FROM TRIANGULATED CATEGORIES TO CLUSTER ALGEBRAS II

PHILIPPE CALDERO AND BERNHARD KELLER

Abstract. In the acyclic case, we etablish a one-to-one correspondence between the tilting objects of the cluster category and the clusters of the associated cluster algebra. This correspondence enables us to solve conjectures on cluster algebras. We prove a positivity theorem, a denominator theorem, and some conjectures on properties of the mutation graph. As in the previous article, the proofs rely on the Calabi-Yau property of the cluster category.

1. Introduction

Cluster algebras are commutative algebras, introduced in [11] by S. Fomin and A. Zelevinsky. Originally, they were constructed to obtain a better understanding of the positivity and multiplicativity properties of Lusztig's dual (semi)canonical basis of the algebra of coordinate functions on homogeneous spaces. Cluster algebras are generated by the so-called *cluster variables* gathered into sets of fixed cardinality called *clusters*. In the framework of the present paper, the cluster variables are obtained by a recurcive process from an antisymmetric square matrix B.

Denote by Q the quiver associated to the matrix B. Assume that Q is connected. A theorem of Fomin and Zelevinsky asserts that the number of cluster variables of the corresponding cluster algebra $\mathcal{A}_Q$ is finite if and only if the graph underlying Q is a simply laced Dynkin diagram. In this case, it is known that the combinatorics of the clusters are governed by the generalized associahedron.

Let Q be any finite quiver without oriented cycles and let k be an algebraically closed field. The cluster category $\mathcal{C} = \mathcal{C}_Q$ was introduced in [8] for type A_n and in [6] in the general case. This construction was motivated by the combinatorial similarities of $\mathcal{C}_Q$ with the cluster algebra $\mathcal{A}_Q$. The cluster category is the category of orbits under an autoequivalence of the bounded derived category $\mathcal{D}^b$ of the category of finite dimensional kQ-modules. By [17], the category $\mathcal{C}_Q$ is a triangulated category. Let us denote its shift functor by S. By construction, the cluster category is Calabi-Yau of CY-dimension 2; in other terms, the functor Ext^1 is symmetric in the following sense:

$$\mathsf{Ext}^1_{\mathcal{C}}(M, N) \simeq D\,\mathsf{Ext}^1_{\mathcal{C}}(N, M).$$

In a series of articles [6], [3], [4], the authors study the tilting theory of the cluster category. More precisely, they describe the combinatorics of the cluster tilting objects of the category $\mathcal{C}$, *i.e.* the objects without self-extensions and with a maximal number of non-isomorphic indecomposable summands. In [4], the authors define a map β between the set of clusters of $\mathcal{A}_Q$ and the set of tilting objects of the category $\mathcal{C}_Q$. A natural question arises: does β provide a one-to-one correspondence between both sets?

In the articles [7] and [10], it is proved that in the finite case, *i.e.* the Dynkin case, the cluster algebra can be recovered from the corresponding cluster category as the so-called *exceptional Hall algebra* of the cluster category. More precisely, in [7], the authors give an explicit correspondence $M \mapsto X_M$ between indecomposable objects of $\mathcal{C}_Q$ and cluster

Date: version du 12/10/2005.

variables of $\mathcal{A}_Q$. In [10], we provide a multiplication rule for the algebra $\mathcal{A}_Q$ in terms of the triangulated category $\mathcal{C}_Q$.

An ingenious application of the methods of [10] can be found in [14], where the authors give a multiplication formula for elements of Lusztig's dual semicanonical basis. Here, the cluster category is replaced by the category of finite-dimensional modules over the preprojective algebra and the rôle of the cluster algebra is played by the coordinate algebra of the maximal unipotent subgroup in the corresponding semisimple algebraic group.

The aim of the present article is to generalize some of the results of [7], [10] to the case where Q is any finite quiver without oriented cycles. Building on the important results obtained in [4] we strengthen here the connections between the cluster category and the cluster algebra by giving an explicit expression for the correspondence β and proving that β is one-to-one. The key ingredient of the proof is a natural analogue of the map $M \mapsto X_M$ of [7]. With the help of a positivity result, we show that $M \mapsto X_M$ defines a bijection between the indecomposable objects without self-extensions of $\mathcal{C}_Q$ and the cluster variables of $\mathcal{A}_Q$.

This correspondence between cluster algebras and cluster categories gives positive answers to some of the conjectures which S. Fomin and A. Zelevinsky formulated in [13]. We prove a positivity conjecture for cluster variables, and connectedness properties of some mutation graphs, *cf.* section 4.3. As a byproduct, we obtain a cluster-categorical interpretation of the passage to a submatrix of the exchange matrix. This strengthens a key result of [4] and may be of independent interest.

Another consequence of the bijectivity of β is that each seed is determined by its cluster. As we have learned recently, this result is obtained independently in [5].

The paper is organized as follows: In the first part, we recall well-known facts on the cluster category. For any object M of the cluster category, we define the Laurent polynomial X_M. Then as a first result, we prove the positivity of the coefficients of X_M, which are obtained from Euler characteristics of Grassmannians of submodules. For this, we need some properties of Lusztig's canonical bases in quantum groups. From the positivity, we deduce that the map $M \mapsto X_M$ is injective when restricted to the set of indecomposable objects of $\mathcal{C}_Q$ without self-extensions. With the techniques of [10], we prove an 'exchange relation' for the X_M. To be more precise, we prove that if M and N are indecomposable objects of the category $\mathcal{C} = \mathcal{C}_Q$ such that $\mathsf{Ext}^1_{\mathcal{C}}(M, N) = k$, then

$$X_M X_N = X_B + X_{B'},$$

where B and B' are the unique objects (up to isomorphism) such that there exist non split triangles

$$N \to B \to M \to SN, \quad M \to B' \to N \to SM.$$

This formula is an analogue of the 'exchange relation' between cluster variables. With the help of a comparison theorem of [4], we prove by induction that the X_M are cluster variables. The injectivity property discussed above gives the one-to-one correspondence β between the set of tilting objects of $\mathcal{C}_Q$ and the set of clusters of $\mathcal{A}_Q$.

Acknowledgements: The first author is indebted to Thomas Brüstle, Ralf Schiffler and Olivier Schiffmann for useful conversations. He also wishes to thank Andrei Zelevinsky for his kind hospitality and for pointing out to him the conjectures of [13].

2. The cluster category and the cluster variable formula

2.1. Let H be a finite dimensional hereditary algebra over an algebraically closed field k. We denote by $H\text{-}\mathsf{mod}$ the category of finitely generated H-modules. We choose representatives S_i, $1 \leq i \leq n$, of the isoclasses of the simple H-modules and denote by I_i the injective hull and by P_i the projective cover of S_i.

The Grothendieck group of $H\text{-}\mathsf{mod}$ is the group $\mathsf{G}_0(H\text{-}\mathsf{mod})$ generated by the isoclasses of modules in $H\text{-}\mathsf{mod}$ and subject to the relations $X = M + N$ obtained from exact sequences $0 \to M \to X \to N \to 0$ in $H\text{-}\mathsf{mod}$. We denote by $[M]$ the class of a module M in $\mathsf{G}_0(H\text{-}\mathsf{mod})$. We put $\alpha_i = [S_i]$. The Grothendieck group is free abelian on the α_i. The dimension vector $\underline{\dim}\,(M)$ of a module M is by definition the vector of the coordinates of $[M]$ in this basis.

We define the Euler form by $< M, N >= \dim \mathsf{Hom}(M, N) - \dim \mathsf{Ext}^1(M, N)$, for any M, N in $H\text{-}\mathsf{mod}$. Since H is hereditary, this form is well-defined on the Grothendieck group.

Let τ be the Auslander-Reiten functor of $H\text{-}\mathsf{mod}$. This functor verifies the Auslander-Reiten formula:

$$D\,\mathsf{Hom}(N, \tau M) = \mathsf{Ext}^1(M, N),$$

where D is the functor $\mathsf{Hom}_k(?, k)$.

2.2. For any H-module M, and any e in $\mathsf{G}_0(H\text{-}\mathsf{mod})$, we denote by $\mathsf{Gr}_e(M)$ the Grassmannian of submodules of M with dimension vector e:

$$\mathsf{Gr}_e(M) = \{N,\, N \in H\text{-}\mathsf{mod},\, N \subset M,\, \underline{\dim}\,(N) = e\}.$$

It is a closed subvariety of the classical Grassmannian of the vector space M. Let χ_c be the Euler-Poincaré characteristic of the etale cohomology with proper support defined by

$$\chi_c(X) = \sum_{i=0}^{\infty} (-1)^i \dim H^i_c(X, \overline{\mathbb{Q}}_l).$$

Let $\mathbb{Q}[x_i^{\pm 1}, 1 \leq i \leq n]$ be the $\mathbb{Q}$-algebra of Laurent polynomials in the variables x_i's. As in [7], for any module M, we set

$$X_M = \sum_e \chi_c(\mathsf{Gr}_e(M)) \prod_i x_i^{-<e,\alpha_i>-<\alpha_i,m-e>} \in \mathbb{Q}[x_i^{\pm 1}, 1 \leq i \leq n],$$

where $m := \underline{\dim}\,(M)$. Note that, as M is finite dimensional, there only exists a finite number of non zero terms in this sum. Remark that X_M only depends on the isoclass of the module M. As in [7] one shows that

$$\chi_c(\mathsf{Gr}_g(M \oplus N)) = \sum_{e+f=g} \chi_c(\mathsf{Gr}_e(M))\chi_c(\mathsf{Gr}_f(N)).$$

Hence, the bilinearity of the Euler form implies that

$$X_{M \oplus N} = X_M X_N.$$

As in [10], for any H-module M, the Laurent polynomial X_M has (integral) positive coefficients. We can be more precise

Proposition 1. *Let M be a finite-dimensional H-module and suppose that $\mathsf{Gr}_e(M)$ is non empty for some dimension vector e. Then we have $\chi_c(\mathsf{Gr}_e(M)) > 0$.*

We will show that this is a consequence of Lusztig's positivity theorem for the canonical basis [18]. We first need to recall some basic facts from theory of quiver representations.

