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POLYNOMIAL DYNAMIC AND LATTICE ORBITS IN S-ARITHMETIC HOMOGENEOUS SPACES

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Consider a homogeneous space under a locally compact group G and a lattice Γ in G. Then the lattice naturally acts on the homogeneous space. Looking at a dense orbit, one may wonder how to describe its repartition. One then adopts a dynamical point of view and compare the asymptotic distribution of points in the orbits with the natural measure on the space. In the setting of Lie groups and their homogeneous spaces, several results show an equidistribution of points in the orbits.

We address here this problem in the setting of p-adic and S-arithmetic groups.

Keywords: Lattice in Lie and p-adic groups; equidistribution.

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Contents

1. Introduction

1.1. *Historical background*

Ten years ago, Ledrappier [13] explained how Ratner's theory (in this particular case, he needed a theorem of Dani [4]) shall be used to understand the asymptotic properties of the action of $\mathrm{SL}(2,\mathbb{Z})$ on the Euclidean plane $\mathbb{R}^2$. He proved the following:

Theorem 1.1. (Ledrappier [13], Nogueira [17]) *Let* Γ *be a lattice of* $\mathrm{SL}(2,\mathbb{R})$ *of finite covolume* $c(\Gamma)$, $|\cdot|$ *the Euclidean norm on the plane* $\mathbb{R}^2$, $\|\cdot\|$ *the Euclidean norm on the algebra of* 2×2*-matrices* $\mathcal{M}(2,\mathbb{R})$, *and* $v\in\mathbb{R}^2$ *with non-discrete orbit under* Γ*.*

Then we have the following limit, for all $\phi\in\mathcal{C}_c(\mathbb{R}^2\backslash\{0\})$:

$$\frac{1}{T}\sum_{\gamma\in\Gamma,\|\gamma\|\le T}\phi(\gamma v)\xrightarrow{T\to\infty}\frac{2}{|v|c(\Gamma)}\int_{\mathbb{R}^2\backslash\{0\}}\phi(w)\frac{dw}{|w|}.$$

Remark. Nogueira [17] proved also the previous theorem for $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ using different techniques.

After that Gorodnik developed the strategy for the space of frames [8] and eventually Gorodnik and Weiss gave an abstract theorem for this problem in Lie groups and then applied it to different situations [10].

Recently, Ledrappier and Pollicott [14], and independently the author in his PhD thesis [11], proved a p-adic analog of the first theorem for lattices of $\mathrm{SL}(2,\mathbb{Q}_p)$ acting on the p-adic plane.

In this paper we adapt this strategy to handle the case of homogeneous space under S-arithmetic groups. Our work can be viewed as the analog of [10] in this setting.

1.2. *The S-arithmetic setting*

We will work in the following arithmetic setting: let K be a number field, $\mathcal{O}$ its integer ring and $\mathcal{V}$ the set of its places. We fix a finite set S in $\mathcal{V}$ containing the Archimedean ones. For all $\nu \in \mathcal{V}$, we note K_ν the completion of K associated to ν and K_S the module product of all K_ν for $\nu \in S$. This ring has a set of integers, noted $\mathcal{O}_S$.

Consider $\mathbf{G}$ a semisimple simply connected K-group. We note $G := \mathbf{G}(K_S)$ its S-points, and we fix Γ an arithmetic lattice, i.e. commensurable to $\mathbf{G}(\mathcal{O}_S)$. Recall that, according to Margulis superrigidity theorem, as soon as the total rank of G is greater than 2, any lattice in G is an arithmetical one. Then let H be a subgroup of G which is a product $\prod_{\nu \in S} H_\nu$ of closed subgroups of $\mathbf{G}(K_\nu)$. For example, one can think to the stabilizer of a point for an action of G defined over K, i.e. $H = g\bar{H}g^{-1}$ where $\bar{H}$ is the K_S-points of a K-group and g an element in G. We will always assume that the subgroup H is unimodular. Some references for these objects are to be found in [18] and [15].

We are interested in the asymptotic distribution of orbits of Γ in $H\backslash G$, so we will always assume this orbit to be dense, or equivalently that $H\Gamma$ is dense in G. This last assumption is quite different of some recent works in the same area [9, 6] where H is supposed to have a closed projection in G/Γ and the dynamic appears by looking at larger and larger orbits. In particular, there will not be any adelic arguments in this work.

1.2.1. *Measures and projections*

Definition 1.1. We say that a triple (G, H, Γ) is *under study* if we are in the precedent case, that is if there is a number field K, a finite set S of places containing the Archimedean ones, and a K-group $\mathbf{G}$, K-reductive and with simply connected semisimple part, such that:

- G is the K_S points of $\mathbf{G}$,
- Γ is an arithmetic lattice in G,
- H is the product of unimodular K_ν-subgroups of $\mathbf{G}(K_\nu)$ for $\nu \in S$,
- $H\Gamma$ is dense in G and H is not compact,
- H is a semidirect product $H^{ss} \rtimes H^u$ of a semisimple part and an unipotent radical.

We now fix some notations for projections and measures: we fix a Haar measure m_G on G; m_H on H; and note m the probability measure on G/Γ locally proportional to m_G. On $H\backslash G$, as H is unimodular, we have a unique — up to scaling — G-invariant measure. We normalize the measure $m_{H\backslash G}$ on $H\backslash G$ such that m_G is locally the product of m_H and $m_{H\backslash G}$. The notations for the projections are as shown:

$$\begin{tikzcd}[column sep=small] & G \arrow[dl, "\tau"'] \arrow[dr, "\pi"] & \\ H\backslash G & & G/\Gamma \end{tikzcd}$$

1.2.2. *Balls and volume*

In order to adopt a dynamical point of view, we need to instillate some evolution in the so far static situation. So we consider families $(G_t)_{t\in R}$ of open and bounded subsets in G (often called balls), and consider the sets $\Gamma_t = \Gamma \cap G_t$. Letting t go to ∞, we may now consider the asymptotic distribution of the sets $H\backslash H\Gamma_t$ in $H\backslash G$. Of course we will consider families (G_t) that are increasing and exhausting (the union of G_t covers G).

We introduce a notation for the intersection of such a family (G_t) and its translates with subsets of G:

Definition 1.2. Fix $(G_t)_{t\in\mathbb{R}}$ a family of open subset G, L a subset of G and g an element of G. Then for all real t, we note $L_t := L \cap G_t$ the intersection of G_t and L and $L_t(g)$ the intersection $L \cap G_t g^{-1}$.

As the restriction of the so-called balls of G, we call the sets L_t *balls* in L, and *skew-balls* the sets $L_t(g)$.

When L is a subgroup, we can compare the growth of volume of its normal subgroup with respect to the sets (G_t). It may happens that a strict subgroup grows as fast as the whole group. Such a subgroup is exhibited in [10, Sec. 12.3]. We will call such a subgroup dominant:

Definition 1.3. Let L be a unimodular subgroup of G and m_L be its Haar measure. Fix G_t a family of open bounded subsets of G, increasing and exhausting.

A normal subgroup L' is said to be dominant in L if for some compact C in L, the volume of $C \cdot L'_t$ grows as fast as the volume of L_t, i.e. $\frac{m_L(C\cdot L'_t)}{m_L(L_t)}$ does not converge to 0 with t.

Eventually we need an explicit way to define balls in Γ. Going back to Ledrappier's theorem, we see that the balls are constructed considering a norm on the algebra of 2×2-matrices. Moreover, Gorodnik and Weiss [10] defined their balls in the same spirit, first representing the group G and then using a norm on the matrix algebra in which G is embedded. Our strategy is the same, but for technical reasons we assume firstly that the norms are "algebraic", secondly that the unipotent radical and the semisimple part are somehow orthogonal with respect to the norm and eventually that the norm on the unipotent part verifies a kind of ellipticity.

Definition 1.4. A *size function* D from G to $\mathbb{R}_+$ is any function constructed in the following way: consider a K-representation ρ of $\mathbf{G}$ in a space $\mathbf{V}$ with compact kernel in G and for all $\nu \in S$ a norm $|\cdot|_\nu$ on the space $\mathrm{End}(\mathbf{V}(K_\nu))$ verifying:

- (Algebraicity) If ν is Archimedean, the norm $|\cdot|_\nu$ may be written in a suitable basis as the L_p-norm for p in $\mathbb{N}^* \cup \{\infty\}$. If ν is ultrametric, we assume that it is the max-norm in some basis.
- (Orthogonality) for all $h_\nu = (h_\nu^{ss}, h_\nu^u)$ in H, its norm $|h_\nu|_\nu$ is an increasing function of both $|h_\nu^{ss}|_\nu$ and $|h_\nu^u|_\nu$.

Now define D for all $g = (g_\nu)_{\nu \in S}$ by the formula $D(g) = \max\{|g_\nu|_\nu \text{ for } \nu \in S\}$.

