Lyapunov and Feedback Characterizations of State Constrained Controllability and Stabilization

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Abstract. A standard control system is considered, in conjunction with a state constraint S and a target set Σ. The properties of open loop S-constrained control to Σ and practical closed loop S-constrained control to Σ are shown to be equivalent, and to be characterizable in terms of the existence of certain types of control Lyapunov functions. Feedback S-constrained stabilizability to Σ can be added to the list of equivalences, when a small time controllability property is posited.

1. Introduction

We consider a control system of the form

$$\dot{x}(t) = f(x(t), u(t)) \quad \text{a.e.}, \quad u(t) \in U. \tag{1}$$

The state trajectory $x(\cdot)$ evolves in $\mathbb{R}^n$ and control functions $u(\cdot)$ are Lebesgue measurable functions $u : \mathbb{R} \to U$, where $U \subset \mathbb{R}^m$ is a control constraint set. The standard hypotheses made on the dynamics will be specified at the end of this section.

A general problem of considerable importance is to determine whether open loop asymptotic controllability of the origin implies closed loop stabilization; we refer to this as problem (P). Closed loop stabilization involves the existence of a feedback law $k : \mathbb{R}^n \to U$ such that the origin is stable (in the classical sense) with respect to the ordinary differential equation

$$\dot{x}(t) = f(x(t), k(x(t))). \tag{2}$$

A minimal condition for the existence of classical solutions to (2) is that the feedback law $k(\cdot)$ be continuous on $\mathbb{R}^n \backslash \{0\}$. However, Sontag and Sussmann [25] and Brockett [4] showed that in problem (P), continuity of feedback laws cannot be expected; see also Ryan [23]. Therefore it is necessary to work with an alternative solution concept for (2), rather than the classical one.

Key words: Nonlinear control system, state constraint, feedback, controllability, stabilization, nonsmooth control Lyapunov function, robust.

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Clarke, Ledyaev, Sontag and Subbotin [7] obtained a positive answer to problem (P) in terms of the following "sample-and-hold" solution concept for (2). Let an initial state $\alpha \in \mathbb{R}^n$ be specified. Then given a partition

$$\pi = \{t_0, t_1, t_2, \ldots\} \tag{3}$$

of $[0, \infty)$ (where $t_0 = 0$), the associated *π-trajectory* $x(\cdot)$ on $[0, \infty)$ with $x(0) = x(t_0) = \alpha$ is the curve satisfying interval-by-interval dynamics as follows: Set $x_0 = \alpha$. Then on the interval $[t_0, t_1]$, $x(\cdot)$ is the classical solution of the differential equation

$$\dot{x}(t) = f(x(t), k(x_0)), \quad x(t_0) = x_0, \quad t \in (t_0, t_1). \tag{4}$$

We then set $x_1 := x(t_1)$, and restart the system on the next interval as follows:

$$\dot{x}(t) = f(x(t), k(x_1)), \quad x(t_1) = x_1, \quad t \in (t_1, t_2). \tag{5}$$

The process is continued in this manner through each interval. Then $x(\cdot)$ is the unique solution on $[0, \infty)$ of the differential equation $\dot{x}(t) = f(x(t), u(t))$ satisfying $x(\tau) = \alpha$, with a certain piecewise constant control determined by the feedback $k(x)$. The constructive methods of [7] relied upon the existence of a nonsmooth control Lyapunov functions.

We refer the reader to Clarke, Ledyaev, Rifford and Stern [6] for more details on the history of problem (P), nonsmooth control Lyapunov functions, Filippov solutions, and related topics. Other relevant references are Clarke, Ledyaev and Stern [8], Clarke, Ledyaev, Stern and Wolenski [9], [10], Clarke and Stern [11], Stern [27], Coron [12], Coron and Rosier [13], Rifford [18], [19], [20], [21], [22], Hermes [14], [15], Sontag [26], [24]), Bacciotti [2], Kokotovic and Sussmann [17], Ancona and Bressan [1], Teel and Praly [29], and Kellett and Teel [16]. As is clear from these references, the complexity of the feedback stabilization problem in the case of general nonlinear systems stems from the fact that it is unavoidable to consider discontinuous feedbacks and nonsmooth Lyapunov functions.