As H is hereditary and finite dimensional, there exists a finite quiver Q without oriented cycles such that H is Morita equivalent to the path algebra kQ of Q. Let Q_0 be the set of vertices and Q_1 the set arrows of Q. Let n be the number of vertices of Q. A representation of Q over a field F is a Q_0-graded F-vector space $V = \oplus_{i \in Q_0} V_i$ together with an element

$x = (x_h)_{h\in Q_1}$ in $E_V := \prod_{h\in Q_1} \mathsf{Hom}(V_{s(h)}, V_{t(h)})$, where $s(h)$ is the source and $t(h)$ the target of the arrow h. The group $G_V := \prod_{i\in Q_0} \mathsf{GL}(V_i)$ acts on E_V by $(g_i).(x_h) = (g_{t(h)}x_h g_{s(h)}^{-1})$. A representation (M, x) over a field F can be considered as an FQ-module and the dimension vector of this module is $\underline{\dim} M = (\dim M_i)$. Clearly, the isoclasses of finite-dimensional H-modules are naturally identified with G_V-orbits of representations of Q.

We can now sketch a proof of the proposition.

Proof. Set $F := \mathbb{F}_q$. For m in $\mathbb{N}^{Q_0}$, and $V_m = \oplus F^{m_i}$, let $\mathcal{H}_m(q)$ be the $\mathbb{C}$-vector space of all G_{V_m}-invariant functions on E_{V_m}. Let $\mathcal{H}(q)$ be the vector space $\oplus_{m\in\mathbb{N}^{Q_0}}\mathcal{H}_m(q)$. We can define on $\mathcal{H}(q)$ a $\mathbb{C}$-algebra structure by

$$(f_1 * f_2)(M) = \sum_{N\subset M} f_1(M/N)f_2(N).$$

For any isoclass X of FQ-module, let $1_X \in \mathcal{H}(q)$ be the caracteristic function of the corresponding orbit. We have

$$(1_X * 1_Y)(M) =| \{N \subset M,\, N \simeq Y,\, M/N \simeq X\} | .$$

For any dimension vector m, let 1_m be the constant function with value 1 on E_{V_m}. Set $m = \underline{\dim} M$. Then it is easily seen that

$$F^M_{m-e,e}(q) := (1_{m-e} * 1_e)(M) =| \{N,\, N \subset M,\, \underline{\dim} N = e\} |, \tag{2.1}$$

Now, we fix an ordering of $Q_0 = [1, n]$ such that if there exists a path from j to i, then $j < i$. Since Q has no oriented cycles, it is possible to construct such an ordering.

Consider the divided power

$$1^{(e_i)}_{\alpha_i} = \frac{1}{[e_i]_q!} 1^{e_i}_{\alpha_i},$$

where $[e_i]_q! = [e_i]_q[e_i - 1]_q \dots [1]_q$, and $[n]_q := \sum_{i=0}^{n-1} q^i$. Then we have

$$1^{(e_1)}_{\alpha_i} * 1^{(e_2)}_{\alpha_i} \dots * 1^{(e_n)}_{\alpha_n} = 1_e.$$

Hence, 1_e is in the composition algebra of the quiver, i.e. the subalgebra of $\mathcal{H}(q)$ generated by the 1_{α_i}'s. Hence, $F^M_{m-e,e}(q)$ is a polynomial in q. Moreover, note that this polynomial is non zero since by hypothesis, the set $\{N,\, N \subset M,\, \underline{\dim} N = e\}$ is not empty.

It is known that $1^{(e_i)}_{\alpha_i}$ is an element of Lusztig's canonical basis. By Lusztig's positivity theorem, the product of elements of the canonical basis has positive coefficients in the canonical basis and moreover, the evaluation of a canonical basis element on an orbit is positive. We can conclude that the polynomial $F^M_{m-e,e}(q)$ has positive coefficients in q (and is non zero). As seen in section 4.3 of [10], the Euler characteristic $\chi_c(\mathsf{Gr}_e(M))$ is $F^M_{m-e,e}(1)$, and so is positive as desired.

□

We deduce a denominator property.

Corollary 1. *Let M be an indecomposable H-module with dimension vector $\underline{\dim} M = (m_i)$. Then the denominator of X_M as an irreducible fraction of integral polynomials in the variables x_i is $\prod_i x_i^{m_i}$.*

Proof. By the positivity theorem, X_M is a linear combination with positive integer coefficients of terms $\prod x_l^{n_l}$, $n_l \in \mathbb{Z}$. These terms are indexed by the set of dimension vectors of submodules N of M, and for each submodule N, we have that

$$n_l = -\langle N, S_l\rangle - \langle S_l, M/N\rangle.$$

Let us put $d = \dim M$ and $e = \dim N$. Then it is sufficient to prove that

1. for all l, we have $n_l \geq -d_l$ and
2. for all l, there exists a submodule N such that the equality holds.

Now a simple computation shows that

$$n_l = -d_l + \sum_{i \to l} e_i + \sum_{l \to j} (d_j - e_j) \, ,$$

where the first sum ranges over all arrows $i \to l$ of Q and the second sum over all arrows $l \to j$ of Q. Since we have $0 \leq e_i \leq d_i$ for all i, the first assertion is clear. For the second one, we simply choose N to be the submodule of M generated by all the spaces M_j such that there exists an arrow $l \to j$. Then the terms in $\sum_{i \to l} e_i$ vanish since Q has no oriented cycles and the terms in $\sum_{l \to j}(d_j - e_j)$ vanish by the construction of N. □

Another corollary is the following injectivity property:

Corollary 2. *If M and M' are non isomorphic indecomposable modules without self-extensions, then $X_M \neq X_{M'}$.*

Proof. This is clear from the corollary above and the fact that M and M' cannot have the same dimension vector since their isoclass corresponds to the unique dense orbit of their representation space. □

2.3. The bounded derived category $\mathcal{D}^b = \mathcal{D}^b(H)$ of H-mod is a triangulated category. We denote its shift functor $M \mapsto M[1]$ by S. The category $\mathcal{D}^b$ is a Krull-Schmidt category and, up to canonical triangle equivalence, it only depends on the underlying graph of Q, see [15]. We identify the category H-mod with the full subcategory of $\mathcal{D}^b$ formed by the complexes whose homology is concentrated in degree 0. We simply call 'modules' the objects in this subcategory. The indecomposable objects of $\mathcal{D}^b$ are the shifts S^iM, $i \in \mathbb{Z}$, of the indecomposable objects of H-mod. We still denote by τ the AR-functor of $\mathcal{D}^b$; it is known that τ is an autoequivalence characterized by the Auslander-Reiten formula.

Let F be the autoequivalence $\tau^{-1}S$ of $\mathcal{D}^b$. The AR-formula implies that

$$\mathsf{Ext}^1_{\mathcal{D}^b}(M,N) = \mathsf{Hom}_{\mathcal{D}^b}(M,SN) = D\,\mathsf{Ext}^1_{\mathcal{D}^b}(FN,M),$$

for any objects M, N of $\mathcal{D}^b$. Let $\mathcal{C} = \mathcal{C}(H)$ be the orbit category $\mathcal{D}^b/F$: the objects of $\mathcal{C}$ are the objects of $\mathcal{D}^b$ and the morphisms of $\mathcal{C}$ are given by

$$\mathsf{Hom}_{\mathcal{C}}(M,N) = \oplus_{i \in \mathbb{Z}} \mathsf{Hom}_{\mathcal{D}^b}(M, F^iN).$$

The category $\mathcal{C}$ is the so-called cluster category, introduced and studied in depth in [6]. Let π be the canonical functor from $\mathcal{D}^b$ to $\mathcal{C}$. We will often omit the functor π from the notations. We have, by [17] and [6]:

Theorem 1. (i) *The category $\mathcal{C}$ is triangulated and*
(ii) *the functor $\pi : \mathcal{D} \to \mathcal{C}$ is a triangle functor.*
(iii) *The category $\mathcal{C}$ is a Krull-Schmidt category and*
(iv) *For any indecomposable object without self-extensions M of $\mathcal{C}$, the ring $\mathsf{End}_{\mathcal{C}}(M)$ is the field k.*

The shift functor of the triangulated category $\mathcal{C}$ will still be denoted by S. For any objects M, N of $\mathcal{C}$, the formulas above imply that there exists an (almost canonical) duality

$$\phi \,:\, \mathsf{Ext}^1_{\mathcal{C}}(M,N) \times \mathsf{Ext}^1_{\mathcal{C}}(N,M) \to k.$$

The set of indecomposable objects of $\mathcal{C}$ is given by

$$\mathsf{Ind}(\mathcal{C}) = \mathsf{Ind}(H\text{-}\mathsf{mod}) \coprod \{SP_i,\, 1 \leq i \leq n\}.$$

Note that $SP_i = S^{-1}\tau^{-1}SP_i = S^{-1}I_i$.

We can extend the definition of X_M to any object M of the category $\mathcal{C}$ by setting $X_{SP_i} = x_i$, $1 \le i \le n$ and $X_{M\oplus N} = X_M X_N$.

