_2; \varphi_1, \ldots, \varphi_b)$ to $(\operatorname{im}(\varphi^c), M_2; \psi_1, \ldots, \psi_b)$, with ψ^c being the natural embedding $\operatorname{im}(\varphi^c) \to M_2^b$.*

Proposition 3.5. *The following conditions on a Q_b-representation M are equivalent:*

(1) *$M = T^-N$ for some representation N.*

(2) *$M = T^- T^+ M$.*
(3) *The map $\varphi^c : M_1 \to M_2^b$ is injective.*
(4) *$\mathbf{dim}\, T^+ M = \sigma s_1(\mathbf{dim}\, M)$.*
(5) *$T^+ M' \neq 0$ for any non-zero subrepresentation M' of M.*
(6) *S_1 is not a direct summand of M.*

Proof. The implication (2) $\Longrightarrow$ (1) is trivial. The implication (1) $\Longrightarrow$ (3) is immediate from the definition of T^-. The equivalence (2) $\Longleftrightarrow$ (3) follows from Proposition 3.4 (3). The equivalences (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) are immediate from the definition of T^+. The implication (5) $\Longrightarrow$ (6) is clear since $T^+ S_1 = 0$. Finally, (6) $\Longrightarrow$ (3) is proved by contradiction as follows. Suppose (3) does not hold, and choose a one-dimensional subspace $M_1' \subset \ker(\varphi^c)$. Let $M_1'' \subset M_1$ be a subspace such that $M_1 = M_1' \oplus M_1''$. Then M is a direct sum of subrepresentations $(M_1', 0)$ and (M_1'', M_2), and $(M_1', 0)$ is isomorphic to S_1, in contradiction to (6). □

Corollary 3.6. *Every indecomposable Q_b-representation M not isomorphic to S_1 satisfies equivalent conditions in Proposition 3.5; furthermore, $T^+ M$ is also indecomposable.*

Proof. The first statement follows from condition (6) in Proposition 3.5. Now let $N = T^+ M$, and suppose that N is the direct sum of two non-zero representations N' and N''. Applying the duality functor, we get $DN = DN' \oplus DN''$. By Proposition 3.4 (1), the representation $DN = DT^+ M = T^- DM$ satisfies condition (1) in Proposition 3.5. Therefore, by condition (5), both $T^+ DN'$ and $T^+ DN''$ are non-zero. Applying condition (2) and Proposition 3.4 (1), we obtain $M = T^- N = T^- N' \oplus T^- N'' = DT^+ DN' \oplus DT^+ DN''$, in contradiction with the assumption that M is indecomposable. □

We now obtain an explicit description of the indecomposable representations $M(m)$ for $m \in \mathbb{Z} - \{1, 2\}$.

Proposition 3.7. *For every $n \geq 0$, we have*

$$M(-n) = (T^+)^n S_2, \quad M(n+3) = (T^-)^n S_1. \tag{3.6}$$

Proof. Remembering the assumption $b \geq 2$, we start by observing that all the roots $\alpha(m)$ are distinct, hence the corresponding indecomposable representations $M(m)$ are mutually non-isomorphic. In particular, all $M(-n)$ for $n \geq 0$ are not isomorphic to S_1. To prove that $M(-n) = (T^+)^n S_2$, we proceed by induction on n. The statement is trivial for $n = 0$. Now assume that it holds for some $n \geq 0$. By Corollary 3.6, the representation $T^+ M(-n)$ is indecomposable. Applying Proposition 3.5 (4) and (3.4), we conclude that

$$\mathbf{dim}\, T^+ M(-n) = \sigma s_1(\mathbf{dim}\, M(-n)) = \sigma s_1(\alpha(-n)) = \alpha(-n-1).$$

Therefore, $T^+ M(-n) = M(-n-1)$, proving the first equality in (3.6). To prove the second equality in (3.6) note that

$$M(n+3) = DM(-n) = D(T^+)^n S_2 = (T^-)^n DS_2 = (T^-)^n S_1.$$

□

Turning to the proof of Theorem 3.2, we start by rewriting (3.5) as

$$X_M(x_1, x_2) = x_1^{-d_1} x_2^{-d_2} P_M(x_1^b, x_2^b), \tag{3.7}$$

where P_M is a polynomial given by

$$P_M(z_1, z_2) = \sum_{\mathbf{e}=(e_1,e_2)} \chi_{\mathbf{e}}(M) z_1^{d_2-e_2} z_2^{e_1}. \tag{3.8}$$

To work with $P_M(z_1, z_2)$, we need to recall some properties of the Euler-Poincaré characteristic. We follow the treatment in [16, Section 4.5], where the Euler-Poincaré characteristic $\chi(X)$ is defined for any complex algebraic variety X (not necessarily smooth, projective or irreducible). The following facts are shown in loc.cit.

(3.9) If $\mathbb{A}$ is a finite dimensional affine space, then $\chi(\mathbb{A}) = 1$.