In order to state the ellipticity condition, let us define some notations: thanks to ρ, we may see H^u_ν as a subgroup of $\mathrm{GL}(\mathbf{V}(K_\nu))$ and its Lie algebra $\mathfrak{h}^u = \prod \mathfrak{h}^u_\nu$ as a Lie subalgebra of $\mathrm{End}(\mathbf{V}(K_\nu))$. We also have for each ν the exponential map $\exp_\nu$ between $\mathfrak{h}^u_\nu$ and H^u_ν and its inverse $\log_\nu$. On each $\mathfrak{h}^u_\nu$, one may choose a basis $B^\nu = (u^\nu_i)$ such that $[u^\nu_i, u^\nu_j]$ only has components on $(u^\nu_l)_{l>i,\, l>j}$ (e.g. a basis adapted to the central filtration). The ellipticity condition states that, up to a diffeomorphism of $\mathfrak{h}_\nu$, the pullback by exp_ν of the norm $|\cdot|_\nu$ in $\mathfrak{h}^u_\nu$ is (a power of) $|\cdot|_\nu$.

Definition 1.5. (Ellipticity) We say moreover that a size function D is *elliptic on* H^u if, for every $\nu \in S$, there exist a $\alpha > 0$ and a polynomial diffeomorphism ϕ_ν of $\mathfrak{h}^u_\nu$ whose differential is upper triangular unipotent in the basis B^ν, for which $\phi_\nu(0) = 0$ and such that $|\exp_\nu(\phi_\nu(u))|_\nu$ is equivalent to $|u|^\alpha_\nu$.

Example 1.1. The canonical example behind the definition is the unipotent radical of a parabolic subgroup in $\mathrm{SL}(n, K_S)$. Consider an integer $1 \leq k < n$ and the following group U (Id_k is the identity matrix of size k):

$$U = \left\{ \begin{pmatrix} \mathrm{Id}_k & (u_{i,j})_{i \leq k,\, j>k} \\ 0 & \mathrm{Id}_{n-k} \end{pmatrix},\ \text{with } u_{i,j} \in K_S \right\}.$$

We choose for all $\nu \in S$ the max-norm on $\mathcal{M}(n, K_\nu)$. It defines a size function on U. Now this size function is elliptic, as we may consider the diffeomorphism ϕ_ν defined by:

$$\exp \circ \phi\nu : \begin{pmatrix} 0 & (u_{i,j}) \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{Id}_k & (u_{i,j}) \\ 0 & \mathrm{Id}_{n-k} \end{pmatrix}.$$

The definition of ellipticity guarantees the following properties of ϕ_u:

Fact 1. The diffeomorphisms ϕ_ν defined above preserve a Haar measure on $\mathfrak{h}^u_\nu$ and there is a line Z_ν in the center $\mathfrak{z}_\nu$ of $\mathfrak{h}_\nu$ on which ϕ_ν is the identity and along which ϕ_ν is affine:

$$\text{For all } u \in \mathfrak{h}_\nu \quad \text{and} \quad z \in Z_\nu,\ \text{we have } \phi_\nu(h+z) = \phi_\nu(h) + z.$$

Proof. This comes directly from the fact that the differential is upper triangular unipotent in the basis B^ν: the Jacobian of ϕ_ν is 1 and ϕ_ν is affine along the direction generated by the last vector of the basis (which belongs to the center by construction). And on this line ϕ_ν is the identity. □

Let us discuss these assumptions before proceeding. The first one (algebraicity) does not seem to be crucial. We will mainly use it in Sec. 3 and the result obtained there may be proved in numerous applications by a direct calculus. The second one (orthogonality) is more important and we do not know whether it is necessary or not. Let us say that for one of the applications (1.3), its verification is not so straightforward. The third one (ellipticity) is stated in a strong way to avoid technicalities. One definitely may relax it. Our main interest in this paper was not to focus on the unipotent part and we stated this only in view of the applications given (where the unipotent part is the radical of a parabolic subgroup of $\mathrm{SL}(n)$). For applications we may verify that these conditions are fulfilled (see Sec. 7). We would like to stress that, when H is semisimple, the only condition is algebraicity. Moreover, every example given in the historical section fit into the framework of this paper.

In this setting, given a size function D, we have an associated family of balls $G_t := \{g \in G \text{ such that } D(g) < t\}$ in G.

1.3. *Statement of the main result*

We prove in this paper the following result:

Theorem 1.2. *Let (G, H, Γ) be a triple under study, D be a size function on G elliptic on H^u and $(G_t)_{t>0}$ be the associated family of balls. Assume that every dominant subgroup H' verifies $H'\Gamma$ is dense in G.*

Then there is a finite partition $I_1, \ldots, I_l$ of $\mathbb{R}_{>0}$, and for each $1 \leq i \leq l$, a function $\alpha_i : H\backslash G \to \mathbb{R}_{>0}$ such that the orbit of the sets $\Gamma_t = G_t \cap \Gamma$ for $t \in I_i$ becomes distributed in $H\backslash G$ according to the density α_i with respect to $m_{H\backslash G}$. That means, for all $\psi \in \mathcal{C}_c(H\backslash G)$, we have:

$$\frac{1}{m_H(H_t)} \sum_{\gamma \in \Gamma_t} \psi(\tau(\gamma)) \xrightarrow[t \in I_i]{t \to +\infty} \int_{H\backslash G} \psi(x)\alpha_i(x) dm_{H\backslash G}(x).$$

The partition of the parameter space in a finite number of subspaces is not needed when there is no non-Archimedean places as in [10] but appears even with very simple examples as soon as ultrametric part is to be taken in consideration. Let us also precise that the densities α_i are explicitly described and effectively computable in examples given afterwards (see Theorem 2.1).

We present here some examples of applications. Of course one may look at numerous situations. We just present here some variations about linear actions of the special linear group on points or subspaces. We believe that these examples show how to apply the previous theorem to specific situations, using algebraic features such as strong approximation in the special linear group. The proofs are postponed to Sec. 7.

1.3.1. *Applications to* $\mathrm{SL}(2)$

Consider the group $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{Q}_p)$ for p a prime number, and fix the lattice $\Gamma = \mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$. We fix here (for sake of simplicity) the standard Euclidean norm $|\cdot|_\infty$ on the matrix algebra $\mathcal{M}(2, \mathbb{R})$ and the max-norm $|\cdot|_p$ on $\mathcal{M}(2, \mathbb{Q}_p)$. For a point v in $\mathbb{R}^2$, we note also $|v|_\infty$ the norm of the matrix whose first column is v and the second one is 0. We define similarly the norm of a point in $\mathbb{Q}_p^2$. We choose a Haar measure $m = m_\infty \otimes m_p$ on G.

First we look at the action on the real plane, proving a result similar to Ledrappier's theorem but for the action of matrices in Γ subject to congruence conditions on their coefficients modulo p:

Application 1.1. Let O be a bounded open subset of $\mathrm{SL}(2, \mathbb{Q}_p)$. Note Γ_T^O the set of elements $\gamma \in \Gamma$ such that $|\gamma|_\infty \leq T$ and $\gamma \in O$ as an element of $\mathrm{SL}(2, \mathbb{Q}_p)$. Let v be a point of the plane $\mathbb{R}^2 \backslash \{0\}$ with coordinates independent over $\mathbb{Q}$.

Then we have the following limit, for any function ϕ continuous with compact support in $\mathbb{R}^2 \backslash \{0\}$:

$$\frac{1}{T} \sum_{\Gamma_T^O} \phi(\gamma(v)) \xrightarrow{T \to \infty} \frac{m_p(O)}{m(G/\Gamma)|v|_\infty} \int_{\mathbb{R}^2} \phi(w) \frac{dw}{|w|_\infty}.$$

Another action of Γ of interest is on the product of real and p-adic planes. A precision: on the p-adic plane, we normalize the measure such that it gives mass 1 to $\mathbb{Z}_p^2$. The result is that if your beginning point generates the whole plane among the $\mathbb{Q}$-subspaces, then its orbit is dense and you get a distribution result (the function E appearing is the integer part):

Application 1.2. Let (v_∞, v_p) be an element of $(\mathbb{R}^2 \backslash 0) \times (\mathbb{Q}_p^2 \backslash 0)$. Suppose that any $\mathbb{Q}$-subspace V of $\mathbb{Q}^2$ verifying $v_\infty \in V \otimes_\mathbb{Q} \mathbb{R}$ and $v_p \in V \otimes_\mathbb{Q} \mathbb{Q}_p$ is $\mathbb{Q}^2$. Denote Γ_T the set of elements $\gamma \in \Gamma$ with $|\gamma|_\infty \leq T$ and $|\gamma|_p \leq T$.