The AR-formula and the fact that τ passes to the Grothendieck group of the derived category of H-mod allow us to rewrite X_M for a module M as

$$X_M = \sum_e \chi_c(\mathsf{Gr}_e(M))\, x^{\tau(e)-\underline{\dim}\, M+e}\,, \tag{2.2}$$

where we have set

$$x^v = \prod_{i=1}^n x_i^{<\underline{\dim}\, S_i, v>},$$

for any v in $\mathbb{Z}^n$. Remark that this notation gives

$$X_{SP_i} = x^{\underline{\dim}\, I_i}.$$

2.4. Each object M of $\mathcal{C}$ can be uniquely decomposed in the following way:

$$M = M_0 \oplus SP_M = M_0 \oplus S^{-1}I_M,$$

where M_0 is the image under π of a module in $\mathcal{D}^b$, and where P_M, respectively I_M, is a uniquely determined projective, respectively injective, module. We will say that an object M of $\mathcal{C}$ is a *module* if $M = M_0$, and that M is the *shift of a projective module* if $M = SP_M$.

From [6], we recall the

Proposition 2. *For any indecomposable modules M and N in $\mathcal{C}$, we have*

$$\mathsf{Ext}^1_{\mathcal{C}}(M,N) = \mathsf{Ext}^1_H(M,N) \oplus D\,\mathsf{Ext}^1_H(N,M).$$

The module M_0 can be recovered using the functor

$$H^0 = \mathsf{Hom}_{\mathcal{C}}(H_H, ?) \,:\, \mathcal{C} \to H\text{-}\mathsf{mod}\,.$$

Indeed, we have

$$H^0(M) = H^0(M_0) \oplus H^0(SP_M) = \mathsf{Hom}_{H\text{-}\mathsf{mod}}(H_H, M_0) \oplus \mathsf{Hom}_{\mathcal{C}}(\oplus_i P_i, SP_M) = M_0\,,$$

as the last factor is zero. The functor H^0 is a homological functor, *i.e.* it maps triangles in $\mathcal{C}$ to long exact sequences of H-modules.

3. A multiplication formula

3.1. The aim of the section is to prove the following theorem:

Theorem 2. *Let M and N be indecomposable objects of the category $\mathcal{C}$ such that $\mathsf{Ext}^1_{\mathcal{C}}(M,N)$ is one-dimensional. Then we have*

$$X_M X_N = X_B + X_{B'},$$

where B and B' are the unique objects (up to isomorphism) such that there exist non split triangles

$$N \to B \to M \to SN, \;\; M \to B' \to N \to SM.$$

Note that when H is the path algebra of a Dynkin quiver, the theorem is a particular case of the cluster multiplication formula of [10]. Actually, we will see that the method of [10] generalizes nicely to the framework of the theorem.

Thanks to the hypotheses of the theorem and the symmetry of Ext^1, we just need to consider the two following cases

1. $N = SP_i$ for a $i \in Q_0$ and M is an indecomposable module.
2. M and N are indecomposable modules.

Indeed, the isomorphisms $M = SP_j$ and $N = SP_i$ would imply

$$\mathsf{Ext}^1_{\mathcal{C}}(M, N) = \mathsf{Ext}^1_{\mathcal{C}}(P_j, P_i) = 0.$$

3.2. We now prove the theorem in the first case. Suppose $N = SP_i$, and let M be an indecomposable module such that $\mathsf{Ext}^1_{\mathcal{C}}(SP_i, M) = k$. Using theorem 1 and the AR-formula, we obtain

$$\underline{\dim}\,(M)_i = \dim \mathsf{Hom}_H(P_i, M) = \dim \mathsf{Hom}_{\mathcal{C}}(P_i, M) = \dim \mathsf{Ext}^1_{\mathcal{C}}(SP_i, M) = 1.$$

Hence, up to a multiplicative scalar, there exists a unique non zero morphism $\zeta : M \to I_i$ and a non zero morphism $\zeta' : P_i \to M$.

Lemma 1. *Let M' be a submodule of M. Then either $M' \subset \ker \zeta$ or $\operatorname{im} \zeta' \subset M'$.*

Proof. By the formula above, the space M'_i is of dimension 0 or 1. We claim that

1. $\dim(M') = 0$ if and only if $M' \subset \ker \zeta$,
2. $\dim(M') = 1$ if and only if $\operatorname{im} \zeta' \subset M'$.

The lemma follows from the claim. Let's prove part 1. The second part is similar and left to the reader. The module $\operatorname{im} \zeta$ is non zero and so it contains the simple socle S_i of I_i. Hence, $\dim(\ker \zeta)_i = 0$, which gives the 'if' part. Conversely, if $\dim(M')_i = 0$, then $\zeta(M') \cap S_i = 0$, hence $\zeta(M') = 0$ as S_i is the socle of I_i. □

Applying the functor H^0 the non split triangle

$$SP_i \xrightarrow{\iota} B \xrightarrow{\pi} M \xrightarrow{\zeta} S^2 P_i = I_i \tag{3.1}$$

we obtain a long exact sequences of H-modules

$$0 \longrightarrow H^0 B \xrightarrow{H^0\pi} M \xrightarrow{H^0\zeta} I_i \xrightarrow{H^1\iota} H^0\tau B \xrightarrow{H^1\pi} H^0\tau M \ ,$$

Now, $H^0\tau B = \tau H^0 B \oplus I_B$, and the first factor is non injective. As the quotient of an injective module is still injective, we have $\operatorname{im}(H^1\iota) \subset I_B$. Moreover, as $H^0\tau M$ is non injective, we have $I_B \subset \ker(H^1\pi)$. Hence, we have equality and so the following exact sequence holds

$$0 \longrightarrow H^0 B \xrightarrow{H^0\pi} M \xrightarrow{H^0\zeta} I_i \xrightarrow{H^1\iota} I_B \longrightarrow 0\,. \tag{3.2}$$

Note that the morphism $H^0\zeta = \zeta$ is non zero.

In the same way, applying the functor H^0, the non split triangle

$$P_i \xrightarrow{\zeta'} M \xrightarrow{\pi'} B' \xrightarrow{\iota'} SP_i, \tag{3.3}$$

we obtain

$$0 \longrightarrow P_{B'} \longrightarrow P_i \xrightarrow{H^0\zeta'} M \xrightarrow{H^0\pi'} H^0 B' \longrightarrow 0\,. \tag{3.4}$$

Note that the morphism $H^0\zeta' = \zeta'$ is non zero.

Now, the lemma implies that for any submodule M' of M, M' is a either submodule of $\operatorname{im} H^0\pi$ or contains $\ker H^0\pi'$. Hence, there is a natural bijection between $\mathsf{Gr}_e M$ and $\mathsf{Gr}_e(H^0 B) \coprod \mathsf{Gr}_{e-k}(H^0 B')$, where

$$k := \underline{\dim}\, \ker H^0\pi' = \underline{\dim}\, P_i - \underline{\dim}\, P_{B'}. \tag{3.5}$$

We want to prove the multiplication formula, which in this case is

$$x^{\underline{\dim}\, I_i} \sum_e \chi_c(\mathsf{Gr}_e\, M) x^{\tau(e) - \underline{\dim}\, M + e} =$$

$$x^{\underline{\dim}\, I_B} \sum_e \chi_c(\mathsf{Gr}_e\, H^0 B) x^{\tau(e) - \underline{\dim}\, H^0 B + e} + x^{\underline{\dim}\, I_{B'}} \sum_e \chi_c(\mathsf{Gr}_e\, H^0 B') x^{\tau(e) - \underline{\dim}\, H^0 B' + e}.$$

So, it remains to prove that

$$\underline{\dim}\, I_i + \tau(e) - \underline{\dim}\, M + e = \underline{\dim}\, I_B + \tau(e) - \underline{\dim}\, H^0 B + e,$$

and

$$\underline{\dim}\, I_i + \tau(e) - \underline{\dim}\, M + e = \underline{\dim}\, I_{B'} + \tau(e-k) - \underline{\dim}\, H^0 B' + e - k.$$

The first formula is a direct consequence of 3.2. The second one comes from 3.4, 3.5 and the formula $\tau(\underline{\dim}\, P_j) = -\underline{\dim}\, I_j$.

3.3. This subsection and the following one are devoted to the proof of the theorem in the second case. In order to simplify notations, we will write (X, Y) for $\mathsf{Hom}_{\mathcal{C}}(X, Y)$.

Let M and N be two indecomposable modules such that $\mathsf{Ext}^1_{\mathcal{C}}(N, M) = k$. By proposition 2, we can suppose that $\mathsf{Ext}^1_H(N, M) = k$ and $\mathsf{Ext}^1_H(M, N) = 0$. In this case, by theorem 1, there exists (up to isomorphism) a unique non split short exact sequence of H-modules

$$0 \longrightarrow M \xrightarrow{\ i\ } B_+ \xrightarrow{\ p\ } N \longrightarrow 0\ ,$$

and two triangles in $\mathcal{C}$

$$M \xrightarrow{\ i\ } B_+ \xrightarrow{\ p\ } N \longrightarrow SM\ ,$$

$$N \xrightarrow{\ i'\ } B_- \xrightarrow{\ p'\ } M \longrightarrow SN\ .$$

Note that B_+ is a 'module' of $\mathcal{C}$ but B_- is just an object; they both are uniquely determined up to isomorphism. We want to prove the formula

$$X_{B_+} + X_{B_-} = X_M X_N,$$

and the idea is first to construct a morphism Ψ between $\mathsf{Gr}\, B_+ \coprod \mathsf{Gr}\, H^0 B_-$ and $\mathsf{Gr}\, M \times \mathsf{Gr}\, N$. For any submodule B'_+ of B_+, set $\Psi(B'_+) = (i^{-1} B'_+, pB'_+)$, and for any submodule B'_- of B_-, set $\Psi(B'_-) = ((H^0 p') B'_-, (H^0 i')^{-1} B'_-)$. As a first step, we want to prove the proposition

Proposition 3. *The variety* $\mathsf{Gr}\, M \times \mathsf{Gr}\, N$ *is the disjoint union of* $\Psi(\mathsf{Gr}\, B_+)$ *and* $\Psi(\mathsf{Gr}\, H^0 B_-)$. *Moreover, the fibers of* Ψ *are affine spaces.*

This proposition will be proved at the end of this subsection.