(3.10) If a variety X is a disjoint union of finitely many locally closed subvarieties X_i, then $\chi(X) = \sum \chi(X_i)$.

(3.11) If $X \to Z$ is a fiber bundle (locally trivial in the Zariski topology) with fiber Y, then $\chi(X) = \chi(Y)\chi(Z)$.

As a consequence of (3.9) and (3.10), the Schubert cell decomposition of the Grassmannian implies that

$$\chi(\mathrm{Gr}_r(V)) = \binom{\dim V}{r}. \tag{3.12}$$

Now let $M = (M_1, M_2; \varphi_1, \ldots, \varphi_b)$ be an arbitrary Q_b-representation of dimension (d_1, d_2). For every two nonnegative integers p and r, we set

$$Z_{p,r}(M) = \{U \in \mathrm{Gr}_r(M_1) : \dim(\sum_{k=1}^{b} \varphi_k(U)) = d_2 - p\}, \tag{3.13}$$

$$Z'_{p,r}(M) = \{U \in \mathrm{Gr}_{d_2-r}(M_2) : \dim(\bigcap_{k=1}^{b} \varphi_k(U)) = p\}. \tag{3.14}$$

Proposition 3.8. *We have*

$$P_M(z_1, z_2) = \sum_{p,r} \chi(Z_{p,r}(M))(z_1 + 1)^p z_2^r = \sum_{p,r} \chi(Z'_{p,r}(M)) z_1^r (z_2 + 1)^p. \tag{3.15}$$

Proof. For every dimension vector $\mathbf{e} = (e_1, e_2)$, we split the Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ into the disjoint union of subvarieties

$$\mathrm{Gr}_{p,\mathbf{e}}(M) = \{(N_1, N_2) \in \mathrm{Gr}_{\mathbf{e}}(M) : N_1 \in Z_{p,e_1}(M)\}. \tag{3.16}$$

The projection $(N_1, N_2) \mapsto N_1$ makes each $\mathrm{Gr}_{p,\mathbf{e}}(M)$ into a fiber bundle over $Z_{p,e_1}(M)$. Since, for a given $N_1 \in Z_{p,e_1}(M)$, the only condition on N_2 is that $\sum_{k=1}^{b} \varphi_k(N_1) \subset N_2$, the fiber of this bundle is the Grassmannian of $(e_2 - d_2 + p)$-dimensional subspaces in a p-dimensional vector space. Applying (3.10), (3.11) and (3.12), we obtain

$$\chi_{\mathbf{e}}(M) = \sum_p \chi(\mathrm{Gr}_{p,\mathbf{e}}(M)) = \sum_p \binom{p}{e_2 - d_2 + p} \chi(Z_{p,e_1}(M)).$$

Substituting this expression into (3.8), we obtain

$$P_M(z_1, z_2) = \sum_{p,e_1,e_2} \binom{p}{e_2 - d_2 + p} \chi(Z_{p,e_1}(M)) z_1^{d_2-e_2} z_2^{e_1} = \sum_{p,r} \chi(Z_{p,r}(M))(z_1+1)^p z_2^r,$$

proving the first equality in (3.15). The second equality can be proved in a similar way. Alternatively, it is easy to see that the correspondence $U \mapsto U^\perp \subset M_1^*$ is an isomorphism between $Z_{p,r}(M)$ and $Z'_{p,r}(DM)$; this implies the second equality in (3.15) in view of the (easily proved) observation

$$P_{DM}(z_1, z_2) = P_M(z_2, z_1). \tag{3.17}$$

□

The key ingredient of the proof of Theorem 3.2 is the following proposition.

Proposition 3.9. *Suppose M is a Q_b-representation of dimension (d_1, d_2) satisfying equivalent conditions in Proposition 3.5. Then we have*

$$P_{T^+M}(z_1, z_2) = (z_1+1)^{-d_1} z_2^{d_2} P_M\left(\frac{(z_1+1)^b}{z_2}, z_1\right) . \tag{3.18}$$

Proof. Consider the representation $T^+M = M^+$ as defined in Definition 3.3. By Proposition 3.5 (4), we have $\mathbf{dim}\, T^+M = (d_2, bd_2 - d_1)$. Since $M_1^+ = M_2$, the statement of the following lemma makes sense.