Then, for all function ϕ continuous with compact support in $(\mathbb{R}^2 \backslash 0) \times (\mathbb{Q}_p^2 \backslash 0)$, we have the following limit:

$$\frac{1}{Tp^{E(\ln_p(T))}} \sum_{\Gamma_T} \phi(\gamma v_\infty, \gamma v_p) \xrightarrow{T \to \infty} \frac{p^2 - 1}{p^2 m(G/\Gamma)|v_\infty|_\infty |v_p|_p} \int_{\mathbb{R}^2 \times \mathbb{Q}_p^2} \phi(v, w) \frac{dv dw}{|w|_\infty |w|_p}.$$

All these results may be extended with the tools presented in the paper for any norm on the matrix algebras and by considering not only a prime number but a finite number of them.

1.3.2. *Applications to* $\mathrm{SL}(n)$

We look here at a generalization in greater dimension. We consider the action of $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ on the kth exterior power $\Lambda^k(\mathbb{R}^n)$, or the space of k-planes equipped with a volume. Once again we fix the standard Euclidean norm $|\cdot|$ on $\mathcal{M}(n, \mathbb{R})$, but this time it is necessary to apply our theorem (see Sec. 7). We consider also the standard Euclidean norm $|\cdot|$ on $\Lambda^k(\mathbb{R}^n)$. And m is a Haar measure on $\mathrm{SL}(n, \mathbb{R})$. We get:

Application 1.3. Let v be a nonzero element of $\Lambda^k(\mathbb{R}^n)$ such that its corresponding k-plane of $\mathbb{R}^n$ contains no rational vector. Denote Γ_T the set of elements $\gamma \in \Gamma$ with $|\gamma| \leq T$.

Then we have a positive real constant c (independent of Γ and v) such that for all function ϕ continuous with compact support on $\Lambda^k(\mathbb{R}^n)\backslash\{0\}$:

$$\frac{1}{T^{n^2+k^2-nk-n}}\sum_{\Gamma_T}\phi(\gamma v)\xrightarrow{T\to\infty}\frac{c}{m(G/\Gamma)|v|}\int_{\Lambda^k(\mathbb{R}^n)}\phi(v')\frac{dv'}{|v'|}.$$

The S-arithmetic generalization of the previous result of course holds. We prefer to postpone its statement and its proof to Sec. 7. Moreover, we do not want to multiply here statements but one may think at examples in special unitary groups or Spin groups instead of the special linear one.

1.4. *Organization of the paper*

The organization of the paper is the following: in Sec. 2 we work out the so-called duality phenomenon, reducing the stated theorem to two results: a statement on volume of balls in the group and an analog of a result of Shah about equidistribution of balls of H in G/Γ. Section 3 is devoted to the study of volume of balls, using p-adic integration. In Sec. 4 we review some tools we need to prove the analog of Shah theorem: mainly Ratner theorem for unipotent flows in a p-adic setting and several results due to G. Tomanov for polynomial dynamics in S-arithmetic homogeneous spaces. Section 5 is the devoted to some technical work. We conclude the proof in the sixth section. Eventually we treat the examples in the last section.

2. Duality

The duality phenomenon, as used by Ledrappier [13] and Gorodnik–Weiss [10], is a consequence of the following idea: a property of the action of Γ on $H\backslash G$ reflects in a property of the action of H on G/Γ. The simplest example is the density of an orbit: Hg has dense orbit under Γ in $H\backslash G$ if and only if $g\Gamma$ has dense orbit under H in G/Γ. This consideration leads to the key point in the proof of Ledrappier: instead of looking at the orbit of the lattice Γ in the space $H\backslash G$, we prefer to translate the problem in terms of the action of H in G/Γ. And then we may use the precise description of unipotent orbits in the space G/Γ, namely Ratner's theory (cf. Sec. 4) to prove some equidistribution results. However, for asymptotic distribution of points, this phenomenon is not granted and requires additional assumptions that we will review in this section.

We may remark that if H is symmetric, Benoist and Oh used other techniques, i.e. the mixing property, to study asymptotic distribution of orbits [1].

In [10, Corollary 2.4], Gorodnik and Weiss presented an axiomatic frame for duality. Unfortunately we cannot use directly their statement as we miss some continuity hypothesis on the distance function — once again the ultrametric part has to be handled specifically, even if the final result holds. So we present a slightly adapted version of their result in Theorem 2.1.

In the setting defined in the precedent section, consider an increasing and exhausting family G_t of open bounded subsets in G. We need a hypothesis of regularity on this family. We choose to state it using the right action of open subsets of G and asking the sets G_t to be uniformly almost invariant by some open set. As we are interested in the intersections with H, the precise (and classical) definition is:

Definition 2.1. Let $(G_t)_{t\in I}$ be a family of open bounded subsets of G. We say that it is *almost (right)-invariant* if for every $\epsilon > 0$ one can find an open neighborhood U_ϵ of id in G such that the two following inequalities hold for every $t \in I$:

- the set G_tU_ϵ is not too large with respect to G_t inside H:

$$m_H(H \cap G_tU_\epsilon \backslash G_t) \leq \epsilon m_H(H \cap G_t).$$

- Not too much points inside G_t are U_ϵ-closed to its complement inside H:

$$m_H(H \cap G_t \backslash G_t^c U_\epsilon) \geq (1-\epsilon) m_H(H \cap G_t).$$

One easily checks that the balls G_t defined by a size function on G are almost invariant. Indeed for the Archimedean part, any norm on the matrix algebra is continuous. And for the ultrametric part, the max-norm is invariant under some open neighborhood of identity.

We also need a result of existence of limits for ratios of volumes of skew-balls in H (Hypothesis D2 in [10]). Recall Definition 1.2: for $g \in G$ and $t \in I$, $H_t(g)$ is the set $H \cap G_t g^{-1}$.

Definition 2.2. We say that a family $(G_t)_{t\in I}$ admits *volume ratio limits for H* if for all g in G the ratio $\frac{m_H(H_t(g))}{m_H(H_t)}$ admits a limit as t goes to $+\infty$ in I.

The Corollary 2.4 of [10] (and its proof) implies the following theorem:

Theorem 2.1. *Let (G, H, Γ) be a triple under study. Let $(G_t)_{t\in I}$ be a family of bounded open subsets of G almost invariant, admitting volume ratio limits for H and such that the volumes of $H_t = H \cap G_t$ go to $+\infty$. Assume moreover that the orbit of H_t in G/Γ becomes equidistributed with respect to $m_{G/\Gamma}$; i.e. for all $\phi \in \mathcal{C}_c(G/\Gamma)$, we have*:

$$\frac{1}{m_H(H_t)} \int_{H_t} \phi(\pi(h)) dm_H(h) \xrightarrow[t\in I]{t\to+\infty} \int_{G/\Gamma} \phi dm_{G/\Gamma}.$$

Then the orbit of $\Gamma_t = G_t \cap \Gamma$ is distributed in $H\backslash G$ according to a density with respect to $m_{H\backslash G}$; i.e. for all $\psi \in \mathcal{C}_c(H\backslash G)$, we have:

$$\frac{1}{m_H(H_t)} \sum_{\gamma\in\Gamma_t} \psi(\tau(\gamma)) \xrightarrow[t\in I]{t\to+\infty} \int_{H\backslash G} \psi(H.g) \frac{m_H(H_T g)}{m_H(H_T)} dm_{H\backslash G}(Hg).$$

In particular the density of the limit measure is described as limit ratio of volumes of balls. We will see in the next section a proof of existence of these ratios. But in this paper we will not go into precise and general estimates of these volumes. Our theorem still benefits of these estimations when available, e.g. in the applications (see Sec. 7). Maucourant [16] gets very precise estimations for H real semisimple.

Proof. The proof is the same as [10, Parts 3 and 4]: the almost invariance replacing the hypothesis of right continuity of the distance function. □

Now we have to understand the right setting to apply this theorem. There are two difficulties: the existence of volume ratio limits and the equidistribution of H-orbits in G/Γ. The next section address the first problem. We will prove the following theorem:

Theorem 2.2. *Let (G, H, Γ) be a triple under study, D a size function on G. Consider $(G_t)_{t\in\mathbb{R}}$ the family of balls for D. Suppose that the volume of H_t goes to $+\infty$.*

Then there exists a finite partition of $\mathbb{R}$ in unbounded subsets $I_1, \ldots, I_k$ such that for all $1 \leq l \leq k$ the family $(G_t)_{t\in I_l}$ admits volume ratio limits for H.

We shall exhibit in the following section a very simple example showing that we really need this partition.

The second part of the paper is to prove the equidistribution property under the hypothesis of Theorem 1.2: H is a semidirect product of a semisimple and a unipotent groups and every dominant subgroup has dense orbit in G/Γ. We will prove in Sec. 6 the following theorem:

Theorem 2.3. *Let (G, H, Γ) be a triple under study, D a size function on G, elliptic on H^u and H_t the induced family of balls in H. Assume that every dominant subgroup H' of H has dense orbit in G/Γ.*

Then the orbits of H_t becomes equidistributed in G/Γ with respect to $m_{G/\Gamma}$; i.e. for all $\phi \in \mathcal{C}_c(G/\Gamma)$, we have:

$$\frac{1}{m_H(H_t)} \int_{H_t} \phi(\pi(h)) dm_H(h) \xrightarrow{t\to+\infty} \int_{G/\Gamma} \phi dm_{G/\Gamma}.$$

Theorem 1.2 is then a direct consequence of the three previous results.

