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UNRAVELING OPEN QUANTUM SYSTEMS: CLASSICAL REDUCTIONS AND CLASSICAL DILATIONS OF QUANTUM MARKOV SEMIGROUPS

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A number of results connecting quantum and classical Markov semigroups, as well as their dilations is reported. The method presented here is based on the analysis of the structure of the semigroup generator. In particular, measure-valued processes appear as a combination of classical reduction and classical dilation of a given quantum Markov semigroup.

Keywords: Quantum Markov semigroups and flows; classical reductions; classical dilations; stochastic Schrödinger equations; unraveling.

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Contents

1. Introduction

Quantum Stochastic Flows and Semigroups extend classical Stochastic Analysis beyond commutativity. This extension is not only an abstract mathematical construction but it is deeply inspired from the theory of Open Quantum Systems in Physics. As a result, subtle questions about time scales and renormalization of space and time, give rise to challenging new branches of Probability Theory.

In particular, a theory of Quantum Markov Semigroups (QMS) has been intensively developed during the last two decades in connection with the Quantum Stochastic Calculus of Hudson and Parthasarathy. One of the major achievements of that calculus has been the theory of quantum stochastic differential equations and related quantum flows. Like in the classical case, the projection (expectation) of a flow defines a semigroup and the generator of the semigroup is characterized by a Chapman–Kolmogorov equation, also referred to as a *master equation* for the *dynamics*. Starting from the master equation, or equivalently, the semigroup, one could say that the flow appears as a *dilation* of the latter. In the quantum case the situation is richer than in the classical framework. Indeed, given a quantum Markov semigroup, one can search for two kinds of dilations. Firstly, a classical one, which consists of solving an infinite dimensional stochastic differential equation driven by classical noises and such that the expectation of the flow of solutions coincides with the given quantum Markov semigroup. Physicists give different names to this dilation procedure, namely, it is said that the flow is a *quantum trajectory* which *unravels* the master equation. Secondly, there is a *quantum dilation* of the semigroup constructed via quantum stochastic differential equations driven by quantum noises as mentioned before.

Moreover, a quantum Markov semigroup is a broad structure which contains numerous classical semigroups which can be obtained by restricting its action to invariant Abelian subalgebras contained in its domain. Taking the verb *to unravel* at its primary meaning, i.e. as a metaphore of undoing woven threads, the *unraveling* of a quantum master equation (or more properly, a quantum Markov semigroup) should consist of *a classical dilation and a classical restriction of the given semigroup.*

Let us start by considering a simple example.

1.1. *Finding a classical Markov chain embedded in an open quantum dynamics*

Consider the space $\mathfrak{h} = \mathbb{C}^2$ and call $\mathfrak{A} = M_2(\mathbb{C})$ the algebra of 2×2 matrices with complex elements. Its canonical basis is denoted

$$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

And introduce a basis in $\mathfrak{A}$ as

$$E_{00} = |e_0\rangle\langle e_0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{01} = |e_0\rangle\langle e_1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

$$E_{10} = |e_1\rangle\langle e_0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{11} = |e_1\rangle\langle e_1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We define an open quantum dynamics on the algebra $\mathfrak{A}$ by means of the generator $\mathcal{L}$ of a semigroup of completely positive maps (see below) $\mathcal{T} = (\mathcal{T}_t)_{t \in \mathbb{R}^+}$.

Take for instance

$$\mathcal{L}(x) = -x + E_{10} x E_{01} + E_{01} x E_{10}, \tag{1.1}$$

for all $x = \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix} \in \mathfrak{A}$. Therefore, $\mathcal{L}$ can be represented by the matrix

$$L = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix},$$

with respect to the basis E_{ij}. And the action of the corresponding semigroup $\mathcal{T}_t(x) = e^{t\mathcal{L}}(x)$ is given by

$$\mathcal{T}_t(x) = \frac{1}{2}\begin{pmatrix} (1 + e^{-2t})x_{00} + (-e^{-2t} + 1)x_{11} & (e^{-t} + e^{t})x_{01} + (-e^{t} + e^{-t})x_{10} \\ (-e^{t} + e^{-t})x_{01} + (e^{-t} + e^{t})x_{10} & (-e^{-2t} + 1)x_{00} + (1 + e^{-2t})x_{11} \end{pmatrix}. \tag{1.2}$$

Now we address a first question:

(Q1) Does there exist an invariant commutative algebra? In that case the restriction of $\mathcal{T}$ to that algebra becomes a classical dynamics. We say that *the semigroup is classically reduced*.

In our simple example consider the observable $K = E_{01} + E_{10}$. A simple computation shows that the spectrum of K is $\Sigma = \{-1, 1\}$. Spectral projections are given by $|b_{-1}\rangle\langle b_{-1}|$ and $|b_1\rangle\langle b_1|$ where

$$b_{-1} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \tag{1.3}$$

provide a basis of eigenvectors.

So, the algebra $W^*(K)$ generated by K contains all elements of the form

$$f(K) = f(-1)|b_{-1}\rangle\langle b_{-1}| + f(1)|b_1\rangle\langle b_1|,$$

for any (bounded) function $f : \Sigma \to \mathbb{R}$.

And K generates an automorphism group $\alpha = (\alpha_t)_{t\in\mathbb{R}^+}$ given by

$$\begin{aligned}\alpha_t(x) &= e^{itK} x e^{-itK} \\ &= e^{t\delta}(x),\end{aligned} \tag{1.4}$$

where $\delta(x) = i[K, x] = i\left(|x^* b_{-1}\rangle\langle b_{-1}| - |b_{-1}\rangle\langle x^* b_{-1}| + |b_1\rangle\langle x^* b_1| - |x^* b_1\rangle\langle b_1|\right)$.

On the other hand, it is easy to verify that $\mathcal{L}(W^*(K)) \subset W^*(K)$. Indeed,

$$\mathcal{L}(f(K)) = Lf(-1)|b_{-1}\rangle\langle b_{-1}| + Lf(1)|b_1\rangle\langle b_1| = Lf(K),$$

where

$$Lf(x) = -\frac{1}{2}(f(x) - f(-x)), \quad x \in \{-1, 1\}. \tag{1.5}$$

This is the generator of a two-state Markov chain on Σ, the transition probability kernel being $Q(x, y) = 1/2$ if $|x - y| = 2$, $Q(x, y) = 0$ otherwise.

To summarize, the commutative invariant algebra here is generated by the self-adjoint operator K which defines an automorphism group α. The fact that $W^*(K)$ is invariant under the action of $\mathcal{T}$ is equivalent to the commutation of α and $\mathcal{T}$ since K is multiplicity-free or nondegenerate so that $W^*(K)$ coincides with the generalized commutant algebra $W^*(K)'$.

1.2. *Introducing some preliminary concepts*

A *Quantum Markov Semigroup* (QMS) arises as the natural noncommutative extension of the well-known concept of Markov semigroup defined on a classical probability space. The motivation for studying a noncommutative theory of Markov semigroups came firstly from Physics. The challenge was to produce a mathematical model to describe the loss-memory evolution of a microscopic system which could be in accordance with the quantum uncertainty principle. Consequently, the roots of the theory go back to the first researches on the so-called *open quantum systems* (for an account see [4]), and have found its main noncommutative tools in much older abstract results like the characterization of *completely positive* maps due to Stinespring (see [28]). Indeed, complete positivity contains a deep probabilistic notion expressed in the language of operator algebras. In many respects it is the core of mathematical properties of (regular versions of) conditional expectations. Thus, complete positivity appears as a keystone in the definition of a QMS. Moreover, in classical Markov Theory, topology plays a fundamental role which goes from the basic setting of the space of states up to continuity properties of the semigroup. In particular, Feller property allows one to obtain stronger results on the qualitative behavior of a Markov semigroup. In the noncommutative framework, Feller property is expressed as a topological and algebraic condition. Namely, a classical semigroup satisfying the Feller property on a locally compact state space leaves invariant the algebra of continuous functions with compact support, which is a particular example of a C^*-algebra. The basic ingredients to start with a non-commutative version of Markov semigroups are then two: firstly, a *-algebra $\mathfrak{A}$, that means an algebra endowed with an involution * which satisfies $(a^*)^* = a$, $(ab)^* = b^*a^*$, for all $a, b \in \mathfrak{A}$, in addition we assume that the algebra contains a unit $\mathbf{1}$; and secondly, we need a semigroup of completely positive maps from $\mathfrak{A}$ to $\mathfrak{A}$ which preserves the unit. We will give a precise meaning to this below. We remind that *positive* elements of the *-algebra are of the form a^*a, $(a \in \mathfrak{A})$. A *state* φ is a linear map $\varphi : \mathfrak{A} \to \mathbb{C}$ such that $\varphi(\mathbf{1}) = 1$, and $\varphi(a^*a) \geq 0$ for all $a \in \mathfrak{A}$.

Definition 1.1. Let $\mathfrak{A}$ be a *-algebra and $\mathcal{P} : \mathfrak{A} \to \mathfrak{A}$ a linear map. $\mathcal{P}$ is *completely positive* if for any finite collection $a_1, \ldots, a_n,\ b_1, \ldots, b_n$ of elements of $\mathfrak{A}$ the element

$$\sum_{i,j} {a_i}^* \mathcal{P}({b_i}^* b_j) a_j$$

is positive.

Throughout this paper, we consider C^*algebras and von Neumann algebras of operators on a complex separable Hilbert space $\mathfrak{h}$.

If $\mathfrak{M}$ is a von Neumann algebra, its predual is denoted by $\mathfrak{M}_*$. The predual contains in particular all the normal states. As a rule, we will only deal with *normal states* φ for which there exists a density matrix ρ, that is, a positive trace-class operator of $\mathfrak{h}$ with unit trace, such that $\varphi(a) = \mathrm{tr}(\rho\, a)$ for all $a \in \mathcal{A}$.

Definition 1.2. A quantum *sub-Markov semigroup, or quantum dynamical semigroup* (QDS) on a *-algebra $\mathfrak{A}$ which has a unit $\mathbf{1}$, is a one-parameter family $\mathcal{T} = (\mathcal{T}_t)_{t \in \mathbb{R}_+}$ of linear maps of $\mathfrak{A}$ into itself satisfying

(M1) $\mathcal{T}_0(x) = x$, for all $x \in \mathfrak{A}$;
(M2) Each $\mathcal{T}_t(\cdot)$ is completely positive;
(M3) $\mathcal{T}_t(\mathcal{T}_s(x)) = \mathcal{T}_{t+s}(x)$, for all $t, s \geq 0$, $x \in \mathfrak{A}$;
(M4) $\mathcal{T}_t(\mathbf{1}) \leq \mathbf{1}$ for all $t \geq 0$.

A quantum dynamical semigroup is called *quantum Markov* (QMS) if $\mathcal{T}_t(\mathbf{1}) = \mathbf{1}$ for all $t \geq 0$.

If $\mathfrak{A}$ is a C^*-algebra, then a quantum dynamical semigroup is *uniformly* (*or norm*) *continuous* if it additionally satisfies

(M5) $\lim_{t \to 0} \sup_{\|x\| \leq 1} \|\mathcal{T}_t(x) - x\| = 0$.

If $\mathfrak{A}$ is a von Neumann algebra, (M5) is usually replaced by the weaker condition:

(M5σ) For each $x \in \mathfrak{A}$, the map $t \mapsto \mathcal{T}_t(x)$ is σ-weak continuous on $\mathfrak{A}$, and $\mathcal{T}_t(\cdot)$ is normal or σ-weak continuous.

The generator $\mathcal{L}$ of the semigroup $\mathcal{T}$ is then defined in the w^* or σ-weak sense. That is, its domain $D(\mathcal{L})$ consists of elements x of the algebra for which the w^*-limit of $t^{-1}(\mathcal{T}_t(x) - x)$ exists as $t \to 0$. This limit is then denoted by $\mathcal{L}(x)$.

The *predual semigroup* $\mathcal{T}_*$ is defined on $\mathfrak{M}_*$ as $\mathcal{T}_{*t}(\varphi)(x) = \varphi(\mathcal{T}_t(x))$ for all $t \geq 0,\ x \in \mathfrak{M},\ \varphi \in \mathfrak{M}_*$. Its generator is denoted $\mathcal{L}_*$.

It is worth noticing that in general a QMS is not a *-homomorphism of algebras. Such a property concerns quantum flows and the concept of *dilation* which we precise below.

Definition 1.3. A *dilation* by a Quantum Markov Flow of a given QMS is a system $(\mathfrak{B}, \mathbb{E}, (\mathfrak{B}_t, \mathbb{E}_t, j_t)_{t\geq 0})$ where

(D1) $\mathfrak{B}$ is a von Neumann algebra with a given state $\mathbb{E}$;
(D2) $(\mathfrak{B}_t)_{t\geq 0}$ is an increasing family of von Neumann subalgebras of $\mathfrak{B}$;
(D3) For any $t \geq 0$, $\mathbb{E}_t$ is a conditional expectation from $\mathfrak{B}$ onto $\mathfrak{B}_t$, such that for all $s, t \geq 0$, $\mathbb{E}_s\mathbb{E}_t = \mathbb{E}_{s\wedge t}$;
(D4) All the maps $j_t : \mathfrak{A} \to \mathfrak{B}_t$ are *-homomorphisms which preserve the identity and satisfy the Markov property:

$$\mathbb{E}_s \circ j_t = j_s \circ \mathcal{T}_{t-s}.$$

$J = (j_t)_{t\geq 0}$ is known as a *Quantum Markov Flow* (QMF) associated to the given QMS.

We call the structure $\mathfrak{B} = (\mathfrak{B}, \mathbb{E}, (\mathfrak{B}_t, \mathbb{E}_t)_{t\geq 0})$ a *quantum stochastic basis.*

The canonical form of a quantum Markov flow is given by $j_t(X) = V_t^* X V_t$, $(t \geq 0)$, where $V_t : \mathfrak{A} \to \mathfrak{B}_t$ is a *cocycle* with respect to a given family of *time-shift* operators $(\theta_t)_{t\geq 0}$. To be more precise

Definition 1.4. Given a quantum stochastic basis $\mathfrak{B}$, a family $(\theta_t)_{t\geq 0}$ of *-homomorphisms of $\mathfrak{B}$ is called a *covariant shift* if

(CS1) $\theta_0(Y) = Y$,
(CS2) $\theta_t(\theta_s(Y)) = \theta_{t+s}(Y)$,
(CS3) $\theta_t^*(\theta_t(Y)) = Y$,
(CS4) $\theta_t(\mathbb{E}_0(\theta_s(Y))) = \mathbb{E}_t(\theta_{t+s}(Y))$,

for any $Y \in \mathfrak{B}$.

A family $(V_t)_{t\geq 0}$ of elements in $\mathfrak{B}$ is a *left cocycle* (resp. *right cocycle*) with respect to a given covariant shift whenever

$$V_{t+s} = V_s\theta_s(V_t),\ (\text{resp. } V_{t+s} = \theta_s(V_t)V_s), \quad s, t \geq 0. \tag{1.6}$$

The connection of quantum Markov semigroups and quantum flows with the commutative case works as follows. Assume E to be a compact Hausdorff space endowed with its Borel σ-algebra $\mathcal{E}$. Take the algebra $\mathfrak{A} = C(E)$ of all complex continuous functions on E endowed with the uniform norm. This is a C^*-algebra which contains a unit $\mathbf{1}$, the constant function 1. Let there be given a Markov semigroup $(T_t)_{t\in\mathbb{R}^+}$ on $(E,\mathcal{E})$ which satisfies the Feller property. Then $\mathfrak{A}$ is invariant under this semigroup so that we may consider that $(T_t)_{t\in\mathbb{R}^+}$ is defined on $\mathfrak{A}$ and satisfies clearly (M1) to (M4), and in addition (M5), that is, it is norm continuous. If a measure μ is given on E endowed with its Borel σ-algebra $\mathcal{E}$, denote $\mathfrak{h} = L^2_{\mathbb{C}}(E,\mathcal{E},\mu)$, then $\mathfrak{A}$ may be interpreted as an algebra of multiplication operators on $\mathfrak{h}$. Thus, $\mathfrak{A}$ is a sub-C^*-algebra of the algebra of all bounded linear operators on $\mathfrak{h}$, $\mathfrak{L}(\mathfrak{h})$.

Within this framework, the quantum flow j is associated to a Markov process $X = (X_t)_{t\in\mathbb{R}^+}$ defined by the semigroup $(T_t)_{t\in\mathbb{R}^+}$, which needs an additional stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}^+},\mathbb{P})$, $X_t : \Omega \to E$, for all $t \geq 0$. That is, in this case $j_t(f) = f(X_t)$, for all Borel bounded complex-valued function defined on E, $t \geq 0$.

As a result, the flow before trivially induces a *random measure process*, that is $j_t(f) = \delta_{X_t}(f)$. The random Dirac measure δ_{X_t} supported by X_t is an extremal point of $M(E,\mathcal{E})$, the set of Radon measures defined on $(E,\mathcal{E})$.

Assume now that E is locally compact and that $\mu \in M(E,\mathcal{E})$. We take again $\mathfrak{h} = L^2_{\mathbb{C}}(E,\mathcal{E},\mu)$. The von Neumann algebra $\mathfrak{M} = \mathfrak{L}(\mathfrak{h})$ of all linear bounded operators on $\mathfrak{h}$ contains an Abelian algebra which is isometrically *-isomorphic to $L^\infty_{\mathbb{C}}(E,\mathcal{E},\mu)$ by the correspondence $f \mapsto M_f$ which associates to each essentially bounded function f the operator of multiplication by f in $\mathfrak{M}$. A Markov semigroup $(T_t)_{t\in\mathbb{R}^+}$ on $(E,\mathcal{E})$ induces now a quantum Markov semigroup $(\mathcal{T}_t)_{t\in\mathbb{R}^+}$ by the natural definition

$$\mathcal{T}_t(M_f) = M_{T_t f}, \quad t \in \mathbb{R}^+.$$

This quantum Markov semigroup satisfies clearly (M1) to (M4), as well as (M5σ), but (M5) is not automatically satisfied.

Numerous examples of purely noncommutative QMS of physical relevance have been considered so far (see e.g. [4, 16, 5]). One way to understand the relationship between classical and quantum Markov semigroups is that the latter contains various classical expressions. That is, if a QMS (respectively QMF) is restricted to an invariant Abelian subalgebra, it becomes isomorphic to a classical Markov semigroup (resp. classical Markov flow associated to a classical process).