Lemma 3.10. *Under the condition in Proposition 3.9, the variety $Z'_{p,r}(M)$ is equal to $Z_{p+br-d_1,d_2-r}(T^+M)$.*

Proof. It suffices to show the following: if $U \subset M_2$ belongs to $Z'_{p,r}(M)$ then

$$\dim(\sum_{k=1}^{b} \varphi_k^+(U)) = b(d_2 - r) - p.$$

By Definition 3.3, we have

$$\sum_{k=1}^{b} \varphi_k^+(U) = (\varphi^+)^r(U^b) = U^b/(U^b \cap \mathrm{im}(\varphi^c)),$$

hence

$$\dim(\sum_{k=1}^{b} \varphi_k^+(U)) = b(d_2 - r) - \dim(U^b \cap \mathrm{im}(\varphi^c)).$$

Remembering the definition of φ^c, we see that

$$U^b \cap \mathrm{im}(\varphi^c) = \varphi^c(\bigcap_{k=1}^{b} \varphi_k^{-1}(U)).$$

Since φ^c is injective by Proposition 3.5 (3), we conclude that

$$\dim(U^b \cap \mathrm{im}(\varphi^c)) = \dim(\bigcap_{k=1}^{b} \varphi_k^{-1}(U)) = p,$$

finishing the proof of Lemma 3.10. □

To prove (3.18), it suffices to combine Lemma 3.10 and formula (3.15):

$$\begin{aligned} P_{T^+M}(z_1,z_2) &= \sum_{p,r} \chi(Z'_{p,r}(M))(z_1+1)^{p+br-d_1} z_2^{d_2-r} \\ &= (z_1+1)^{-d_1} z_2^{d_2} \sum_{p,r} \chi(Z'_{p,r}(M))(z_1+1)^p \left(\frac{(z_1+1)^b}{z_2}\right)^r \\ &= (z_1+1)^{-d_1} z_2^{d_2} P_M\left(\frac{(z_1+1)^b}{z_2}, z_1\right), \end{aligned}$$

as claimed. □

Proof of Theorem 3.2. To show that $x_{-n} = X_{M(-n)}(x_1,x_2)$ for $n \geq 0$, we proceed by induction on n. The check for $n=0$ is straightforward. Thus we assume that $x_{-n} = X_{M(-n)}(x_1,x_2)$ for some $n \geq 0$, and will show that $x_{-n-1} = X_{M(-n-1)}(x_1,x_2)$. To see this, we first note that $x_{-n-1} = X_{M(-n)}(x_0,x_1)$ by the obvious symmetry of the exchange relations (3.1). Now we apply Proposition 3.9 to $M = M(-n)$ and $T^+M = M(-n-1)$ (see Corollary 3.6 and Proposition 3.7). Using (3.7), (3.18) and (3.1), we obtain

$$\begin{aligned} X_{M(-n-1)}(x_1,x_2) &= x_1^{-d_2} x_2^{-bd_2+d_1} P_{M(-n-1)}(x_1^b, x_2^b) \\ &= x_1^{-d_2} x_2^{d_1} (x_1^b+1)^{-d_1} P_{M(-n)}\left(\left(\frac{x_1^b+1}{x_2}\right)^b, x_1^b\right) \\ &= X_{M(-n)}\left(\frac{x_1^b+1}{x_2}, x_1\right) = X_{M(-n)}(x_0,x_1) = x_{-n-1}, \end{aligned}$$

as desired.

To show that $x_{n+3} = X_{M(n+3)}(x_1,x_2)$ for $n \geq 0$, we note that by (3.17) the equality $M(n+3) = DM(-n)$ implies that $X_{M(n+3)}(x_1,x_2) = X_{M(-n)}(x_2,x_1)$. On the other hand, an obvious symmetry of the relations (3.1) implies that the automorphism of $\mathbb{Q}(x_1,x_2)$ that interchanges x_1 and x_2, sends x_{-n} to x_{n+3}; therefore, the Laurent expansion of x_{n+3} in x_1 and x_2 is also obtained from that of x_{-n} by interchanging x_1 and x_2. This completes the proof of Theorem 3.2. □

4. Cluster variables associated with the Kronecker quiver

In this section we sharpen the results in Section 3 in the special case $b=2$. Thus we work with the cluster algebra $\mathcal{A}(B)$ associated with the matrix

$$B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}. \tag{4.1}$$

In this case, we obtain the following explicit expression for cluster variables.

Theorem 4.1. *In the cluster algebra associated to the matrix B in* (4.1)*, the cluster variables are given by*

$$x_{-n} = x_1^{-n} x_2^{-n-1} \left(x_1^{2(n+1)} + \sum_{q+r \le n} \binom{n+1-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2r} \right), \tag{4.2}$$

$$x_{n+3} = x_1^{-n-1} x_2^{-n} \left(x_2^{2(n+1)} + \sum_{q+r \le n} \binom{n-r}{q} \binom{n+1-q}{r} x_1^{2q} x_2^{2r} \right) \tag{4.3}$$

for all $n \ge 0$.

Proof. First of all, an obvious symmetry of the relations (3.1) implies that the map $x_m \mapsto x_{3-m}$ extends to an automorphism of our cluster algebra. Applying this automorphism to both sides of (4.2) yields (4.3), so it suffices to prove (4.2).

We use Theorem 3.2 to express x_{-n} in terms of the representations of the Kronecker quiver Q_2 consisting of two vertices 1 and 2 and two arrows from 1 to 2. As a consequence of (3.2), we have $S_n(2) = n+1$ for all $n \ge -1$. Thus, by Proposition 3.1, x_{-n} has the denominator vector $(n, n+1)$, and so the corresponding indecomposable Q_2 representation $M(-n)$ is of dimension $(n, n+1)$ as well. An explicit form of this representation can be given as follows.

