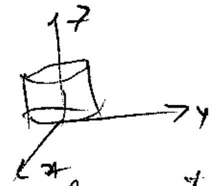
Corrigé Partiel 2 MMI 2022-2023

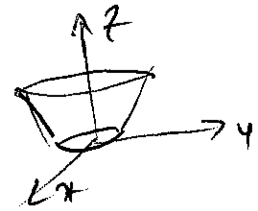
Exo 1

I a)



Cylindre avec base le cercle centré en (0,0) et rayon 1 et axe l'axe Oz

I b)



Cône tronqué d'axe Oz.

II a) $\varphi(\theta, t) = \begin{pmatrix} (at+b)\cos\theta \\ (at+b)\sin\theta \\ t \end{pmatrix}$, $\forall (\theta, t) \in [0, 2\pi) \times [0,1]$.

$\frac{\partial \varphi}{\partial \theta} = \begin{pmatrix} -(at+b)\sin\theta \\ (at+b)\cos\theta \\ 0 \end{pmatrix}$; $\frac{\partial \varphi}{\partial t} = \begin{pmatrix} a\cos\theta \\ a\sin\theta \\ 1 \end{pmatrix}$

$\frac{\partial \varphi}{\partial \theta} \wedge \frac{\partial \varphi}{\partial t} = \begin{pmatrix} (at+b)\cos\theta \\ (at+b)\sin\theta \\ -a(at+b) \end{pmatrix} = (at+b)\begin{pmatrix} \cos\theta \\ \sin\theta \\ -a \end{pmatrix}$

Comme $a \geq 0$, $t \geq 0$ et $b > 0$ alors $at+b > 0$ donc $|at+b| = at+b$. Alors

$\left\| \frac{\partial \varphi}{\partial \theta} \wedge \frac{\partial \varphi}{\partial t} \right\| = |at+b| \cdot \left\| \begin{pmatrix} \cos\theta \\ \sin\theta \\ -a \end{pmatrix} \right\| = (at+b)\sqrt{a^2+1} > 0$

Alors $\frac{\partial \varphi}{\partial \theta} \wedge \frac{\partial \varphi}{\partial t} \neq 0$ $\forall (\theta, t)$ donc φ regulière donc

$\nu = \frac{1}{\left\| \frac{\partial \varphi}{\partial \theta} \wedge \frac{\partial \varphi}{\partial t} \right\|} \left(\frac{\partial \varphi}{\partial \theta} \wedge \frac{\partial \varphi}{\partial t} \right)$

$\nu = \frac{1}{\sqrt{a^2+1}} \begin{pmatrix} \cos\theta \\ \sin\theta \\ -a \end{pmatrix}$

II b) $\text{Aire}(S) = \int_\varphi 1 \, d\sigma = \int_D \left\| \frac{\partial \varphi}{\partial \theta} \wedge \frac{\partial \varphi}{\partial t} \right\| (\theta, t) \, d\theta dt$

$= \sqrt{a^2+1} \int_D (at+b)\, d\theta dt = \sqrt{a^2+1} \int_0^1 \left[\underbrace{\int_0^{2\pi} (at+b) d\theta}_{= 2\pi(at+b)} \right] dt$

$= 2\pi \sqrt{a^2+1} \int_0^1 (at+b) dt$

D'autre part $\int_0^1 (at+b)\,dt = a\left[\frac{t^2}{2}\right]_0^1 + b = \frac{a}{2}+b$

Alors

Aire(S) $= 2\pi\left(\frac{a}{2}+b\right)\sqrt{a^2+1} = \pi(a+2b)\sqrt{a^2+1}$

II c) $\int_\varphi V(x,y,z)\,d\sigma = \int_D V(\varphi(\theta,t))\left\|\frac{\partial\varphi}{\partial\theta}\wedge\frac{\partial\varphi}{\partial t}\right\|(\theta,t)\,d\theta\,dt$

$= \iint_D \left[(at+b)^2\sin^2\theta - t\right](at+b)\sqrt{a^2+1}\,d\theta\,dt$

$= \sqrt{a^2+1}\left[\int_D (at+b)^3\sin^2\theta\,d\theta\,dt - \int_D t(at+b)\,d\theta\,dt\right.$

Mais

$\int_D (at+b)^3\sin^2\theta\,d\theta\,dt = \int_0^1\int_0^{2\pi}(at+b)^3\sin^2\theta\,d\theta\,dt$

$= \int_0^1 (at+b)^3\,dt \int_0^{2\pi}\underbrace{\sin^2\theta}_{=\frac{1-\cos(2\theta)}{2}}\,d\theta = \left[\frac{1}{4a}(at+b)^4\right]_0^1 \left[\pi - \underbrace{\int_0^{2\pi}\cos(2\theta)\,d\theta}_{=\left[\frac{\sin(2\theta)}{2}\right]_0^{2\pi}=0}\right]$

$= \frac{\pi}{4a}\left[(a+b)^4 - b^4\right]$

D'autre part

$\int_D t(at+b)\,d\theta\,dt = \int_0^1\left(\underbrace{\int_0^{2\pi} t(at+b)\,d\theta}_{=2\pi t(at+b)}\right)dt = 2\pi\int_0^1 t(at+b)\,dt$

$= 2\pi\left(\int_0^1 at^2\,dt + \int_0^1 bt\,dt\right) = 2\pi\left(\left[\frac{at^3}{3}\right]_0^1 + \left[b\frac{t^2}{2}\right]_0^1\right) =$