1.3. *Is classical reduction inherent to open quantum systems?*

As it has been noticed by Accardi *et al.* [3] as well as Kossakowski in several papers, the Markov approach to open quantum systems determines the property of commutation of the dissipative part with the system Hamiltonian. As explained in [27], this fact is connected with the phenomenon of decoherence. Indeed, one can detect a good self-adjoint operator reducing the open system dynamics as soon as some particular kind of limit is used to construct the model. In [3] the so-called *stochastic limit* is used by the authors to prove their claim under two important hypotheses: the *Rotating Wave Approximation* and the *Dipole Form* of the interaction Hamiltonian. In [6], Attal and Joye explore the connections between *Repeated Interactions* and the *Weak Coupling* and *Continuous Limits*. They do not refer specifically to classical reduction of the open system dynamics, however their approach provides a nice asymptotic formula from which the reader can easily understand why this phenomenon occurs.

Let us consider Attal–Joye's setting which uses a perturbation method in the following manner. Let there be given a complex separable Hilbert space $\mathfrak{h}_S$ which is used to describe the *system* dynamics. The *reservoir* is assumed to be represented by the Hilbert space $\mathfrak{h}_R = \bigotimes_{j\geq 1} \mathbb{C}_j^{n+1}$, where $\mathbb{C}_j^{n+1}$ is simply a copy of $\mathbb{C}^{n+1}$. On each copy of $\mathbb{C}^{n+1}$ consider the canonical basis $(e_0, e_1, \ldots, e_n)$, and define the vacuum vector on $\mathfrak{h}_R$ as $|0\rangle = e_0 \otimes \cdots \otimes e_0 \otimes \cdots$. Identify $\mathfrak{h}_S$ with $\mathfrak{h}_S \otimes \mathbb{C}|0\rangle$ and define the projection $P = \mathbf{1} \otimes |0\rangle\langle 0| : \mathfrak{h}_S \otimes \mathfrak{h}_R \to \mathfrak{h}_S$. In which follows we assume that all the operators are bounded, to avoid technicalities at this stage. The method followed by Attal and Joye consists of assuming a repeated interaction dynamics described by evolution operators $U_k = U_\tau(\lambda) := e^{-i\tau H(\lambda)}$ acting on small time intervals of length τ, so that if one takes $t \in \lambda^2\mathbb{N}$, $(\lambda > 0)$, the reduced dynamics on $\mathfrak{h}_S$ is given by $PU(t/\lambda^2, 0)P = PU_{t/\lambda^2}U_{t/\lambda^2-1}\cdots U_1 P$. After that, the weak coupling limit is obtained letting $\lambda \to 0$ with $t/\lambda^2 \in \mathbb{N}$.

Let $H(\lambda) = H(0) + \lambda W$, where $H(0) = H_S \otimes \mathbf{1} + \mathbf{1} \otimes H_R$ is a self-adjoint operator acting on $\mathfrak{h}_S \otimes \mathfrak{h}_R$ which generates the *free dynamics*, W is also self-adjoint and describes the *interaction* on a small time interval of length $\tau > 0$. In [6] the following hypothesis is explicitly assumed to perform the perturbation estimates:

(H1) Assume in general $H(\lambda)$ to be a self-adjoint operator of the form $H(\lambda) = H(0) + \lambda W$, where $H(0)$ and W are bounded and $0 \leq \lambda \leq \lambda_0$ for some $\lambda_0 > 0$. Moreover, it is assumed that $[H(0), P] = 0$, and that $W = PWQ + QWP$, where $Q = \mathbf{1} - P$.

Call $U_\tau(\lambda) = \exp(-i\tau H(\lambda))$. Under the above hypothesis, Corollary 3.1 of [6] states that for $\lambda, \tau > 0$ small it holds

$$U_\tau(\lambda) = U_\tau(0) + \lambda F(\tau, W) + \lambda^2 G(\tau, W) + O(\lambda^3\tau^3), \tag{1.7}$$

where

$$F(\tau, W) = \sum_{n\geq 1} \frac{(-i\tau)^n}{n!} \sum_{m=0}^{n-1} H(0)^m W H(0)^{n-1-m}, \tag{1.8}$$

$$G(\tau, W) = \sum_{n\geq 2} \frac{(-i\tau)^n}{n!} \sum_{k,m=0}^{n-2} H(0)^k W H(0)^m W H(0)^{n-2-k-m}. \tag{1.9}$$

Consider the flow $j_t^0 : \mathfrak{L}(\mathfrak{h}_S \otimes \mathfrak{h}_R) \to \mathfrak{L}(\mathfrak{h}_S \otimes \mathfrak{h}_R)$, such that $x \mapsto e^{itH(0)} x e^{-itH(0)}$, $t \geq 0$. Call $\mathcal{F}(j^0)$ the set of fixed points of j^0, that is $\mathcal{F}(j^0) = \left\{x \in \mathfrak{L}(\mathfrak{h}_S \otimes \mathfrak{h}_R) : j_t^0(x) = x, \text{ for all } t \geq 0\right\}$. Here we assume that the operator $H(0)$ is nondegenerate, i.e. it has a pure point spectrum with eigenvalues of multiplicity one. As a result, $\mathcal{F}(j^0)$ coincides with the von Neumann algebra $W^*(H(0))$

generated by $H(0)$ and for any $x \in \mathfrak{L}(\mathfrak{h}_S \otimes \mathfrak{h}_R)$ its conditional expectation $\mathbb{E}^{\mathcal{F}(j^0)}(x)$ is computed as

$$\mathbb{E}^{\mathcal{F}(j^0)}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T j_t^0(x) dt, \tag{1.10}$$

in norm.

From (1.7), Proposition 3.1 in [6] yields

$$U_\tau(0)^{-\frac{t}{\lambda^2}} U_\tau(\lambda)^{\frac{t}{\lambda^2}} = e^{t \exp(iH(0)) F(\tau, \mathbb{E}^{\mathcal{F}(j^0)}(W))} + O(\lambda^2), \tag{1.11}$$

in norm, as $\lambda \to 0$, with $t/\lambda^2 \in \mathbb{N}$.

Noticing that $U_\tau(0), F(\tau, \mathbb{E}^{\mathcal{F}(j^0)}(W)) \in \mathcal{F}(j^0)$, one obtains

Proposition 1.1. *If $\lambda \to 0$, with $t/\lambda^2 \in \mathbb{N}$, then*

$$\|\mathbb{E}^{\mathcal{F}(j^0)}(U_\tau(\lambda)^{\frac{t}{\lambda^2}}) - U_\tau(\lambda)^{\frac{t}{\lambda^2}}\| \to 0. \tag{1.12}$$

As a result, if $(V_t)_{t \in \mathbb{R}^+}$ denotes the family of unitary operators obtained by means of the weak coupling limit before, then for any $t \geq 0$, $V_t \in \mathcal{F}(j^0)$ and the flow j, where $j_t(x) := V_t^* x V_t$, $(x \in \mathfrak{L}(\mathfrak{h}_S \otimes \mathfrak{h}_R)$, $t \geq 0)$, commutes with j^0 which is generated by the free dynamics.

So that, to summarize, the existence of a self-adjoint operator whose generated Abelian algebra remains invariant under the action of a flow or a given semigroup seems to be inherent to certain limiting procedures. In [26] a weaker limit leading to a classical reduction was studied.

1.4. *A class of adiabatic limit*

Assume that there exists a faithful normal stationary state for an automorphism group $\alpha = (\alpha_t)_{t \in \mathbb{R}^+}$ defined on $\mathfrak{L}(\mathfrak{h})$ where $\mathfrak{h}$ is an arbitrary complex separable Hilbert space. Then the algebra of its fixed points, $\mathcal{F}(\alpha)$, is a von Neumann algebra, and there exists a conditional expectation $\mathbb{E}^{\mathcal{F}(\alpha)}(\cdot)$. In this case, we obtain the following property.

Proposition 1.2. *Under the assumption before, given any bounded completely positive linear map $\mathcal{T}$ on $\mathfrak{L}(\mathfrak{h})$ such that $\|\mathcal{T}(x)\| \leq \|x\|$, the family*

$$\mathcal{C}(x) = \left(\frac{1}{T} \int_0^T \alpha_t \circ \mathcal{T} \circ \alpha_{-t}(x) dt; \ T \geq 0 \right),$$

admits a w^*-limit on $\mathfrak{L}(\mathfrak{h})$, denoted $\mathcal{T}^\alpha(x)$, for all $x \in \mathfrak{L}(\mathfrak{h})$. The map $x \mapsto \mathcal{T}^\alpha(x)$ is completely positive and bounded.

Proof. We first use Banach–Alaglou Theorem. Given $x \in \mathfrak{L}(\mathfrak{h})$,

$$\left\| \frac{1}{T} \int_0^T \alpha_t \circ \mathcal{T} \circ \alpha_{-t}(x) dt \right\| \leq \|x\| .$$

So that, given any state ω,

$$\omega \left(\frac{1}{T} \int_{-T}^T \alpha_t \circ \mathcal{T} \circ \alpha_{-t}(x) dt \right) \leq \|x\| ,$$

for all $T \geq 0$, so that $\mathcal{C}(x)$ is w^*-compact. Take two different w^*-limit points $\mathcal{T}_1^\alpha(x)$ and $\mathcal{T}_2^\alpha(x)$ of $\mathcal{C}(x)$. Since complete positivity is preserved by composition of maps and by w^*-limits, $x \mapsto \mathcal{T}_j^\alpha(x)$ is completely positive for $j = 1, 2$.

If ω_0 is a faithful normal invariant state, it follows easily that

$$\omega_0(\mathcal{T}_1^\alpha(x)) = \omega_0(\mathcal{T} \circ \mathbb{E}^{\mathcal{F}(\alpha)}(x)) = \omega_0(\mathcal{T}_2^\alpha(x)).$$

Since ω_0 is faithful, the above equation, together with the positivity of maps imply that $\mathcal{T}_1^\alpha(x) = \mathcal{T}_2^\alpha(x)$. □

This proposition allows one to introduce a procedure for building up *adiabatic approximations* of a given semigroup.

Definition 1.5. Given any quantum Markov semigroup $\mathcal{T} = (\mathcal{T}_t)_{t \in \mathbb{R}^+}$ on $\mathfrak{L}(\mathfrak{h})$, and any automorphism group α, we define the α-*adiabatic limit* of $\mathcal{T}$ as the semigroup $(\mathcal{T}_t^\alpha)_{t \in \mathbb{R}^+}$ where each $\mathcal{T}_t^\alpha$ is obtained through Proposition 1.2.

Proposition 1.3. *If there exists a faithful normal invariant state for the automorphism group α, the α-adiabatic limit of any QMS commutes with α.*

Proof. Let us keep the notations of Proposition 1.2. Then, for any $x \in \mathcal{F}(\alpha)$, $\alpha_t \circ \mathcal{T}_s(\alpha_{-t}(x)) = \alpha_t \circ \mathcal{T}_s(x)$, for all $s, t \geq 0$. Therefore, $\mathcal{T}_s^\alpha(x) = \mathbb{E}^{\mathcal{F}(\alpha)}(\mathcal{T}_s(x)) \in \mathcal{F}(\alpha)$, for all $s \geq 0$. □

Adiabatic limits are used in numerous phenomenological descriptions of open quantum systems. As such, in view of the previous proposition, those approaches yield to quantum Markov semigroups endowed with a property of classical reduction by the algebra of fixed points of the main dynamics, the automorphism group generated by its bare Hamiltonian.

2. Classical Reductions of Quantum Markov Flows and Semigroups

Within this section we start by obtaining some elementary consequences of von Neumann's Spectral Theorem.

2.1. *Some general results*

Consider a C^*-algebra $\mathfrak{B}$ which contains a unit $\mathbf{1}$. States are elements of the dual $\mathfrak{B}^*$ of $\mathfrak{B}$. A state φ is *pure* if the only positive linear functionals majorized by φ are of the form $\lambda\varphi$ with $0 \leq \lambda \leq 1$. For an Abelian C^*-algebra, the set of pure states coincides with that of all *characters*, also called spectrum of the algebra (see [8], Prop. 2.3.27, p. 62). A character φ of an Abelian C^*-algebra $\mathfrak{A}$ is a state which satisfies $\varphi(ab) = \varphi(a)\varphi(b)$, for all $a, b \in \mathfrak{A}$; the set of all these elements is usually denoted $\sigma(\mathfrak{A})$ (for *spectrum*) or $P_{\mathfrak{A}}$ (for *pure states*).

Definition 2.1. We say that a completely positive map $\mathcal{P}$ defined on $\mathfrak{B}$ is *reduced* by an Abelian *-subalgebra $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ is invariant under the action of $\mathcal{P}$.

Analogously, a quantum dynamical (resp. Markov) semigroup $(\mathcal{T}_t)_{t\in\mathbb{R}^+}$ defined on $\mathfrak{B}$ is *reduced* by $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ is invariant under the action of $\mathcal{T}_t$ for all $t \geq 0$.

We start by an elementary lemma on the classical reduction of a completely positive map.

Lemma 2.1. *Let $\mathcal{P}$ be a completely positive map defined on $\mathfrak{B}$ such that $\mathcal{P}(\mathbf{1}) = 1$. If $\mathcal{P}$ is reduced by an Abelian sub C^*-algebra $\mathfrak{A}$ of the algebra $\mathfrak{B}$, then its restriction $\mathbf{P}$ to $\mathfrak{A}$ defines a norm-continuous kernel.*

Proof. Since $\mathfrak{A}$ is an Abelian C^*-algebra which contains the unit, its set of characters $\sigma(\mathfrak{A})$ is a w^*-compact Hausdorff space (see [8], Thm. 2.1.11A, p. 62).

$\mathfrak{A}$ is isomorphic to the algebra of continuous functions $C(\sigma(\mathfrak{A}))$ via the Gelfand transform: $a \mapsto \hat{a}$, where $\hat{a}(\gamma) = \gamma(a)$, for all $a \in \mathfrak{A}$, $\gamma \in \sigma(\mathfrak{A})$.

Define $\mathbf{P}\hat{a} = \widehat{\mathcal{P}(a)}$. Then $\mathbf{P} : C(\sigma(\mathfrak{A})) \to C(\sigma(\mathfrak{A}))$ is linear, positive and continuous in norm since for all $a \in \mathfrak{A}$:

$$\|\mathbf{P}\hat{a}\| = \|\widehat{\mathcal{P}(a)}\| \leq \|a\| = \|\hat{a}\| \, .$$

Therefore, by the disintegration of measures property (see [14]), there exists a kernel $P : \sigma(\mathfrak{A}) \times \mathcal{B}(\sigma(\mathfrak{A})) \to [0, 1]$ such that $P(\psi, \cdot)$ is a (Radon) probability measure for all $\psi \in \sigma(\mathfrak{A})$, $P(\cdot, A)$ is a continuous function, for all $A \in \mathcal{B}(\sigma(\mathfrak{A}))$ and

$$\mathbf{P}\hat{a}(\psi) = \int_{\sigma(\mathfrak{A})} P(\psi, d\varphi)\varphi(a). \qquad \square$$

Corollary 2.1. *Under the hypotheses of the previous lemma, suppose in addition that $\mathcal{P}$ is a *-homomorphism classically reduced by the Abelian C^*-subalgebra $\mathfrak{A}$. Then the restriction $\boldsymbol{P}$ of $\mathcal{P}$ to $\mathfrak{A}$ is a Dirac kernel, that is, $\boldsymbol{P}\hat{a}(\psi) = \delta_{X(\psi)}(\hat{a})$, for all $a \in \mathfrak{A}$, where X is a Borel-measurable function on $\sigma(\mathfrak{A})$.*

Proof. With the notations of the previous lemma it holds that for all $\psi \in \sigma(\mathfrak{A})$, $\boldsymbol{P}(\hat{a}\hat{b})(\psi) = \widehat{\mathcal{P}(ab)}(\psi) = \widehat{\mathcal{P}(a)}(\psi)\widehat{\mathcal{P}(b)}(\psi) = \boldsymbol{P}\hat{a}(\psi)\boldsymbol{P}\hat{b}(\psi)$, therefore, $\hat{a} \mapsto \boldsymbol{P}\hat{a}(\psi)$ is a character of the Abelian algebra $C(\sigma(\mathfrak{A}))$ and it is an extremal point of the cone of probability measures defined on $\sigma(\mathfrak{A})$, so that there exists a Borel-measurable map $X : \sigma(\mathfrak{A}) \to \sigma(\mathfrak{A})$ such that

$$\boldsymbol{P}\hat{a}(\psi) = \delta_{X(\psi)}(\hat{a}) = \hat{a}(X(\psi)), \tag{2.1}$$

for all $a \in \mathfrak{A}$. □

Proposition 2.1. *If a norm continuous quantum Markov semigroup $\mathcal{T}$ defined on $\mathfrak{B}$ is reduced by an Abelian C^*-subalgebra $\mathfrak{A}$ of $\mathfrak{B}$, there exists a classical Feller semigroup which is isomorphic to the restriction of $\mathcal{T}$ to $\mathfrak{A}$, called the* reduced *semigroup.*

Proof. As in the previous lemma, a semigroup $(\mathbf{T}_t)_{t\in\mathbb{R}_+}$ is defined on $C(\sigma(\mathfrak{A}))$ through the relation:

$$\mathbf{T}_t\hat{a} = \widehat{\mathcal{T}_t(a)}, \tag{2.2}$$

for all $a \in \mathfrak{A}$.

The above semigroup preserves the identity. Moreover, $\|\mathcal{T}_t(x)\| \leq \|x\|$ $(x \in \mathfrak{B})$ implies that $\mathbf{T}_t$ is a contraction. Therefore, $(\mathbf{T}_t)_{t\in\mathbb{R}_+}$ is a Markov semigroup on $C(\sigma(\mathfrak{A}))$ which is strongly continuous, hence it is a Feller semigroup. □

We denote $\mathcal{L}(\cdot)$ the generator of the QMS. The semigroup is norm-continuous if and only if $\mathcal{L}(\cdot)$ is a bounded operator on $\mathfrak{B}$.

Theorem 2.1. *Assume that K is a normal operator in the C^*-algebra $\mathfrak{B}$ and call $C^*(K)$ the Abelian C^*-algebra generated by K. Then a norm-continuous quantum Markov semigroup $\mathcal{T}$ defined on $\mathfrak{B}$ is reduced by $C^*(K)$ if and only if $\mathcal{L}(K^n) \in C^*(K)$ for all $n \in \mathbb{N}$.*

Proof. Let $\mathcal{K}$ denote the *-subalgebra generated by the commuting variables $\mathbf{1}$, K, K^*. $\mathcal{K}$ is norm dense in $C^*(K)$. On the other hand, $\mathcal{T}$ is norm-continuous, so that $C^*(K)$ is invariant under $\mathcal{T}$ if and only if $\mathcal{L}(\mathcal{K}) \subseteq C^*(K)$. Since $\mathcal{L}(\mathbf{1}) = 0$ and $\mathcal{L}(K^{*n}) = \mathcal{L}(K^n)^*$, it follows easily that $\mathcal{L}(\mathcal{K}) \subseteq C^*(K)$ if and only if $\mathcal{L}(K^n) \in C^*(K)$, for all $n \in \mathbb{N}$. □

In [12], Christensen and Evans provided an expression for the infinitesimal generator $\mathcal{L}$ of a norm-continuous quantum dynamical semigroup defined on a C^*-algebra, extending previous results obtained by Lindblad [21], Gorini, Kossakowski, Sudarshan [29]. We recall their result here below.