Proposition 4.2. *Let $M(-n) = (M_1, M_2; \varphi_1, \varphi_2)$ be the indecomposable Q_2 representation of dimension $(n, n+1)$. Then there exist a basis $\{u_1, \dots, u_n\}$ in M_1 and a basis $\{v_1, \dots, v_{n+1}\}$ in M_2 such that $\varphi_1(u_k) = v_k$ and $\varphi_2(u_k) = v_{k+1}$ for $k \in [1, n]$.*

This result is due to L. Kronecker [19]; for a modern treatment see [15, Section 5.4]. A self-contained proof can be given by induction on n with the help of Proposition 3.7.

The key ingredient for the proof of (4.2) is the following result.

Proposition 4.3. *Let $M(-n)$ be a Q_2-representation in Proposition 4.2. For every nonnegative integers p, r, we have (see* (3.13)*)*

$$\chi(Z_{p,r}(M(-n))) = \binom{r-1}{n-p-r} \binom{n+1-r}{p} \tag{4.4}$$

(with the convention that the right hand side is equal to $\delta_{p,n+1}$ for $r = 0$).

Proof. The statement is trivial for $r = 0$, so we assume that $r > 0$. We use the Schubert cell decomposition of the Grassmannian $\mathrm{Gr}_r(M_1)$. We label the Schubert cells by r-element subsets $J \subset [1, n]$. The elements of the cell $\mathcal{O}(J)$ are parameterized by the arrays of complex numbers

$$C = (c_{ij}) \ (j \in J, \ i \in [1, n] - J, \ i < j),$$

with the corresponding $U(C) \in \mathcal{O}(J)$ being an r-dimensional subspace of M_1 with the basis

$$\{u_j + \sum_i c_{ij} u_i : j \in J\}.$$

Breaking the subvariety $Z_{p,r}(M(-n)) \subset \mathrm{Gr}_r(M_1)$ into the disjoint union of its intersections with the Schubert cells, and using (3.10), we see that

$$\chi(Z_{p,r}(M(-n))) = \sum_J \chi(Z_{p,r}(M(-n)) \cap \mathcal{O}(J)).$$

Let $c(J)$ denote the number of connected components of a subset $J \in [1, n]$ (by a connected component of J we mean a maximal interval $[a, b] = \{a, a+1, \ldots, b\}$ contained in J). The desired formula is a consequence of the following two results:

$$\chi(Z_{p,r}(M(-n)) \cap \mathcal{O}(J)) = \delta_{c(J),n+1-p-r}; \tag{4.5}$$

(4.6) the number of r-element subsets $J \subset [1, n]$ with $c(J) = t$ is equal to $\binom{r-1}{t-1}\binom{n+1-r}{t}$.

Since (4.6) is a purely combinatorial statement, let us dispose of it first. Let us write an r-element subset J as the union of its connected components:

$$J = [a_1, b_1] \cup \cdots \cup [a_t, b_t],$$

so we have

$$1 \le a_1 < b_1 + 1 < a_2 < b_2 + 1 < \cdots < a_t < b_t + 1 \le n + 1,$$

and

$$(b_1 + 1 - a_1) + \cdots + (b_t + 1 - a_t) = r.$$

Let $b_k + 1 - a_k = r_k$ for $k = 1, \ldots, t$. The number of t-tuples $(r_1, \ldots, r_t)$ of positive integers with sum r is known to be $\binom{r-1}{t-1}$ (these tuples are in a bijection with $(t-1)$-element subsets of $[1, r-1]$ via $(r_1, \ldots, r_t) \mapsto \{r_1, r_1 + r_2, \ldots, r_1 + \cdots + r_{t-1}\}$). And for every given such tuple, the number of corresponding subsets J is equal to $\binom{n+1-r}{t}$: they are in a bijection with t-element subsets of $[1, n+1-r]$ via

$$[a_1, b_1] \cup \cdots \cup [a_t, b_t] \mapsto \{a_1, a_2 - r_1, \ldots, a_t - r_1 - \cdots - r_{t-1}\}.$$

This proves (4.6).