3. Asymptotic Developments of Volumes

3.1. *An example*

The following part is slightly technical and may be misunderstood without any example in mind. Let us show on a very simple example that we have to be careful in describing the asymptotics of volumes of balls.

We will take here $G = \mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{Q}_p)$ for some prime p and H the image under the adjoint representation of $\mathrm{SL}(2)$ of the upper triangular nilpotent subgroup:

$$H = \left\{ h(t_\infty, t_p) = \left(\begin{pmatrix} 1 & 2t_\infty & t_\infty^2 \\ 0 & 1 & t_\infty \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2t_p & t_p^2 \\ 0 & 1 & t_p \\ 0 & 0 & 1 \end{pmatrix} \right) ;\ t_\infty \in \mathbb{R} \text{ and } t_p \in \mathbb{Q}_p \right\}.$$

We choose the max-norm on both $\mathcal{M}_3(\mathbb{R})$ and $\mathcal{M}_3(\mathbb{Q}_p)$ such that:

$$H_{p^n} = \{h(s_\infty, s_p) \text{ for } s_\infty \in \mathbb{R} \text{ with } |s_\infty^2| \leq p^n \text{ and } s_p \in \mathbb{Q}_p \text{ with } |s_p^2|_p \leq p^n\}.$$

Hence the volume of H_{p^n} is equal to $p^{\frac{n}{2}+E(\frac{n}{2})}$ (E is the *integer part*).

Now let us have a look on a specific skew-ball: $H_{p^n}\left(\mathrm{Id}, \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^{-1} \end{pmatrix}\right)$, and we note $g = \left(\mathrm{Id}, \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^{-1} \end{pmatrix}\right)$. Then the skew-ball is described by:

$$H_{p^n}(g) = \{h(s_\infty, s_p) \text{ for } |s_\infty|^2 \leq p^n \text{ and } |p^{-1}s_p^2|_p \leq p^n\},$$

hence its volume $m_H(H_{p^n}(g))$ is equal to $p^{\frac{n}{2}+E(\frac{n-1}{2})}$. We see that the ratio $\frac{m_H(H_{p^n}(g))}{m_H(H_{p^n})}$ is equal to $p^{E(\frac{n}{2})-E(\frac{n-1}{2})}$. This sequence does not admit any limit as n goes to ∞. But we can split it in two subsequences: n odd or even. And then both subsequences admit a limit (respectively p and 1).

Keeping this example in mind we will now explain why we are always able to do this: split the space of parameters t in a finite number of subspaces in which the hypothesis of admitting volume ratio limits is fulfilled.

3.2. *Volume ratio limits*

We will prove here Theorem 2.2 stated above. We will use the fact that if two functions have an asymptotic development on the same (reasonable) scale and their ratio is bounded, then this ratio admits a limit.

In order to get this asymptotic behavior, we use the algebraic hypothesis on the norm. Then, following Benoist–Oh [1, Part 16], we get the expected result as a consequence of resolution of singularities in the Archimedean case and Denef's Cell decomposition theorem in the non-Archimedean one. These results are the two following propositions:

Proposition 3.1. (Benoist–Oh [1], Proposition 7.2) *Let H be the group of $\mathbb{R}$-points of an algebraic $\mathbb{R}$-group, $\rho : H \to GL(V)$ a $\mathbb{R}$-representation of H, m_H the Haar measure on H and $|\cdot|$ an algebraic norm on* $\mathrm{End}(V)$.

Then, for all $g \in \mathrm{GL}(V)$, the volume $m_H(H_t(g)) = m_H\{h \in H \,|\rho(h)g| \leq t\}$ has an asymptotic development on the scale $t^a \ln(t)^b$ with $a \in \mathbb{Q}^+$ and $b \in \mathbb{N}$.

For the ultrametric part, we do not get exactly an asymptotic development rather a finite number of asymptotic developments. This was already noted in [1] but we need here a slightly more precise result, namely a uniformity on the number of simple functions needed:

Proposition 3.2. (Benoist-Oh) *Let k be a finite extension of $\mathbb{Q}_p, q$ be the norm of an uniformizer, H the group of k-points of an algebraic k-group, $\rho : H \to \mathrm{GL}(V)$ a k-representation of H, m_H the Haar measure on H and $|\cdot|$ a* max*-norm on* $\mathrm{End}(V)$*. Let $S_t(g)$ be the sphere of radius t: $S_t(g) := \{h \in H \text{ such that } |hg| = t\}$.*

Then there exist N_0 an integer such that for all $g \in G$ and for each $0 \leq j_0 \leq N_0$ one of the following holds:

(1) *$S_{q^j}(g)$ is empty for all $j = j_0 \bmod N_0$.*
(2) *There exist $d_{j_0} \in \mathbb{Q}_{\geq 0}$, e_{j_0} an integer and $c_{j_0} > 0$ such that $m_H(S_{q^j}(g)) \sim c_{j_0} q^{d_{j_0} j} j^{e_{j_0}}$ for all $j = j_0 \bmod N_0$.*

Proof. We will not go into details as the proof is the same as [1, Corollary 16.7]. We will just say that applying a theorem of Denef [5, Theorem 3.1 and remark below], we get the following:

For any polynomial map $f(x, \lambda)$ from $\mathbb{Q}_p^{m+d}$ to some $\mathrm{GL}(V)$, for any semialgebraic measure μ on a semialgebraic set $S \subset \mathbb{Q}_p^m$, there are some functions $\gamma_i(\lambda, n)$ and $\beta_i(\lambda, n)$ for $1 \leq i \leq e$ such that the measure $I(\lambda, n)$ of the set of elements $x \in S$ with $|f(x, \lambda)| = q^n$ is of the form:

$$I(\lambda, n) = \sum_{i=1}^{e} \gamma_i(\lambda, n) p^{\beta_i(\lambda, n)}.$$

Moreover, the functions γ_i and β_i are simple in the following sense: for any of these functions (hereafter denoted α) there exists an integer N such that for all λ, the map $n \mapsto \alpha(\lambda, n)$ is affine along at most N arithmetic progressions in $\mathbb{N}$ which cover $\mathbb{N}$ up to a finite set.

Now, the above proposition is just this result in the case where S is the image under the representation ρ of H, μ is the Haar measure on H and $f(\lambda, x) = \lambda \cdot x$ for $\lambda \in \mathrm{GL}(V)$ and $x \in H$. □

We may go on with the proof of Theorem 2.2. Let us write more explicitly the information we get on the function $m_H(H_t(g))$ from these two results. Fix some g in G. Consider the set S_f of finite places in S. For each $\nu \in S_f$, we note q_ν the norm of the uniformizer of K_ν. The previous proposition gives us an integer N_ν and for all $0 \leq j \leq N_\nu - 1$ some $d_{\nu,j} \in \mathbb{Q}$, $e_{\nu,j} \in \mathbb{N}$ and $c_{\nu,j} > 0$ describing the volume of spheres in the group H_ν. Moreover, for the Archimedean part, Proposition 3.1 gives some triple $d_\infty \in \mathbb{Q}_{>0}$, $e_\infty \in \mathbb{N}$ and $c_\infty > 0$ such that the volume of $(H_\infty)_t$ is equivalent to $c_\infty t^{e_\infty} e^{d_\infty t}$. With this data we are able to describe the volume of H_t:

Lemma 3.3. *With the data above*, $m_H(H_t(g))$ *is equivalent*, *as t goes to* ∞, *to*:

$$c_\infty t^{d_\infty} (\ln t)^{E_\infty} \prod_{\nu \in S_f} \left(\sum_{j=0}^{E(\ln_{q_\nu} t)} c_{\nu, j[N_\nu]} q_\nu^{d_{\nu, j[N_\nu]} j} j^{e_{\nu, j[N_\nu]}} \right). \tag{3.1}$$

Moreover, let

$$d = d_\infty \times \prod_{\nu \in S_f} \max_{0 \le j \le N_\nu} d_{\nu,j} \text{ and } e = e_\infty \times \prod_{\nu \in S_f} \max_{0 \le j \le N_\nu} e_{\nu,j}.$$

Then $m_H(H_t(g))$ lies between two constants times $t^e e^{dt}$.

Proof. By definition of the size function, the ball $H_t(g)$ is the product for all ν in S of the balls $(H_\nu)_t(g_\nu)$ in the group H_ν. For each of these balls the two previous theorems give us an equivalent for the volume in H_ν (all functions are positive so there is no trouble summing equivalent). Now the Haar measure on H is the product of the Haar measures on the H_ν's. And formula (3.1) is just the product of these equivalences.

The second part directly comes from the first one. □

The following lemma is the last step:

Lemma 3.4. *Under the hypothesis of Theorem 2.2 fix an element g in G.*

Then there exists a constant $c > 1$ such that the ratio $\frac{m_H(H_t(g))}{m_H(H_t)}$ lies between c^{-1} and c for all t.