In [11], the authors considered a state constrained version of problem (P), with a general (closed) target set. For a given state constraint set $S \subset \mathbb{R}^n$ and target set Σ such that $S \cap \Sigma \neq \emptyset$, the following definitions are relevant:

Definition 1.1.

(A) One has *open loop S-controllability to Σ prior to time $T > 0$* provided that for any initial state $\alpha \in S$, there exists a control function $u(\cdot)$ and a time $t(\alpha) \in [0, T]$ such that

$$x(t) = x(t; 0, \alpha, u(\cdot)) \in S \quad \forall\, t \in [0, t(\alpha)] \tag{6}$$

and

$$x(t(\alpha)) \in \Sigma. \tag{7}$$

(B) One has *practical closed loop S-controllability to Σ prior to time $T > 0$* provided that for each $\gamma > 0$, there exists a feedback law $k_\gamma : \mathbb{R}^n \to U$ along with a scalar $\beta(\gamma) > 0$, such that the following holds: If

$$\operatorname{diam}(\pi) := \max\{t_{i+1} - t_i \ : i = 0, 1, \ldots\} \leq \beta(\gamma),$$

then for every $\alpha \in S$, there exists $t(\alpha) \in [0, T]$ such that the π-trajectory associated with the ordinary differential equation

$$\dot{x}(t) = f(x(t), k(x(t))$$

and initial condition $x(0) = \alpha$, satisfies

$$x(t) \in S \quad \forall\, t \in [0, t(\alpha)] \tag{8}$$

and

$$x(t(\alpha)) \in \Sigma + \gamma\overline{B}. \tag{9}$$

(Here B denotes the open unit ball in $\mathbb{R}^n$, and $\overline{B}$ its closure.) The sense of this definition is that for any given tolerance, there is a feedback whose associated trajectories respect the state constraint (exactly) and attain the target prior to time T (up to the prescribed tolerance), provided that the sampling rate is sufficiently high.

(C) One has *practical closed loop S-stabilizability to Σ prior to time $T > 0$* provided that practical closed loop S-controllability to Σ prior to time $T > 0$ holds, with (9) strengthened to

$$x(t) \in S \cap (\Sigma + \gamma\overline{B}) \quad \forall\, t \geq t(\alpha), \tag{10}$$

where $t(\alpha) \in [0, T]$.

The distinction with (B) is of course that not only is the target attained (to the prescribed tolerance), but the state remains there.

In Theorem 2.3 below, it is shown that under certain geometric assumptions on S, open loop S-control to Σ and practical closed loop S-control to Σ are equivalent and characterizable in terms of the existence of control Lyapunov functions with certain infinitesimal decrease properties, expressed in terms of nonsmooth Hamilton-Jacobi inequalities. Practical closed loop S-stabilizability to Σ is added to the list of equivalences, when a state constrained small time controllability hypothesis is posited.

We shall assume that the following standard hypotheses hold for the dynamics:

(F1) The function $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuous and is locally Lipschitz in the state variable x, uniformly for $u \in U$ where $U \subset \mathbb{R}^m$ is assumed to be compact; that is, for each bounded set $\Gamma \subset \mathbb{R}^n$, there exists $K_\Gamma > 0$ such that

$$||f(x, u) - f(y, u)|| \leq K_\Gamma ||x - y||,$$

whenever (x, u) and (y, u) are in $\Gamma \times U$.

(F2) The function f possesses linear growth; that is, there exist positive numbers c_1, c_2 such that

$$||f(x, u)|| \leq c_1||x|| + c_2 \quad \forall\, (x, u) \in \mathbb{R}^n \times U,$$

where $|| \cdot ||$ denotes the Euclidean norm.

(F3) The velocity set

$$f(x,U) := \{f(x,u) : u \in U\}$$

is convex for every $x \in \mathbb{R}^n$.