Suppose that $\mathcal{T}$ is a norm-continuous quantum dynamical semigroup on $\mathfrak{B}$ and denote $\bar{\mathfrak{B}}$ the σ-weak closure of the C^*-algebra $\mathfrak{B}$. Then there exist a completely positive map $\Psi : \mathfrak{B} \to \bar{\mathfrak{B}}$ and an operator $G \in \bar{\mathfrak{B}}$ such that the generator $\mathcal{L}(\cdot)$ of the semigroup is given by

$$\mathcal{L}(x) = G^*x + \Psi(x) + xG, \quad x \in \mathfrak{B}. \tag{2.3}$$

The map Ψ can be represented by means of Stinespring Theorem [28] as follows. There exist a representation $(\mathfrak{k}, \pi)$ of the algebra $\mathfrak{B}$ and a bounded operator V from

$\mathfrak{h}$ to the Hilbert space $\mathfrak{k}$ such that

$$\Psi(x) = V^*\pi(x)V, \quad x \in \mathfrak{B}. \tag{2.4}$$

Notice that $\Psi(\mathbf{1}) = V^*V = -(G^* + G) = -2\Re(G) \in \bar{\mathfrak{B}}$, where $\Re(G)$ denotes the real part of G, since $\mathcal{L}(\mathbf{1}) = 0$. So that, if we call H the self-adjoint operator $2^{-1}i(G - G^*) = -\Im(G) \in \bar{\mathfrak{B}}$, where $\Im(G)$ stands for the imaginary part of G, then $\mathcal{L}(\cdot)$ can also be written as

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2}(V^*Vx - 2V^*\pi(x)V - xV^*V), \quad x \in \mathfrak{B}. \tag{2.5}$$

The representation of $\mathcal{L}(x)$ in terms of G and Ψ is not unique. However, given a generator through an expression like (2.3), we look for sufficient conditions to have the semigroup reduced by the Abelian C^*-algebra $C^*(K)$ generated by a normal element $K \in \mathfrak{B}$.

Corollary 2.2. *Given a normal operator K in the C^* algebra $\mathfrak{B}$, suppose that the generator of a norm-continuous quantum dynamical semigroup $\mathcal{T}$ on $\mathfrak{B}$ is implemented by* (2.5), *where H and V satisfy*:

(i) $[H, K] \in C^*(K)$;
(ii) $V^*V \in C^*(K)$;
(iii) *For each $n \in \mathbb{N}$, there exists a constant $\alpha_n \in \mathbb{C}$ such that*

$$VK^n - \pi(K^n)V = \alpha_n V.$$

Then, the semigroup $\mathcal{T}$ is reduced by the algebra $C^*(K)$.

Proof. From hypothesis (i) and the derivation property of $[H, \cdot]$ it follows that $[H, K^n] \in C^*(K)$ for all $n \in \mathbb{N}$. On the other hand, hypothesis (iii) yields

$$V^*VK^n - V^*\pi(K^n)V = \alpha_n V^*V,$$

so that $V^*\pi(K^n)V$ belongs to $C^*(K)$ as well as V^*VK^n and K^nV^*V, applying (ii). As a result, $\mathcal{L}(K^n) \in C^*(K)$ for all $n \in \mathbb{N}$, and the proof is complete. □

We recall that a *-Abelian algebra $\mathfrak{A} \subseteq \mathfrak{L}(\mathfrak{h})$ is maximal if $\mathfrak{A} = \mathfrak{A}'$. In this case $\mathfrak{A}$ becomes a von Neumann algebra.

Proposition 2.2. *Suppose that $\mathcal{T}$ is a quantum Markov semigroup defined on $\mathfrak{L}(\mathfrak{h})$ which is reduced by a maximal Abelian von Neumann subalgebra $\mathfrak{A}$. Then there exists a compact Hausdorff space E endowed with a Radon measure μ such that the restriction of $\mathcal{T}$ to $\mathfrak{A}$ is *-isomorphic to a classical Markov semigroup $(\mathbf{T}_t)_{t\in\mathbb{R}_+}$ on $L^\infty(E, \mu)$. Moreover, if the semigroup $\mathcal{T}$ is uniformly continuous, then $(\mathbf{T}_t)_{t\in\mathbb{R}_+}$ is in particular a Feller semigroup.*

Proof. Since $\mathfrak{A}$ is maximal Abelian, there exists a triple (E, μ, U), where E is a compact second countable Hausdorff space, μ a Radon measure on E and U is an isometry from $L^2(E,\mu)$ onto $\mathfrak{h}$. E is in fact the space of characters of $\mathfrak{A}$ which is w^*-compact.

So that $\mathcal{U} : L^\infty(E,\mu) \to \mathfrak{A}$, defined by $\mathcal{U}(f) = UM_fU^*$, where $f \in L^\infty(E,\mu)$ and M_f denotes the multiplication operator by f in $L^2(E,\mu)$, is an isometric *-isomorphism of algebras.

A semigroup $(\mathbf{T}_t)_{t\in\mathbb{R}_+}$ is defined on $L^\infty(E,\mu)$ through the relation

$$M_{\mathbf{T}_t f} = U^*\mathcal{T}_t(UM_fU^*)U, \tag{2.6}$$

for all $f \in L^\infty(E,\mu)$.

The semigroup $(\mathbf{T}_t)_{t\in\mathbb{R}_+}$ preserves the identity, since U is an isometry. Moreover, $\|\mathcal{T}_t(x)\| \leq \|x\|$ $(x \in \mathfrak{M})$ implies that $\mathbf{T}_t$ is a contraction. Therefore, $(\mathbf{T}_t)_{t\in\mathbb{R}_+}$ is a Markov semigroup on $L^\infty(E,\mu)$.

If $\mathcal{T}$ is uniformly continuous, the Feller property is obtained through the previous proposition. □

Given a bounded normal operator K, we denote $W^*(K)$ its generated von Neumann algebra which coincides with the weak closure of $C^*(K)$. The Abelian algebra $W^*(K)$ is maximal if and only if K is *multiplicity-free* or *nondegenerate*, which means that there exists a cyclic vector for $C^*(K)$, i.e. $\{f(K)\mathbf{w} :\ f \in C(\mathbf{Sp}(K))\}$ is dense in $\mathfrak{h}$ for some vector $\mathbf{w}$, where $\mathbf{Sp}(K)$ denotes the spectrum of K.

Remark 2.1. Consider a von Neumann algebra $\mathfrak{M}$ on the Hilbert space $\mathfrak{h}$. The representation of the generator $\mathcal{L}(\cdot)$ of a norm-continuous QMS on $\mathfrak{M}$ is then improved as follows. There exists a set of operators $(L_k)_{k\in\mathbb{N}}$ such that $\sum_k L_k^*L_k$ is a bounded operator in $\mathfrak{M}$; $\sum_k L_k^*xL_k \in \mathfrak{M}$ whenever $x \in \mathfrak{M}$ and there exists a self-adjoint operator $H = H^* \in \mathfrak{M}$, such that

$$\mathcal{L}(x) = i[H,x] - \frac{1}{2}\sum_k (L_k{}^*L_kx - 2L_k{}^*xL_k + xL_k{}^*L_k). \tag{2.7}$$

We recover the expression (2.3) if we put

$$G = -iH - \frac{1}{2}\sum_k L_k^*L_k; \quad \Psi(x) = \sum_k L_k^*xL_k. \tag{2.8}$$

Theorem 2.2. *Assume that the quantum Markov semigroup $\mathcal{T}$ is norm-continuous on a von Neumann algebra $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{h})$ and denote $\mathcal{L}(\cdot)$ its generator represented in the Christensen–Evans form* (2.3)*. Call $H = -\Im(G)$ and denote α the group of automorphisms with generator $\delta(x) = i[H,x]$.*

(1) *If for all $x \in \mathcal{F}(\alpha)$ one has $\Phi(x) \in \mathcal{F}(\alpha)$, then $\mathcal{T}$ commutes with α.*
(2) *Reciprocally, assume that $\Phi(\mathbf{1}) \in \mathcal{F}(\alpha)$ and that the semigroup commutes with α, then $\Phi(x) \in \mathcal{F}(\alpha)$ for all $x \in \mathcal{F}(\alpha)$.*

Proof. Since the semigroup is Markovian, it holds that $\mathcal{L}(\mathbf{1}) = G^* + \Phi(\mathbf{1}) + G = 0$, so that $\Re(G) = -\frac{1}{2}\Phi(\mathbf{1})$.

(1) Given any $x \in \mathcal{F}(\alpha)$, x commutes with $H = -\Im(G)$. Moreover, by hypothesis, $\Phi(x)$ commutes with H, in particular $[H, \Re(G)] = 0$, from which we obtain that $[H, G] = 0 = [H, G^*]$ and $\mathcal{L}(x)$ commutes with H as well. Thus, $\mathcal{L}(x) \in \mathcal{F}(\alpha)$ if $x \in \mathcal{F}(\alpha)$ and the semigroup commutes with α.
(2) If $\Phi(\mathbf{1})$ commutes with H, it follows that $[H, G] = 0$. Now, given any $x \in \mathcal{F}(\alpha)$, $\mathcal{L}(x) - G^*x - xG = \Phi(x)$ is also an element of $\mathcal{F}(\alpha)$ since the semigroup commutes with α. □

Corollary 2.3. *Assume that the semigroup $\mathcal{T}$ is norm-continuous. Suppose in addition that all the coefficients L_k of* (2.7) *satisfy the following hypothesis*:

(H1) *There exists a matrix $(c_{k,\ell})$ of bounded maps from $W^*(H)' \to W^*(H)'$ such that $c_{k,\ell}(a)^* = c_{\ell,k}(a^*)$, for all $a \in W^*(H)'$ and such that*

$$[H, L_k] = \sum_{\ell \in \mathbb{N}} c_{k,\ell}(H) L_\ell, \quad k \in \mathbb{N}. \tag{2.9}$$

Then $\mathcal{T}$ commutes with α.

Proof. The algebra $\mathcal{F}(\alpha)$ coincides with the generalized commutator algebra $W^*(H)'$ of the von Neumann algebra $W^*(H)$ generated by H. Notice that (2.9) implies that

$$[H, L_k^* x L_k] = 0,$$

for all k if x commutes with H. Indeed, since $[H, \cdot]$ is a derivation, it turns out that

$$[H, {L_k}^* x L_k] = -\sum_\ell {L_\ell}^* c_{\ell,k}(H) x L_k + \sum_\ell {L_k}^* x c_{k,\ell}(H) L_\ell.$$

Now, exchanging ℓ by k in the first sum and noticing that x and $c_{k,\ell}(H)$ commute yields the result.

Therefore, $\Phi(x) = \sum_k {L_k}^* x L_k$ satisfies the hypothesis (1) of the previous theorem and the proof is complete. □

Remark 2.2. Recently, Fagnola and Skeide have proved that (H) is also a necessary condition to leave the algebra $W^*(H)'$ invariant under the action of the semigroup $\mathcal{T}$. Their method of proof is based on Hilbert modules [19].

Corollary 2.4. *If a self-adjoint operator $K \in \mathfrak{M}$ is nondegenerate, $W^*(K)$ reduces the norm-continuous quantum Markov semigroup $\mathcal{T}$ if and only if $\mathcal{L}(x)$ commutes with K for any $x \in W^*(K)$.*

In particular, assume that

(i) $[H, K] \in W^*(K)$, *and*
(ii) $[L_k, K] = c_k L_k$, *where $c_k = c_k^* \in W^*(K)$, for all $k \in \mathbb{N}$.*

Then $W^*(K)$ reduces the semigroup $\mathcal{T}$.

Proof. If K is nondegenerate, then $W^*(K)$ is maximal Abelian and coincides with its commutator $W^*(K)'$. Thus, $\mathcal{L}(W^*(K)) \subseteq W^*(K)$ if and only if $\mathcal{L}(x)$ lies in $W^*(K)'$ for any element $x \in W^*(K)'$. □

Remark 2.3. The previous results suggest that a "good candidate" for an operator K reducing the dynamics is $H - \frac{1}{2}\sum_j {L_j}^* L_j$, especially if one derives the semigroup through a limiting procedure as those analyzed in the Introduction. This corresponds to the so-called *effective Hamiltonian* in some physical models. Also, when $H = 0$ there is an interesting class of models for which one can exhibit al least two "natural" self-adjoint operators which classically reduce the semigroup.

Corollary 2.5. *Assume that $H = 0$ and that the coefficients $(L_j)_{j\in\mathbb{N}}$ are bounded and satisfy the commutation relations*

$$[L_j, L_\ell] = [{L_j}^*, {L_\ell}^*] = 0, \tag{2.10}$$

$$[L_j, {L_\ell}^*] = \delta_{j\ell}\mathbf{1}. \tag{2.11}$$

Suppose in addition that the self-adjoint operators $K_1 = \sum_j {L_j}^* L_j$, and $K_2 = \sum_j (L_j + {L_j}^*)$ are bounded. Then the QMS with generator (2.7) *is classically reduced by K_1 and K_2.*

Proof. Consider $\ell \in \mathbb{N}$ fixed. Then

$$\begin{aligned} [K_1, L_\ell] &= \sum_j [{L_j}^* L_j, L_\ell] \\ &= -\sum_j [{L_j}^*, L_\ell] L_j \\ &= \sum_j \delta_{\ell j} L_j \\ &= -L_\ell. \end{aligned} \tag{2.12}$$

Thus, (2.12) shows that K_1 satisfies Corollary 2.4.

Since K_2 is bounded, we prove that its generated C^*-algebra reduces the semigroup. To do this, one first shows that for any $x \in C^*(K_2)$ it holds

$$[x, L_\ell] \in C^*(K_2), \tag{2.13}$$

for all $\ell \in \mathbb{N}$.

A simple computation yields

$$[K_2, L_\ell] = [L_\ell^*, L_\ell] = -\mathbf{1} \in C^*(K_2),$$

for any $\ell \in \mathbb{N}$. Suppose that for $m = 1, \dots, n$ it holds

$$[K_2^m, L_\ell] \in C^*(K_2),$$

take $m = n + 1$, then for any $\ell \in \mathbb{N}$,

$$\begin{aligned} [{K_2}^{n+1}, L_\ell] &= K_2[K_2^n, L_\ell] + [K_2, L_\ell] K_2^n \\ &= K_2[K_2^n, L_\ell] - K_2^n, \end{aligned}$$

and the second member is an element of $C^*(K_2)$ by the induction hypothesis. As a result (2.13) holds for any $x \in C^*(K_2)$.

Consider any $x \in C^*(K_2)$, and write $\mathcal{L}(x)$ in the form:

$$\mathcal{L}(x) = -\frac{1}{2}\sum_{\ell}\left(L_\ell^*[L_\ell, x] + [x, L_\ell^*]L_\ell\right).$$

Then

$$\begin{aligned}[\mathcal{L}(x), K_2] &= -\frac{1}{2}\sum_{\ell}\left([L_\ell^*, K_2][L_\ell, x] + [x, L_\ell^*][L_\ell, K_2]\right)\\ &= \frac{1}{2}\sum_{\ell}\left([L_\ell, x] - [x, L_\ell^*]\right)\\ &= \frac{1}{2}[x, K_2]\\ &= 0.\end{aligned}$$

So that $\mathcal{L}(C^*(K_2)) \subset C^*(K_2)$ and the proof is complete. □

Remark 2.4. The previous results can be improved to consider an unbounded self-adjoint operator K affiliated with the von Neumann algebra $\mathfrak{M}$. Assume that K is nondegenerate which means here that there is a vector $\mathbf{w}$ in the intersection of all domains $D(K^n)$, $(n \geq 1)$, such that the subspace spanned by the vectors $(K^n\mathbf{w};\ n \geq 1)$ is dense in $\mathfrak{h}$. We denote ξ the spectral measure of K. In addition, given a Radon measure on the measurable space $(\mathbf{Sp}(K), \mathcal{B}(\mathbf{Sp}(K)))$ we denote $L(\mathbf{Sp}(K), \mu)$ the *-algebra which is obtained as the quotient of the set of Borel functions by null functions under the given Radon measure.

Lemma 2.2. *Assume that $\mathcal{P}$ is a normal linear completely positive map defined on $\mathfrak{M}$ and such that $\mathcal{P}(\mathbf{1}) = \mathbf{1}$. Let K to be a nondegenerate self-adjoint operator affiliated with $\mathfrak{M}$. Then the following propositions are equivalent*:

(i) *$W^*(K)$ is invariant under $\mathcal{P}$.*
(ii) *For any projection $p \in W^*(K)$, $\mathcal{P}(p) \in W^*(K)$.*
(iii) *For all $A \in \mathcal{B}(\mathbf{Sp}(K))$, $\mathcal{P}(\xi(A)) \in W^*(K)$.*
(iv) *There exists a kernel $P : \mathbf{Sp}(K) \times \mathcal{B}(\mathbf{Sp}(K)) \to \mathbb{R}^+$ such that $P(x, \mathbf{Sp}(K)) = 1$, for all $x \in \mathbf{Sp}(K)$ and*

$$\mathcal{P}(\xi(A)) = \int_{\mathbf{Sp}(K)} \xi(dx) P(x, A),$$

for all $A \in \mathcal{B}(\mathbf{Sp}(K))$.

Proof. Clearly, (i) implies (ii) which in turn implies (iii). The equivalence of (i) and (iii) follows from a straightforward application of the Spectral Theorem. So that (i), (ii), and (iii) are equivalent. To prove that (iii) implies (iv), we first notice that $\mathcal{P} \circ \xi$ is an operator valued measure. Indeed, since ξ is the spectral measure of

K and $\mathcal{P}$ is linear and completely positive, the map $\mathcal{P}\circ\xi$ is additive on $\mathcal{B}(\mathbf{Sp}(K))$. Moreover, take any pairwise disjoint sequence $(A_n)_{n\in\mathbb{N}}$ of Borel subsets of $\mathbf{Sp}(K)$. The projection $\sum_n \xi(A_n)$ exists as a strong limit of the partial sums $\sum_{k\le n}\xi(A_k)$. Moreover, $\sum_n \xi(A_n) = \text{l.u.b.} \sum_{k\le n}\xi(A_k)$ in the order of positive operators. Thus, the normality of the map $\mathcal{P}$ yields $\mathcal{P}\big(\sum_n \xi(A_n)\big) = \text{l.u.b.}\mathcal{P}\big(\sum_{k\le n}\xi(A_k)\big)$, and $\mathcal{P}\circ\xi$ is an operator-valued measure.

