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A Lipschitz regularity theorem

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PROOF

Abstract. This paper gives a direct and elementary proof of the fact that under hypotheses of Tonelli type, solutions to the basic problem in the calculus of variations are Lipschitz when the Lagrangian is autonomous. This fact was first proved by Clarke and Vinter in 1985, using other methods.

1. *The classical problem in one dimension*

We are given an interval $[a, b] \subset \mathbb{R}$ and a differentiable function $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of three variables. L is referred to as the *Lagrangian*, and the generic notation for its three variables is (t, x, v). We assume that L is *coercive* (in v): there is a function $\theta : \mathbb{R} \to [0, +\infty)$ with

$$\lim_{r \to +\infty} \theta(r)/r = +\infty$$

such that

$$L(t, x, v) \geq \theta(|v|) \quad \text{for all } (t, x, v) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n.$$

We also assume that L is *convex* as a function of v; that is, that the function $v \mapsto L(t, x, v)$ is convex for any choice of $(t, x) \in [a, b] \times \mathbb{R}^n$. For a given absolutely continuous function $x : [a, b] \to \mathbb{R}$, we set

$$J(x) := \int_a^b L(t, x(t), \dot{x}(t))\, dt.$$

Observe that $J(x)$ is well defined, possibly as $+\infty$. Given two points A and B in $\mathbb{R}^n$, we consider now the basic problem in the calculus of variations:

$$\text{minimize } J(x) : \quad x \in AC([a, b], \mathbb{R}^n), \quad x(a) = A, \quad x(b) = B, \tag{P}$$

where $AC([a, b], \mathbb{R}^n)$ signifies the class of absolutely continuous functions mapping $[a, b]$ to $\mathbb{R}^n$. Tonelli's celebrated theorem asserts that under the present hypotheses, a solution to (P) exists. It is known [**2**] that solutions may fail to be Lipschitz in general, under the hypotheses posited so far.

1.1. *The regularity theorem.* The problem, or its Lagrangian L, are said to be *autonomous* when L has no dependence on the t variable.

THEOREM 1.1. *Under the above hypotheses, if in addition the Lagrangian is autonomous, then any solution x_* of the problem (P) is Lipschitz on $[a, b]$.*

Proof. (A) Let us consider any measurable function $\alpha : [a, b] \to [1/2, 3/2]$ satisfying the equality $\int_a^b \alpha(t)\, dt = b - a$. For any such α, the relation

$$\tau(t) := a + \int_a^t \alpha(s)\, ds$$

defines a bi-Lipschitz one-to-one mapping from $[a, b]$ to itself; the inverse mapping $t(\tau)$ satisfies

$$\frac{d}{d\tau} t(\tau) = \frac{1}{\alpha(t(\tau))} \quad \text{almost everywhere.}$$

Proceed now to define an arc y by $y(\tau) := x_*(t(\tau))$. Then y is admissible for the problem (P), whence

$$\int_a^b L(y(\tau), y'(\tau))\, d\tau \geq J(x_*).$$

Applying the change of variables $\tau = \tau(t)$ to the integral on the left, and noting that $y'(\tau) = x'(t(\tau))/\alpha(t(\tau))$ almost everywhere, we obtain

$$\int_a^b L(x_*(t), x_*'(t)/\alpha(t))\alpha(t)\, dt \geq J(x_*).$$

Note that equality holds when α is the function $\alpha_* \equiv 1$, so we see that α_* solves a certain minimization problem.

Let us formulate this problem more explicitly by introducing

$$\Phi(t, \alpha) := L(x_*(t), x_*'(t)/\alpha)\alpha.$$

It is straightforward to verify that for each t, the function $\Phi(t, \cdot)$ is convex on the interval $(0, +\infty)$. Consider now an optimization problem (Q) defined on the Banach space $X := L^\infty([a, b], \mathbb{R})$. It consists of minimizing the functional f defined by

$$f(\alpha) := \int_a^b \Phi(t, \alpha(t))\, dt$$

over the subset Ω of X consisting of all measurable functions $\alpha : [a, b] \to [1/2, 3/2]$ satisfying the equality constraint

$$h(\alpha) := \int_a^b \alpha(t)\, dt = b - a.$$

We remark that $f(\alpha)$ is well defined on the convex set Ω, possibly as $+\infty$, and that f is convex (we take $f(\alpha) = +\infty$ when α does not lie in Ω). The argument given above shows that the function $\alpha_* \equiv 1$ solves (Q).