Under (F1)-(F2), for every initial phase $(\tau, \alpha) \in \mathbb{R} \times \mathbb{R}^n$ and every control function $u(\cdot)$, there exists a unique trajectory $x(t) = x(t; \tau, \alpha, u(\cdot))$ defined for $t \geq \tau$ and satisfying $x(\tau) = \alpha$. Property (F3) is required so as to have available the familiar sequential compactness property for trajectories of (1) on compact time intervals. (On the other hand, in the absence of (F3), the results of this article could be framed in the context of relaxed controls.)

2. Characterization theorem

We refer the reader to [10] as a basic reference on nonsmooth analysis. For a lower semicontinuous extended real valued function $g : W \to \mathbb{R} \cup \{+\infty\}$ (where $W \subset \mathbb{R}^n$ is open), we denote the proximal and limiting subdifferentials of $g(\cdot)$ at x by $\partial_P g(x)$ and $\partial_L g(x)$, respectively. For a closed set $S \subset \mathbb{R}^n$ and a point $x \in S$, we denote by $T_S^C(x)$, $N_S^L(x)$ and $N_S^C(x)$ the Clarke tangent cone, the limiting normal cone, and the Clarke normal cone to S at x, respectively.

The requisite geometric hypotheses on S will now be stated.

(S1) S is a compact subset of $\mathbb{R}^n$ which is *wedged* at each $x \in S$; that is, $\operatorname{int} T_S^C(x) \neq \emptyset$.

(S2) The following "strict inwardness" condition holds: for all $x \in \operatorname{bdry}(S)$, for all $\zeta \in N_S^C(x)$ different from, 0, we have

$$\min_{v \in f(x,U)} \langle v, \zeta \rangle < 0, \tag{11}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product.

- Hypotheses (F1)-(F3) and (S1)-(S2) will be assumed to hold in all that follows.

We remark that (S1) holds automatically if S is a convex body (a compact convex set with nonempty interior). It also holds if S is defined by (nondegenerate) inequalities $g_i(x) \leq 0$ where the g_i are smooth functions. The sense of (S2) is that at the boundary of S, there is available some "inward pointing" velocity direction.

The target set Σ is assumed to be closed with $S \cap \Sigma \neq \emptyset$. For a closed set Γ and $\alpha > 0$, we adopt the neighborhood notation $\Gamma^\alpha = \Gamma + \alpha B$ and $\overline{\Gamma^\alpha} = \Gamma + \alpha \overline{B}$. The indicator of S is the lower semicontinuous extended real valued function

$$I_S(x) := \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{if } x \notin S \end{cases}$$

Prior to formulating the main result, certain types of CLF families will be specified.

Definition 2.1.

(A) A *strong CLF family* is a family of functions $\{\varphi_\gamma(\cdot)\}$, for which there exist $\varepsilon > 0$, $c > 0$ such that for every $\gamma > 0$, one has

(a) φ_γ is Lipschitz and *locally semiconcave* on S^ε; that is, there exists $a > 0$ such that for any $x \in S^\varepsilon$, the function $x \to \varphi_\gamma(x) - a||x||^2$ is concave on an open neighborhood of x contained in S^ε.

(b) For every $x \in S^\varepsilon \backslash \overline{\Sigma^{2\gamma}}$, one has

$$\min_{v \in T_S^C(x) \cap f(x,U)} \langle v, \zeta \rangle \leq -c \quad \forall \zeta \in \partial_L \varphi_\gamma(x) \quad \text{if} \quad x \in S \tag{12}$$

and

$$\min_{f(x,U)} \langle v, \zeta \rangle \leq -c \quad \forall \zeta \in \partial_L \varphi_\gamma(x) \quad \text{if} \quad x \notin S. \tag{13}$$

(c) $\varphi_\gamma > 0$ on $S \backslash \overline{\Sigma^\gamma}$ and $\varphi_\gamma = 0$ on $S \cap \overline{\Sigma^\gamma}$.

(B) A *weak CLF family* is a family of functions $\{\varphi_\gamma(\cdot)\}$, for which there exist $\varepsilon > 0$, $c > 0$ such that for every $\gamma > 0$, one has

(a) φ_γ is lower semicontinuous on S^ε and bounded above on S.