Given a submodule M' of M, a submodule N' of N, and the corresponding embeddings i_M and i_N, we have a diagram

$$\begin{array}{ccccc} S^{-1}M & \xrightarrow{\ \varepsilon'\ } & N & \xrightarrow{\ \varepsilon\ } & SM \\ {\scriptstyle S^{-1} i_M}\uparrow & & {\scriptstyle i_N}\uparrow & & \uparrow{\scriptstyle S i_M} \\ S^{-1}M' & & N' & & SM' \end{array}$$

and two complexes

$$(S^{-1}M, N') \xrightarrow{\ \alpha'\ } (S^{-1}M, N) \oplus (S^{-1}M', N') \xrightarrow{\ \beta'\ } (S^{-1}M', N)$$

$$(N', SM) \xleftarrow{\ \alpha\ } (N, SM) \oplus (N', SM') \xleftarrow{\ \beta\ } (N, SM')$$

where

$$\alpha' = \begin{bmatrix} (i_{N'})_* \\ (S^{-1}i_{M'})^* \end{bmatrix}, \ \beta' = \left[(S^{-1}i_{M'})^*, -(i'_N)_*\right], \ \alpha = [(i_{N'})_*, (Si_{M'})^*], \ \beta = \begin{bmatrix} (Si_{M'})^* \\ -(i_{N'})_* \end{bmatrix}.$$

The two sequences are dual to each other via the canonical duality ϕ.

The following proposition is straightforward by using basic properties of triangulated categories.

Proposition 4. *The following conditions are equivalent:*

(i) *There exists a submodule $B'_+ \subset B_+$ such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & B_+ & \longrightarrow & N & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & B'_+ & \longrightarrow & N' & \longrightarrow & 0. \end{array}$$

commutes.

(ii) *There exists a morphism $\eta : N' \to SM'$ such that the square*

$$\begin{array}{ccc} N & \xrightarrow{\varepsilon} & SM \\ {\scriptstyle i_N}\uparrow & & \uparrow{\scriptstyle Si_M} \\ N' & \xrightarrow{\eta} & SM' \end{array}$$

commutes.

(iii) *The composed morphism*

$$\ker \alpha \hookrightarrow (N', SM') \oplus (N, SM) \to (N, SM)$$

is non zero.

(iv) *The composed morphism*

$$\operatorname{cok} \alpha' \leftarrow (S^{-1}M', N') \oplus (S^{-1}M, SN) \hookleftarrow (S^{-1}M, N)$$

is non zero.

The following proposition sheds light on the situation when the conditions of proposition 4 do not hold.

Proposition 5. *The following conditions are equivalent:*

(i) *The composition*

$$\operatorname{cok} \alpha' \leftarrow (S^{-1}M', N'') \oplus (S^{-1}M, SN) \hookleftarrow (S^{-1}M, N)$$

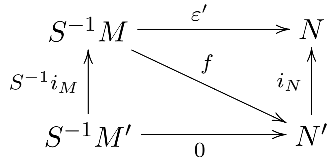
vanishes, i.e. *$(S^{-1}M, N)$ is contained in the image of α'.*

(ii) *There exist a submodule $B'_- \hookrightarrow H^0B_-$ and a commutative diagram*

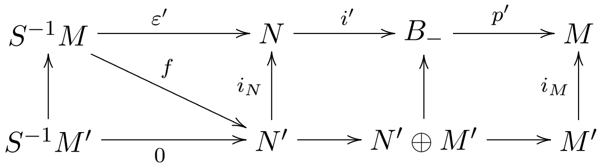
$$\begin{array}{ccccccc} N & \xrightarrow{H^0 i'} & H^0B' & \xrightarrow{H^0 p'} & M & & \\ \uparrow & & \uparrow & & \uparrow & & \\ N' & \longrightarrow & B'_- & \longrightarrow & M' & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

where $M' := H^0p'(B'_-)$ and $N' = (H^0i')^{-1}(B'_-)$.

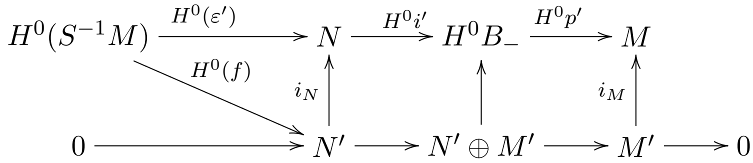
Proof. Let us show that (i) implies (ii). By the assumption, we can find a commutative square



We complete it to a morphism between triangles:



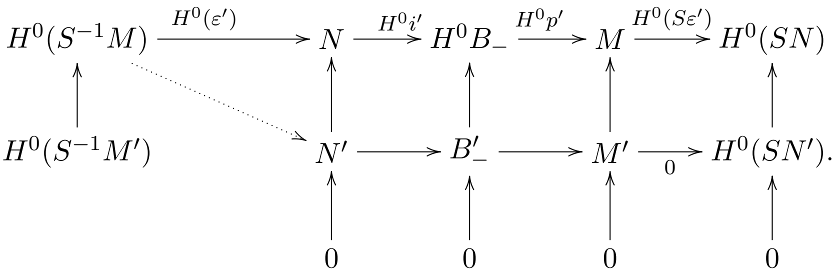
We take the homology:



We take B'_- as the image of $N' \oplus M' \to H^0B_-$. Let us show that $N' \subset N$ is $H^0(i')^{-1}(B'_-)$. Indeed, clearly the image of N' is contained in B'_-. Conversely, if we have $x \in N$ whose image lies in B'_-, then the image is the image of (x', y') in $N' \oplus M'$, and the image of $x \in N$ under $N \to H^0B_- \to M$ vanishes. So, the image of y' in M vanishes. But $M' \to M$ is mono. So y' vanishes and we get x' in N' such that x in N and x' have the same image in H^0B_-. Then $x = x' + (H^0(\varepsilon')(z)$ for some z in $S^{-1}M$. But $H^0\varepsilon' = (H^0 i_N) \circ (H^0 f)$. So $(H^0\varepsilon')(z)$ lies in fact in $N' \subset N$ and x lies in N'.

Let us show that M' is the image of B'_-. Clearly, the image of B'_- is contained in M'. Conversely, if $x' \in M'$, we consider the image y in B'_- of $(0, x') \in N' \oplus M'$. Then clearly, the image of y is x'.

Let us prove that (ii) implies (i). The hypothesis yields the following diagram



As the composition $H^0(S^{-1}M) \to N \to H^0B_-$ vanishes, the image of $H^0(S^{-1}M)$ is contained in N', which is the inverse image of B'_-.

As the composition $H^0(B_-) \to M \to H^0(SN)$ vanishes, M' is contained in the kernel of $M \to H^0(SN)$. We know that M is not injective, so, $S^{-1}M = \tau^{-1}M$ is still a module. Moreover, we have

$$D\,\mathsf{Ext}^1(N, M) = \mathsf{Hom}_H(\tau^{-1}M, N) = \mathsf{Hom}_H(M, \tau N) = k.$$

We obtain the commutative diagrams

$$
\begin{array}{ccc}
S^{-1}M & \xrightarrow{\varepsilon'} & N \\
 & \overset{f}{\searrow} & \uparrow i_N \\
 & & N'
\end{array}
\qquad\qquad
\begin{array}{ccc}
M & \xrightarrow{S\varepsilon'} & SN \\
i_M \uparrow & \overset{0}{\nearrow} & \\
M' & &
\end{array}
$$

The module M' has no injective direct summand, because M is indecomposable and non injective. So, $S^{-1}M'$ is still a module. Consider

$$
\begin{array}{ccc}
S^{-1}M & \xrightarrow{\varepsilon'} & N \\
S^{-1}i_M \uparrow & \overset{f}{\searrow} & \uparrow i_N \\
S^{-1}M & \xrightarrow[0]{} & N'
\end{array}\ .
$$

We have $i_N \circ f \circ S^{-1}i_M = \varepsilon' \circ S^{-1}i_M = 0$. As i_N is injective, this gives $f \circ S^{-1}M = 0$, which implies (i). $\square$

Propositions 4 and 5 imply the first part of proposition 3. The second part is a well-known fact, *cf.* Lemma 3.8 of [7].