If we assume (iii), given any $A\in\mathcal{B}(\mathbf{Sp}(K))$, there exists $P(\cdot,A)\in L(\mathbf{Sp}(K))$ such that

$$\mathcal{P}\circ\xi(A) = \int_{\mathbf{Sp}(K)} \xi(dx)P(x,A). \tag{2.14}$$

Denote $(e_n)_{n\in\mathbb{N}}$ an orthonormal basis of $\mathfrak{h}$ and define the positive measure $\mu = \sum_n 2^{-n}\langle e_n, \xi(\cdot)e_n\rangle$. Since $\mathcal{P}\circ\xi$ is an operator-valued measure, it follows that $A\mapsto P(x,A)$ satisfies

$$P\left(x, \bigcup_n A_n\right) = \sum_n P(x,A_n),$$

for μ-almost all $x\in\mathbf{Sp}(K)$.

Since μ is a probability measure, it is tight on $\mathbf{Sp}(K)\subseteq\mathbb{R}$. Therefore, for each $n\ge 1$, there exists a compact K_n such that $\mu(K_n)\ge 1-2^{-n}$, so that $J=\bigcup_n K_n\subseteq \mathbf{Sp}(K)$ satisfies $\mu(J)=1$. We imbed J in $[-\infty,\infty]$, and consider μ as a probability measure defined on $[-\infty,\infty]$, supported by J. Let denote $\mathfrak{k}$ a vector space over the field of rational numbers, closed for lattice operations $\vee,\wedge$, dense in $C([-\infty,\infty])$ and such that $1\in\mathfrak{k}$. Define

$$A = \{x\in J : f\mapsto P(x,f) \text{ is a positive } \mathbb{Q}\text{-linear form on } \mathfrak{k} \text{ and } P(x,1)=1\}.$$

For all $x\in A$, $P(x,\cdot)$ can be extended as a positive linear form to all of $C([-\infty,\infty])$, and then to $L([-\infty,\infty])$. Moreover, $\mu(A)=1$ and $\mu(\{x\in\mathbf{Sp}(K) : P(x,J^c)=0\})=1$. We can complete the definition of the kernel P choosing $P(x,\cdot)=\theta(\cdot)$ for all $x\notin A$, where θ is an arbitrary probability measure.

Finally to prove that (iv) implies (iii), it suffices to apply the Spectral Theorem again which yields $\int\xi(dx)P(x,A)\in W^*(K)$. □

For any quantum Markov semigroup $\mathcal{T}$ there exist $M>0$ and $\beta\in\mathbb{R}$ such that $\|\mathcal{T}_t\|\le M\exp(\beta t)$ for all $t\ge 0$ (see [8], Prop. 3.1.6, p. 166). As a result, the resolvent $\mathcal{R}_\lambda(\cdot)$ of the semigroup is given by the Laplace transform

$$\mathcal{R}_\lambda(x) = (\lambda\mathbf{1}-\mathcal{L})^{-1}(x) = \int_0^\infty dt e^{-\lambda t}\mathcal{T}_t(x),$$

for all $x\in\mathfrak{M}$, whenever $\Re\lambda>\beta$. The above lemma yields the following characterization.

Theorem 2.3. *Let $\mathcal{T}$ be a quantum Markov semigroup on the von Neumann algebra $\mathfrak{M}$ and K a nondegenerate self-adjoint operator affiliated with $\mathfrak{M}$. Then the* *following propositions are equivalent*:

(i) *The semigroup is reduced by* $W^*(K)$.
(ii) *For all* $A \in \mathcal{B}(\mathbf{Sp}(K))$ *and any* $t \geq 0$, $\mathcal{T}_t(\xi(A)) \in W^*(K)$.
(iii) *The manifold* $D(\mathcal{L}) \cap W^*(K)$ *is nontrivial and for all* $x \in D(\mathcal{L}) \cap W^*(K)$, *it holds that* $\mathcal{L}(x) \in W^*(K)$.
(iv) *There exists a classical Markov semigroup* $(\mathbf{T}_t)_{t\in\mathbb{R}^+}$ *on* $\mathbf{Sp}(K)$ *such that for all* $f \in L(\mathbf{Sp}(K))$,

$$\mathcal{T}_t(f(K)) = \int_{\mathbf{Sp}(K)} \xi(dx)\mathbf{T}_t f(x).$$

(v) *For all* λ *such that* $\Re\lambda > \beta$ *and all* $A \in \mathcal{B}(\mathbf{Sp}(K))$ $\mathcal{R}_\lambda(\xi(A)) \in W^*(K)$.

Proof. We clearly have the equivalence of (i), (ii) and (iii). Furthermore, the equivalence of (i) with (iv) follows from the previous lemma. (v) is equivalent to the existence of a family of kernels R_λ on the spectrum of K which defines a classical semigroup $\mathbf{T}$. Thus (v) and (iv) are equivalent and this completes the proof. □

The particular case of norm-continuous semigroups enjoys a richer characterization in terms of the generator.

Corollary 2.6. $W^*(K)$ *reduces a norm-continuous quantum Markov semigroup* $\mathcal{T}$ *if and only if one of the following equivalent conditions is satisfied*:

(i) $\mathcal{L}(\xi(A)) \in W^*(K)$ *for all* $A \in \mathcal{B}(\mathbf{Sp}(K))$.
(ii) $[\mathcal{L}(\xi(A)), \xi(B)] = 0$ *for all* $A, B \in \mathcal{B}(\mathbf{Sp}(K))$.
(iii) *There exists a dense domain* $D \subseteq L(\mathbf{Sp}(K))$ *and an operator* $\mathbf{L} : D \to L(\mathbf{Sp}(K))$, *such that for all* $f \in D$, $f(K) \in D(\mathcal{L})$ *and*

$$\mathcal{L}(f(K)) = \int_{\mathbf{Sp}(K)} \xi(dx)\mathbf{L}f(x).$$

In particular, suppose that the generator $\mathcal{L}(\cdot)$ *is given by* (2.7) *which in addition satisfies the two conditions below*:

(a) $[H, \xi(A)] \in W^*(K)$, *and*
(b) $[L_k, \xi(A)] = c_k(A)L_k$, *where* $c_k(A)$ *is a self-adjoint element in* $W^*(K)$, *for all* $k \in \mathbb{N}$ *and* $A \in \mathcal{B}(\mathbf{Sp}(K))$.

Then $W^*(K)$ *reduces the semigroup* $\mathcal{T}$.

Proof. The generator $\mathcal{L}(\cdot)$ is everywhere defined since the semigroup is norm-continuous. Thus, the equivalence of (i), (ii), (iii) with (i) of the previous result is a simple consequence of the Spectral Theorem.

The last part follows from the first and Theorem 2.4 applied to $\xi(A)$. □

2.2. *The case of a form-generator*

In most of the applications $\mathcal{L}(\cdot)$ is not known as an operator directly but through a sesquilinear form. To discuss this case we restrict ourselves to the von Neumann algebra $\mathfrak{M} = \mathfrak{L}(\mathfrak{h})$ and we rephrase, for easier reference the crucial result which allows to construct a quantum dynamical semigroup starting from a generator given as a sesquilinear form. For further details on this matter we refer to [17], Sec. 3.3, see also [10].

Let G and L_ℓ, ($\ell \geq 1$) be operators in $\mathfrak{h}$ which satisfy the following hypothesis:

- (H-min) G is the infinitesimal generator of a strongly continuous contraction semigroup in $\mathfrak{h}$, $D(G)$ is contained in $D(L_\ell)$, for all $\ell \geq 1$, and, for all $u, v \in D(G)$, we have

$$\langle Gv, u\rangle + \sum_{\ell=1}^{\infty} \langle L_\ell v, L_\ell u\rangle + \langle v, Gu\rangle = 0.$$

Under the above assumption (H-min), for each $x \in \mathfrak{L}(\mathfrak{h})$ let $\pounds(x) \in \mathfrak{F}(D(G) \times D(G))$ be the sesquilinear form with domain $D(G) \times D(G)$ defined by

$$\pounds(x)(v, u) = \langle Gv, xu\rangle + \sum_{\ell=1}^{\infty} \langle L_\ell v, xL_\ell u\rangle + \langle v, xGu\rangle. \tag{2.15}$$

It is well known (see e.g. [13] Sec. 3, [17] Sec. 3.3) that, given a domain $D \subseteq D(G)$, which is a core for G, it is possible to build up a quantum dynamical semigroup, called the *minimal* QDS, satisfying the equation:

$$\langle v, \mathcal{T}_t(x)u\rangle = \langle v, xu\rangle + \int_0^t \pounds(\mathcal{T}_s(x))(v, u)ds, \tag{2.16}$$

for $u, v \in D$.

This equation, however, in spite of the hypothesis (H-min) and the fact that D is a core for G, does not necessarily determine a unique semigroup. The minimal QDS is characterized by the following property: for any w^*-continuous family $(\mathcal{T}_t)_{t\geq 0}$ of positive maps on $\mathfrak{L}(\mathfrak{h})$ satisfying (2.16) we have $\mathcal{T}_t^{(\min)}(x) \leq \mathcal{T}_t(x)$ for all positive $x \in \mathfrak{L}(\mathfrak{h})$ and all $t \geq 0$ (see e.g. [17] Th. 3.22).

For simplicity we will drop the superscript "min" in the notation of the minimal QDS in what follows. In which follows we assume that the operator K is an unbounded nondegenerate self-adjoint operator affiliated with $\mathfrak{L}(\mathfrak{h})$, that (H-min) holds and the domain D denotes a core for G.

Theorem 2.4. *Assume that for all $x \in W^*(K)$, and all spectral projection $\xi(A)$, where $A \in \mathcal{B}(\mathbf{Sp}(K))$ is such that $\xi(A)(D) \subseteq D$ it holds*

$$\pounds(x)(v, \xi(A)u) = \pounds(x)(\xi(A)v, u), \tag{2.17}$$

for all $(u, v) \in D \times D$. Then the minimal semigroup $\mathcal{T}$ is reduced by K.

In particular, this is the case when the following two conditions hold:

(a) *The operator G is affiliated with $W^*(K)$*,
(b) *Given any $x \in W^*(K)$, $(u, v) \in D \times D$, $\ell \geq 1$*,

$$\langle L_\ell v, x L_\ell \xi(A) u\rangle = \langle L_\ell \xi(A) v, x L_\ell u\rangle, \tag{2.18}$$

for all $A \in \mathcal{B}(\mathbf{Sp}(K))$ such that $\xi(A)(D) \subseteq D$.

Proof. The proof follows the construction of the minimal quantum dynamical semigroup associated to the form $\mathfrak{L}(\cdot)$, as presented by Chebotarev (see [10] and extensively used by him and Fagnola in their joint research on the Markov property of this minimal semigroup (see [11]).

Define $\mathcal{T}_t^{(0)}(x) = x$. Then, clearly $\langle v, \mathcal{T}_t^{(0)}(x)pu\rangle = \langle pv, \mathcal{T}_t^{(0)}(x)u\rangle$, for all $x \in W^*(K)$, all projection $p = \xi(A)$ leaving D invariant, $(u, v) \in D \times D$. We follow by defining $\mathcal{T}_t^{(1)}(x)$ as follows: for each $(u, v) \in D \times D$,

$$\langle v, \mathcal{T}_t^{(1)}(x)u\rangle = \langle v, xu\rangle + \int_0^t \mathfrak{L}(\mathcal{T}_s^{(0)}(x))(v, u)ds.$$

Take $x \in W^*(K)$ a projection $p = \xi(A)$ as before, and apply hypothesis (2.17). Then it follows that

$$\langle v, \mathcal{T}_t^{(1)}(x)pu\rangle = \langle pv, \mathcal{T}_t^{(1)}(x)u\rangle.$$

This yields that $\mathcal{T}_t^{(1)}(x) \in W^*(K)$ if $x \in W^*(K)$.

By induction, suppose $\mathcal{T}_t^{(0)}(\cdot), \ldots, \mathcal{T}_t^{(n)}(\cdot)$ constructed and reduced by $W^*(K)$, then define $\mathcal{T}_t^{(n+1)}(\cdot)$ through the relation

$$\langle v, \mathcal{T}_t^{(n+1)}(x)u\rangle = \langle v, xu\rangle + \int_0^t \mathfrak{L}(\mathcal{T}_s^{(n)}(x))(v, u)ds.$$

By the induction hypothesis, and (2.17) again, it follows that

$$\langle v, \mathcal{T}_t^{(n+1)}(x)pu\rangle = \langle pv, \mathcal{T}_t^{(n+1)}(x)u\rangle,$$

for all projection $p = \xi(A)$ such that $p(D) \subseteq D$ and $(u, v) \in D \times D$, whenever $x \in W^*(K)$. Therefore, K reduces the whole sequence $(\mathcal{T}^{(n)})_{n\in\mathbb{N}}$. This sequence is used in the construction of the minimal quantum dynamical semigroup as follows. It is proved that $\langle u, \mathcal{T}_t^{(n)}(x)u\rangle$ is increasing with n and $\langle u, \mathcal{T}_t(x)u\rangle$ is defined as its limit, for all $u \in \mathfrak{h}$, $x \in \mathfrak{L}(\mathfrak{h})$ (see [17]). Then by polarization $\langle v, \mathcal{T}_t(x)u\rangle$ is obtained. Thus, the minimal quantum dynamical semigroup $\mathcal{T}$ satisfies $\mathcal{T}_t(\mathbf{1}) \leq \mathbf{1}$, and given any other σ-weakly continuous family $(\mathcal{S}_t)_{t\in\mathbb{R}^+}$ satisfying (2.16) and every positive operator $x \in \mathfrak{L}(\mathfrak{h})$, it holds $\mathcal{T}_t(x) \leq \mathcal{S}_t(x)$, for all $t \geq 0$. Moreover, since $\mathcal{T}_t^{(n)}(W^*(K)) \subseteq W^*(K)$, for all $n \in \mathbb{N}$ and $t \geq 0$, it follows that K reduces the minimal quantum dynamical semigroup.

Assume now hypotheses (a) and (b). Condition (a) implies that $G\xi(A) = \xi(A)G$ for all projections $\xi(A)$ leaving D invariant. Moreover, (b) yields $\sum_\ell \langle L_\ell v, x L_\ell \xi(A)u\rangle = \sum_\ell \langle L_\ell \xi(A)v, x L_\ell u\rangle$ and this, together with (a), clearly determine (2.17) and the proof is complete. □

Corollary 2.7. *With the notations and previous assumptions to the above theorem, suppose that in addition the two hypotheses below are satisfied:*

(a) *G is affiliated with $W^*(K)$,*
(b) *For all $\ell \geq 1$ and any $A \in \mathcal{B}(\mathbf{Sp}(K))$ such that $\xi(A)$ leaves D invariant, there exists a self-adjoint operator $c_\ell(A) \in W^*(K)$, such that*

$$L_\ell \xi(A) = (\xi(A) + c_\ell(A)) L_\ell. \tag{2.19}$$

Then K reduces the minimal quantum dynamical semigroup $\mathcal{T}$.

Proof. Hypothesis (a) is identical to condition (a) of the previous theorem. On the other hand, if $x \in W^*(K)$, and $A \in \mathcal{B}(\mathbf{Sp}(K))$ is such that $\xi(A)$ leaves D invariant

$$\begin{aligned}
\langle L_\ell \xi(A) v, x L_\ell u \rangle &= \langle \xi(A) L_\ell v, x L_\ell u \rangle + \langle c_\ell(A) L_\ell v, x L_\ell u \rangle \\
&= \langle L_\ell v, x \xi(A) L_\ell u \rangle + \langle L_\ell v, x c_\ell(A) L_\ell u \rangle \\
&= \langle L_\ell v, x (L_\ell \xi(A) - c_\ell(A) L_\ell) u \rangle + \langle L_\ell v, x c_\ell(A) L_\ell u \rangle \\
&= \langle L_\ell v, x L_\ell \xi(A) u \rangle,
\end{aligned}$$

for all $(u, v) \in D \times D$. Thus, condition (b) of Theorem 2.4 is satisfied and the proof is complete. □

Remark 2.5. It is worth noticing that Corollary 2.5 admits also an extension within this framework. Indeed, if $H = 0$, and the formal generator contains only a finite number of operators L_ℓ, say $L_1, \ldots, L_n$, then under the hypothesis (H-min), D is also a core for both operators K_1 and K_2 below:

$$K_1 = \sum_{j=1}^{n} {L_j}^* L_j, \tag{2.20}$$

$$K_2 = \sum_{j=1}^{n} (L_j + {L_j}^*). \tag{2.21}$$

If moreover the coefficients L_j satisfy the commutation relations of Corollary 2.5, then both operators K_1 and K_2 reduce classically the Quantum Markov Semigroup given by the formal generator with coefficients L_j. This, in particular, is the case for the semigroup of the harmonic oscillator which we precise in the following corollary.

Corollary 2.8. *Let $\mathfrak{h} = L^2(\mathbb{R}^d; \mathbb{C})$ and let $\mathcal{A} = \mathfrak{L}(\mathfrak{h})$. By a Quantum Brownian Semigroup we mean a Quantum Markov Semigroup $\mathcal{T}$ on $\mathfrak{L}(\mathfrak{h})$ which is the minimal semigroup with form generator*

$$\pounds(x) = -\frac{1}{2} \sum_{j=1}^{d} (a_j a_j^* x - 2 a_j x a_j^* + x a_j a_j^*) - \frac{1}{2} \sum_{j=1}^{d} (a_j^* a_j x - 2 a_j^* x a_j + x a_j^* a_j),$$

where a_j^*, a_j are the creation and annihilation operators

$$a_j = (q_j + \partial_j)/\sqrt{2}, \quad a_j^* = (q_j - \partial_j)/\sqrt{2},$$

∂_j being the partial derivative with respect to the jth coordinate q_j.