(B) The next step is to write the Lagrange multiplier rule for the problem (Q) and its solution, which we do from first principles. To this end, we recall the finite-dimensional

PROOF

case of the Hahn–Banach separation theorem, which asserts that if a point $p \in \mathbb{R}^k$ does not lie in the interior of a non-empty convex set $S \subset \mathbb{R}^k$, then there exists a non-zero $\zeta \in \mathbb{R}^k$ such that

$$\langle \zeta, s \rangle \geq \langle \zeta, p \rangle \quad \text{for all } s \in S.$$

Applying this fact to the convex subset

$$S := \{(r, h(\alpha)) : \alpha \in \Omega, r \geq f(\alpha)\}$$

of $\mathbb{R}^2$ and the point $p := (f(\alpha_*), b - a)$, we obtain a non-zero vector $\zeta = (\lambda_0, -\lambda)$ in $\mathbb{R}^2$ such that

$$\lambda_0 r - \lambda h(\alpha) \geq \lambda_0 f(\alpha_*) - \lambda(b - a)$$

whenever $\alpha \in \Omega$ and $r \geq f(\alpha)$. It follows that λ_0 is non-negative and, in fact, strictly positive. We may normalize by taking $\lambda_0 = 1$, and in so doing arrive at the following conclusion: for any measurable function α with values in $[1/2, 3/2]$, we have

$$\int_a^b \{L(x_*(t), x_*'(t)/\alpha(t))\alpha(t) - \lambda\alpha(t)\}\, dt \geq \int_a^b \{L(x_*(t), x_*'(t)) - \lambda\}\, dt.$$

As a consequence of standard measurable selection theorems (or see the next section), it follows that, for almost every t, the function

$$\alpha \mapsto L(x_*(t), x_*'(t)/\alpha)\alpha - \lambda\alpha$$

attains a minimum over the interval $[1/2, 3/2]$ at the interior point $\alpha = 1$. Setting the derivative equal to zero gives

$$L(x_*(t), x_*'(t)) - \langle x'(t), L_v(x_*(t), x_*'(t))\rangle = \lambda \quad \text{almost everywhere.} \tag{$*$}$$

(C) The last step in the proof consists of showing that $(*)$ implies that $x_*'(t)$ is essentially bounded. Let t be such that $x_*'(t)$ exists, and such that $(*)$ holds. We have

$$\begin{aligned} &L(x_*(t), x_*'(t)\{1 + |x_*'(t)|\}^{-1}) - L(x_*(t), x_*'(t)) \\ &\qquad \geq [\{1 + |x_*'(t)|\}^{-1} - 1]\langle x_*'(t), L_v(x_*(t), x_*'(t))\rangle \end{aligned}$$

by the subgradient inequality in the v variable, and is equal to

$$[\{1 + |x_*'(t)|\}^{-1} - 1]\{L(x_*(t), x_*'(t)) - \lambda\},$$

in light of $(*)$. Letting M be a uniform bound for all values of L at points of the form $(x_*(t), w)$ with $t \in [a, b]$ and w in the unit ball, this leads to

$$\theta(|x_*'(t)|) \;\leq\; L(x_*(t), x_*'(t)) \;\leq\; M + (M + |\lambda|)|x_*'(t)|.$$

Since $\lim_{r\to+\infty} \theta(r)/r = +\infty$, we deduce that $|x_*'(t)|$ is essentially bounded, as required. □

1.2. *Remarks and extensions.*

1. The presence of an additional state constraint $x(t) \in E$ in the problem (P) has no effect on the proof, nor on the conclusions, which include the *second Erdmann condition*: for some constant λ, the solution x_* satisfies

$$L(x_*(t), x_*'(t)) - \langle x_*'(t), L_v(x_*(t), x_*'(t))\rangle = \lambda \quad \text{almost everywhere.}$$

When the existence of the solution x_* is postulated, then the proof of Lipschitz regularity given above requires only the coercivity of $L(x_*(t), \cdot)$, locally in t. Thus, the theorem extends naturally to the context of arcs on a C^1 manifold, through the use of local charts.

2. It is easy to adapt the proof (as well as this conclusion) to the case in which L is merely continuous (say) rather than differentiable, through the use of the familiar subdifferential of convex analysis (only the v variable is involved).
3. The proof extends readily to the case in which x_* is merely a *strong local minimum* for (P) rather than a global minimum; that is, a minimum relative to the admissible arcs x satisfying $\|x - x_*\|_\infty \le \epsilon$ for some $\epsilon > 0$ (the set Ω must be redefined accordingly).
4. In their original result, Clarke and Vinter [**3**] supposed that L is locally Lipschitz in (x, v), and convex and coercive in v. By invoking necessary conditions in [**1**], it is possible to show that Theorem 1.1 remains valid if L is merely lower semicontinuous, bounded above on bounded sets, and coercive in v (convexity in v is not required). Both of these sources discuss various other criteria for regularity of the solution.
5. The proof of Theorem 1.1 can be modified to allow (restricted) dependence on t. A simple example is the following: in the presence of all of the other hypotheses, the requirement that L be autonomous can be replaced by the growth condition

$$|L_t(t, x, v)| \le c(1 + L(t, x, v)) \quad \text{for all } (t, x, v) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n.$$

1.3. *A measurable selection result.* In order to make this note self-contained, we give here a simple proof of the measurable selection result invoked in the proof of Theorem 1.1.