3.4. We want to prove the multiplication formula for the second. It reads as follows:

$$
\sum_e \chi_c(\mathsf{Gr}_e\, M)x^{\tau(e)-\underline{\dim}\, M+e}\sum_f \chi_c(\mathsf{Gr}_f\, N)x^{\tau(f)-\underline{\dim}\, N+f} =
$$

$$
\sum_g \chi_c(\mathsf{Gr}_g\, H^0B_+)x^{\tau(g)-\underline{\dim}\, B_+ +g} + x^{\underline{\dim}\, I_{B_-}}\sum_g \chi_c(\mathsf{Gr}_g\, H^0B_-)x^{\tau(g)-\underline{\dim}\, H^0B_- +g}.
$$

By combining Proposition 3 with Proposition 3.6 of [7], we can compare Euler characteristics on both sides of the equality. What we need to prove now is

$$
\tau(e)-\underline{\dim}\, M+e+\tau(f)-\underline{\dim}\, N+f=\tau(g)-\underline{\dim}\, B_+ +g, \tag{3.6}
$$

with $e=\underline{\dim}\, M'$, $f=\underline{\dim}\, N'$, $g=\underline{\dim}\, B'_+$, in the setting of Proposition 4 (i), and then

$$
\tau(e)-\underline{\dim}\, M+e+\tau(f)-\underline{\dim}\, N+f=\underline{\dim}\, I_{B_-}+\tau(g)-\underline{\dim}\, H^0B_- +g, \tag{3.7}
$$

with $e=\underline{\dim}\, M'$, $f=\underline{\dim}\, N'$, $g=\underline{\dim}\, B'_-$, in the setting of Proposition 5 (ii).

The formula 3.6 is clear since $g=e+f$ in this case.

In order to prove the second formula, we need to complete the diagram of proposition 5 by adding kernels and cokernels

$$
\begin{array}{ccccccccc}
 & & N/N' & \longrightarrow & (H^0B_-)/B'_- & \longrightarrow & M/M' & \longrightarrow & C \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
H^0(S^{-1}M) & \longrightarrow & N & \xrightarrow{H^0(i)} & H^0B_- & \xrightarrow{H^0(p)} & M & \longrightarrow & H^0(SN) \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 \longrightarrow K & \longrightarrow & N' & \longrightarrow & B'_- & \longrightarrow & M' & &
\end{array}
$$

With the notation above, the diagram implies the equalities

$$
\tau(e)+\tau(f)=\tau(g)+\tau(\underline{\dim}\, K),\quad (\underline{\dim}\, M-e)+(\underline{\dim}\, N-f)=(\underline{\dim}\, H^0B_- -g)+\underline{\dim}\, C.
$$

So, in order to prove formula 3.7, we remains to show that

$$\tau\underline{\dim}\,K - \underline{\dim}\,C = \underline{\dim}\,I_{B_-}. \tag{3.8}$$

For this, we first note that we have the three triangles

$$N \xrightarrow{H^0(i)} H^0B_- \longrightarrow \mathsf{cok}(H^0 i) \oplus SK \longrightarrow SN$$

$$SP_{B_-} \longrightarrow B_- \longrightarrow H^0B_- \xrightarrow{0} I_{B_-}$$

$$N \to Y \to M \to SN$$

in $\mathcal{C}_Q$. Note that $H^0 i$ is the composition of the morphism $N \to B_-$ with the projection $B_- \to H^0B_-$. If we form the octahedron associated with this composition, the three triangles we have just mentioned appear among its faces, as well as a new triangle, namely

$$SP_{B_-} \longrightarrow M \longrightarrow \mathsf{cok}(H^0 i) \oplus SK \longrightarrow I_{B_-}\ .$$

If we apply H^* to this triangle, we obtain the exact sequence of H-modules

$$0 \longrightarrow M \longrightarrow \mathsf{cok}(H^0 i) \oplus H^0(\tau K) \longrightarrow I_{B_-} \longrightarrow H^0(\tau M)\ .$$

Since M is an indecomposable module, τM is either an indecomposable non injective module or zero. The image of $I_{B_-} \to \tau M = H^0\tau M$ is injective (as a quotient of an injective module). Hence it is zero and we get an exact sequence

$$0 \longrightarrow M \longrightarrow \mathsf{cok}(H^0 i) \oplus H^0(\tau K) \longrightarrow I_{B_-} \longrightarrow 0\ .$$

In the Grothendieck group, this yields

$$0 = \underline{\dim}\,M - \underline{\dim}\ \mathsf{cok}(H^0 i) - \underline{\dim}\,H^0(\tau K) + \underline{\dim}\,I_{B_-} = \underline{\dim}\,C - \underline{\dim}\,H^0(\tau K) + \underline{\dim}\,I_{B_-}.$$

Now, by the third triangle, K is a quotient of $H^0(S^{-1}M) = H^0(\tau^{-1}M)$. As M is a non injective indecomposable module, $H^0(\tau^{-1}M) = \tau^{-1}M$, so τK is a quotient of M, and hence, τK is a module. Thus, we get formula 3.8 as desired. This ends the proof of theorem 2.

4. Application to a class of cluster algebras

4.1. We recall some terminology on cluster algebras. The reader can find more precise and complete information in [12].

Let n be a positive integer. We fix the *ambient field* $\mathcal{F} = \mathbb{Q}(x_1, \ldots, x_n)$, where the x_i's are indeterminates. Let $\mathbf{x}$ be a free generating set of $\mathcal{F}$ over $\mathbb{Q}$ and let $B = (b_{ij})$ be an $n \times n$ antisymmetric matrix with coefficients in $\mathbb{Z}$. Such a pair $(\mathbf{x}, B)$ is called *a seed.*

Let $(\mathbf{u}, B)$ be a seed and let u_j, $1 \le j \le n$, be in $\mathbf{u}$. We define a new seed as follows. Let u'_j be the element of $\mathcal{F}$ defined by the *exchange relation*:

$$u_j u'_j = \prod_{b_{ij}>0} u_i^{b_{ij}} + \prod_{b_{ij}<0} u_i^{-b_{ij}}. \tag{4.1}$$

Set $\mathbf{u}' = \mathbf{u} \cup \{u'_j\} \backslash \{u_j\}$. Let B' be the $n \times n$ matrix given by

$$b'_{ik} = \begin{cases} -b_{ik} & \text{if } i = j \text{ or } k = j \\ b_{ik} + \frac{1}{2}(\,|b_{ij}|\, b_{jk} + b_{ij}\,|b_{jk}|\,) & \text{otherwise.} \end{cases}$$

By a result of Fomin and Zelevinsky, $(\mathbf{u}', B') = \mu_j(\mathbf{u}, B)$ is a seed. It is called the *mutation* of the seed $(\mathbf{u}, B)$ in the direction u_j (or j). We consider all the seeds obtained by iterated mutations. The free generating sets occurring in the seeds are called *clusters*, and the variables they contain are called *cluster variables.* By definition, the *cluster algebra* $\mathcal{A}(\mathbf{x}, B)$

associated to the seed $(\mathbf{x}, B)$ is the $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by the set of cluster variables. The graph whose vertices are the seeds and whose edge are the mutations between two seeds is called the *mutation graph* of the cluster algebra.

The *Laurent phenomenon*, see [11], asserts that the cluster variables are Laurent polynomials with integer coefficients in the x_i, $1 \leq i \leq n$. So, we have $\mathcal{A}(\mathbf{x}, B) \subset \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Note that an antisymmetric matrix B defines a quiver $Q = Q_B$ with vertices corresponding to its rows (or columns) and which has b_{ij} arrows from the vertex i to the vertex j whenever $b_{ij} \geq 0$. The quivers Q thus obtained are precisely the finite quivers without oriented cycles of length 1 or 2. For such quivers Q, we denote by B_Q the corresponding antisymmetric matrix. The cluster algebra associated to the seed $(\mathbf{x}, B)$ will be also denoted by $\mathcal{A}(Q)$. In the sequel, we will be concerned with cluster algebras associated to a quiver Q without oriented cycles.

4.2. We fix a quiver Q without oriented cycles and we set $H = kQ$. We consider the cluster category $\mathcal{C} = \mathcal{C}_H$ associated to the quiver Q, *cf.* [6]. An object T of $\mathcal{C}$ is called *exceptional* if it has no self-extensions, *i.e.* if $\mathsf{Ext}^1(T, T) = 0$. An exceptional object is called *cluster-tilting* or simply *tilting* (although this is an abuse of language) if it has n non isomorphic indecomposable direct summands, where n is the number of vertices of Q. In the sequel, we will often identify a tilting object with the datum of its indecomposable summands. An exceptional object is called *almost tilting* if it has $n-1$ non isomorphic indecomposable direct summands. It was shown in [6] that any almost tilting object $\overline{T}$ can be completed to precisely two non isomorphic tilting objects T and T^*.

For any tilting object T of $\mathcal{C}$, let Q_T be the quiver associated to the algebra $\mathsf{End}_{\mathcal{C}}(T)$. To be explicit, fix an ordering of the indecomposable objects $T_1, \ldots, T_n$ of T and let A be the endomorphism algebra of the sum of the T_i. Let $e_i \in A$ the idempotent corresponding to T_i. Then the vertices of Q_T are $1, \ldots, n$, and the number of arrows from i to j is equal to $\dim e_j((\mathsf{rad}\, A)/(\mathsf{rad}\, A)^2)e_i$. A pair (T, Q_T) is called a *cluster seed*.

For $1 \leq i \leq n$, we define, following [4], the mutation of the cluster seed (T, Q_T) in direction i by

$$\delta_i(T, Q_T) := (T^*, Q_{T^*}),$$

where T and T^* are the two complements of the almost tilting object

$$\overline{T} = T_1 \oplus \ldots \oplus T_{i-1} \oplus T_{i+1} \oplus \ldots \oplus T_n.$$

Note that there exists an indecomposable object T_i^*, unique up to isomorphism, such that

$$T^* = T_1 \oplus \ldots \oplus T_{i-1} \oplus T_i^* \oplus T_{i+1} \oplus \ldots \oplus T_n,$$

which provides a natural ordering of the indecomposable summands of T^*.

The following theorem is the main result of this article. The first assertion is a refinement of Conjecture 9.1 of [6] and the second assertion strengthens the main result of [4].