Proof. The element g acts continuously on the module $\mathrm{End}(\mathbf{V}(K_S))$ (recall that in order to define balls in G we fixed some representation of $\mathbf{G}$ in a vector space $\mathbf{V}$). So there are two constants A and B such that we have for all h in H (recall that D denotes the size function):

$$A \cdot D(h) \le D(hg) \le B \cdot D(h).$$

That implies that the set $H_t(g)$ contains H_{At} and is contained in H_{Bt}.

But the second part of the previous lemma implies that the ratios $\frac{m_H(H_{At})}{m_H(H_t)}$ and $\frac{m_H(H_{Bt})}{m_H(H_t)}$ are bounded. Hence we have proven the lemma. □

We now have the tools to proceed with the proof of Theorem 2.2:

Proof. Each finite place leads to a finite partition of the space of parameters in the following way: For $\nu \in S_f$ we have q_ν the norm of the uniformizer and the integer N_ν given by Proposition 3.2. For $0 \le j \le N_\nu - 1$, we call $I_{\nu,j}$ the set of real numbers t such that $E(\ln_{q_\nu} t)$ is equal to j modulo N_ν. Proposition 3.2 implies that *on the sets* $I_{\nu,j}$ and for all $g \in G$ we have an asymptotic development of the volume of $(H_\nu)_t(g)$ of the form: $m_H((H_\nu)_t(g)) \sim C_{\nu,j} t^{E_{\nu,j}} e^{D_{\nu,j} t}$.

Now consider the finite partition $I_1, \ldots, I_l$ of $\mathbb{R}$ given by the intersection of all these partitions. Then on a set I_j of this partition and for all g in G, the volume $m_H(H_t(g))$ is equivalent to some $C_j(g) t^{E_j(g)} e^{D_j(g)t}$. But we know by the previous lemma that the ratio $\frac{m_H(H_t(g))}{m_H(H_t)}$ is bounded.

At this point we are done: since the ratio is bounded, we have $E_j(g) = E_j(\mathrm{Id})$ and $D_j(g) = D_j(\mathrm{Id})$. Hence the ratio admits a limit (depending on the set I_j), namely $\frac{C_j(g)}{C_j(\mathrm{Id})}$. □

4. Polynomial Dynamic in Homogeneous Spaces

We here recall some facts about polynomial dynamic in S-arithmetic groups. The result we need can mainly be found in Tomanov [21]. They are also used in [9]. The main difference here, which is only a technical one, is that we need to extend all the results to orbit of polynomials in several variables. This does not change deeply the proof of the theorems. The interested reader may refer to the author's PhD thesis [11] for details.

4.1. *Measure on G/Γ invariant under the action of a unipotent subgroup*

4.1.1. *Measure rigidity in an S-arithmetic setting*

We need the rigidity theorem for measures invariant under a unipotent group, often called Ratner's theorem. For p-adic groups, it has been proved by Ratner and by Margulis and Tomanov. However in an S-arithmetic setting a more precise version can be found in [21].

Accordingly to [21], we define the notion of subgroup of class $\mathcal{F}$:

Definition 4.1. Let $\mathbf{A}$ be a $\mathbb{Q}$-subgroup of $\mathbf{G}$. Then $\mathbf{A}$ belongs to the class $\mathcal{F}$ if and only if $\mathbf{A}(K_S)$ is the Zariski closure of the group generated by the unipotent elements of $\mathbf{A}(K_S)$.

Recall from [21] that for a class $\mathcal{F}$-group $\mathbf{P}$, the subgroup $P \cap \Gamma$ is a lattice in P. It implies that the projection of P in G/Γ is closed.

We can now state the measure rigidity theorem:

Theorem 4.1. (Ratner, Margulis-Tomanov, Tomanov) *Let $\mathbf{G}$ be a $\mathbb{Q}$-group, Γ an arithmetic subgroup of $G = \mathbf{G}(K_S)$ and U a subgroup of G generated by its one-parameter unipotent subgroups.*

Then for all probability measure μ on G/Γ which is U-invariant and U-ergodic, there exist a class $\mathcal{F}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ and P' a finite index subgroup of $P = \mathbf{P}(K_S)$ such that the probability μ is the P'-invariant probability on a translate of a P'-orbit in G/Γ.

This theorem allows a complete description of U-invariant probability measures.

4.1.2. *The non-ergodic case*

Let U be a subgroup of G generated by its one-parameter unipotent subgroups and μ be a U-invariant probability measure on G/Γ.

For each class $\mathcal{F}$ subgroup of $\mathbf{G}$, the precedent theorem defines a class of U-ergodic probability measures. To understand the decomposition of μ into ergodic components, we have to define some subsets of G:

Definition 4.2. Let $\mathbf{P}$ be a class $\mathcal{F}$ subgroup of $\mathbf{G}$. Then the sets $X(P,U)$ and $S(P,U)$ are defined in the following way:

$$X(P,U) = \{g \in G \text{ such that } Ug \subset gP\},$$
$$S(P,U) = \bigcup_{\mathbf{P}' \in \mathcal{F}, \mathbf{P}' \subset \mathbf{P}} X(P',U).$$

We remark that $X(P,U)$ is an algebraic subvariety of G.

For each class $\mathcal{F}$ subgroup $\mathbf{P}$ of $\mathbf{G}$, let $\mu_{\mathbf{P}}$ be the restriction of μ to $\pi(X(P,U)-S(P,U))$. Then each ergodic component of $\mu_{\mathbf{P}}$ is of the form given by the precedent theorem for this group $\mathbf{P}$. Moreover, since the sets $\pi(X(P,U)-S(P,U))$ are disjoint, we get the following decomposition of μ in a denombrable sum:

$$\mu = \sum_{\mathbf{P} \in \mathcal{F}} \mu_{\mathbf{P}}.$$

This decomposition enlightens the following fact: in order to understand a measure U-invariant, we have to understand the behavior of trajectories near the variety $\pi(X(P,U) - S(P,U))$. The goal of this section is to get a such a result. But first of all, we will define some useful representations of the group G.

4.2. *A suitable representation*

We fix here a class $\mathcal{F}$-subgroup $\mathbf{P}$. Chevalley's theorem [2, 5.1] grants the existence of a K-representation ρ_P of $\mathbf{G}$ such that $\mathbf{P}$ is the stabilizer of a line $\mathbf{D}$ in the space $\mathbf{V}_P$ of the representation.

We fix a point v_P in $\mathbf{D}(K)$. Moreover, we consider v_P as a point of the K_S-module $V_P = \mathbf{V}_P(K_S)$. We now get a function η_P from G to V given by the following formula:

$$\eta_P(g) = \rho_P(g) \cdot v_P.$$

The normalizer $\mathbf{N}(\mathbf{P})$ of $\mathbf{P}$ fix the line $\mathbf{D}$ but not the point v_P. So we define $\mathbf{N_1}(\mathbf{P})$ to be the fixator of the point v_P.

The following lemma will be useful, as a link between properties of subset in G/Γ and in V_P:

Lemma 4.2. • *The set $\eta_P(\Gamma)$ is discrete in V_P.*
• *The set $N_1(P)\Gamma/\Gamma$ is closed in G/Γ.*

Proof. First the subgroup $\mathbf{V}_P(\mathcal{O}_S)$ is discrete in $V_P = \mathbf{V}_P(K_S)$ and ρ_P is a K-representation. So the set $\rho_P(\mathbf{G}(\mathcal{O}_S)) \cdot v_P$ is discrete in V_P. Moreover, Γ is supposed to be arithmetic, so $\eta_P(\Gamma)$ is contained in a finite number of translates of $\rho_P(\mathbf{G}(\mathcal{O}_S)) \cdot v_P$. Hence it is a discrete set.

Second, let $g_k = n_k \gamma_k$ be a sequence of points in $N_1(P)\Gamma$ and assume that g_k converges to a point g. We want to prove that $g\Gamma/\Gamma$ belongs to $N_1(P)\Gamma/\Gamma$. We rewrite the definition of g_k: $\gamma_k^{-1} = g_k^{-1} n_k$. By definition of $N_1(P)$, we then get

$\eta_P(\gamma_k^{-1}) = \eta_P(g_k^{-1})$. We just showed that $\eta_P(\Gamma)$ is discrete. So the sequence γ_k is stationary equal to a γ for k large enough. Then $g_k\gamma^{-1}$ fixes v_P for k large enough. That is $g_k\gamma^{-1}$ belongs to $N_1(P)$. So does its limit and we can conclude: g belongs to $N_1(P)\Gamma$. □

We conclude with a last definition involving the group U. The set $X(P,U)$ is $N(P)$-invariant hence $N_1(P)$-invariant by right multiplication and it is a Zariski closed set of G. So its image by the function η_P, which is Zariski-open and surjective on $\eta_P(G)$, is Zariski-closed in $\eta_P(G)$. However, there is no reason for it to be Zariski-closed as well in V_P. So we define $F(P,U)$ as the Zariski-closure of $\eta_P(X(P,U))$ in V_P.