The commutative von Neumann subalgebra $W^*(q)$ *of* $\mathfrak{L}(\mathfrak{h})$ *whose elements are multiplication operators* M_f *by a function* $f \in L^\infty(\mathbb{R}^d;\mathbb{C})$ *is* $\mathcal{T}$*-invariant and* $\mathcal{T}_t(M_f) = M_{T_t f}$ *where*

$$(T_t f)(x) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(y) e^{-|x-y|^2/2t} dy. \tag{2.22}$$

The same conclusion holds for the commutative algebra $W^*(p) = F^* W^*(q) F$, *where* F *denotes the Fourier transform. Therefore, this semigroup deserves the name of* quantum Brownian semigroup *since it contains a couple of noncommuting classical Brownian semigroups.*

Moreover, notice that the von Neumann algebra $W^*(N)$ *generated by the number operator* $N = \sum_j a_j^* a_j$ *is also* $\mathcal{T}$ *invariant and the classical semigroup obtained by restriction of* $\mathcal{T}$ *to* $W^*(N)$ *is a birth and death semigroup on* $\mathbb{N}$ *with birth rates* $(n+1)_{n\geq 0}$ *and death rates* $(n)_{n\geq 0}$.

3. Dilations and Unravelings

3.1. *Structure maps and quantum flows*

In addition to the separable complex Hilbert space $\mathfrak{h}$, let $\mathfrak{g}$ be another Hilbert space of the like which will serve to introduce a countable family of quantum noises as follows. Firstly, denote $\Phi = \Gamma_b\left(L^2(\mathbb{R}^+;\mathfrak{g})\right)$ the bosonic Fock space associated to $L^2(\mathbb{R}^+;\mathfrak{g})$, and $(g_n)_{n\in\mathbb{N}}$ an orthonormal basis of $\mathfrak{g}$. The total space is then $\mathfrak{h}_T = \mathfrak{h} \otimes \Phi$ which has a generating family $\Xi := \{u \otimes e(f);\ u \in \mathfrak{h},\ f \in\}$ where $e(f) := \sum_{n\in\mathbb{N}} f^{\otimes n}/\sqrt{n!}$ and denote $\mathfrak{M} = \mathfrak{L}(\mathfrak{h}_T)$. Also, we consider the increasing family (in the sense of immersions) of spaces $\Phi_t = \Gamma_b(L^2_{\mathfrak{g}}([0,t])$ and $\mathfrak{M}_t = \mathfrak{L}(\mathfrak{h} \otimes \Phi_t)$. With these notations, the main quantum noises are defined as

$$A_\ell^\dagger(t) u \otimes e(f) = \frac{d}{d\epsilon} u \otimes e(f + \epsilon 1_{[0,t]} g_\ell)|_{\epsilon=0}, \tag{3.1}$$

$$A_\ell(t) = \langle g_\ell 1_{[0,t]}, f\rangle u \otimes e(f), \tag{3.2}$$

$$N_{\ell,m}(t) = A_\ell^\dagger(t) A_m(t), \tag{3.3}$$

for all $\ell, m \in \mathbb{N}$, $t \geq 0$.

To shorten the writing of these noises they will be denoted using Belavkin convention:

$$\begin{aligned} \Lambda_\ell^0(t) &= A_\ell^\dagger(t), && \text{if } \ell > 0 \\ \Lambda_0^m(t) &= A_m(t), && \text{if } m > 0 \\ \Lambda_\ell^m(t) &= N_{\ell,m}(t), && \text{if } \ell, m > 0 \\ \Lambda_0^0(t) &= t\mathbf{1}, \end{aligned}$$

for all $t \geq 0$.

Let there be given an automorphism group α on $\mathfrak{L}(\mathfrak{h})$ with generator $\delta = i[K,\cdot]$. α is extended to $\mathfrak{L}(\mathfrak{h}\otimes\Phi)$ via the generator $i[K\otimes\mathbf{1},\cdot]$. Consider a quantum cocycle

$V = (V_t)_{t \in \mathbb{R}^+}$ which is the solution of a Hudson–Parthasarathy stochastic differential equation (see [24] Chap. III, Sec. 27) of the form

$$dV_t = \left(\sum_{\ell, m \geq 0} L_\ell^m d\Lambda_m^\ell \right) V_t, \quad V_0 = \mathbf{1}, \tag{3.4}$$

where $\sum_{\ell,m\geq 0} \|L_\ell^m\| < \infty$ in $\mathfrak{L}(\mathfrak{h})$, and denote $j_t(x) = V_t^*(x \otimes \mathbf{1})V_t$.

Theorem 3.1. *Under the above assumptions suppose in addition that the cocycle V satisfies the Rotating Wave Hypothesis (RWH) with respect to α, that is, there exists $\omega \in \mathbb{R}$ such that for all $t, s \geq 0$,*

$$\alpha_s(V_t) = e^{-i\omega s} V_t. \tag{3.5}$$

Then the flow j and the automorphism α commutent.

Proof. Let there be given an element $x \in \mathcal{F}(\alpha)$, then for all $s, t \geq 0$,

$$\begin{aligned} \alpha_s(j_t(x)) &= \alpha_s(V_t^*)(x \otimes \mathbf{1})\alpha_s(V_t) \\ &= e^{i\omega s} V_t^*(x \otimes \mathbf{1}) e^{-i\omega s} V_t \\ &= j_t(x). \end{aligned}$$

So that $j_t(x) \in \mathcal{F}(\alpha)$ for all $t \geq 0$. □

Remark 3.1. One can write stochastic differential equations for quantum Markov flows as introduced by Belavkin, Evans, Hudson and Parthasarathy (cf. [24]). That is, using the representation of noises Λ_m^ℓ given on the Boson Fock space, the generic stochastic differential equation for a quantum flow is then written as

$$dj_t(x) = \sum_{\ell, m} j_t(\theta_\ell^m(x)) d\Lambda_m^\ell. \tag{3.6}$$

The maps θ_ℓ^m are called the *structure maps* of the flow. In the previous notations, one has

$$\theta_m^\ell(x) = xL_m^\ell + L_\ell^{m*} x + \sum_{k \geq 1} L_\ell^{k*} x L_m^k. \tag{3.7}$$

In particular, $\theta_0^0 = \mathcal{L}(\cdot)$, corresponds to the generator of a Quantum Markov semigroup (generated by the flow), though the notations are slightly different to those previously used in (2.7) and (2.8). This question is at the hearth of the proof of the existence and uniqueness of unitary cocycle solutions given by Hudson and Parthasarathy to (3.4) and is related to the construction of a *quantum dilation* of a given QMS. Indeed, suppose that the generator of the QMS is given in the form (2.7), more precisely, one assumes that there exists bounded operators $H = H^*$, S_j^i, L_i, for $i, j \geq 1$, and a constant $c > 0$ such that

$$\sum_{i \geq 1} \|L_i u\|^2 \leq c^2 \|u\|^2, \tag{3.8}$$

for all $u \in \mathfrak{h}$ and $\sum_{i,j\geq 1} S^i_j \otimes |g_i\rangle\langle g_j|$ is a unitary operator in $\mathfrak{h}\otimes\mathfrak{g}$. Moreover, define

$$L^i_j = \begin{cases} S^i_j - \delta_{ij}\mathbf{1}, & \text{if } i,j \geq 1; \\ L_i, & \text{if } i \geq 1,\ j = 0; \\ -\sum_{k\geq 1} L_k{}^* S^k_j, & \text{if } j \geq 1,\ i = 0; \\ -iH - \frac{1}{2}\sum_{k\geq 1} L_k{}^* L_k, & \text{if } i = j = 0, \end{cases} \tag{3.9}$$

then Theorem 27.8 in [24] proves that (3.4) has a unique unitary cocycle solution.

As a result, one obtains that

$$\mathcal{T}_t(x) = \mathbb{E}\left(j_t(x)\right) = EV_t{}^*(x\otimes\mathbf{1})V_tE, \quad t\geq 0,\ x\in\mathfrak{L}(\mathfrak{h}), \tag{3.10}$$

where E is the projection of $\mathfrak{h}\otimes\Phi$ onto $\mathfrak{h}$ identified with $\mathfrak{h}\otimes\mathbb{C}e(0)$.

To summarize, the structure maps θ^m_ℓ play a role of generators of the flow so that, under the above conditions on the operators L^ℓ_m, a straightforward modification of Theorem 28.8 in [24] yields the following:

Theorem 3.2. *Suppose that $K \in \mathfrak{L}(\mathfrak{h})$ is a multiplicity-free self-adjoint operator and $\theta^\ell_m(W^*(K)) \subset W^*(K)$, then the algebra $\mathfrak{J}$ generated by $\{j_t(x); x \in W^*(K), t\geq 0\} \subset \mathfrak{L}(\mathfrak{h}\otimes\Phi)$ is Abelian.*

For the proof see [24], Thm. 28.8, p. 244. Thus, the quantum flow restricted to the Abelian algebra $W^*(K)$ defines a classical stochastic process.

Remark 3.2. The canonical classical stochastic process $X = (X_t)_{t\in\mathbb{R}^+}$ referred before is constructed on the space $\Omega = \mathbf{Sp}(K)^{\mathbb{R}^+}$ with the σ-field of cylinders. The Spectral Theorem associates $W^*(K)$ with $L^\infty(\mathbf{Sp}(K))$ and $j_t(f(K))$ with $f(X_t)$. The latter is equivalent to state that the flow restricted to the Abelian algebra $W^*(K)$ is identified with the measure-valued stochastic process δ_{X_t}, $t\geq 0$.

3.2. *Classical dilations and classical reductions*

A quantum dilation of a given norm continuous QMS has been constructed in the previous subsection. This was done via a quantum stochastic differential equation. However, a number of open quantum systems in Physics are described by QMS with generators given as a sesquilinear form and the theory of quantum stochastic differential equations with unbounded coefficients is not sufficiently developed to provide in general the suitable quantum dilation.

On the other hand, the Theory of Measurement in Open Quantum Systems naturally introduces classical stochastic equations in infinite dimensions to construct stochastic processes associated to a given QMS. Let us continue the simple Example 1.1 to illustrate the main features of this kind of dilation.

3.3. *An example of a classical dilation*

We keep the notations of Sec. 1.1. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where two Brownian motions $W^{(0)}$ and $W^{(1)}$ are defined. Assume that $L^2(\Omega, \mathcal{F}, \mathbb{P})$ represents now the *environment* or *reservoir*.

Our goal is to build up a stochastic process $(k_t)_{t\in\mathbb{R}}$ such that $k_t(\omega, \cdot) : \mathfrak{A} \to \mathfrak{A}$ be a linear completely positive map, $k_t(\omega, a^*) = k_t(\omega, a)^*$ for all $t \geq 0$, $\omega \in \Omega$ and

$$\mathcal{T}_t(x) = \mathbb{E}(k_t(x)).$$

The complete positivity of k determines that $k_t(x) = V_t^* x V_t$. The dynamics is given through a stochastic differential equation

$$dV_t = \left(-\frac{1}{2}\mathbf{1}\, dt + E_{01} dW_t^{(1)} + E_{10} dW_t^{(0)}\right) V_t. \tag{3.11}$$

To obtain the equation of the flow k, use Itô's formula:

$$\begin{aligned} dk_t(x) &= d(V_t^* x V_t) \\ &= dV_t^* x V_t + V_t^* x dV_t + dV_t^* x dV_t, \end{aligned}$$

and compute the formal product of differentials $dW_t^{(i)} dW_t^{(j)} = \delta_{ij} dt$. This yields to

$$dk_t(x) = k_t(\mathcal{L}(x))dt + k_t(\theta_{01}(x))dW_t^{(1)} + k_t(\theta_{10}(x))dW_t^{(0)}, \tag{3.12}$$

where

$$\begin{aligned} \mathcal{L}(x) &= -x + E_{10} x E_{01} + E_{01} x E_{10}; \\ \theta_{i,j}(x) &= x E_{ij} + E_{ji} x, \quad i, j = 0, 1. \end{aligned}$$

Taking expectation in Eq. (3.12), Brownian terms disappear and one can check that $\mathcal{T}_t(x) = \mathbb{E}\left(k_t(x)\right)$.

Which kind of dynamics induces k on $W^*(K)$, the algebra which is invariant to the action of the QMS?

Take first a unitary vector $u \in \mathbb{C}^2$, and call $\psi_t(u) = V_t u$, with components ψ_t^0, ψ_t^1. From (3.11) we obtain

$$d\psi_t(u) = -\frac{1}{2}\psi_t(u)dt + E_{01}\psi_t dW_t^{(1)} + E_{10} dW_t^{(0)}. \tag{3.13}$$

Define $X_t(f) = \langle \psi_t(u), f(K)\psi_t(u)\rangle = \langle u, k_t(f(K))u\rangle$.

From (3.12) one easily derives

$$\begin{aligned} dX_t(f) = &\langle \psi_t(u), \mathcal{L}(f(K))\psi_t(u)\rangle dt \\ &+ \langle \psi_t(u), \theta_{01}(f(K))\psi_t(u)\rangle dW_t^{(1)} \\ &+ \langle \psi_t(u), \theta_{10}(f(K))\psi_t(u)\rangle dW_t^{(0)}. \end{aligned}$$

We have

$$\begin{aligned} \langle \psi_t(u), \mathcal{L}(f(K))\psi_t(u)\rangle &= \langle \psi_t(u), Lf(K)\psi_t(u)\rangle \\ &= X_t(Lf) \end{aligned}$$

$$\begin{aligned}\langle \psi_t(u), \theta_{01}(f(K))\psi_t(u)\rangle &= \psi_t^1(\psi_t^0 + \psi_t^1)f(1)\\ &\quad + \psi_t^1(\psi_t^0 - \psi_t^1)f(-1)\\ \langle \psi_t(u), \theta_{01}(f(K))\psi_t(u)\rangle &= \psi_t^0(\psi_t^0 + \psi_t^1)f(1)\\ &\quad - \psi_t^0(\psi_t^0 - \psi_t^1)f(-1).\end{aligned}$$

So that $f \mapsto X_t(f)$ is a *random measure*, moreover, $X_t(f) - \int_0^t X_s(Lf)ds$ is a martingale with quadratic variation process $\int_0^t Q_s(f)ds$ where

$$\begin{aligned}Q_t(f) = {}&[(\psi_t^1)^2 + (\psi_t^0)^2](\psi_t^1 + \psi_t^0)^2 f(1)^2\\ &- 2[(\psi_t^1)^2 - (\psi_t^0)^2]f(1)f(-1)\\ &+ [(\psi_t^1)^2 + (\psi_t^0)^2](\psi_t^1 - \psi_t^0)^2 f(-1)^2.\end{aligned}$$

Notice that this measure-valued process is no more a Dirac kernel because the flow k is not a homomorphism. Thus, the combination of classical reduction and classical dilation of a quantum Markov semigroup leads to a *measure-valued process* (this process runs over the set of generators L of Markov chains defined on the set $\{-1, 1\}$).

3.4. *A view on classical dilations*

Here we state without proof one of the main results of [22] (see also [23]), which gives the solution to a wide class of Stochastic Schrödinger equations in infinite dimensions. These equations allow one to construct a classical dilation of a given QMS. This dilation combined with a classical reduction provide a class of measure-valued stochastic processes.

Consider the following linear stochastic differential equation.

$$\psi_t(\xi) = \xi + \int_0^t G\psi_s(\xi)ds + \sum_{k=1}^{\infty}\int_0^t L_k\psi_s(\xi)dW_s^k. \tag{3.14}$$

Here, $W^1, W^2, \ldots$, are real-valued independent Wiener processes on a filtered complete probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$. This time $G, L_1, L_2, \ldots$, are (possibly) unbounded linear operators in $\mathfrak{h}$ satisfying $\mathrm{Dom}(G) \subset \mathrm{Dom}(L_k)$, with $k \in \mathbb{N}$, such that

$$2\Re\langle x, Gx\rangle + \sum_{k=1}^{\infty}\|L_k x\|^2 = 0 \tag{3.15}$$

for any $x \in \mathrm{Dom}(G)$.

LSSE-Hypotheses. There exists a linear a self-adjoint positive operator $C : \mathfrak{h} \to \mathfrak{h}$ such that:

(i) $\mathrm{Dom}(C) \subset \mathrm{Dom}(G) \cap \mathrm{Dom}(G^*)$.
(ii) There exists an orthonormal basis $(e_n)_{n\in\mathbb{Z}_+}$ of $\mathfrak{h}$ formed by elements of $\mathrm{Dom}(C)$ such that for all $n \in \mathbb{Z}_+$, $\sum_{k=1}^{\infty}\|L_k^* e_n\|^2 < \infty$.

(iii) Let $P_n : \mathfrak{h} \to \mathfrak{h}_n$ be the orthogonal projection of $\mathfrak{h}$ over $\mathfrak{h}_n$, where $\mathfrak{h}_n$ is the linear manifold spanned by $e_0, \ldots e_n$. Then, there exist constants $\alpha, \beta \in [0, \infty[$ satisfying

$$2\Re \langle Cx, CP_nGx \rangle + \sum_{k=1}^{\infty} \|CP_nL_kx\|^2 \leq \alpha(\|Cx\|^2 + \|x\|^2 + \beta), \tag{3.16}$$

for any $n \in \mathbb{Z}_+$ and $x \in \mathfrak{h}_n$.

(iv) $\sup_{n\in\mathbb{Z}_+} \|CP_nx\| \leq \|Cx\|$ for all $x \in \mathrm{Dom}(C)$.

Definition 3.1. Let C satisfy the LSSE-Hypotheses. Suppose that $\mathbb{T}$ is either $[0, \infty[$ or $[0, T]$ with $T \in [0, \infty[$. We say that the stochastic process $(\psi_t(\xi))_{t\in\mathbb{T}}$ is a strong solution of class C of (3.14) on the interval $\mathbb{T}$ (for simplicity, C-strong solution) if:

- $(\psi_t(\xi))_{t\in\mathbb{T}}$ is an adapted process taking values in $\mathfrak{h}$ with continuous sample paths.
- For any $t \in \mathbb{T}$, $\mathbb{E}\|\psi_t(\xi)\|^2 \leq \mathbb{E}\|\xi\|^2$, $\psi_t(\xi) \in \mathrm{Dom}(C)$ $\mathbb{P}$-a.s. and

$$\sup_{s\in[0,t]} \mathbb{E}\|C \circ \pi_C(\psi_s(\xi))\|^2 < \infty,$$

  where

$$\pi_C(x) = \begin{cases} x, & \text{if } x \in \mathrm{Dom}(C), \\ 0, & \text{if } x \notin \mathrm{Dom}(C). \end{cases}$$

- $\mathbb{P}$-a.s., for all $t \in \mathbb{T}$,

$$\psi_t(\xi) = \xi + \int_0^t G \circ \pi_C(\psi_s(\xi))ds + \sum_{k=1}^{\infty} \int_0^t L_k \circ \pi_C(\psi_s(\xi))dW_s^k. \tag{3.17}$$

Theorem 3.3. (Mora-Rebolledo [22]) *Let C satisfy the LSSE-Hypotheses. Suppose that ξ is a $\mathfrak{F}_0$-random variable taking values in $\mathfrak{h}$ such that $\xi \in \mathrm{Dom}(C)$ a.s. and $\mathbb{E}(\|\xi\|_C^2) < \infty$. Assume that $\mathbb{T}$ is either $[0, \infty[$ or $[0, T]$ whenever $T \in [0, \infty[$. Then, there exists a unique C-strong solution $(\psi_t(\xi))_{t\in\mathbb{T}}$ of (3.14). In addition, for all $t \in \mathbb{T}$ we have*

$$\mathbb{E}\|C\psi_t(\xi)\|^2 \leq \exp(\alpha t)\,(\mathbb{E}\|C\xi\|^2 + \alpha t(\mathbb{E}\|\xi\|^2 + \beta)). \tag{3.18}$$

And moreover,

$$\mathbb{E}\|\psi_t(\xi)\|^2 = \mathbb{E}\|\xi\|^2. \tag{3.19}$$

Corollary 3.1. *Assume the LSSE-Hypotheses with H nondegenerate and define a classical flow k through the relation*

$$\langle v, k_t(x)u \rangle = \langle \psi_t(v), x\psi_t(u) \rangle, \tag{3.20}$$

for all $u, v \in \mathfrak{h}$.