Turning to the proof of (4.5), we restate it as follows. Fix an r-element subset $J \subset [1, n]$ and break the Schubert cell $\mathcal{O}(J)$ into the disjoint union of the fibers of the function $d : \mathcal{O}(J) \to \mathbb{Z}_{\ge 0}$ given by

$$d(U(C)) = \dim(\varphi_1(U(C)) + \varphi_2(U(C))).$$

We need to show that $\chi(d^{-1}(r + c(J)) = 1$, while all the other fibers have Euler characteristic 0. By Proposition 4.2, the subspace $\varphi_1(U(C)) + \varphi_2(U(C)) \subset M_2$ is spanned by the vectors $\{e_j(C), e_j^+(C) : j \in J\}$, where we use the notation

$$e_j(C) = v_j + \sum_i c_{ij} v_i \quad e_j^+(C) = v_{j+1} + \sum_i c_{ij} v_{i+1}.$$

Denote $J^+ = \{j + 1 : j \in J\} \subset [2, n+1]$. Note that $J - J^+$ is a set of representatives of the connected components of J, so $|J - J^+| = c(J)$. Consider the set of spanning vectors

$$E(C) = \{e_j^+(C)\,(j \in J),\ e_j(C)\,(j \in J - J^+)\}$$

of cardinality $r + c(J)$. The vectors from $E(C)$ are linearly independent since they have distinct leading terms in the expansion in the basis $v_1, \ldots, v_{n+1}$ of M_2. Using these leading terms, it is easy to see that for each remaining spanning vector $e_j(C)$ with $j \in J \cap J^+$, there is a unique vector of the form

$$e'_j(C) = \sum_{i<j,\ i \notin J \cup J^+} c'_{ij} v_i$$

obtained from $e_j(C)$ by adding a linear combination of the vectors $e_{j'}(C), e^+_{j'}(C) \in E(C)$ with $j' < j$. Furthermore, each coefficient c'_{ij} is of the form

$$c'_{ij} = c_{ij} + P_{ij}, \tag{4.7}$$

where P_{ij} is a polynomial in the variables $c_{i',j'}$ for $i' < j' < j$. Clearly, replacing each $e_j(C)$ for $j \in J \cap J^+$ by $e'_j(C)$ does not change the rank of the collection $\{e_j(C), e^+_j(C) : j \in J\}$, hence we have

$$d(U(C)) = r + c(J) + \mathrm{rk}(e'_j(C) : j \in J \cap J^+); \tag{4.8}$$

in particular, we see that $r + c(J)$ is the minimal value of the function d on $\mathcal{O}(J)$.

In more geometric terms, the above statements can be rephrased as follows. For each $j \in J \cap J^+$ let $V(j)$ denote the coordinate subspace of M_2 spanned by the vectors v_i for $i < j$, $i \notin J \cup J^+$. The correspondence $U(C) \mapsto (e'_j(C) : j \in J \cap J^+)$ defines a map $\pi : \mathcal{O}(J) \to \prod_{j \in J \cap J^+} V(j)$. As a consequence of (4.7), π is a fiber bundle with fibers being finite dimensional affine spaces. By (3.9) and (3.11), we have $\chi(X) = \chi(\pi(X))$ for every subvariety $X \subset \mathcal{O}(J)$. In particular, in view of (4.8), $\chi(d^{-1}(r + c(J) + s))$ is equal to the Euler characteristic of the subvariety of $\prod_{j \in J \cap J^+} V(j)$ consisting of all collections of vectors having rank s. For $s = 0$, the latter subvariety is just one point, implying $\chi(d^{-1}(r + c(J)) = 1$. And for $s > 0$, the subvariety of rank s collections in $\prod_{j \in J \cap J^+} V(j)$ has Euler characteristic 0 by (3.11) since it has an obvious free $\mathbb{C}^*$-action, and $\chi(\mathbb{C}^*) = 0$. This completes the proof of Proposition 4.3. □

To finish the proof of (4.2), we first use Theorem 3.2 and (3.7) to obtain

$$x_{-n} = X_{M(-n)}(x_1, x_2) = x_1^{-n} x_2^{-n-1} P_{M(-n)}(x_1^2, x_2^2).$$

Using (4.4) and (3.15), we get

$$\begin{aligned} x_{-n} = x_1^{-n} x_2^{-n-1} \, &((x_1^2+1)^{n+1} \\ &+ \sum_{p \geq 0,\ r \geq 1} \binom{r-1}{n-p-r} \binom{n+1-r}{p} (x_1^2+1)^p x_2^{2r}). \end{aligned} \tag{4.9}$$

The desired formula (4.2) follows from (4.9) by elementary manipulations with binomial coefficients. Expanding the powers of $(x_1^2 + 1)$, we obtain

$$x_{-n} x_1^n x_2^{n+1} = \sum_{q \geq 0} \binom{n+1}{q} x_1^{2q} + \sum_{q \geq 0,\ r \geq 1} a_{q,r} x_1^{2q} x_2^{2r},$$