Remark. To avoid confusion, let us describe the Zariski topology in K_S-modules: a polynomial Q of $K_S[X_1,\dots,X_n]$ is nothing but a collection of polynomials Q_ν for all ν in S. A Zariski-closed subset of a K_S-module $M = \prod_{\nu\in S} m_\nu$ is then naturally an intersection of products of Zariski-closed subsets of each M_ν.

4.3. *Behavior of polynomial functions*

We now state a theorem allowing to control polynomial dynamics along the sets $\pi(X(P,U) - S(P,U))$. Let us begin by the definition of a polynomial function in the K_S-points G of a K-group $\mathbf{G}$ with a faithful linear representation ρ: a function $f = (f_\nu)_{\nu\in S}$ from $(K_S)^m$ to G is said to be polynomial of degree d if for all $\nu \in S$, the matrix entries of $\rho\circ f_\nu$ are all polynomial of degree d. The set of functions from K_S^m to G polynomial of degree at most d will be noted $\mathcal{P}_{(d,m)}(G)$. Moreover, we note $\theta = \bigotimes_{\nu\in S}\theta_\nu$ the Haar measure on K_S normalized such that the volume of $K_S/\mathcal{O}_S$ equals 1 and $\theta_m = \bigotimes^m \theta$ the induced measure on K_S^m.

Recall the definition of η from G to some K-module V_G given by Chevalley's theorem. Moreover, $F(P,U)$ has been defined as the Zariski closure of $\eta(X(P,U))$ inside V_G. Hereafter, we call *cube* in $(K_S)^m$ a product of balls $\prod_{i=1}^m \prod_{\nu\in S} B_{i,\nu}$.

Theorem 4.3. (Tomanov) *Let $\mathbf{G}$ be a K-group, Γ an arithmetic subgroup of $G = \mathbf{G}(K_S)$, U be a subgroup of G generated by its one-parameter unipotent subgroups and $\mathbf{P}$ a class $\mathcal{F}$-subgroup. Let C be a compact subset of $X(P,U)\Gamma/\Gamma$, d and m two integers and $\varepsilon > 0$.*

Then there exists a compact subset D of $F(P,U)$ such that for all relatively compact neighborhood W_0 of D in V_G, there exists a neighborhood W of C in G/Γ, such that for all m, for all cube B in $(K_S)^m$, and all function f in $\mathcal{P}_{(d,m)}(G)$ we have:

- *either we can find γ in Γ such that $\eta(f(B)\gamma) \subset W_0$*
- *or $\theta_m(\{t \in B$ such that $(f(t)\Gamma/\Gamma) \in W\}) < \varepsilon\theta_m(B)$.*

In [21] the theorem was not stated for functions in $\mathcal{P}_{(d,m)}$ but for one-parameter unipotent orbits. However, there is no conceptual jump in the proof of the above theorem. Moreover, the real cases of this theorem (and of all this section) is well known [19]. The interested reader may find more technical details in the author's PhD thesis [11].

4.4. *Non-divergence of polynomial orbits*

We need a last result in order to control the divergence of polynomial orbits. The following theorem is a kind of analog of a result of Eskin–Margulis–Shah [7]. However, we will not need the whole precision of their result, we may just use a slight adaptation of [12, Theorems 8.4 and 9.1]:

Theorem 4.4. (Kleinbock–Tomanov) *Let* $\mathbf{G}$ *be a* K*-group,* Γ *an arithmetic subgroup of* $G = \mathbf{G}(K_S)$*. Fix* d *and* m *two integers.*

Then there are a finite number of parabolic subgroups $\mathbf{P_k}$ *of* $\mathbf{G}$ *and their associated Chevalley representations* ρ_k *in a space* V_k *with a marked point* $v_k \in V_k$ *in a line stabilized by* $\mathbf{P_k}$ *such that*:

for all $\varepsilon > 0$ *there are a compact* D *in* G/Γ *and compact subsets* D_k *in each* V_k *verifying*: *for all* f *in* $\mathcal{P}_{(d,m)}(G)$, *for all cube* B *in* $(K_S)^m$, *one of the following holds*:

(1) $\theta_m(\{t \in B \text{ such that } (f(t)\Gamma/\Gamma) \notin D\}) < \varepsilon\theta_m(B)$.
(2) *There is an integer* k *such that there exists* $\gamma \in \Gamma$ *with*: $\rho_k(f(B)\gamma) \cdot v_k \subset D_k$.

5. Some Tools: Cartan Decomposition, Decomposition of Measures and Representations

Our proof of Theorem 2.3 requires some technical tools. The first one is more than classical: the Cartan decomposition in the semisimple part, which we recall to settle some notations. The second one is merely a way to note all the measures (and their translates) we will consider in the sequel, together with some basic lemmas. The third and last one is a lemma on representations of H. It is an extension of [19, Part 5] to our setting.

5.1. *Cartan decomposition in* H^{ss}

The group H is a semidirect product of a semisimple part H^{ss} and a unipotent one H^u. For the semisimple part we have a Cartan decomposition: for all ν in S such that H_ν is non-compact we choose a maximal K_ν-split torus A_ν in H_ν. We choose then a system of positive simple restricted roots Φ_ν thus defining the associated sub-semigroup A_ν^+ of A_ν. Then there exists maximal compact subgroups C_ν and finite sets D_ν in the normalizer of A_ν such that the following Cartan decomposition holds: H_ν is the disjoint union of the double class $C_\nu d a C_\nu$ for $a \in A_\nu^+$ and $d \in D_\nu$. For the existence of these objects we refer to [20]. When H_ν is compact we just choose $C_\nu = H_\nu$, A_ν and D_ν are reduced to the identity.

Let $A^+ = \prod_{\nu\in S} A_\nu^+$ and similarly C and D are the products of the C_ν's and D_ν's. Let Φ be the union of the Φ_ν. For $\alpha \in \Phi_\nu \subset \Phi$ and $a = (a_\nu)_{\nu\in S}$ we define $\alpha(a) = \alpha(a_\nu)$.

Consider a sequence a_n of elements of A^+.

Definition 5.1. A sequence a_n of elements of A^+ is *simplified* if for all α in Φ we have the alternative:

- either $\alpha(a_n)$ is bounded,
- or $\alpha(a_n)$ goes to $+\infty$.

Associated to such a simplified sequence, we consider the contracted unipotent subgroup of H^{ss}.

$$U^+ = \left\{ h \in H^{ss} \text{ such that } \lim_{n\to+\infty} a_n^{-1} h a_n = e \right\}.$$

Remark. We did not assume that a simplified sequence a_n is unbounded. So the group U^+ associated may be equal to the trivial group.

5.2. *Decomposition of measures*

The idea is simple: given some measure μ on the ball $(H^{ss})_t$, we want to define a probability measure on the ball H_t which disintegrates (in the product $H = H^{ss} \rtimes H^u$) on μ and the Haar measure in the fibers. The notations may seem tedious as we must work at each place in parallel. But it will prove useful later.

The assumptions made on the norm ensure the following: for all h^{ss} in H_ν^{ss}, the set of elements in H_ν^u such that $h^{ss}h^u$ belongs to $(H_\nu)_t$ is a ball of radius some $l_{[\nu,t]}(h^{ss})$ in H_ν^u and moreover depends continuously on h^{ss} and t. So for all t, there is a continuous function $l_{[\nu,t]}$ from H_ν^{ss} to $\mathbb{R}^+$ such that:

$$(H_\nu)_t = \bigcup_{h\in H_\nu^{ss}} \{h\} \times (H_\nu^u)_{l_{[\nu,t]}(h^{ss})}.$$

This in turn translates in terms of measures. We note $m_\nu^u(l)$ the restriction of the Haar measure $m_{H_\nu^u}$ to the ball $(H_\nu^u)_l$. And for measure μ_ν in H_ν^{ss}, we may define the measure $m_\nu(\mu_\nu, t)$ by the formula, for all ϕ continuous with compact support on H_ν:

$$\int_{H_\nu} \phi dm_\nu(\mu_\nu, t) = \int_{H_\nu^{ss}} \int_{(H_\nu^u)_{l_{[\nu,t]}(o)}} \phi(ob) dm_\nu^u(l_{[\nu,t]}(o))(b) d\mu_\nu(o).$$

For $\mu = \bigotimes \mu_\nu$ a product measure on H^{ss} of finite total mass and t positive, we note $m(\mu, t)$ the product $\bigotimes_{\nu\in S} m_\nu(\mu_\nu, t)$. Eventually we note $\mathbb{P}(\mu, t)$ the renormalized probability measure and $\mathrm{Supp}(\mu, t)$ its support. Remark that, if μ proportional to the Haar measure of some subgroup S in H^{ss}, then $m(\mu, t)$ is proportional to the Haar measure in $S \rtimes H^u$ restricted to $(S \rtimes H^u)_t$.