Theorem 3. *Let Q be a quiver with n vertices and no oriented cycles, and let $H = kQ$ be the hereditary algebra associated to Q. Then*

(i) *The correspondence $M \mapsto X_M$ provides a bijection between the set of indecomposable objects without self-extensions of $\mathcal{C}_H$ and the set of cluster variables of $\mathcal{A}(Q)$.*
(ii) *The correspondence $\{T_1, \ldots, T_n\} \mapsto \{X_{T_1}, \ldots, X_{T_n}\}$ provides a bijection compatible with mutations between the set of tilting objects of $\mathcal{C}_H$ and the set of clusters of $\mathcal{A}(Q)$.*

Proof. By construction, any cluster variable belongs to a cluster. As the map $M \mapsto X_M$ is injective on the set of indecomposable objects without self-extensions by corollary 2, it is enough to prove (ii).

Let us prove (ii). Suppose that $T = T_1 \oplus \ldots \oplus T_n$ is a tilting object of $\mathcal{C}$ and let T^* be its mutation in direction i. Then $\mathsf{Ext}^1(T_i, T_i^*)$ is one-dimensional by [6]. Hence, by theorem 2, we have

$$X_{T_i} X_{T_i^*} = \prod_j X_{T_j}^{a_{ij}} + \prod_j X_{T_j}^{c_{ij}}, \tag{4.2}$$

where a_{ij} and c_{ij} are integers defined by the following non split triangles (unique up to isomorphism)

$$T_i \to \oplus a_{ij} T_j \to T_i^* \to ST_i$$

$$T_i^* \to \oplus c_{ij} T_j \to T_i \to ST_i^*.$$

By a theorem 6.2 b) of [4], the quiver Q_T is determined by these triangles: for any i and j, there are a_{ij} arrows from i to j and c_{ij} arrows from j to i. Moreover, if there exists an arrow from i to j , then there is no arrow from j to i, by Proposition 3.2 of [4].

We now define, as in [4], a correspondence β between tilting seeds and cluster seeds. First note that the shift of H, is a tilting object and that (SH, Q) is a tilting seed. For a given word $i_1 \ldots i_t$, we can define

$$\beta(SH, Q) = (\mathbf{x}, B_Q), \tag{4.3}$$

$$\beta(\delta_{i_t} \ldots \delta_{i_1}(SH, Q)) = \mu_{i_t} \ldots \mu_{i_1}(\mathbf{x}, B_Q). \tag{4.4}$$

Set $(T, Q_T) := \delta_{i_t} \ldots \delta_{i_1}(SH, Q))$. By [4], the quiver obtained from Q by the sequence of *tilting* mutation in direction $i_1, \ldots, i_t$ is equal to the quiver obtained from Q by the sequence of *cluster* mutation in direction $i_1, \ldots, i_t$. Hence, by comparing the cluster exchange relation 4.1 and the tilting exchange relation 4.2, we obtain by induction that

$$\beta(\delta_{i_t} \ldots \delta_{i_1}(SH, Q)) = (\{X_{T_1}, \ldots, X_{T_n}\}, B_{Q_T}).$$

In particular, $\beta(\delta_{i_t} \ldots \delta_{i_1}(SH, Q))$ does not depend on the choice of the word $i_1 \ldots i_t$.

By Proposition 3.5 of [6], the mutation graph on the set of tilting seeds is connected. Hence, equalities 4.3 and 4.4 define a map β from the complete set of tilting seeds to the set of cluster seeds. The surjectivity of β follows from the fact that its image is stable under mutation. The injectivity of β follows from corollary 2. □

4.3. This section is devoted to the proof of some of the conjectures formulated by S. Fomin and A. Zelevinsky in [13]. The first corollary follows from theorem 3 and proposition 1. It corresponds to [13, Conjecture 4.19 (1)] in the acyclic case.

Corollary 3. *Let Q be an finite quiver without oriented cycles. Then the cluster variables of $\mathcal{A}(Q)$ are in $\mathbb{N}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.*

The next corollary is a straightforward consequence of theorem 3. It corresponds to [13, Conjecture 4.14 (2)] in the acyclic case.

Corollary 4. *Let Q be an finite quiver without oriented cycles. Then a cluster seed $(\mathbf{u}, B)$ of $\mathcal{A}(Q)$ only depends on $\mathbf{u}$.*

This corollary is [13, Conjecture 4.14 (3)] in the acyclic case.

Corollary 5. *For any cluster variable x, the set of seeds whose clusters contain x form a connected subgraph of the exchange graph.*

Proof. Indeed, the cluster variable x corresponds to an exceptional indecomposable object T_1 of $\mathcal{C}_Q$. Without restriction of generality, we assume that T_1 is non projective. The seeds containing x are in bijection with the *completions of* T_1, *i.e.* the sets $\{T_2, \dots, T_n\}$ of indecomposables such that the sum of the T_i is cluster tilting. Two seeds are joined by an edge of the exchange graph iff the corresponding sets of exceptional indecomposables are obtained from one each other by a mutation. By [6], this occurs iff they differ by precisely two indecomposables T_i and T_i^* and these satisfy

$$\dim \mathsf{Ext}^1(T_i, T_i^*) = 1.$$

This makes it clear that theorem 4 below yields a bijection compatible with mutations

$$\{T_2, \dots, T_n\} \mapsto \{PT_2, \dots, PT_n\}$$

between the completions of T_1 and the basic tilting sets of $\mathcal{C}_{Q'}$, where Q' is the quiver of the endomorphism ring of a projective generator of the category $\mathcal{H}' \subset \mathsf{mod}\, kQ$ of modules L with

$$\mathsf{Hom}(M, L) = 0 = \mathsf{Ext}^1(M, L).$$

Thus, by theorem 3 (ii), the subgraph of the exchange graph of Q formed by the seeds containing x is isomorphic to the exchange graph of Q', which is connected by definition. □

A consequence of Theorem 3 is also the proof of [13, Conjecture 4.14 (4)] in the general case.

Corollary 6. *The set of seeds whose matrix is acyclic form a connected subgraph (possibly empty) of the exchange graph.*

Proof. A seed with an acyclic matrix corresponds to a cluster tilting object T whose endomorphism algebra $A = \mathsf{End}_{\mathcal{C}_Q}(T)$ has a quiver without oriented cycles. Let us show that A is hereditary. After replacing Q we may assume that T lifts to a tilting module $\tilde{T}$ of projective dimension ≤ 1 in $\mathsf{mod}\, kQ$. Let B be the endomorphism algebra of $\tilde{T}$. We know that there is a triangle equivalence $\mathcal{D}^b(\mathsf{mod}\, B) \to \mathcal{D}^b(\mathsf{mod}\, kQ)$ taking the free module B to T. Using this, it is not hard to show that we have an isomorphism

$$A \xrightarrow{\sim} B \oplus \mathsf{Ext}^2_B(DB, B)\,, \tag{4.5}$$

where $\mathsf{Ext}^2_B(DB, B)$ appears as a square-zero ideal with the natural bimodule structure. It follows that the quiver of B is a subquiver of that of A. In particular, the quiver of B is also directed so that we have a natural order on the set of indecomposable projectives such that $\mathsf{Hom}_B(P, Q) \neq 0$ implies $P \leq Q$. This induces the corresponding order on the simples. Let S_1, S_2 be two simple (right) B-modules such that $\mathsf{Ext}^2_B(S_1, S_2) \neq 0$. By examining the possible terms of a minimal projective resolution of S_1, we see that we have $S_1 > S_2$. Now B is of global dimension ≤ 2 so that Ext^3_B vanishes. This implies that $\mathsf{Ext}^2_B(\nu P_1, P_2) \neq 0$, where νP_1 is the injective hull of S_1 and P_2 the projective cover of S_2. It follows by 4.5 that the quiver of A contains a cycle passing through the vertices corresponding to P_1 and P_2. This contradiction shows that we must have $\mathsf{Ext}^2_B(S_1, S_2) = 0$ for all simple B modules so that B is hereditary and $A \xrightarrow{\sim} B$ as well.

So we obtain a triangle equivalence $\mathcal{C}_A \xrightarrow{\sim} \mathcal{C}_Q$ which takes A to T. Such an equivalence induces an isomorphism

$$\Gamma_A \to \Gamma_B$$

of the Auslander-Reiten quivers of the two cluster categories. We refer to [6] for the description of the Auslander-Reiten quivers. Since A is hereditary, the quiver of its indecomposable projectives forms a slice of the component Γ_A^{pr} of Γ_A containing the projectives (recall that a slice is a full connected subquiver whose vertices are a system of representatives of the τ-orbits in the component). The isomorphism must take Γ_A^{pr} to Γ_B^{pr} since this is the only components isomorphic to the repetition $\mathbb{Z}R$ of a finite quiver R. It is clear that any slice of Γ_B^{pr} can be transformed to the slice of the projectives by finitely many reflections at sources or sinks. □

4.4. **Cluster tilting objects containing a given summand.** Here, we refine a technique pioneered in section 2 of [4]: Let H be a finite-dimensional hereditary algebra and $\mathcal{H}$ the category of finite-dimensional right H-modules. Let $M \in \mathcal{H}$ be a non projective indecomposable with $\mathsf{Ext}^1(M, M) = 0$. Then $\mathsf{End}(M)$ is a (possibly non commutative) field. Let $\mathcal{H}'$ be the full subcategory on the modules L such that

$$\mathsf{Hom}(M, L) = 0 \text{ and } \mathsf{Ext}^1(M, L) = 0.$$

We know from [16] and [15] that $\mathcal{H}'$ is a hereditary abelian category with enough projectives and that a projective generator G of $\mathcal{H}'$ is obtained by choosing an exact sequence

$$0 \to H \to G \to M^r \to 0$$

which induces an isomorphism

$$\mathsf{Hom}(M, M^r) \xrightarrow{\sim} \mathsf{Ext}^1(M, H).$$

Let $\mathcal{C}_{\mathcal{H}}$ and $\mathcal{C}_{\mathcal{H}'}$ be the cluster categories associated with $\mathcal{H}$ and $\mathcal{H}'$. The following theorem is an elaboration on Theorem 2.13 of [4].