where the coefficients $a_{q,r}$ are given by

$$a_{q,r} = \sum_p \binom{p}{q} \binom{r-1}{n-p-r} \binom{n+1-r}{p}.$$

Using an obvious identity

$$\binom{p}{q}\binom{n+1-r}{p} = \binom{n+1-r-q}{p-q}\binom{n+1-r}{q},$$

we can rewrite the last sum as

$$a_{q,r} = \binom{n+1-r}{q}\sum_p \binom{r-1}{n-p-r}\binom{n+1-r-q}{p-q}.$$

Using the well-known Vandermonde identity

$$\sum_k \binom{a}{k}\binom{b}{c-k} = \binom{a+b}{c},$$

we conclude that

$$a_{q,r} = \binom{n+1-r}{q}\binom{n-q}{r},$$

implying (4.2). This completes the proof of Theorem 4.1. □

5. Regular representations of the Kronecker quiver

In this section we complement Theorem 4.1 by computing the Laurent polynomials X_M associated with the regular indecomposable representations of the Kronecker quiver Q_2, i.e., those whose dimension vectors are imaginary positive roots. In our case, the dimension vectors in question are (n,n) for $n \geq 1$, and for each n, the indecomposable representations up to isomorphism are parameterized by $\mathbb{C}P^1$ (see [18] or [15]). It is easy to see that all the regular indecomposable representations M of the same dimension (n,n) have the same Laurent polynomial X_M. To compute it, we choose the following representative $M^{\mathrm{reg}}(n)$ (cf. Proposition 4.2).

Definition 5.1. Let $M^{\mathrm{reg}}(n)$ be a Q_2-representation of dimension (n,n) defined as follows: the space M_1 (resp. M_2) has a basis $\{u_1,\dots,u_n\}$ (resp. $\{v_1,\dots,v_n\}$) such that $\varphi_1(u_k)=v_k$ and $\varphi_2(u_k)=v_{k+1}$ for $k \in [1,n]$, with the convention that $v_{n+1}=0$. We denote

$$s_n = X_{M^{\mathrm{reg}}(n)}(x_1,x_2). \tag{5.1}$$

We prove the following analogue of Theorem 4.1.

Theorem 5.2. *The Laurent polynomials s_n are given by*

$$s_n = x_1^{-n}x_2^{-n}\sum_{q+r\leq n}\binom{n-r}{q}\binom{n-q}{r}x_1^{2q}x_2^{2r} \tag{5.2}$$

for all $n \geq 1$.

Proof. The proof follows that of Theorem 4.1. The analogue of Proposition 4.3 is as follows.

Proposition 5.3. *For every nonnegative integers p,r, we have (see* (3.13)*)*

$$\chi(Z_{p,r}(M^{\mathrm{reg}}(n))) = \binom{r}{n-p-r}\binom{n-r}{p}. \tag{5.3}$$

Proof. The proof of Proposition 5.3 follows that of Proposition 4.3 almost verbatim with obvious modifications coming from the fact that $\varphi_2(u_n) = 0$. First, (4.5) gets replaced by

$$\chi(Z_{p,r}(M^{\mathrm{reg}}(n)) \cap \mathcal{O}(J)) = \delta_{c(J)-\varepsilon(J),n-p-r}, \tag{5.4}$$

where we set

$$\varepsilon(J) = \begin{cases} 1 & \text{if } n \in J; \\ 0 & \text{if } n \notin J. \end{cases} \tag{5.5}$$

We then show the following analogue of (4.6):

(5.6) the number of r-element subsets $J \subset [1,n]$ with $c(J) - \varepsilon(J) = t$ is equal to $\binom{r}{t}\binom{n-r}{t}$.

This can be proved by a slight modification of the proof of (4.6). Alternatively, one can deduce (5.6) from (4.6) by the following simple argument. Let $c(r,n,t)$ denote the number of subsets J in (4.6), that is, the number of r-element subsets $J \subset [1,n]$ with $c(J) = t$. Then it is easy to see that the number of subsets J in (5.6) is equal to $c(r,n-1,t) + c(r,n,t+1) - c(r,n-1,t+1)$. Using (4.6), we see that the number in question is equal to

$$\binom{r-1}{t-1}\binom{n-r}{t} + \binom{r-1}{t}\binom{n+1-r}{t+1} - \binom{r-1}{t}\binom{n-r}{t+1}$$
$$= \binom{r-1}{t-1}\binom{n-r}{t} + \binom{r-1}{t}\binom{n-r}{t} = \binom{r}{t}\binom{n-r}{t},$$

as desired.

Formula (5.3) is an immediate consequence of (5.4) and (5.6). □

Arguing as in Section 4, we obtain the following analogue of (4.9):

$$s_n = \frac{1}{x_1^n x_2^n} \sum_{p,r \geq 0} \binom{r}{n-p-r}\binom{n-r}{p} (x_1^2+1)^p x_2^{2r}. \tag{5.7}$$