(b) For every $x \in S^\varepsilon \backslash \overline{\Sigma^{2\gamma}}$, one has

$$\min_{v \in f(x,U)} \langle v, \eta \rangle \leq -c \quad \forall \eta \in \partial_P(\varphi_\gamma + I_S)(x). \tag{14}$$

(c) $\varphi_\gamma > 0$ on $S \backslash \overline{\Sigma^\gamma}$ and $\varphi_\gamma = 0$ on $S \cap \overline{\Sigma^\gamma}$.

The distinction between the CLF families described in (A) and (B) lies in two things: the regularity of the functions, and the form in which the infinitesimal decrease is expressed. In (A), the functions are semiconcave (and hence Lipschitz continuous). This class of functions has emerged as the "nicest" that can be hoped for, since smooth Lyapunov functions do not exist in general. The Hamilton-Jacobi inequality (12),(13) is relatively explicit and lends itself to feedback synthesis. In contrast, the functions in (B) need not be continuous, and (14) is a more abstract expression of the infinitesimal decrease property that lies at the heart of the Lyapunov approach.

When it is desired to confirm controllability, it is better to have a criterion involving weak CLFs as in (B); it is easier to exhibit such a weak family of functions. But when one has controllability and wishes to employ a family of CLFs (to synthesize a stabilizing feedback, for example), then it is better to know that a strong CLF family exists. One of the principal features of the theorem below is its assertion that the existence of a weak family is equivalent to the existence of a strong one.

We also require below an S-constrained version of the small time controllability property. The unconstrained case (that is, $S = \mathbb{R}^n$) is discussed in [3], Cannarsa and Sinestrari [5], and Wolenski and Zhuang [30]. The state constrained version of the property is characterized in Stern [28].

Definition 2.2. We say that (S, Σ)-*small time controllability* holds provided that there exists $\varepsilon > 0$ such that the following S-constrained minimum time function $\tau : S \to \mathbb{R}$ to the target Σ,

$$\tau_\Sigma(\alpha) := \min\{\tilde{t} \geq 0 : x(\tilde{t}) \in \Sigma,\ x(t) \in S\ \forall\, t \in [0, \tilde{t}],\ x(0) = \alpha\}, \tag{15}$$

is continuous on $S \cap \{\Sigma^\varepsilon\}$, where trajectories $x(\cdot)$ are understood to be solutions of the control system (1).

The following theorem summarizes, unifies and extends, in a general state constrained setting, a number of recent results.

Theorem 2.3. *The following four properties are equivalent:*

(i) There exists $T > 0$ such that one has open loop S-controllability to Σ prior to time T.

(ii) There exists a strong CLF family with respect to S and Σ.

(iii) There exists a weak CLF family with respect to S and Σ.

(iv) There exists $T > 0$ such that one has practical closed loop S-controllability to Σ prior to time T.

Furthermore, suppose that (S, Σ)-small time controllability holds. Then (i)-(iv) are equivalent to

(v) There exists $T > 0$ such that one has practical closed loop S-stabilizability to Σ prior to time T.

Several of the implications of the theorem follow immediately from previously cited results. The proof of Theorem 4.1 in [11] shows that $(i) \Longrightarrow (ii)$. The method is rather technical but constructive in nature. The specific family $\{\varphi_\gamma(\cdot)\}$ constructed in that result is obtained via infimal convolution of what might be termed a "proto-CLF" family, which is obtained via a constraint removal technique. As was shown in [11], the local semiconcavity property of the constructed strong CLF family facilitates the construction of a feedback law such that (iv) holds, and allows one to show that (iv) is equivalent to (v) when (Σ, S)-small time controllability is assumed. (See also Remark 4.7 in [11].) The implication $(iv) \Longrightarrow (i)$ is a straightforward consequence of the sequential compactness of trajectories of (1) on compact time intervals.