Theorem 4. *Let $\mathcal{C}(\mathcal{H}, M)$ be the full additive subcategory of $\mathcal{C}_{\mathcal{H}}$ whose objects are the sums of indecomposables L of $\mathcal{C}_{\mathcal{H}}$ such that $\mathsf{Ext}^1(M, L) = 0$. There is a canonical equivalence of k-linear categories*

$$P : \mathcal{C}(\mathcal{H}, M)/(M) \xrightarrow{\sim} \mathcal{C}_{\mathcal{H}'} ,$$

where (M) denotes the ideal of morphisms factoring through a sum of copies of M. Moreover, we have

$$\mathsf{Ext}^1(L_1, L_2) \cong \mathsf{Ext}^1(PL_1, PL_2)$$

for all $L_1, L_2 \in \mathcal{C}(\mathcal{H}, M)$.

Note that $\mathcal{C}(\mathcal{H}, M)$ is not a triangulated subcategory and not even stable under the shift functor. The theorem merely claims that as a k-linear category, $\mathcal{C}_{\mathcal{H}'}$ is a 'subquotient' of $\mathcal{C}_{\mathcal{H}}$. To construct the equivalence P, we choose a 'fundamental domain' for the action of the autoequivalence $F = \tau^{-1}S$ on $\mathcal{D}$.

Let $\mathcal{P}$ be the full subcategory of the projectives of $\mathcal{H}$ and $\mathcal{H}^+$ the full additive subcategory of $\mathcal{D} = \mathcal{D}^b(\mathcal{H})$ each of whose indecomposables lies in $\mathcal{H}$ or $S\mathcal{P}$. Let $\pi : \mathcal{D} \to \mathcal{C}_{\mathcal{H}}$ be the projection functor. We know from [6] that π induces a bijection from the set of isoclasses of indecomposables of $\mathcal{H}^+$ to that of $\mathcal{C}_{\mathcal{H}}$ and that we have

$$\mathsf{Ext}^1(\pi(L_1), \pi(L_2)) \xrightarrow{\sim} \mathsf{Ext}^1(L_1, L_2) \oplus D\,\mathsf{Ext}^1(L_2, L_1)$$

for any two indecomposables of $\mathcal{H}^+$. Moreover, the category $\mathcal{C}_{\mathcal{H}}$ is equivalent to the category whose objects are those of $\mathcal{H}^+$ and whose morphisms are given by

$$\mathsf{Hom}(L_1, L_2) \oplus \mathsf{Hom}(L_1, FL_2)$$

with the natural composition. Therefore, theorem 4 follows from

Theorem 5. *There is a canonical bijection $L \mapsto L'$ from the set of isoclasses of indecomposables L of $\mathcal{H}^+$ with*

$$L \not\cong M\ ,\ \mathsf{Ext}^1(L, M) = 0 \textit{ and } \mathsf{Ext}^1(M, L) = 0 \qquad (*)$$

to the set of isoclasses of indecomposables of $\mathcal{H}'^+$*. Moreover, for any two objects* L_1, L_2 *of* $\mathcal{H}^+$ *satisfying* $(*)$*, there is a canonical isomorphism*

$$\mathsf{Ext}^1(L_1, L_2) \xrightarrow{\sim} \mathsf{Ext}^1(L_1', L_2')$$

and there are canonical isomorphisms

$$\mathsf{Hom}(L_1, L_2)/(M) \xrightarrow{\sim} \mathsf{Hom}(L_1', L_2') \text{ and } \mathsf{Hom}(L_1, FL_2)/(M) \xrightarrow{\sim} \mathsf{Hom}(L_1', F'L_2')$$

compatible with compositions.

Several of the arguments needed in the proof are contained in section 2 of [4]. For the convenience of the reader, we nevertheless include them below.

Proof. Let $\mathcal{U} \subset \mathcal{D}$ be the full triangulated subcategory generated by M. Since $\mathsf{Ext}^1(M, M)$ vanishes and $\mathsf{Hom}(M, M)$ is a field, its objects are the sums of shifted copies of M. Let $\mathcal{V}$ be the full subcategory of $\mathcal{D}$ whose objects are the $L \in \mathcal{D}$ such that $\mathsf{Hom}(U, L) = 0$ for all $U \in \mathcal{U}$. Then $\mathcal{U}, \mathcal{V}$ form a semiorthogonal decomposition [1] of $\mathcal{D}$, *i.e.* for each object X of $\mathcal{D}$, there is a triangle

$$X_{\mathcal{U}} \to X \to X^{\mathcal{V}} \to SX_{\mathcal{U}}$$

with $X_{\mathcal{U}} \in \mathcal{U}$ and $X^{\mathcal{V}} \in \mathcal{V}$. This triangle is unique up to unique isomorphism; the functor $X \to X_{\mathcal{U}}$ is right adjoint to the inclusion of $\mathcal{U}$ and the functor $X \mapsto X^{\mathcal{V}}$ is left adjoint to the inclusion of $\mathcal{V}$. We have $\mathcal{H}' = \mathcal{H} \cap \mathcal{V}$ and the inclusion $\mathcal{H}' \subset \mathcal{V}$ extends canonically to an equivalence $\mathcal{D}^b(\mathcal{H}') \to \mathcal{V}$. In particular, each object of $\mathcal{V}$ is a direct sum of shifts of objects of $\mathcal{H}'$. We have $\mathcal{U} \cap \mathcal{H} = \mathcal{M}$, the full subcategory on the direct sums of copies of M. The inclusion $\mathcal{H}' \subset \mathcal{H}$ commutes with kernels, cokernels and preserves Ext^1-groups. We will show that $L \mapsto L' = L^{\mathcal{V}}$ yields the bijection announced in the assertion.

Let L be indecomposable in $\mathcal{H}^+$ such that $(*)$ holds. Let us first show that $\mathsf{Hom}(S^iM, L)$ vanishes if $i \neq 0$. Indeed, if L belongs to $\mathcal{H}$, then this group clearly vanishes if $i \neq 0, -1$ and if $i = -1$, it vanishes because $\mathsf{Ext}^1(M, L) = 0$. If $L = SP$ for a projective $P \in \mathcal{H}$, then $\mathsf{Hom}(S^iM, L) = \mathsf{Hom}(S^iM, SP)$ clearly vanishes for $i \neq 0, 1$ and it vanishes for $i = 1$ because M is a non projective indecomposable.

Now let us show that $L^{\mathcal{V}}$ is indecomposable: Consider the canonical triangle

$$L_{\mathcal{U}} \to L \to L^{\mathcal{V}} \to SL_{\mathcal{U}}.$$

Since $\mathsf{Hom}(S^iM, L)$ vanishes for $i \neq 0$, we have $L_{\mathcal{U}} \in \mathcal{M}$. Therefore, in the associated exact sequence

$$\mathsf{Hom}(L, L) \to \mathsf{Hom}(L, L^{\mathcal{V}}) \to \mathsf{Hom}(L, SL_{\mathcal{U}})$$

the third term vanishes. Thus the composition

$$\mathsf{Hom}(L, L) \to \mathsf{Hom}(L, L^{\mathcal{V}}) \xrightarrow{\sim} \mathsf{Hom}(L^{\mathcal{V}}, L^{\mathcal{V}})$$

is surjective and $\mathsf{End}(L^{\mathcal{V}})$ is local as a quotient of the local ring $\mathsf{End}(L)$.

Let us show that L' belongs to $\mathcal{H}'^+$. Since $L_{\mathcal{U}}$ belongs to $\mathcal{M}$, the canonical morphism $f : L_{\mathcal{U}} \to L$ is a morphism of $\mathcal{H}$ and therefore its cone $L^{\mathcal{V}}$ in $\mathcal{D}$ is isomorphic to $\mathsf{cok}(f) \oplus S\,\mathsf{ker}(f)$. Since $L^{\mathcal{V}}$ is indecomposable, one of the two summands vanishes. If $\mathsf{ker}(f)$ vanishes, then $L^{\mathcal{V}}$ belongs to $\mathcal{H}' \subset \mathcal{H}'^+$. If $\mathsf{cok}(f)$ vanishes, we have to show that $\mathsf{ker}(f)$ is projective in $\mathcal{H}'$. Now indeed the short exact sequence

$$0 \to \mathsf{ker}(f) \to L_{\mathcal{U}} \to L \to 0$$

induces a surjection

$$\mathsf{Ext}^1_{\mathcal{H}}(L_{\mathcal{U}}, U) \to \mathsf{Ext}^1_{\mathcal{H}}(\mathsf{ker}(f), U) \to 0$$

for each $U \in \mathcal{H}$. The left hand term vanishes since $L_{\mathcal{U}}$ is a sum of copies of M and the right hand term is isomorphic to $\mathsf{Ext}^1_{\mathcal{H}'}(\mathsf{ker}(f), U)$ because the inclusion $\mathcal{H}' \subset \mathcal{H}$ preserves extension groups. Thus, $\mathsf{ker}(f)$ is projective in $\mathcal{H}'$.