Suppose that there exists a collection of real numbers λ_j such that

$$[H, L_j] = \lambda_j \mathbf{1}, \tag{3.21}$$

for all j.

Then $(k_t)_{t\in\mathbb{R}^+}$ is classically reduced by $W^*(H)$.

Fix $u \in \mathfrak{h}$. If $E_H(dx)$ is the spectral family of H, denote $\mu(dx) = \langle u, E_H(dx)u\rangle$. Define

$$X_t(f) = \langle \psi_t(u), f(H)\psi_t(u)\rangle, \tag{3.22}$$

where f is a bounded measurable function defined on the spectrum of H.

Then X is a classical measure-valued stochastic process. If L denotes the restriction of the generator $\mathcal{L}$ to the algebra $W^*(H)$, it holds that $X_t(f) - \int_0^t X_s(Lf)ds$ is a martingale for all f such that $f(K) \in \mathrm{Dom}(\mathcal{L})$.

Proof. First of all, notice that an application of the classical Itô's formula yields

$$dk_t(x) = k_t(\mathcal{L}(x))dt + \sum_\ell k_t(\vartheta_\ell(x))dW_t^\ell, \tag{3.23}$$

where $\vartheta_\ell(x) = L_\ell^* x + xL_\ell$. It is important to notice that k_t is not an homomorphism. From (3.21) it holds that $\vartheta(W^*(H)) \subset W^*(H)$. Therefore k is classically reduced by H. As a result, the process X is measure-valued and

$$X_t(f) - \int_0^t X_s(Lf)ds = \sum_\ell \int_0^t X_s(\vartheta_\ell(f))dW_s^\ell,$$

where we used the same notation ϑ for the restriction of that map to $W^*(H)$. The right-hand side of the above equation is clearly a martingale, and its quadratic variation process is

$$\int_0^t \sum_\ell X_s(\vartheta_\ell(f))^2 ds.$$

□

Example 3.1. Consider a typical quantum harmonic oscillator:

$$\begin{aligned} d\psi_t(u) = &-\frac{1}{2}(a^*a + aa^*)\psi_t(u)dt \\ &+ a\psi_t(u)dW_t^1 + a^*\psi_t(u)dW_t^2. \end{aligned}$$

This produces a measure-valued process X which solves the equation

$$\begin{aligned} X_t(f) = \mu(f) &+ \int_0^t X_s\left(\frac{1}{2}f''\right)ds \\ &+ \int_0^t X_s\left(-\frac{1}{\sqrt{2}}f' + \sqrt{2}I\ f\right)dW_s^1 \\ &+ \int_0^t X_s\left(\frac{1}{\sqrt{2}}f' + \sqrt{2}I\ f\right)dW_s^2, \end{aligned}$$

where $I(x) = x$.

The above equation implies that

$$X_t(f) - \mu(f) - \int_0^t X_s\left(\frac{1}{2}f''\right)ds$$

is a martingale, whose quadratic variation is

$$\int_0^t \left[\left(X_s \left(-\frac{1}{\sqrt{2}} f' + \sqrt{2} I\ f \right) \right)^2 + \left(X_s \left(\frac{1}{\sqrt{2}} f' + \sqrt{2} I\ f \right) \right)^2 \right] ds.$$

4. Stationary States

We end by giving some applications of classical reductions to the search of invariant states for QMS.

Theorem 4.1. *Assume that a QMS $\mathcal{T}$ commutes with an automorphism group generated by a non-degenerate bounded self-adjoint operator H. Then the restriction of $\mathcal{T}$ to the algebra $\mathcal{F}(\alpha)$ is isomorphic to a classical Markov semigroup $T = (T_t)_{t\in\mathbb{R}^+}$ defined on the algebra $L^\infty(\sigma(H), \nu)$ where $\sigma(H)$ is the spectrum of H and ν is the measure determined by its spectral decomposition. Moreover, if the semigroup $\mathcal{T}$ is norm-continuous, then T satisfies the Feller property, that is, the C^*-algebra $C_0(\sigma(H))$ of all continuous functions vanishing at infinity is invariant under T.*

In particular, suppose that the semigroup $\mathcal{T}$ satisfies the hypothesis (H). Then a density matrix ρ which commutes with H defines an invariant state for $\mathcal{T}$ if and only if:

$$\operatorname{tr}(\rho\Psi(x)) = \operatorname{tr}(\rho\Psi(\mathbf{1})x), \tag{4.1}$$

for all $x \in \mathfrak{L}(\mathfrak{h})$, where Ψ is related to the generator of $\mathcal{T}$ by (2.3).

Proof. If H is nondegenerate, then the algebra $W^*(H)$ is maximal Abelian (see [25] Chap. 4), therefore, $W^*(H) = W^*(H)' = \mathcal{F}(\alpha)$. Thus, by the Spectral Theorem, $\mathcal{F}(\alpha)$ is isomorphic to the space $L^\infty(\sigma(H), \mu)$ where μ is a Radon measure obtained from the spectral decomposition of H. More precisely, there exists an isometry $U : L^2(\sigma(H), \mu) \to \mathfrak{h}$ such that $f \mapsto UM_fU^*$ is an isometric *-isomorphism of $L^\infty(\sigma(H), \mu)$ on $\mathcal{F}(\alpha)$, where M_f denotes the operator multiplication by f. We define the semigroup T by

$$M_{T_t f} = \mathcal{T}_t(UM_fU^*).$$

This is a Markov semigroup, since the complete positivity is preserved, $\mathbf{1} \in \mathcal{F}(\alpha)$ and $T_t 1 = 1$. If the original semigroup is norm continuous, then T is a contraction for the uniform norm, so that $C_0(\sigma(H))$ is invariant under the action of T and the semigroup is Feller.

Finally, notice that if ρ commutes with H, the hypothesis (H) implies that it also commutes with $\Psi(\mathbf{1})$ which is an element of the Abelian algebra $W^*(H)$. So that,

$$\begin{aligned}
\operatorname{tr}(\mathcal{L}_*(\rho)x) &= \operatorname{tr}(\rho\mathcal{L}(x)) \\
&= \operatorname{tr}(\rho G^* x) + \operatorname{tr}(\rho\Psi(x)) + \operatorname{tr}(\rho x G) \\
&= \operatorname{tr}(\rho G^* x) + \operatorname{tr}(\rho\Psi(x)) + \operatorname{tr}(\rho G x) \\
&= \operatorname{tr}(\rho(G^* + G)x) + \operatorname{tr}(\rho\Psi(x)) \\
&= -\operatorname{tr}(\rho\Psi(\mathbf{1})x) + \operatorname{tr}(\rho\Psi(x))
\end{aligned}$$

for all $x \in \mathfrak{L}(\mathfrak{h})$. Now, ρ is a fixed point for the predual semigroup $\mathcal{T}_*$ (invariant state) if and only if $\mathcal{L}_*(\rho) = 0$. The previous computation shows that the latter is equivalent to have (5.49) holds true. □

Consider again a QMS $\mathcal{T}$ defined on a general von Neumann algebra $\mathfrak{M}$. Suppose that $\mathcal{T}$ has a stationary state φ, so that

$$\varphi(\mathcal{T}_t(x)) = \mathcal{T}_{t*}(\varphi)(x) = \varphi(x),$$

for all $t \geq 0$, $x \in \mathfrak{M}$.

We adopt the hypotheses of Theorem 2.3, that is, K is a nondegenerate self-adjoint operator affiliated with $\mathfrak{M}$ which satisfies one of the equivalent conditions to have $\mathcal{T}$ reduced by K. Then for any $x = f(K) \in W^*(K)$, we obtain

$$\varphi(x) = \int f(y)\varphi(\xi(dy)) = \varphi(\mathcal{T}_t(x)) = \int \mathbf{T}_t f(y)\varphi(\xi(dy)),$$

so that

$$\mu_\varphi(dx) = \varphi(\xi(dx)) \tag{4.2}$$

is a stationary measure for the reduced semigroup $(\mathbf{T}_t)_{t\in\mathbb{R}^+}$.

Suppose now that μ is a stationary measure for $(\mathbf{T}_t)_{t\in\mathbb{R}}$. One want to use this measure to determine a stationary state for $\mathcal{T}$. However, a stationary state could even not exist.

In [27] a partial answer is provided to this question, i.e. whether the knowledge of stationary measures for the classical semigroup completely determines a stationary state for the quantum Markov semigroup. Indeed, a classical reduction obtained through a pure point spectrum self-adjoint operator K dramatically simplifies the previous question. In that case, $W^*(K)$ is isomorphic to $\ell^\infty(\mathbb{N})$ which contains the space $\ell^1(\mathbb{N})$ and is in turn isomorphic to the predual space $W^*(K)_*$. Let rephrase Theorem 6 of [27]. We assume that the orthonormal basis of eigenvectors of K can be chosen on the dense subset D which is a core for the operator G and all the operators L_k defining the form-generator $\mathfrak{L}(\cdot)$ of the quantum Markov semigroup. Then, as proved in [13], Secs. 2 and 3, the linear space spanned by all the projections $|e_n\rangle\langle e_m|$, $(n, m \in \mathbb{N})$, is a core for the predual generator $\mathcal{L}_*$. As a result, all the operators $\mathcal{L}_*(|e_n\rangle\langle e_n|)$ are well defined.

Theorem 4.2. *Suppose that the self-adjoint operator K is nondegenerate and that $W^*(K)$ reduces the minimal quantum Markov semigroup $\mathcal{T}$ defined on $\mathfrak{L}(\mathfrak{h})$, obtained through a form-generator which satisfies the hypotheses of Sec.* 2.2. *Assume that there exists a faithful probability density $(p(\lambda))_{\lambda\in\mathbf{Sp}(K)}$ on the spectrum of K which is stationary for the reduced semigroup. If for all $n \in \mathbb{N}$, $\mathcal{L}_*(|e_n\rangle\langle e_n|)$ commutes with K, then $p(K)$ is a stationary state of $\mathcal{T}$.*

Proof. Call $\rho = p(K) = \sum_n p(\lambda_n)|e_n\rangle\langle e_n|$. We will prove that $\rho \in D(\mathcal{L}_*)$ and that $\mathcal{L}_*(\rho) = 0$. Given any bounded function f on $\mathbf{Sp}(K)$, call L the reduction of the generator, that is $\langle e_n, \mathcal{L}(f(K))e_n\rangle = Lf(\lambda_n)$. The hypothesis on the stationarity of p is then expressed as $\sum_n p(\lambda_n)Lf(\lambda_n) = 0$.

Define

$$\rho_N = \sum_{n\leq N} p(\lambda_n)|e_n\rangle\langle e_n|, \quad N \in \mathbb{N}.$$

Notice that $\rho_N \in D(\mathcal{L}_*)$, since each projection $|e_n\rangle\langle e_n|$ belongs to $D(\mathcal{L}_*)$. Moreover, $\mathcal{L}_*(\rho_N)$ is a trace-class operator as well (see Sec. 2.2). Since $\mathcal{L}_*(|e_n\rangle\langle e_n|)$ commutes with K and K is nondegenerate, $\mathcal{L}_*(\rho_n) \in W^*(K)$ too.

Taking $N, M \in \mathbb{N}$, with $N > M$ say, one obtains

$$|\mathrm{tr}((\mathcal{L}_*(\rho_N) - \mathcal{L}_*(\rho_M))x)| \leq C(x) \sum_{n=M+1}^{N} p(\lambda_n),$$

for any fixed $x \in D(\mathcal{L})$, for a constant $C(x) > 0$. Since $\sum_n p(\lambda_n) = 1$, we obtain that $\mathcal{L}_*(\rho_N)$ weakly converges as $N \to \infty$. On the other hand, ρ_N converges in the norm of the trace to ρ. Since $\mathcal{L}_*(\cdot)$ is weakly closed, then $\mathcal{L}_*(\rho) = \lim_N \mathcal{L}_*(\rho_N)$ and $\rho \in D(\mathcal{L}_*)$. Moreover, $\mathcal{L}_*(\rho)$ is a trace-class operator which commutes with K.

To prove that $\mathcal{L}_*(\rho) = 0$ it suffices to show that $\langle e_k, \mathcal{L}_*(\rho)e_k\rangle = \mathrm{tr}(\mathcal{L}_*(\rho)|e_k\rangle\langle e_k|) = 0$ for all $k \in \mathbb{N}$. Now,

$$\begin{aligned}\mathrm{tr}(\mathcal{L}_*(\rho)|e_k\rangle\langle e_k|) &= \mathrm{tr}(\rho\mathcal{L}(|e_k\rangle\langle e_k|)) \\ &= \sum_n p(\lambda_n)\langle e_n, \mathcal{L}(|e_k\rangle\langle e_k|)e_n\rangle \\ &= \sum_n p(\lambda_n)L1_{\{\lambda_k\}}(\lambda_n) \\ &= 0, \quad \text{(since } p \text{ is stationary for the reduced semigroup).}\end{aligned}$$

Thus, $\mathcal{L}_*(\rho) = 0$ and ρ is a stationary state for $\mathcal{T}$. □

Notice that $\mathcal{L}_*(|e_n\rangle\langle e_n|)$ commutes with K in the above theorem as soon as the operators H and L_j of the form-generator satisfy the hypothesis (a) of Theorem 2.4 and (b*) below:

(b*) Given any $(u, v) \in D \times D$, $\ell \geq 1$,

$$\langle {L_\ell}^* v, e_n\rangle\langle e_n, {L_\ell}^* e_p\rangle\langle e_p, u\rangle = \langle e_p, v\rangle\langle {L_\ell}^* e_p, e_n\rangle\langle e_n, {L_\ell}^* u\rangle, \tag{4.3}$$

for all $n, p \in \mathbb{N}$.

Indeed, G is affiliated with $W^*(K)$ according to Theorem 2.4(a), so that $|e_n\rangle\langle e_n|$ commutes with each operator ${L_\ell}^* L_\ell$. Therefore, to verify that $\mathcal{L}_*(|e_n\rangle\langle e_n|)$ commutes with K it suffices to check that for each $\ell \geq 1$, the operator $L_\ell|e_n\rangle\langle e_n|{L_\ell}^*$ commutes with each spectral projection $|e_r\rangle\langle e_r|$ and this follows from (b*) via a straightforward computation.

Corollary 4.1. *Let K be as in the previous theorem and suppose that H and $(L_j,\, j \geq 1)$ satisfy the hypotheses* (a), (b) *of Theorem* 2.4 *and in addition* (b*) *here before. Then K reduces the quantum Markov semigroup $\mathcal{T}$ and if there exists a faithful probability density $(p(\lambda))_{\lambda\in\mathbf{Sp}(K)}$ on the spectrum of K which is stationary for the reduced semigroup, the density operator $p(K)$ defines a stationary state of $\mathcal{T}$.*

Proof. By Theorem 2.4, K reduces the semigroup. The previous remark shows that $\mathcal{L}_*(|e_n\rangle\langle e_n|)$ commutes with K for each $n \in \mathbb{N}$, so that Theorem 4.2 yields the conclusion. □

If we assume the semigroup to be continuous in norm and defined in a general von Neumann algebra like in Corollary 2.6, we can specialize the previous result to

Corollary 4.2. *Suppose that K is a pure-point spectrum non degenerate self-adjoint operator affiliated with $\mathfrak{M}$, with eigenvectors $(e_n)_{n\in\mathbb{N}}$ providing an orthonormal basis of $\mathfrak{h}$. Assume that the QMS is norm-continuous, defined on the von Neumann algebra $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{h})$ and such that the coefficients H and L_k, $k \in \mathbb{N}$, of its generator satisfy*

(a) $[H, |e_n\rangle\langle e_n|] = 0$, *and*
(b) $[L_k, |e_n\rangle\langle e_n|] = c_k(n)L_k$, *where $c_k(n)$ is a self-adjoint element in $W^*(K)$, for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$.*

Then the semigroup is reduced by K and if the reduced semigroup admits a stationary probability p, then $p(K)$ defines a stationary density matrix for the quantum Markov semigroup.

Proof. To apply Theorem 4.2 it remains to prove that for all $n \in \mathbb{N}$, $\mathcal{L}_*(|e_n\rangle\langle e_n|)$ commutes with K. This is an easy consequence of (a) and (b). Indeed, the above conditions imply that $|e_n\rangle\langle e_n|$ commutes with H as well as with each operator ${L_k}^*L_k$. Moreover, given any $m \in \mathbb{N}$,

$$\begin{aligned}[|e_m\rangle\langle e_m|, L_k|e_n\rangle\langle e_n|{L_k}^*] &= [|e_m\rangle\langle e_m|, L_k]|e_n\rangle\langle e_n|{L_k}^* \\ &\quad + L_k|e_n\rangle\langle e_n|[|e_m\rangle\langle e_m|, {L_k}^*] \\ &= c_k(m)L_k|e_n\rangle\langle e_n|{L_k}^* - c_k(m)L_k|e_n\rangle\langle e_n|{L_k}^* \\ &= 0.\end{aligned}$$

□

5. Applications

5.1. *Squeezed reservoirs*

A class of open quantum dynamics appearing in recent studies in Quantum Optics is addressed within this section. The form of generators was obtained in [18] by application of a weak coupling limit approach, this is summarized here below.