Formula (5.2) follows from (5.7) in the same way as (4.2) follows from (4.9). □

Theorem 5.2 allows us to sharpen the results in [22] on the canonical basis in the cluster algebra $\mathcal{A}$ associated to the Kronecker quiver. Following [22], we call a non-zero element $x \in \mathcal{A}$ *positive* if its Laurent expansion in terms of every cluster $\{x_m, x_{m+1}\}$ has positive (integer) coefficients; furthermore, a positive element x is *indecomposable* if it cannot be written as a sum of two positive elements. In [22, Theorem 2.3] it is proved that all indecomposable positive elements form a $\mathbb{Z}$-basis in $\mathcal{A}$, referred to as the *canonical basis*. As shown in [22, Theorem 2.8], the canonical basis consists of all *cluster monomials* $x_m^p x_{m+1}^q$ ($m \in \mathbb{Z}$, $p, q \geq 0$) together with a sequence of elements z_n ($n \geq 1$) defined as follows. First of all, let

$$z_1 = \frac{x_1^2 + x_2^2 + 1}{x_1 x_2}; \tag{5.8}$$

the fact that $z_1 \in \mathcal{A}$ follows from an easily checked equality $z_1 = x_0x_3 - x_1x_2$. The elements z_n for all $n \geq 1$ are defined by

$$z_n = P_n(z_1), \tag{5.9}$$

where the P_n are *normalized Chebyshev polynomials of the first kind*, related to the polynomials $S_n(x)$ in (3.2) by

$$P_n(x) = S_n(x) - S_{n-2}(x) \quad (n \geq 0) \tag{5.10}$$

(with the convention that $S_{-2}(x) = 0$).

We can now state an explicit formula (unnoticed in [22]) for the Laurent expansion of each z_n.

Theorem 5.4. *The Laurent expansion of each z_n for $n \geq 1$ in terms of x_1 and x_2 is given by*

$$z_n = x_1^{-n}x_2^{-n}\,(x_1^{2n} + x_2^{2n} + \sum_{q+r \leq n-1} \frac{n}{n-q-r}\binom{n-1-r}{q}\binom{n-1-q}{r} x_1^{2q}x_2^{2r}). \tag{5.11}$$

Proof. First of all, in view of (5.8), formula (5.11) holds for $n = 1$, and we also have $z_1 = s_1$. A direct calculation using (5.2) shows that the right hand side of (5.11) is equal to $s_n - s_{n-2}$ for $n \geq 2$ (with the convention that $s_0 = 1$). Taking into account (5.9) and (5.10), we see that it remains to show that

$$s_n = S_n(z_1) \quad (n \geq 0). \tag{5.12}$$

By (3.2), it suffices to show that the elements s_n satisfy the recursion

$$s_{n+1} = z_1s_n - s_{n-1} \quad (n \geq 1).$$

This is an easy consequence of (5.2), finishing the proof. $\square$

We conclude with three remarks.

Remark 5.5. Explicit expressions (4.2), (4.3) and (5.11) make obvious the facts about the Newton polygons of the elements of the canonical basis in [22, Propositions 3.5, 5.1 and 5.2].

Remark 5.6. In view of (5.9), (5.10) and (5.12), the elements z_n and s_n of the cluster algebra $\mathcal{A}$ are related by

$$z_1 = s_1, \; z_n = s_n - s_{n-2} \quad (n \geq 2). \tag{5.13}$$

It follows that replacing each z_n by s_n transforms the canonical basis into another $\mathbb{Z}$-basis of $\mathcal{A}$. The relationship between this new basis and the canonical basis is analogous to the relationship between the (dual) semicanonical and the (dual) canonical basis for quantum groups, cf. [17].

Remark 5.7. In view of formula (3.5), Theorems 4.1 and 5.2 provide a simple closed expression for the Euler-Poincaré characteristic $\chi(\mathrm{Gr}_{\mathbf{e}}(M))$ of each "quiver Grassmannian" in an arbitrary indecomposable representation M of the Kronecker quiver. One can also use the proofs of these theorems to obtain a nice combinatorial interpretation for $\chi(\mathrm{Gr}_{\mathbf{e}}(M))$. Namely, if we realize M as in Proposition 4.2 and Definition 5.1, then in each case, the spaces M_1 and M_2 are supplied with the distinguished bases $\{u_i\}$ and $\{v_i\}$, respectively. The calculations in the proofs of

Theorems 4.1 and 5.2 imply that $\chi(\mathrm{Gr}_{\mathbf{e}}(M))$ is equal to the (finite) number of points $(N_1, N_2) \in \mathrm{Gr}_{\mathbf{e}}(M)$ such that both N_1 and N_2 are coordinate subspaces with respect to these distinguished bases. This expression for $\chi(\mathrm{Gr}_{\mathbf{e}}(M))$ can be rephrased as the following combinatorial expression for the polynomial $P_M(z_1, z_2)$ (see (3.8)). Consider the polynomials $F(w_1, \dots, w_N)$ given by

$$F(w_1, \dots, w_N) = \sum_D \prod_{k \in D} w_k, \tag{5.14}$$

where D runs over all subsets of $[1, N]$ containing no two consecutive integers (these polynomials appear in a different context in [13, Example 2.15]). Then, for every $n \geq 0$, we have