From what we have shown, we conclude that the map $L \to L'$ is well-defined. Let us show that it is injective. For this, we show that the morphism $L^{\mathcal{V}} \to SL_{\mathcal{U}}$ occuring in the canonical triangle is a minimal left $S\mathcal{M}$-approximation. Then L is determined up to isomorphism as the shifted cone over this morphism. To show that $L^{\mathcal{V}} \to SL_{\mathcal{U}}$ is a minimal left approximation, consider the canonical triangle

$$L_{\mathcal{U}} \to L \to L^{\mathcal{V}} \to SL_{\mathcal{U}}$$

and the induced sequence

$$\mathsf{Hom}(L, SM) \leftarrow \mathsf{Hom}(L^{\mathcal{V}}, SM) \leftarrow \mathsf{Hom}(SL_{\mathcal{U}}, SM).$$

Since $\mathsf{Hom}(L, SM)$ vanishes by assumption, we do get a surjection $\mathsf{Hom}(L^{\mathcal{V}}, SM) \leftarrow \mathsf{Hom}(SL_{\mathcal{U}}, SM)$. If it is not minimal, then there is a retraction $r : SL_{\mathcal{U}} \to SM$ whose composition with $L^{\mathcal{V}} \to SL_{\mathcal{U}}$ vanishes. Then r extends to a retraction $\tilde{r} : SL \to SM$. This is impossible since L is indecomposable and not isomorphic to M.

Let us show now that $L \mapsto L'$ is surjective. Let N be indecomposable in $\mathcal{H}'^{+}$. Let $N \to SM'$ be a minimal $S\mathcal{M}$-approximation and form the triangle

$$M' \to L \to N \to SM'.$$

Let us show that L is indecomposable. Since $M' \in \mathcal{U}$, we have $L^{\mathcal{V}} \xrightarrow{\sim} N^{\mathcal{V}}$ and since $N \in \mathcal{V}$, we have $L_{\mathcal{U}} \xrightarrow{\sim} M'$. If L is decomposable, say $L = L_1 \oplus L_2$, then we get

$$L_1^{\mathcal{V}} \oplus L_2^{\mathcal{V}} \xrightarrow{\sim} N$$

and, say, $L_1^{\mathcal{V}}$ vanishes. Then L_1 belongs to $\mathcal{U}$ and thus $M' \xrightarrow{\sim} L_1 \oplus (L_2)_{\mathcal{U}}$. Since $N \to SM'$ is a minimal $S\mathcal{M}$-approximation, we have $L_1 = 0$. So L is indecomposable.

Let us show that L belongs to $\mathcal{H}^{+}$. It is clear from the above triangle that L has homology at most in degrees 0 and 1. Since L is indecomposable, its homology is concentrated in one degree. If the homology is concentrated in degree 0, then L belongs to $\mathcal{H} \subset \mathcal{H}^{+}$. Suppose that L has its homology concentrated in degree 1. Then we must have $N = SQ$ for some indecomposable projective Q of $\mathcal{H}'$. We know that if P_H is a projective generator for $\mathcal{H}$, then $P_H^{\mathcal{V}}$ is is a projective generator for $\mathcal{H}'$. Thus, there is a projective P of $\mathcal{H}$ and a section $s : Q \to P^{\mathcal{U}}$ which identifies Q with a direct factor of $P^{\mathcal{U}}$. Since $N \to SM'$ is an $S\mathcal{M}$-approximation, the composition

$$N \xrightarrow{Ss} SP^{\mathcal{U}} \longrightarrow SP_{\mathcal{M}}$$

extends to SM' so that we obtain a morphism of triangles

$$\begin{array}{ccccccc}
M' & \longrightarrow & L & \longrightarrow & N & \longrightarrow & SM' \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle Ss} & & \downarrow \\
SP_{\mathcal{U}} & \longrightarrow & SP & \longrightarrow & SP^{\mathcal{U}} & \longrightarrow & SP_{\mathcal{M}}.
\end{array}$$

The morphism $L \to SP$ is non zero since its composition with $SP \to SP^{\mathcal{U}}$ equals the composition of the non zero morphism $L \to N$ with the section Ss. So we obtain a non zero morphism $S^{-1}L \to P$ in $\mathcal{H}$. Since $S^{-1}L$ is indecomposable and P is projective, $S^{-1}L$ is projective and we have $L \in \mathcal{H}^{+}$.

Finally, let us show that L satisfies the condition $(*)$. If L was isomorphic to M, we would have $N = L^{\mathcal{V}} = 0$ contrary to our hypothesis that N is indecomposable.

The triangle

$$M' \to L \to N \to SM'$$

yields an exact sequence

$$\mathsf{Hom}(M', SM) \leftarrow \mathsf{Hom}(L, SM) \leftarrow \mathsf{Hom}(N, SM) \leftarrow \mathsf{Hom}(SM', SM).$$

Here the leftmost term vanishes since $\mathsf{Ext}^1(M, M) = 0$ and the rightmost map is surjective since $N \to SM'$ is a left $S\mathcal{M}$-approximation. Thus we have $\mathsf{Ext}^1(L, M) = 0$. The triangle also yields the sequence

$$\mathsf{Hom}(S^{-1}M, M') \to \mathsf{Hom}(S^{-1}M, L) \to \mathsf{Hom}(S^{-1}M, N).$$

The left hand term vanishes since $\mathsf{Ext}^1(L_1, M) = 0$ and the right hand term vanishes since N belongs to $\mathcal{V}$. Thus we have $\mathsf{Ext}^1(M, L) = 0$.

Now let L_1, L_2 be indecomposables of $\mathcal{H}^+$ satisfying condition $(*)$. Consider the triangle

$$(L_2)_{\mathcal{U}} \to L_2 \to L_2^{\mathcal{V}} \to S(L_2)_{\mathcal{U}}.$$

It induces an exact sequence

$$\mathsf{Hom}(S^{-1}L_1, (L_2)_{\mathcal{U}}) \to \mathsf{Hom}(S^{-1}L_1, L_2) \to \mathsf{Hom}(S^{-1}L_1, L_2^{\mathcal{V}}) \to \mathsf{Hom}(S^{-1}L_1, S(L_2)_{\mathcal{U}}).$$

The leftmost term vanishes since $\mathsf{Ext}^1(L_1, M) = 0$ and the rightmost term vanishes since $\mathsf{Ext}^2(L_1, M) = 0$. Thus we have

$$\mathsf{Hom}(L_1, SL_2) \xrightarrow{\sim} \mathsf{Hom}(L_1, SL_2^{\mathcal{V}}) \xrightarrow{\sim} \mathsf{Hom}(L_1^{\mathcal{V}}, SL_2^{\mathcal{V}})\ ,$$

which proves the assertion on the extension groups. The above triangle also induces an exact sequence

$$\mathsf{Hom}(L_1, (L_2)_{\mathcal{U}}) \to \mathsf{Hom}(L_1, L_2) \to \mathsf{Hom}(L_1, L_2^{\mathcal{V}}) \to \mathsf{Hom}(L_1, S(L_2)_{\mathcal{U}}).$$

The last term vanishes since $\mathsf{Ext}^1(M, M) = 0$. Thus the kernel of the map

$$\mathsf{Hom}(L_1, L_2) \to \mathsf{Hom}(L_1, L_2^{\mathcal{V}}) \xrightarrow{\sim} \mathsf{Hom}(L_1^{\mathcal{V}}, L_2^{\mathcal{V}})$$

is formed by the morphisms factoring through sums of M. Put $F = \tau^{-1}S$. Consider the triangle

$$(FL_2)_{\mathcal{U}} \to FL_2 \to (FL_2)^{\mathcal{V}} \to S(FL_2)_{\mathcal{U}}.$$

Note that the functor F does not take $\mathcal{V}$ to itself. We have

$$\mathsf{Hom}(S^iM, FL_2) \xrightarrow{\sim} D\,\mathsf{Hom}(L_2, S^i\tau^2M).$$

This can be non zero only if i equals 0 or 1. Thus $(FL_2)_{\mathcal{U}}$ is a sum of copies of M and SM. Therefore, in the exact sequence

$$\mathsf{Hom}(L_1, (FL_2)_{\mathcal{U}}) \to \mathsf{Hom}(L_1, FL_2) \to \mathsf{Hom}(L_1, (FL_2)^{\mathcal{V}}) \to \mathsf{Hom}(L_1, S(FL_2)_{\mathcal{U}})$$

the last term vanishes and

$$\mathsf{Hom}(L_1, (FL_2)^{\mathcal{V}}) \xrightarrow{\sim} \mathsf{Hom}(L_1^{\mathcal{V}}, (FL_2)^{\mathcal{V}})$$

identifies with the quotient of $\mathsf{Hom}(L_1, FL_2)$ by the subspace of morphisms factoring through a sum of copies of M and SM. Since $\mathsf{Hom}(L_1, SM)$ vanishes, this is also the subspace of morphisms factoring through a sum of copies of M. To finish the proof, it remains to be noticed that under the canonical equivalence $\mathcal{D}^b(\mathcal{H}') \xrightarrow{\sim} \mathcal{V}$, if $L_2^{\mathcal{V}}$ corresponds to L_2', then the object $(FL_2)^{\mathcal{V}}$ does correspond to $\tau_{\mathcal{H}'}^{-1}SL_2'$, by Lemma 2.14 of [4] or section 8.1 of [17].

□

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Institut Camille Jordan, Université Claude Bernard Lyon I, 69622 Villeurbanne Cedex, France

E-mail address: `caldero@igd.univ-lyon1.fr`

UFR de Mathématiques, Institut de Mathématiques, Case 7012, Université Paris 7 - Denis Diderot, 2, place Jussieu, 75251 Paris Cedex 05, France

E-mail address: `keller@math.jussieu.fr`