Let us immediately state two lemmas showing that these probability measures behave well with respect to μ as soon as the support of μ does not approach the frontier of the ball H_t. First look at translations:

Lemma 5.1. *Let μ_n be a sequence of probability measure on H^{ss} and t_n go to ∞. Let h_n go to* Id *in H^{ss}. Assume that the support of μ_n is included in a ball of radius $H^{ss}_{(1-\varepsilon)t_n}$ for some $\varepsilon > 0$.*

Then the sequence of (signed) measure $\mathbb{P}(((h_n)_*\mu_n), t_n) - \mathbb{P}(\mu_n, t_n)$ converges to 0*.*

Proof. The assumption on the supports of μ_n ensures that the supports of $(h_n)_*\mu_n$ are included in $(H^{ss})_{t_n}$ for n large enough. Moreover (by left-uniform continuity of the norms), we have for every sequence g_n in the support of μ and for all place ν (here we forget some subscripts ν to keep the formula readable):

$$\frac{l_{[\nu,t_n]}(h_n g_n)}{l_{[\nu,t_n]}(g_n)} \xrightarrow{n\to\infty} 1.$$

As, eventually, the sequence of signed measures $(h_n)_*\mu_n - \mu_n$ goes to 0 as $n \to \infty$, the lemma is proven by a straightforward calculus. $\square$

The second lemma allows one to handle also a sequence of measure μ_n:

Lemma 5.2. *Let μ_n be a sequence of probability measures on H^{ss} converging to μ as $n \to \infty$ with all these measures supported in a given compact set and absolutely continuous with respect to some λ. Let t_n be a sequence of real numbers going to $+\infty$ and h_n a sequence of elements in H^{ss}.*

Then the sequence of (signed) measure $\mathbb{P}(((h_n)_*\mu_n), t_n) - \mathbb{P}((h_n)_*\mu, t_n)$ goes to 0 *as $n \to \infty$.*

Proof. By hypothesis, the signed measure $\mu_n - \mu$ has a density going to zero in $L^1(\lambda)$ as $n \to \infty$. But all these densities are supported inside a compact set. Hence $\mu_n - \mu$ has a total variation going to zero, i.e. for all $\epsilon > 0$ and n large enough, for all functions on H^{ss}, we get:

$$\left| \int f d\mu_n - \int f d\mu \right| \leq \epsilon \max(|f|).$$

This ensures that its translates under h_n go to zero, i.e. that

$$\mathbb{P}((h_n)_*(\mu_n, t_n)) - \mathbb{P}(((h_n)_*\mu), t_n) \xrightarrow{n\to\infty} 0. \qquad \square$$

5.3. *A lemma on linear representation*

The first equidistribution result we will prove is for projections of probability measures of the form $\mathbb{P}((a_n)_*l, t_n)$ where l is a probability measure on U^+ absolutely continuous with respect to the Haar measure. But we need a result on the action of its support $S((a_n)_*l, t_n)$: it sends every non-invariant point to ∞.

The situation of this section is the following: let (a_n) be a simplified sequence. Let Ω be an open and relatively compact subset of U^+. Let (t_n) be a sequence of real numbers tending to ∞ such that the sets $a_n\Omega$ are included in balls $H^{ss}_{t_n}$. Let N^{ss} be the smallest normal subgroup of H such that the projection of a_n is bounded in H/N^{ss}.

Lemma 5.3. *Let $\rho = (\rho_\nu)_{\nu\in S}$ be a K_S-representation of H in a finite dimensional K_S-module $V = \prod V_\nu$. Let O_n be the set $\{o \in \Omega \times H^u, D(a_n o) \leq t_n\}$. Let N be the smallest subgroup of H such that $a_n O_n$ stay in a compact in H/N.*

Let Λ be a discrete subset of V with no N-invariant points and v_n a sequence of elements of Λ.

Then the sequence of sets $\rho(a_n O_n)v_n$ is not contained in any compact subset of V.

This whole subsection will be the proof of this lemma.

Proof. We split this proof into two cases: whether the sequence a_n is bounded or not.

Case 1. *a_n is bounded.*

We may assume that every a_n equals Id. Then U^+ is trivial, O_n is the ball $D(o) \leq t_n$ in H^u and N is the group H^u. As t_n go to ∞, we may extract an increasing subsequence of balls covering H^u. If v_n is not bounded, as Id belongs to O_n, then the lemma is proven. If not, as Λ is discrete, we may assume that v_n is constantly equal to some v which is not N-invariant. Now the exponential function composed of $g \mapsto \rho(g)v$ gives us a polynomial function from the Lie algebra $\mathfrak{h}^u$ to V, and $\rho(H^u)v$ is the image of this polynomial function. That means that this function is constant or unbounded. As it is not constant, it is unbounded, proving the lemma in this case.

Case 2. *a_n is not bounded.*

In this situation, the action of a_n and U^+ alone send non-invariant points to ∞ (remark that Ω is included in O_n by definition).

First of all, let $V^{N^{ss}}$ be the N^{ss}-invariant sub-module of V and W an N^{ss}-invariant complement. Write $v_n = v_n^{N^{ss}} + w_n$. If w_n goes to 0, by discreteness of Λ, v_n^N goes to ∞. Let C be a compact of G such that $a_n O_n$ is included in CN. Then, by definition of U^+ and semisimplicity of H^{ss}, the sets $a_n U^+$ are included in CN^{ss}. And for any $\omega \in \Omega$, the sequence $\rho(a_n\omega)v_n = \rho(a_n\omega)(v_n^N) + \rho(a_n\omega)w_n$ belongs to $\rho(C)v_n^N + W$. Hence this sequence goes to ∞, proving the lemma in this case.

So we may assume that w_n does not go to zero. Up to a renormalization and an extraction, we assume that w_n converges to some nonzero element $w \in W$. It is enough to prove that the sets $\rho(a_n\Omega)w$ leave every compact of V. Making this reduction we lose the discreteness hypothesis on Λ but we will not need it anymore.

We now prove the lemma by contradiction: suppose that the above sets stay in some compact. We prove first that w is N^{ss}-invariant and then N-invariant.

The first step is to show that we may assume that w is U^+-invariant: let V^+ be the module of U^+-invariant points and V^- its a_n-invariant complement. Note p^+ the projection on V^+ in the direction V^-. We have the following:

Lemma 5.4. *Let $\rho = (\rho_\nu)_{\nu \in S}$ be a K_S-representation of H in a finite dimensional K_S-module $V = \prod V_\nu$. Let U be a nontrivial unipotent subgroup, and Ω an open subset of U.*

Then the set $\rho(\Omega)w$ is not contained in any complement of the submodule V^U of U-invariant points.

Proof. Once again we prove it by contradiction: suppose $\rho(\Omega)w$ generates some submodule V' in direct sum with V^U. And let $\omega_1, \ldots, \omega_k$ be elements of Ω such that the $\rho(\omega_i)w$ generate V'. Then there is a neighborhood Ω' of the identity in U such that all the $\Omega'\omega_i$ are included in Ω.

And V' is Ω' invariant. So it is invariant by the Zariski closure of Ω', i.e. by U. The Lie–Kolchin theorem implies that there is a nonzero U-invariant element in the U-invariant module V' (to be very precise, you have to apply the Lie–Kolchin theorem at each place, restricting the representation in the obvious way). This is the contradiction: V' cannot be in direct sum with V^U. □

So, there is some $\omega \in \Omega$ such that $p^+(\rho(\omega)w)$ is not zero. But we know that $\rho(a_n\omega)w$ is bounded. Hence $\rho(a_n)p^+(\rho(\omega)w)$ is bounded. Let us show that it implies that N^{ss} is contained in the kernel of the representation:

Lemma 5.5. *Let v be a U^+-invariant and nonzero point of V such that $\rho(a_n)v$ is bounded. Then N^{ss} is contained in the kernel of the representation ρ.*

Proof. We may assume that at each place ρ_ν is an irreducible representation. First of all, let W be the sub-K_S-module of V containing all the vectors w such that $\rho(a_n)w$ is bounded. Consider P^- the opposite parabolic subgroup in H^{ss}:

$$P^- = \{h \in H \text{ such that } a_n h a_n^{-1} \text{ is a bounded sequence}\}.$$

Then it is clear that $\rho(P^-)v$ is included in W. By U^+ invariance of v, we even get that $\rho(P^-U^+)v$ is included in W. But P^-U^+ is open in H; so Zariski-dense. We deduce that $\rho(H)v$ is included in W and by irreducibility that $W = V$.