$$\begin{aligned} P_{M(-n)}(z_1, z_2) &= F(w_1, \dots, w_{2n+1})|_{w_k = z_{\langle k \rangle}}, \\ P_{M(n+3)}(z_1, z_2) &= F(w_1, \dots, w_{2n+1})|_{w_k = z_{\langle k+1 \rangle}}, \\ P_{M^{\mathrm{reg}}(n+1)}(z_1, z_2) &= F(w_1, \dots, w_{2n+2})|_{w_k = z_{\langle k \rangle}}, \end{aligned} \tag{5.15}$$

where $\langle k \rangle$ stands for the element of $\{1, 2\}$ congruent to k modulo 2. In view of (3.7), we also have

$$\begin{aligned} x_{-n} &= x_1^{-n} x_2^{-n-1} F(w_1, \dots, w_{2n+1})|_{w_k = x^2_{\langle k \rangle}}, \\ x_{n+3} &= x_1^{-n-1} x_2^{-n} F(w_1, \dots, w_{2n+1})|_{w_k = x^2_{\langle k+1 \rangle}}, \\ s_{n+1} &= x_1^{-n-1} x_2^{-n-1} F(w_1, \dots, w_{2n+2})|_{w_k = x^2_{\langle k \rangle}}. \end{aligned} \tag{5.16}$$

These formulas are easily seen to be equivalent to the combinatorial expressions for cluster variables in [21].

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References

[1] A. Berenstein, S. Fomin, A. Zelevinsky, Cluster algebras III: Upper bounds and double Bruhat cells, *Duke Math. J.* **126** (2005), 1–52.
[2] J. Bernstein, I. Gelfand, V. Ponomarev, Coxeter functors, and Gabriel's theorem, *Uspehi Mat. Nauk* **28** (1973), no. 2(170), 19-33.
[3] A. Buan, R. Marsh, I. Reiten, Cluster tilted algebras, `math.RT/0402075`.
[4] A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, Tilting theory and cluster combinatorics, `math.RT/0402054`.
[5] P. Caldero, F. Chapoton, Cluster algebras as Hall algebras of quiver representations, `math.RT/0410184`.
[6] P. Caldero, B. Keller, From triangulated categories to cluster algebras, `math.RT/0506018`.
[7] P. Caldero, B. Keller, From triangulated categories to cluster algebras II, `math.RT/0510251`.
[8] H. Derksen, J. Weyman, On canonical decomposition for quiver representations, *Compositio Math.* **133** (2002), no. 3, 245-265.
[9] V. Fock, A. Goncharov, Dual Teichmuller and lamination spaces, `math.DG/0510312`.
[10] S. Fomin, N. Reading, Root systems and generalized associahedra, Lecture notes for the IAS/Park City Graduate Summer School in Geometric Combinatorics (July 2004), `math.CO/0505518`.

[11] S. Fomin, A. Zelevinsky, Cluster algebras. I. Foundations, *J. Amer. Math. Soc.* **15** (2002), no. 2, 497–529.
[12] S. Fomin, A. Zelevinsky, Cluster algebras. II. Finite type classification, *Invent. Math.* **154** (2003), no. 1, 63–121.
[13] S. Fomin, A. Zelevinsky, Y-systems and generalized associahedra, *Ann. in Math.* **158** (2003), 977–1018.
[14] S. Fomin, A. Zelevinsky, Cluster algebras: Notes for the CDM-03 conference, in: *CDM 2003: Current Developments in Mathematics*, International Press, 2004.
[15] I. Frenkel, A. Malkin, M. Vybornov, Affine Lie algebras and tame quivers, *Selecta Math. (N.S.)* **7** (2001), no. 1, 1–56.
[16] W. Fulton, Introduction to Toric Varieties, Ann. of Math. Stud., vol. 131, Princeton University Press, 1993.
[17] C. Geiss, B. Leclerc, J. Schröer, Semicanonical bases and preprojective algebras, *Ann. Sci. École Norm. Sup. (4)* **38** (2005), no. 2, 193–253.
[18] V. Kac, Infinite root systems, representations of graphs and invariant theory, *Invent. Math.* **56** (1980), no. 1, 57-92.
[19] L. Kronecker, Algebraische reduction der schaaren bilinearer formen, *Sitzungsberichte Akad. Berlin* (1890), 1225-1237.
[20] R. Marsh, M. Reineke, A. Zelevinsky, Generalized associahedra via quiver representations. *Trans. Amer. Math. Soc.* **355** (2003), no. 10, 4171–4186.
[21] G. Musiker, J. Propp, Combinatorial interpretations for rank-two cluster algebras of affine type, `math.CO/0602408`.
[22] P. Sherman, A. Zelevinsky, Positivity and canonical bases in rank 2 cluster algebras of finite and affine types, *Moscow Math. J.* **4** (2004), no. 4, 947–974.

Institut Camille Jordan, Université Claude Bernard Lyon I, 69622 Villeurbanne Cedex, France
E-mail address: caldero@igd.univ-lyon1.fr

Department of Mathematics, Northeastern University, Boston, MA 02115, USA
E-mail address: andrei@neu.edu