Interactions between quantum systems and broadband squeezed reservoirs yield a variety of *modified* quantum dynamics. Strong modification of spontaneous emission rates [20, 15], and the resonance fluorescent spectrum [9] have been predicted for a two-level atom. The case of a quantum system consisting of one atom interacting with one mode of the electromagnetic field inside a cavity has been studied as well. The modified dynamics arise because of the reduction of noise in one of the reservoir quadratures (squeezing), which is, roughly speaking, transferred to the quantum system. That means obtaining a state which reduces the variance in the measurement of a particular observable at the expense of increased uncertainty in the measurement of a second noncommuting observable. Depending on the reservoir, theoretically two specific models may be considered. The first case consists of a continuum of modes of the electromagnetic field interacting with one atom. In the second case, the interaction takes place between the electromagnetic field and the atoms of the cavity. An effective squeezed-like reservoir is obtained by an adiabatic elimination of additional atomic levels under a convenient choice of the system parameters. Here, an arbitrary quantum system interacting with a *squeezed vacuum* is considered as a physical framework. The reservoir is assumed to be composed of an infinite set of radiation modes, which has a *flat* spectrum around the characteristic frequency of the quantum system, that is, a so-called *broadband squeezed vacuum.* The choice of radiation modes for the reservoir is motivated by the simplicity of the commutation relation between creation and annihilation operators.

5.1.1. *Squeezing the vacuum*

Here we consider again a complex separable Hilbert space $\mathfrak{h}_S$ describing the main system dynamics, and a reservoir composed of an infinite number of bosonic particles, that is, the Canonical Commutation Relations (CCR) are satisfied. Denote $\mathfrak{h}_R$ another complex Hilbert space and let denote $\Gamma_b(\mathfrak{h}_R)$ the bosonic Fock space which will be used to represents the dynamics of the reservoir. Moreover, let $\mathfrak{R}(\mathfrak{h}_R)$ the C^*-algebra generated by the Weyl operators $(W(f);\ f \in \mathfrak{h}_R)$, which is included in the algebra $\mathfrak{L}(\Gamma_b(\mathfrak{h}_R))$ of all the endomorphisms of $\Gamma_b(\mathfrak{h}_R)$. These operators satisfy $W(-f) = W(f)^*$ and the CCR expressed in the form:

$$W(f)W(g) = e^{-i\Im\langle f,g\rangle} W(f+g), \quad f, g \in \mathfrak{h}_R. \tag{5.1}$$

The *field operators* are defined as

$$\Phi(f) = -i\frac{d}{dt}W(tf)|_{t=0}, \quad f \in \mathfrak{h}_R, \tag{5.2}$$

that is, $\Phi(f)$ is the generator of the unitary group $(W(tf))_{t\in\mathbb{R}}$.

In terms of field operators, one can introduce the customary *creation* and *annihilation* operators as follows:

$$A^\dagger(f) = \frac{1}{\sqrt{2}}\left(\Phi(f) - i\Phi(if)\right), \tag{5.3}$$

$$A(f) = \frac{1}{\sqrt{2}}\left(\Phi(f) + i\Phi(if)\right), \tag{5.4}$$

for all $f \in \mathfrak{h}_R$.

Given any *real linear* invertible operator T on $\mathfrak{h}_S$ such that

$$\Im\langle Tf, Tg\rangle = \Im\langle f, g\rangle, \tag{5.5}$$

a unique $*$-automorphism γ_T of $\mathfrak{R}(\mathfrak{h}_R)$ is induced by

$$\gamma_T(W(f)) = W(Tf).$$

The mean-zero quasi-free state φ on $\mathfrak{R}(\mathfrak{h}_R)$

$$\varphi(W(f)) = \exp\left(-\frac{1}{2}\|f\|^2\right), \quad f \in \mathfrak{h}_R,$$

called the *vacuum state*, is transformed by the automorphism γ_T into the state $\varphi \circ \gamma_T$ determined by

$$(\varphi \circ \gamma_T)(W(f)) = \varphi(W(Tf)) = \exp\left(-\frac{1}{2}\Re\langle f, Qf\rangle\right), \quad f \in \mathfrak{h}_R,$$

where Q is a *real linear* operator in $\mathfrak{h}_R$ such that

$$\Re\langle f, Qf\rangle = \|Tf\|^2. \tag{5.6}$$

To stress the dependence of the state $\varphi \circ \gamma_T$ on Q, φ_Q will be preferred as a notation. We call Q the *squeezing operator*.

The GNS representation of the algebra $\mathfrak{R}(\mathfrak{h}_R)$ with state φ_Q allows to represent Weyl operators $W(f)$ as unitary operators $W_Q(f)$ in the Fock space over $\mathfrak{h}_R$. Moreover the unitary one-parameter groups $(W_Q(tf))_{t\in\mathbb{R}}$ are strongly continuous. Also, the associated field operators are correspondingly written Φ_Q.

Denoting φ_Q too the pure state corresponding to the given quasi-free state, a straightforward computation yields

$$\varphi_Q(\Phi_Q(f)) = 0,\ \varphi_Q(\Phi_Q(f)\Phi_Q(g)) = i\Im\langle f, g\rangle + \frac{1}{2}\Re\left(\langle f, Qg\rangle + \langle g, Qf\rangle\right). \tag{5.7}$$

The Canonical Commutation Relation (5.1) yields

$$[\Phi_Q(f), \Phi_Q(if)] = i\|f\|^2\mathbf{1}.$$

It is worth noticing that, if the operator Q is complex linear, then both $\varphi_Q(A_Q(f)A_Q(f))$ and $\varphi_Q(A_Q^\dagger(f)A_Q^\dagger(f))$ vanish and the variances $\varphi_Q(|\Phi_Q(f)|^2)$, $\varphi_Q(|\Phi_Q(if)|^2)$ coincide with $\Re\langle f, Qf\rangle$.

However, since Q is only *real linear*, the variances of the conjugate fields $\Phi_Q(f)$, $\Phi_Q(if)$ could be different. In particular, one might be small at the expenses of the other so that the inequality (Heisenberg principle)

$$\varphi_Q(|\Phi_Q(f)|^2) \cdot \varphi_Q(|\Phi_Q(if)|^2) \geq \frac{1}{4}\left|\varphi_Q\left([\Phi_Q(f), \Phi_Q(if)]\right)\right|^2 \tag{5.8}$$

is fulfilled.

Definition 5.1. A *quasi-free* state φ_Q is a *squeezed vacuum* if it has *zero mean*, that is $\varphi_Q(\Phi_Q(f)) = 0$ for every $f \in \mathfrak{h}_R$, and $\varphi(|\Phi_Q(f)|^2) \neq \varphi_Q(|\Phi_Q(if)|^2)$ for some $f \in \mathfrak{h}_R$, and it satisfies the identity

$$\|f\|^4 = \varphi_Q(|\Phi_Q(f)|^2) \cdot \varphi_Q(|\Phi_Q(if)|^2) = \frac{1}{4} \left|\varphi_Q\left([\Phi_Q(f), \Phi_Q(if)]\right)\right|^2, \tag{5.9}$$

which corresponds to the equality in inequality (5.8).

The "one-particle" free evolution of the reservoir is characterized by a unitary group S_t acting on the algebra of all bounded linear operator on $\mathfrak{h}_R$, denoted $\mathfrak{L}(\mathfrak{h}_R)$, with a generator H_1, for which it is assumed that

$$S_t Q = Q S_t, \quad t \in \mathbb{R}, \tag{5.10}$$

therefore the second quantization $\Gamma(S_t)$ leaves φ_Q invariant and is implemented by a semigroup with generator $H_R = d\Gamma(H_1)$. This generator is customarily called the *Hamiltonian of the free evolution of the reservoir.*

5.1.2. *Weak coupling limit and the master equation*

The evolution of the system coupled with the reservoir is associated with the unitary group generated by the total Hamiltonian $H^{(\lambda)}$ given by

$$H^\lambda = H_S \otimes 1_R + 1_S \otimes H_R + \lambda V, \tag{5.11}$$

where H_S is the Hamiltonian of the system; λ is a coupling constant and

$$V = i(D \otimes A_Q^\dagger(g) - D^* \otimes A_Q(g)), \tag{5.12}$$

D and its adjoint D^*, being linear operators on $\mathfrak{h}_S$.

To perform the weak coupling limit (see e.g. [1, 2]), several hypotheses are required, which we precise here below:

(H1) There exists a dense subset of elements $\xi \in \mathfrak{h}_S$ for which D satisfies

$$\sum_{n=1}^{\infty} \frac{|\langle \xi, D^n \xi\rangle|}{[n/2]!} < \infty. \tag{5.13}$$

(H2) In addition, it is assumed the *rotating wave approximation* hypothesis, which postulates the existence of a frequency ω_0 for which

$$\exp(itH_S) D \exp(-itH_S) = \exp(-i\omega_0 t) D. \tag{5.14}$$

(H3) As in [2] assume there exists a linear nonzero subspace of the domain of Q such that, for any two elements f and g of it, the function $t \mapsto \exp(-i\omega_0 t)\langle f, S_t g\rangle$ is Lebesgue-integrable on the whole real line. The integral of the above function allows to introduce a structure of a pre-Hilbert space inside the domain of Q (see Lemma (3.2) in [2]). By a customary procedure which consists in taking the quotient of the above pre-Hilbert space by the zero-norm elements, one obtains a Hilbert space denoted K with the scalar product:

$$(f|g) = \int_{-\infty}^{\infty} \exp(-i\omega_0 t)\langle f, S_t g\rangle dt. \tag{5.15}$$

Moreover define

$$(f|g)_- = \int_{-\infty}^{0} \mathrm{e}^{-i\omega_0 t}\langle f, S_t g\rangle dt, \quad (f|g)_+ = \int_{0}^{+\infty} \mathrm{e}^{-i\omega_0 t}\langle f, S_t g\rangle dt.$$

Define an approximated wave operator $U^\lambda(t)$ by:

$$U^\lambda(t) = \exp(itH^0)\exp(-itH^\lambda). \tag{5.16}$$

The weak coupling limit is studied under the Friedrichs–van Hove time rescaling, that is, one is interested in characterizing the asymptotic behavior of $U^\lambda(t/\lambda^2)$ as $\lambda \to 0$. For convenience, we recall here the following theorem proved in [2].

Theorem 5.1. *Under the above assumptions* (H1) *to* (H3), *a limit* $U(t)$ *exists. More precisely, for each* $f_1, f_2 \in K, u_1, u_2 \in h_0$ *and each* $t_1, s_1, t_2, s_2 \in \mathbb{R}$ *the matrix elements*

$$\left\langle u_1 \otimes W_Q\left(\lambda \int_{s_1/\lambda^2}^{t_1/\lambda^2} S_r f_1 dr\right), U^\lambda(t/\lambda^2) u_2 \otimes W_Q\left(\lambda \int_{s_2/\lambda^2}^{t_2/\lambda^2} S_r f_2 dr\right)\right\rangle$$

converge as λ *tends to* 0 *towards the corresponding matrix elements of the unitary solution* U *of the quantum stochastic differential equation*:

$$\begin{aligned} dU(t) = \big(&D \otimes dA^\dagger(t,g) - D^* \otimes dA(t,g) \\ &- \frac{1}{4}(F_{-,+}DD^* \otimes 1 + F_{+,-}D^*D \otimes 1 \\ &+ F_{-,-}D^*D^* \otimes 1 + F_{+,+}DD \otimes 1)dt\big)U(t). \end{aligned} \tag{5.17}$$

The noises A *and* $A^\dagger$ *can be regarded as a quantum Brownian motion in the Fock space* $\Gamma(L^2(\mathbb{R}_+) \otimes K)$. *They satisfy the following Itô table*:

$$dA(t,g)dA^\dagger(t,g) = \frac{1}{2}\Re F_{+,-}dt, \tag{5.18}$$

$$dA(t,g)dA(t,g) = -\frac{1}{2}F_{-,-}dt, \tag{5.19}$$

$$dA^\dagger(t,g)dA^\dagger(t,g) = -\frac{1}{2}F_{+,+}dt, \tag{5.20}$$

$$dA^\dagger(t,g)dA(t,g) = \frac{1}{2}\Re F_{-,+}dt. \tag{5.21}$$

The terms $F_{-,-}, F_{+,+}, F_{-,+}, F_{+,-}$ are numerical coefficients given by

$$\begin{aligned} 2F_{+,-} &= \Re((g|Qg) + (ig|Q(ig))) + 2(g|g) + 4i\Im(g|g)_- \\ &\quad + i\Re((ig|Qg)_- + (g|Q(ig))_+ - (ig|Qg)_+ - (g|Q(ig))_-), \qquad (5.22) \\ 2F_{-,-} &= \Re((g|Qg) - (ig|Q(ig))) + i\Re((ig|Qg) + (g|Q(ig))), \qquad (5.23) \\ 2F_{+,+} &= \Re((g|Qg) - (ig|Q(ig))) - i\Re((ig|Qg) + (g|Q(ig))), \qquad (5.24) \\ 2F_{-,+} &= \Re((g|Qg) + (ig|Q(ig))) - 2(g|g) - 4i\Im(g|g)_+ \\ &\quad - i\Re((ig|Qg)_- + (g|Q(ig))_+ - (ig|Qg)_+ - (g|Q(ig))_-). \qquad (5.25) \end{aligned}$$

The coefficients $F_{\pm,\pm}$ can be interpreted when compared to the covariances $\varphi_Q(A_Q(f)A_Q^\dagger(f)$, $\varphi_Q(A_Q(f)A_Q(f), \ldots$.

The evolution of system observables under the unitary transformations $U(t)$

$$x \mapsto U(t)^*(x \otimes 1)U(t)$$

projected on the system space $\mathfrak{h}_S$ yields a quantum dynamical semigroup $(\mathcal{T}_t)_{t\geq 0}$ on $\mathfrak{L}(\mathfrak{h}_S)$ with infinitesimal generator

$$\begin{aligned} \mathcal{L}(x) = &-\frac{\nu+\eta}{2}\left(D^*Dx - 2D^*xD + xD^*D\right) \\ &- \frac{\nu}{2}\left(DD^*x - 2DxD^* + xDD^*\right) \\ &- \frac{\zeta}{2}(D^{*2}x - 2D^*xD^* + xD^{*2}) \\ &- \frac{\bar{\zeta}}{2}\left(D^2x - 2DxD + xD^2\right) \\ &- i\xi[DD^* - D^*D, x] + i\delta[DD^* + D^*D, x], \qquad (5.26) \end{aligned}$$

where the parameters are:

$$2\nu = \Re F_{-,+}, \quad \eta = (g|g), \quad 2\zeta = F_{-,-} = \overline{F_{+,+}}, \quad \delta = \frac{1}{2}\Im(g|g)_-,$$
$$\xi = \frac{1}{8}\Re((ig|Qg)_- + (g|Q(ig))_+ - (ig|Qg)_+ - (g|Q(ig))_-).$$

Here before we used the easily proved identities

$$\Re(g|g)_- = \Re(g|g)_+ = (g|g)/2, \quad \Im(g|g)_- = -\Im(g|g)_+.$$

5.1.3. *Classical unravelings*

(a) *The CCR case*

As a first case, consider D and D^* satisfying the CCR, that is $DD^* - D^*D = \mathbf{1}$ and that the function g is chosen so that $F_{-,-} = F_{+,+} = 0$.

In that case, $\zeta = 0$ and the term with coefficient ξ vanishes in (5.30) so that the generator becomes

$$\mathcal{L}(x) = i[H,x] - \frac{\nu+\eta}{2}\left(D^*Dx - 2D^*xD + xD^*D\right) - \frac{\nu}{2}(DD^*x - 2DxD^* + xDD^*), \tag{5.27}$$

where $H = \delta(2D^*D + \mathbf{1})$. A simple formal computation proves that the operator $K_1 = D^*D$, which commutes with H, satisfies in addition

$$[K_1, D] = -D, \tag{5.28}$$

so that K_1 classically reduces the QMS.

An example of this kind of model is given by the following choice of representation: $\mathfrak{h}_S = \ell^2(\mathbb{N})$, $D = a$, where a is the customary annihilation operator defined on $\mathfrak{h}_S$. Thus K_1 is the number operator N. Therefore, if f is any bounded function $f : \mathbb{N} \to \mathbb{C}$,

$$\mathcal{L}(f(N)) = (Lf)(N),$$

where

$$Lf(n) = -(\nu+\eta)n\,(f(n) - f(n-1)) - \nu(n+1)\,(f(n) - f(n+1)), \tag{5.29}$$

is the generator of a birth and death semigroup.

Moreover, if $\delta = 0$, a straightforward application of Corollary 2.5 and Remark 2.5 yields another operator $K_2 = D + D^*$ which reduces the semigroup too.

(b) *The CAR case*

As a second case, let us assume this time that D and D^* are bounded and satisfy the CAR, i.e. $\{D, D^*\} = DD^* + D^*D = \mathbf{1}$, $\{D, D\} = \{D^*, D^*\} = 0$. Then the term with coefficient δ vanishes. Moreover, if $\zeta \neq 0$, the generator has the form

$$\begin{aligned}\mathcal{L}(x) = i[H,x] &- \frac{\nu+\eta}{2}\left(D^*Dx - 2D^*xD + xD^*D\right)\\ &- \frac{\nu}{2}\left(DD^*x - 2DxD^* + xDD^*\right)\\ &+ \zeta D^*xD^*\\ &+ \bar{\zeta}DxD,\end{aligned} \tag{5.30}$$

where $H = \xi(2D^*D - \mathbf{1})$. Choosing again $K_1 = D^*D$, an elementary computation using the CAR yields $[D, (D^*D)^n] = D$, thus $[(D^*D)^n, D^*] = D^*$, and moreover $D^*(D^*D)^nD^* = 0 = D(D^*D)^nD$. Therefore,

$$[\mathcal{L}(K_1^n), K_1] = 0, \tag{5.31}$$

so that $\mathcal{L}(C^*(K_1)) \subset C^*(K_1)$.