$= 2\pi\left(\frac{a}{3}+\frac{b}{2}\right)$

Alors

$\int_\varphi V(x,y,z)\,d\sigma = \sqrt{a^2+1}\left[\frac{\pi}{4a}\left[(a+b)^4-b^4\right] - 2\pi\left(\frac{a}{3}+\frac{b}{2}\right)\right]$

II d)

$([0,2\pi], f)$ avec $f : [0,2\pi] \to \mathbb{R}^3$

$\theta \mapsto f(\theta) = \begin{pmatrix} \left(\frac{a}{2}+b\right)\cos\theta \\ \left(\frac{a}{2}+b\right)\sin\theta \\ \frac{1}{2}\end{pmatrix}$

C'est le cercle ~~centré en~~ dans le plan $z=\frac{1}{2}$ centré en $(0,0,\frac{1}{2})$ et de rayon $\left(\frac{a}{2}+b\right)$.

$$f'(\theta) = \begin{pmatrix} -\left(\frac{a}{2}+b\right)\sin\theta \\ \left(\frac{a}{2}+b\right)\cos\theta \\ 0 \end{pmatrix} = \left(\frac{a}{2}+b\right)\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}.$$

$$\|f'(\theta)\| = \underbrace{\left|\frac{a}{2}+b\right|}_{>0} \cdot \underbrace{\left\|\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}\right\|}_{=1} = \frac{a}{2}+b$$

Alors $\displaystyle\int_f V(x,y,z)\,d\sigma = \int_0^{2\pi} V(f(\theta))\,\|f'(\theta)\|\,d\theta = \left(\frac{a}{2}+b\right)\left[\left(\frac{a}{2}+b\right)^2 \int_0^{2\pi} \sin^2\theta - \int_0^{2\pi}\frac{1}{2}\,d\theta\right]$

$\displaystyle = \int_0^{2\pi}\left[\left(\frac{a}{2}+b\right)^2 \sin^2\theta - \frac{1}{2}\right]\left(\frac{a}{2}+b\right)d\theta$

$\left(\frac{a}{2}+b\right)\cos\theta$; $\displaystyle\int_0^{2\pi}\left(\frac{a}{2}+b\right)\cos\theta = 0$

Mais $\displaystyle\int_0^{2\pi} \sin^2\theta\, d\theta = \int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\,d\theta = \int_0^{2\pi}\frac{1}{2}\,d\theta - \underbrace{\int_0^{2\pi}\frac{\cos(2\theta)}{2}\,d\theta}_{=0} = \pi$

Alors

$$\int_f V(x,y,z)\,d\sigma = \left(\frac{a}{2}+b\right)\left[\left(\frac{a}{2}+b\right)^2\pi - \pi\right] = \pi\left[\left(\frac{a}{2}+b\right)^3 - \left(\frac{a}{2}+b\right)\right]$$

<u>Problème</u>

a) $h(z) = z^3 = (x+iy)^3 = (x+iy)^2(x+iy) = (x^2-y^2+2xyi)(x+iy) =$

$x(x^2-y^2) + iy(x^2-y^2) + 2x^2yi - 2xy^2$

$= f(x,y) + i\,g(x,y)$ avec $f(x,y) = x(x^2-y^2) - 2xy^2$

$g(x,y) = y(x^2-y^2) + 2x^2y$

b) Soit $z \in \mathbb{C}$ fixé $z = x+iy$ $x,y \in \mathbb{R}$ $k \to 0$.

et ~~ke~~ $k \in \mathbb{C}$, $k = u+iv$ $u,v \in \mathbb{R}$,

On ~~pose h(z+k)~~ étudie $\displaystyle\lim_{k\to 0} \frac{h(z+k)-h(z)}{k}$.

$$\frac{h(z+k)-h(z)}{k} = \frac{h((x+u)+i(y+v)) - h(x+iy)}{u+iv} = \cancel{f(x+u}$$

$$= \frac{f(x+u,\, y+v) - f(x,y)}{u+iv}$$

si on pose $v=0$ et $u \to 0$, $u \neq 0$ alors

$$\frac{h(z+k)-h(z)}{k} = \frac{f(x+u,y) - f(x,y)}{u} \xrightarrow{u\to 0} \frac{\partial f}{\partial x}(x,y)$$

(on pourrait prendre $u = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

donc $v = \frac{1}{n}$

D'autre part si on pose $u = 0$ et $v \to 0$, $v \neq 0$ alors

$$\frac{h(z+k) - h(z)}{k} = \frac{f(x, y+v) - f(x,y)}{iv} \xrightarrow[v \to 0]{} \frac{\partial f}{\partial y}(x,y)(-i)$$

(on pourrait prendre $v = \frac{1}{n}$ donc $k = \frac{i}{n}$

La seule possibilité d'avoir $\frac{\partial f}{\partial x}(x,y) = -\frac{\partial f}{\partial y}(x,y)\, i$

est d'avoir $\frac{\partial f}{\partial x}(x,y) = 0$ et $\frac{\partial f}{\partial y}(x,y) = 0$

c'est à dire $\nabla f(x,y) = 0$ et ceci $\forall (x,y) \in \mathbb{R}^2$

Si on fixe $(x_0, y_0) \in \mathbb{R}^2$ tel que $\nabla f(x_0, y_0) \neq 0$ alors

h n'est pas dérivable en $x_0 + i y_0$

Donc h n'est pas holomorphe sur $\mathbb{C}$.