Completing the proof of the theorem: In view of the preceding discussion, it suffices to verify the two implications

$$(ii) \Longrightarrow (iii). \tag{16}$$

and

$$(iii) \Longrightarrow (i). \tag{17}$$

In order to verify implication (16), first note that for the functions φ_γ occurring in (ii), one has

$$\begin{array}{rcll} \partial_L(\varphi_\gamma + I_S)(x) & \subset & \partial_L\varphi_\gamma(x) + \partial_L I_S(x) & [\text{L} - \text{sum rule of Proposition 1.10.1 in [10]}] \\ & = & \partial_L\varphi_\gamma(x) + N_S^L(x) & [\text{Exercise 1.10.3 in [10]}] \end{array}$$

for any $x \in S^\varepsilon\backslash\overline{\Sigma^{2\gamma}}$. Now, (12) and the fact that $\langle v, \psi\rangle \leq 0$ for every $v \in T_C^S(x)$, $\psi \in N_S^L(x)$, together imply that

$$\min_{v \in T_S^C(x)\cap f(x,U)} \langle v, \zeta\rangle \leq -c \quad \forall\, \zeta \in \partial_L(\varphi_\gamma + I_S)(x). \tag{18}$$

Then (16) follows readily from (18), (13), and the general fact that the P-subdifferential is a subset of the L-subdifferential.

We now turn to the verification of (17). Let us define

$$g(t,x) := \varphi_\gamma(x) + I_S(x) + ct.$$

By the "semismooth" proximal sum rule given by Proposition 1.2.11 in the reference [10], one has the formula

$$\partial_P g(t,x) = \partial_P(\varphi_\gamma + I_S)(x) + c \quad \forall\, (t,x) \in \mathbb{R} \times S^\varepsilon. \tag{19}$$

In view of condition (14), it follows that

$$\min_{(1,v)\in\{1\}\times f(x,U)} \langle (1,v)\ ,\ \psi\rangle \leq 0 \quad \forall\, \psi \in \partial_P g(t,x), \quad \forall\, (t,x) \in \mathbb{R} \times \{S^\varepsilon\backslash\overline{\Sigma^{2\gamma}}\}. \tag{20}$$

Then according to Theorem 4.6.1 of [10], the system $g(\cdot,\cdot)$ is *weakly decreasing* on $\mathbb{R} \times \{S^\varepsilon\backslash\overline{\Sigma^{2\gamma}}\}$, with respect to the dynamics $\{1\}\times f(x,U)$. This implies that for any startpoint $\alpha \in S^\varepsilon\backslash\overline{\Sigma^{2\gamma}}$, there exists a trajectory $x(\cdot)$ of the control system (1) such that $x(0) = \alpha$ and

$$\varphi_\gamma(x(t)) + I_S(x(t)) + ct \leq \varphi_\gamma(\alpha) + I_S(\alpha) \tag{21}$$

on any interval $[0,\hat{T}]$ such that $x(t) \in S^\varepsilon\backslash\overline{\Sigma^{2\gamma}}$. If we take $\alpha \in S$, then since $I_S(\alpha) = 0$ and $\varphi_\gamma(\alpha)$ is finite (by (iii)(a)), we deduce that $x(t) \in S$ and

$$\varphi_\gamma(x(t)) \leq \varphi_\gamma(\alpha) - ct \tag{22}$$

on any such interval. Furthermore, since $\varphi_\gamma(\cdot)$ is bounded above on S, say by $M > 0$, (22) together with (iii)(c) implies that $x(t)$ enters $\overline{\Sigma^{2\gamma}}$ not later than $T := \frac{M}{c}$, which shows that (i) holds and completes the proof of the theorem. $\square$

Remark 2.4. We refer the reader to §5 of [11] in regard to the following points:

- The (discontinuous) feedback laws in parts (iv) and (v) of the theorem possess a robustness property with respect to state measurement errors which are small in an appropriate sense. See also [24] and [6].

- Suppose that the function f in the dynamics (1) is only defined for state values $x \in S$. Such a restricted definition is reasonable in many models, since the dynamics might not make sense or break down when $x \notin S$. In this situation, it is possible to extend f from $S \times U$ to $\mathbb{R}^n \times U$ in a suitable way, so that Definition 1.1 as well as the statement and proof of Theorem 2.3 remain the same.
- While Theorem 2.3 deals with the case of compact S, this can be relaxed to mere closedness, if in the corresponding version of Definition 1.1 the open and closed loop properties to target Σ are provided not for any $\alpha \in S$, but for any α in a specified bounded subset of S.

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