We illustrate this case with a two-level atom, that is $\mathfrak{h}_S = \mathbb{C}^2$ and D, D^* are the lowering and raising operators in $\mathbb{C}^2$:

$$D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad D^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Using the notations of Example 1.1, we have $D = E_{10} = |e_1\rangle\langle e_0|$, $D^* = E_{01} = |e_0\rangle\langle e_1|$, and $K_1 = D^*D = |e_0\rangle\langle e_0|$. Thus, for any function $f : \{0,1\} \to \mathbb{R}$, one obtains

$$\mathcal{L}(f(K_1)) = \begin{pmatrix} -(\nu+\eta)(f(1)-f(0)) & 0 \\ 0 & \nu(f(1)-f(0)) \end{pmatrix}, \tag{5.32}$$

that is $\mathcal{L}(f(K_1)) = Lf(K_1)$, where L is the classical Markovian generator

$$Lf = -(\nu+\eta)(f(1)-f(0))1_{\{0\}} + \nu(f(1)-f(0))1_{\{1\}}. \tag{5.33}$$

5.2. *The quantum exclusion semigroup*

Consider a self-adjoint bounded operator H_0 defined on a separable complex Hilbert space $\mathfrak{h}_0$. H_0 will be thought of as describing the dynamics of a single fermionic particle. We assume that there is an orthonormal basis $(\psi_n)_{n\in\mathbb{N}}$ of eigenvectors of H_0, and denote E_n the eigenvalue of ψ_n $(n \in \mathbb{N})$. The set of all finite subsets of $\mathbb{N}$ is denoted $\mathfrak{P}_f(\mathbb{N})$ and for any $\Lambda \in \mathfrak{P}_f(\mathbb{N})$, we denote $\mathfrak{h}_0^\Lambda$ the finite-dimensional Hilbert subspace of $\mathfrak{h}_0$ generated by the vectors $(\psi_n;\ n \in \Lambda)$. To deal with a system of infinite particles we introduce the fermionic Fock space $\mathfrak{h} = \Gamma_f(\mathfrak{h}_0)$ associated to $\mathfrak{h}_0$ whose construction we recall briefly (see [7]).

The Fock space associated to $\mathfrak{h}_0$ is the direct sum

$$\Gamma(\mathfrak{h}_0) = \bigoplus_{n\in\mathbb{N}} \mathfrak{h}_0^{\otimes n},$$

where $\mathfrak{h}_0^{\otimes n}$ is the n-fold tensor product of $\mathfrak{h}_0$, with the convention $\mathfrak{h}_0^{\otimes 0} = \mathbb{C}$. Define an operator $\mathbf{P}_\mathrm{a}$ on the Fock space as follows,

$$\mathbf{P}_\mathrm{a}(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_\pi \varepsilon_\pi f_{\pi_1} \otimes \cdots \otimes f_{\pi_n}.$$

The sum is over all permutations $\pi : \{1, \ldots, n\} \to \{\pi_1, \ldots, \pi_n\}$ of the indices and ε_π is 1 if π is even and -1 if π is odd. Define the antisymmetric tensor product on the Fock space as $f_1 \wedge \cdots \wedge f_n = \mathbf{P}_\mathrm{a}(f_1 \otimes f_2 \otimes \cdots \otimes f_n)$. In this manner, the Fermi–Fock space $\mathfrak{h}$ is obtained as

$$\mathfrak{h} = \Gamma_f(\mathfrak{h}_0) = \mathbf{P}_\mathrm{a}\left(\bigoplus_{n\in\mathbb{N}} \mathfrak{h}_0^{\otimes n}\right) = \bigoplus_{n\in\mathbb{N}} \mathfrak{h}_0^{\wedge n}.$$

We follow [7] to introduce the so-called fermionic *Creation* $b^\dagger(f)$ and *Annihilation* $b(f)$ operators on $\mathfrak{h}$, associated to a given element f of $\mathfrak{h}_0$. Firstly, on $\Gamma(\mathfrak{h}_0)$

we define $a(f)$ and $a^\dagger(f)$ by initially setting $a(f)\psi^{(0)} = 0$, $a^\dagger(f)\psi^{(0)} = f$, for $\psi = (\psi^{(0)}, \psi^{(1)}, \ldots) \in \Gamma(\mathfrak{h}_0)$ with $\psi^{(j)} = 0$ for all $j \geq 1$, and

$$a^\dagger(f)(f_1 \otimes \cdots \otimes f_n) = \sqrt{n+1} f \otimes f_1 \otimes \cdots \otimes f_n. \tag{5.34}$$

$$a(f)(f_1 \otimes \cdots \otimes f_n) = \sqrt{n} \langle f, f_1 \rangle f_2 \otimes f_3 \otimes \cdots \otimes f_n. \tag{5.35}$$

Finally, define annihilation and creation on $\Gamma_f(\mathfrak{h}_0)$ as $b(f) = \mathbf{P}_\mathrm{a} a(f) \mathbf{P}_\mathrm{a}$ and $b^\dagger(f) = \mathbf{P}_\mathrm{a} a^\dagger(f) \mathbf{P}_\mathrm{a}$. These operators satisfy the Canonical Anticommutation Relations (CAR) on the Fermi–Fock space:

$$\{b(f), b(g)\} = 0 = \{b^\dagger(f), b^\dagger(g)\}, \tag{5.36}$$

$$\{b(f), b^\dagger(g)\} = \langle f, g \rangle \mathbf{1}, \tag{5.37}$$

for all $f, g \in \mathfrak{h}_0$, where we use the notation $\{A, B\} = AB + BA$ for the two operators A and B.

Moreover, $b(f)$ and $b^\dagger(g)$ have bounded extensions to the whole space $\mathfrak{h}$ since $\|b(f)\| = \|f\| = \left\|b^\dagger(f)\right\|$.

To simplify notations, we write $b_n^\dagger = b^\dagger(\psi_m)$ (respectively $b_n = b(\psi_n)$) the creation (respectively annihilation) operator associated with ψ_n in the space $\mathfrak{h}_0$, $(n \in \mathbb{N})$.

The C^*-algebra generated by $\mathbf{1}$ and all the $b(f)$, $f \in \mathfrak{h}_0$, is denoted $\mathfrak{A}(\mathfrak{h}_0)$ (the canonical CAR algebra).

Remark 5.1. The algebra $\mathfrak{A}(\mathfrak{h}_0)$ is the unique, up to *-isomorphism, C^*-algebra generated by elements $b(f)$ satisfying the anti-commutation relations over $\mathfrak{h}_0$ (see e.g. [7], Theorem 5.2.5).

The family $(b(f), b^\dagger(g);\ f, g \in \mathfrak{h}_0)$ is *irreducible* on $\mathfrak{h}$, that is, the only operators which commute with this family are the scalar multiples of the identity ([7], Prop. 5.2.2). Clearly, the same property is satisfied by the family $(b_n, b_n^\dagger;\ n \in \mathbb{N})$, since $(\psi_n)_{n \in \mathbb{N}}$ is an orthonormal basis of $\mathfrak{h}_0$.

$\mathfrak{A}(\mathfrak{h}_0)$ is the strong closure of $\mathfrak{D} = \bigcup_{\Lambda \in \mathfrak{P}_f(\mathbb{N})} \mathfrak{A}(\mathfrak{h}_0^\Lambda)$ (see [7], Proposition 5.2.6), this is the *quasi-local* property. Moreover, the finite dimensional algebras $\mathfrak{A}(\mathfrak{h}_0^\Lambda)$ are isomorphic to algebras of matrices with complex components.

An element η of $\{0,1\}^\mathbb{N}$ is called a *configuration* of particles. For each n, $\eta(n)$ takes the value 1 or 0 depending on whether the n-th site has been occupied by a particle in the configuration η. In other terms, we say that the site n is occupied by the configuration η if $\eta(n) = 1$. We denote $\mathbf{S}$ the set of configurations η with a finite number of 1's, that is $\sum_n \eta(n) < \infty$. Each $\eta \in \mathbf{S}$ is then identifiable to the characteristic function $1_{\{s_1, \ldots, s_m\}}$ of a finite subset of $\mathbb{N}$, which, in addition, we will suppose ordered as $0 \leq s_1 < s_2 < \cdots < s_m$. For simplicity we write 1_k the configuration $1_{\{k\}}$, $(k \in \mathbb{N})$. Furthermore, we define

$$\mathbf{b}^\dagger(\eta) = b_{s_m}^\dagger b_{s_{m-1}}^\dagger \cdots b_{s_1}^\dagger, \tag{5.38}$$

$$\mathbf{b}(\eta) = b_{s_m} b_{s_{m-1}} \cdots b_{s_1}, \tag{5.39}$$

for all $\eta = 1_{\{s_1, \ldots, s_m\}}$. Clearly, $\mathbf{b}^\dagger(1_k) = b_k^\dagger$, $\mathbf{b}(1_k) = b_k$, $(k \in \mathbb{N})$.

To obtain a cyclic representation of $\mathfrak{A}(\mathfrak{h}_0)$ we call $|0\rangle$ the vacuum vector in $\mathfrak{h}$, and $|\eta\rangle = \mathbf{b}^\dagger(\eta)|0\rangle$, $(\eta \in \mathbf{S})$. Then $(|\eta\rangle,\ \eta \in \mathbf{S})$ is an orthonormal basis of $\mathfrak{h}$. In this manner, any $x \in \mathfrak{A}(\mathfrak{h}_0)$ can be represented as an operator in $\mathfrak{L}(\mathfrak{h})$. Moreover, call $\mathfrak{v}$ the vector space spanned by $(|\eta\rangle,\ \eta \in \mathbf{S})$.

An elementary computation based on the CAR shows that for any $\eta, \zeta \in \mathbf{S}$, it holds

$$b_k^\dagger|\eta\rangle = (1-\eta(k))|\eta + 1_k\rangle, \tag{5.40}$$

$$b_k|\eta\rangle = \eta(k)|\eta - 1_k\rangle, \quad k \in \mathbb{N}. \tag{5.41}$$

We assume in addition that H_0 is *bounded from below*, so that there exists $b \in \mathbb{R}$ such that $b < E_n$ for all $n \in \mathbb{N}$. Then, the second quantization of H_0 becomes a self-adjoint operator H acting on $\mathfrak{h}$, with domain $D(H)$ which includes $\mathfrak{v}$ and can formally be written as

$$H = \sum_n E_n b_n^\dagger b_n. \tag{5.42}$$

It is worth mentioning that the restriction $H^\Lambda = \sum_{n\in\Lambda} E_n b_n^\dagger b_n$ of H to each space $\Gamma_f(\mathfrak{h}_0^\Lambda)$ is an element of the algebra $\mathfrak{A}(\mathfrak{h}_0^\Lambda)$, $\Lambda \in \mathfrak{P}_f(\mathbb{N})$, so that H^Λ is a bounded operator. Moreover, for each $\eta \in \mathbf{S}$, $\left\| H|\eta\rangle - H^\Lambda|\eta\rangle \right\| \to 0$ as Λ increases to $\mathbb{N}$, for each $\eta \in \mathbf{S}$.

The *transport* of a particle from a site i to a site j, at a rate $\gamma_{i,j}$ is described by an operator $L_{i,j}$ defined as

$$L_{i,j} = \sqrt{\gamma_{i,j}}\, b_j^\dagger b_i. \tag{5.43}$$

This corresponds to the action of a reservoir on the system of fermionic particles pushing them to jump between different sites. Each operator $L_{i,j}$ is an element of $\mathfrak{A}(\mathfrak{h}_0)$ and $\|L_{i,j}\| = \sqrt{\gamma_{i,j}}$. Notice that these operators satisfy

$$[H, L_{i,j}] = \gamma_{i,j} L_{i,j}, \tag{5.44}$$

for all i, j, which is a sufficient condition for the existence of a classical reduction.

We additionally assume that

$$\sup_i \sum_j \gamma_{i,j} < \infty. \tag{5.45}$$

The following proposition has been proved in [27].

Proposition 5.1. *For each $x \in \mathfrak{A}(\mathfrak{h}_0)$ the unbounded operator*

$$\mathcal{L}(x) = i[H,x] - \frac{1}{2}\sum_{i,j}\left(L_{i,j}^* L_{i,j} x - 2L_{i,j}^* x L_{i,j} + x L_{i,j}^* L_{i,j}\right), \tag{5.46}$$

whose domain contains the dense manifold $\mathfrak{v}$, is the generator of a quantum Feller semigroup $\mathcal{T}$ on the C^*-algebra $\mathfrak{A}(\mathfrak{h}_0)$. This semigroup is extended into a σ-weak continuous QMS defined on the whole algebra $\mathfrak{L}(\mathfrak{h})$.

Moreover, the semigroup is reduced by the algebra $W^*(H)$. The reduced semigroup $\mathbf{T}$ corresponds to a classical exclusion process with generator

$$Lf(\eta) = \sum_{i,j} c_{i,j}(\eta)\,(f(\eta + 1_j - 1_i) - f(\eta)), \tag{5.47}$$

for all bounded cylindrical function $f : \mathbf{S} \to \mathbb{R}$, where $c_{i,j}(\eta) = \gamma_{i,j}\eta(i)(1 - \eta(j))$.

Now consider a density matrix which is of the form $p(H)$, that is:

$$\rho = \sum_{\eta} p(\eta)|\eta\rangle\langle\eta|, \tag{5.48}$$

where $\eta \mapsto p(\eta)$ is a summable function with $\sum_\eta p(\eta) = 1$.

Proposition 5.2. *Let us assume that*

$$\pi(i)\gamma_{i,j} = \pi(j)\gamma_{j,i}, \quad i, j \in \mathbb{N} \tag{5.49}$$

where $(\pi(i))_{i\in\mathbb{N}}$ is any sequence of positive numbers. Then a normal state ω with density matrix ρ given by (5.48) is stationary if

$$p(\eta) = \prod_{i\in\mathbb{N}} \alpha_i(\eta(i)),$$

for all $\eta \in \mathbf{S}$, where $\alpha_i : \{0, 1\} \to [0, 1]$ is, for each $i \in \mathbb{N}$, a probability measure given by

$$\alpha_i(x) = \frac{(\pi(i))^x}{1 + \pi(i)}, \quad i \in \mathbb{N}, \; x \in \{0, 1\}. \tag{5.50}$$

Moreover, H induces decoherence of the semigroup $\mathcal{T}$.

The above proposition was proved in [27] using a classical result due to Liggett.

6. Conclusions

Within this paper it has been shown that invariant Abelian subalgebras under the action of a QMS appear naturally in a number of physical models. This phenomenon, connected as well with decoherence, provides an interesting field of interactions between classical and quantum probability. From one side, classical probability can be used to perform numerical computations through the unraveling of an open quantum system for instance, or can be a valuable support to analyze the large time behavior of a QMS (providing explicit ways of obtaining invariant states, for example). On the other hand, quantum probability provides a large theoretical framework where very different classical stochastic processes can be associated to a single quantum Markov semigroup.

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References

1. L. Accardi, A. Frigerio and Y. G. Lu, On the weak coupling limit problem, in *Quantum Probability and Applications*, IV (Rome, 1987) (Springer, 1989), pp. 20–58.
2. L. Accardi, A. Frigerio and Y. G. Lu, The weak coupling limit as a quantum functional central limit, *Comm. Math. Phys.* **131** (1990) 537–570.
3. L. Accardi, Y. G. Lu and I. Volovich, *Quantum Theory and Its Stochastic Limit* (Springer-Verlag, 2002), 493 pp.
4. R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications*, Lect. Notes in Physics, Vol. 286 (Springer-Verlag, 1987).
5. G. Alli and G. L. Sewell, New methods and structures in the theory of the multimode dicke laser model, *J. Math. Phys.* **36** (1995) 5598–5626.
6. S Attal and A. Joye, Weak coupling and continuous limits for repeated quantum interactions, *J. Statist. Phys.* **126** (2007) 1241–1283.
7. O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, 2*, 2nd edn. (Springer-Verlag, 1997).
8. O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, 1*, 2nd edn. (Springer-Verlag, 1987).
9. H. J. Carmichael, Spectrum of squeezing and photocurrent shot noise: A normally ordered treatment, *J. Opt. Soc. Am. B* **4** (1987) 1588–1603.
10. A. Chebotarev, *Lectures on Quantum Probability*, Textos, Vol. 4 (Aportaciones Matem. S.M.M., 2000).
11. A. Chebotarev and F. Fagnola, Sufficient conditions for conservativity of minimal quantum dynamical semigroups, *J. Funct. Anal.* **153** (1998) 382–404.
12. E. Christensen and D. E. Evans, Cohomology of operator algebras and quantum dynamical semigroups, *J. Lon. Math. Soc.* **20** (1979) 358–368.
13. E. B. Davies, Quantum dynamical semigroups and the neutron diffusion equation, *Rep. Math. Phys.* **11** (1977) 169–188.
14. C. Dellacherie and P.-A. Meyer, *Probabilités et Potentiel*, Vols. I–III (Hermann, 1975).
15. L. M. Duan, A. Sørensen, J. I. Cirac and P. Zoller, Squeezing and entanglement of atomic beams, *Phys. Rev. Lett.* **85** (2000) 3991–3994.
16. R. Rebolledo, F. Fagnola and C. Saavedra, Quantum flows associated to master equations in quantum optics, *J. Math. Phys.* **35** (1994) 1–12.
17. F. Fagnola, Quantum markov semigroups and quantum flows, *Proyecciones, J. Math.* **18** (1999) 1–144.
18. F. Fagnola, R. Rebolledo and C. Saavedra, Reduction of noise by squeezed vacuum, in *Stochastic Analysis and Mathematical Physics* (Viña del Mar, 1996) (World Scientific, 1998), pp. 61–71.
19. F. Fagnola and M. Skeide, Restrictions of CP-semigroups to maximal commutative subalgebras, in *Noncommutative Harmonic Analysis with Applications to Probability*, Banach Center Publ., Vol. 78 (Polish Acad. Sci., 2007), pp. 121–132.
20. C. Gardiner, Inhibition of atomic phase decays by squeezed light: A direct effect of squeezing, *Phys. Rev. Lett.* **56** (1986) 1917–1920.
21. G. Lindblad, On the generators of quantum dynamical semigroups, *Comm. Math. Phys.* **48** (1976) 119–130.
22. C. M. Mora and R. Rebolledo, Regularity of solutions to linear stochastic Schrödinger equations, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **10** (2007) 237–259.
23. C. M. Mora and R. Rebolledo, Basic properties of nonlinear stochastic Schrödinger equations driven by Brownian motions, *Ann. Appl. Probab.* **18** (2008) 591–619.
24. K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, Monographs in Mathematics, Vol. 85 (Birkhaüser-Verlag, 1992).

25. G. Pedersen, *Analysis Now*, Graduate Texts in Mathematics, Vol. 118 (Springer-Verlag, 1989).
26. R. Rebolledo and D. Spehner, Adiabatic limits and quantum decoherence, in *Stochastic Analysis in Mathematical Physics* (World Scientific, 2008), pp. 94–108.
27. R. Rebolledo, Decoherence of quantum Markov semigroups, *Ann. Inst. H. Poincaré Probab. Statist.* **41** (2005) 349–373.
28. W. F. Stinespring, Positive functions on C^*-algebras, *Proc. Am. Math. Soc.* **6** (1955) 211–216.
29. A. Kossakowski, V. Gorini and E. C. G. Sudarshan, Completely positive dynamical semigroups of n-level systems, *J. Math. Phys.* **17** (1976) 821–825.