PROPOSITION 1.2. *Let S be a subset of $\mathbb{R}^n$, and let $\phi : [a, b] \times S \to \mathbb{R}$ be such that $\phi(\cdot, p)$ is (Lebesgue) measurable for each p, and $\phi(t, \cdot)$ is continuous for each t. Let Σ denote the set of all bounded measurable functions $p : [a, b] \to S$ for which the integral over $[a, b]$ of the function $t \mapsto \phi(t, p(t))$ is well-defined, either finitely or as $-\infty$. We assume that Σ is non-empty. Then we have*

$$\int_a^b \inf_{p \in S} \phi(t, p)\, dt = \inf_{p(\cdot) \in \Sigma} \int_a^b \phi(t, p(t))\, dt,$$

where both sides may equal $-\infty$.

Proof. Let p_0 be an element of Σ, and set $\sigma(t) := \inf_{p \in S} \phi(t, p)$. Let $\{d_i\}$ be a countable dense set in S. Then we have $\sigma(t) = \inf_i \phi(t, d_i)$, which shows that σ is measurable. Since, in addition, $\sigma(t)$ is bounded above by the function $\phi(t, p_0(t))$, it follows that the integral of σ that appears on the left in the conclusion of the proposition is well defined, possibly as $-\infty$.

If the integral over $[a, b]$ of the function $\phi(t, p_0(t))$ is $-\infty$, then the required conclusion is evident. We may assume, therefore, that the integral in question is finite. We fix $\epsilon > 0$, and for each $t \in [a, b]$, let $i(t)$ be the first index i for which we have

$$\phi(t, d_i) < \sigma(t) + \epsilon$$

(such an index exists because ϕ is continuous in x and the set $\{d_i\}$ is dense in S). We now show that the function $t \mapsto d_{i(t)}$ is measurable. Let V be any set open in $\mathbb{R}^n$. We need to show that the set

$$\Gamma := \{t \in [a, b] : d_{i(t)} \in V\}$$

is measurable. Let $\{d_{i_j}\}$ be the subsequence of points in $\{d_i\}$ lying in V. We have

$$\Gamma = \bigcup_{j \geq 1} \{t : d_{i(t)} = d_{i_j}\},$$

so it suffices to prove that each term in this countable union is measurable. However, this is a direct consequence of the identity

$$\{t : d_{i(t)} = d_{i_j}\} = \bigcap_{k=1}^{j-1} \{t : \phi(t, d_{i_k}) \geq \sigma(t) + \epsilon\} \cap \{t : \phi(t, d_{i_j}) < \sigma(t) + \epsilon\},$$

since ϕ is measurable in t. A similar argument establishes that $t \to i(t)$ is measurable. For $m > 0$, we now set

$$\Omega_m := \{t \in [a, b] : i(t) < m\},$$

and we define

$$p_m(t) = \begin{cases} d_{i(t)} & \text{if } t \in \Omega_m, \\ p_0(t) & \text{otherwise.} \end{cases}$$

Then it follows that p_m is measurable and belongs to Σ. We deduce

$$\begin{aligned} \inf_{p(\cdot) \in \Sigma} \int_a^b \phi(t, p(t))\, dt &\leq \int_a^b \phi(t, p_m(t))\, dt \\ &\leq \int_{\Omega_m} [\sigma(t) + \epsilon]\, dt + \int_{\Omega_m^c} \phi(t, p_0(t))\, dt. \end{aligned}$$

As $m \to +\infty$, the right-hand side of this last inequality tends to

$$\int_a^b \sigma(t)\, dt + (b - a)\epsilon,$$

and since $\epsilon > 0$ is arbitrary, we obtain

$$\inf_{p(\cdot) \in \Sigma} \int_a^b \phi(t, p(t))\, dt \leq \int_a^b \inf_{p \in S} \phi(t, p)\, dt.$$

However, the opposite inequality is evident, so the result follows. □

The following fact is needed in the proof of Theorem 1.1.

COROLLARY 1.3. *Under the given hypotheses on ϕ, suppose that the function $\bar{p} \in \Sigma$ minimizes (finitely) $\int_a^b \phi(t, p(t))\, dt$ over Σ. Then, for almost every t, the point $\bar{p}(t)$ minimizes $\phi(t, \cdot)$ over S.*

This follows from the fact that (in view of Proposition 1.2), we have

$$\int_a^b \phi(t, \bar{p}(t))\, dt = \int_a^b \inf_{p \in S} \phi(t, p)\, dt,$$

whence $\phi(t, \bar{p}(t)) = \inf_{p \in S} \phi(t, p)$ almost everywhere.

REFERENCES

Q2 **[1]** F. Clarke. Necessary conditions in dynamic optimization. *Mem. Amer. Math. Soc.* **173**(816) (2005).

[2] F. H. Clarke and R. B. Vinter. On the conditions under which the Euler equation or the maximum principle hold. *Appl. Math. Optim.* **12** (1984), 73–79.

[3] F. H. Clarke and R. B. Vinter. Regularity properties of solutions to the basic problem in the calculus of variations. *Trans. Amer. Math. Soc.* **289** (1985), 73–98.

Q3 **[4]** R. B. Vinter. *Optimal Control*. Birkhäuser, Boston, MA, 2000.

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