Let us now prove that all the elements of V are U^+-invariant. We just have to prove it on eigenvectors for the action of a_n (V is the sum of the eigenspaces for this action). Remind that, as a_n has determinant one and all the vectors have a bounded orbit under the action of a_n, all the eigenvalues of this action are of modulus 1. So let v' be in V with $\rho(a_n)v' = \lambda_n v'$ and ω be some element of U^+. Fix an open neighborhood of the identity Ω in U^+. Then by definition there is some integer i such that $a_i^{-1}\omega a_i$ belongs to Ω. Hence $\rho(\omega)v'$ belongs to $\rho(a_i)(\rho(\Omega)\lambda_i^{-1}v')$. But the latter is included in some compact B independent of i because we have seen that all elements of V have bounded orbit in V and the sets $\rho(\Omega)\lambda_i^{-1}v'$ are contained in some compact. So $\rho(U^+)v'$ is included in B. But U^+ is an unipotent subgroup hence $\rho(U^+)v'$ is the whole image of a polynomial function. It can be bounded if and only if it is constant. Hence v' is U^+-invariant.

We have just proven that every element of V is U^+-invariant. Hence the kernel of the representation contains the normal subgroup generated by U^+, hence contains N^{ss}. □

Let us proceed with the proof of Lemma 5.3. The situation is now simple: we may forget about the semisimple part because it acts trivially. And we just have an element w of V such that $\rho(O_n)w$ is bounded. Now for each n, the projection of O_n in H^u is a ball (by the hypothesis of orthogonality on the norm). If the O_n are bounded, then $N = N^{ss}$ is semisimple and we are done. If not we may as in case 1 assume that the projections of O_n on H^u are increasing balls and $\rho(O_n)w$ may be bounded only if w is H^u-invariant. Here N is the subgroup generated by N^{ss} and H^u and w is N-invariant.

In both cases we found the contradiction: w is N-invariant. Hence Lemma 5.3 is proved. □

6. Equidistribution of Dense Orbits

The aim of this section is to prove Theorem 2.3 (see p. 10). We use the rigidity of the dynamic of unipotent flows reviewed in the previous sections. The article of Shah [19] is the main source of inspiration for this proof.

6.1. *Equidistribution over unipotent subgroups*

The first equidistribution result is the following: if a_n is simplified and l a probability measure on U^+, then the projections of $\mathbb{P}((a_n)_*l, t_n)$ in G/Γ become equidistributed with respect to the Haar measure $m_{G/\Gamma}$ if its support $\mathrm{Supp}((a_n)_*l, t_n)$ does not stay close to a normal subgroup with closed orbit:

Proposition 6.1. *Let (G, H, Γ) be a triple under study with a size function D elliptic on H^u. Let t_n be a sequence of positive number going to $+\infty$ and (a_n) be a simplified sequence in A^+, U^+ the contracted unipotent subgroup of H^{ss} and l a measure on U^+ compactly supported and absolutely continuous with respect to the Haar measure. Let N be the smallest normal subgroup of H such that the projections of* $\mathrm{Supp}((a_n)_*l, t_n)$ *remain in a compact subset in H/N. Assume eventually that $N\Gamma$ is dense in G.*

Then we have the following limit in the space of probability on G/Γ:

$$\lim_{n\to\infty} \pi_*(\mathbb{P}((a_n)_*l, t_n)) = m_{G/\Gamma},$$

that is, for every function ϕ continuous with compact support on G/Γ, we have:

$$\int_H \phi(x\Gamma/\Gamma)d\mathbb{P}((a_n)_*l, t_n)(x) \xrightarrow{n\to\infty} \int_{G/\Gamma} \phi dm_{G/\Gamma}.$$

The proof of this proposition is the core of Theorem 2.3. We will use here the theory of polynomial orbits and Ratner's theorem exposed above, together with Lemma 5.3. The derivation of Theorem 2.3 from this proposition will not present any major difficulty.

Proof. We first show that any weak limit of the sequence studied in the proposition is a probability measure invariant by some unipotent subgroup. Then we will use the theory developed and the previous lemma to show that it can only be the Haar probability measure on G/Γ.

So consider the sequence of probability measures $\pi_*(\mathbb{P}((a_n)_*l, t_n))$ and μ a weak limit. The first step will be to prove that μ is a probability measure on G/Γ. The second one will be an invariance of μ by some unipotent subgroup, thus allowing the use of the tools reviewed. Eventually we will prove that this μ is the Haar measure on G/Γ, proving the proposition.

The group U^+ is a unipotent subgroup of G of dimension say m. Hence the exponential map from its Lie algebra $\mathfrak{u}$ to it is a polynomial map which send the Haar measure on the Haar measure.

Now, look at the projections of $\mathrm{Supp}((a_n)_*l, t_n)$ in H^u. They are product of balls $(H^u_\nu)_{r_n(\nu)}$ of radius some $r_n(\nu)$. As before, we have the exponential map from $\mathfrak{h}^u$ to H^u. Recall that we assumed that there is a polynomial diffeomorphism ϕ of $\mathfrak{h}^u$ which is measure preserving, affine along a line Z on which ϕ is identity, and such that the pullback of D by $\exp\circ\phi$ is equivalent to the max of some power of norms $|\cdot|_\nu$. Hence the preimage of a ball H^u_t is almost a cube: there is a $0 < A < 1$ such that the preimage of a ball H^u_t is of volume at least A in a cube. From now on, $\exp_2$ is the composition $\exp\circ\phi$.

And the measure $(a_n^{-1})_*[\mathbb{P}((a_n)_*l, t_n)]$ is absolutely continuous with respect to the image under $\exp_1\times\exp_2$ of the Haar measure on $\mathfrak{u}\times\mathfrak{h}^u$. The first result we get is a uniformity on the absolute continuity. Up to adding variables, we see $\exp_1\times\exp_2$ as a map from $K_S^m\times K_S^r$ to $U^+\times H^u$. Recall that θ_m (resp. θ_r) is a Haar measure on K_S^m (resp. K_S^r).

Lemma 6.2. *Let C be a positive real number. There exist a cube B in K_S^m, a sequence of cubes B_n in K_S^r and an $\varepsilon > 0$ such that for all measurable subset E in G/Γ we have*:

If $\frac{1}{\theta_m(B)\theta_r(B_n)}\pi'_*((\exp_1)_*(\theta_m)\otimes(\exp_2)_*(\theta_r))(E)\leq\varepsilon$,
then $\pi'_*((a_n^{-1})_*[\mathbb{P}((a_n)_*l, t_n)])(E)\leq\frac{C}{2}$.

Proof. We choose B to be a cube in K_S^m such that Ω is included in $\exp_1(B)$ and B_n to be a cube in which the preimage of $\prod_\nu H^u_{r_n(\nu)}$ under $\exp_2$ is of measure at least A.

We claim that for a set of positive measure of element ω in Ω, the ball $\{u\in H^u$ such that $D(a_n\omega u)\leq t_n\}$ contains the product of balls of radius $\frac{r_n(\nu)}{2}$. This is a direct consequence of the fact that Ω is a compact and the hypothesis made on D; namely the so-called orthogonality between the semisimple part and the unipotent part.

Hence in the set $\exp_1(B)\times\exp_2(B_n)$, the set $a_n^{-1}\mathrm{Supp}((a_n)_*l,t_n)$ is positive and bounded from 0 relative measure, i.e. for some $C>0$, for all n:

$$\frac{((\exp_1)_*(\theta_m)\otimes(\exp_2)_*(\theta_r))[a_n^{-1}\mathrm{Supp}((a_n)_*l,t_n)]}{\theta_m(B)\theta_r(B_n)}\geq C.$$

As $(a_n^{-1})_*[\mathbb{P}((a_n)_*l,t_n)]$ is the restriction of the measure $l\otimes(\exp_2)_*(\theta_r)$ to its support $\mathrm{Supp}((a_n)_*l,t_n)$ renormalized to be a probability measure, and l is absolutely continuous with respect to $(\exp_1)_*(\theta_m)$ the conclusion of the lemma follows. $\square$

Step 1. The measure μ is a probability measure on G/Γ.

Proof. This result is quite classical, at least in the setting of Lie groups. We will of course use Theorem 4.4. Moreover, it is enough to prove it for the sequence of measures $(a_n)_*(\exp_1)_*(\theta_m)\otimes(\exp_2)_*(\theta_r)$ restricted to $B\times B_n$ thanks to the previous lemma.

Consider the functions $\Theta_n(t,s)=a_n\exp_1(t)\exp_2(s)$. They are polynomials of fixed degree. Fix some $0<\epsilon<1$. We want to find a compact set D in G/Γ such that the images of all (but a finite number) the function Θ_n are included inside this compact except for a set of relative measure at most ϵ.

We claim now that the subset D given to us by Theorem 4.4 is convenient. The strategy seems clear: apply Theorem 4.4 and then show that the second part of the alternative is impossible for all but finitely many n.

Hence we may apply Theorem 4.4 to the functions Θ_n restricted to $B\times B_n$ which is a cube. And we know, using Lemma 5.3, that the action of $a_n\exp_1(B)\times\exp_2(B_n)$ sends the points v_k outside of the compact D_k unless it is invariant by the group N.

So for n large enough, either all the points v_k appearing in Theorem 4.4 are invariant under N^{ss} and H^u or the whole cube $B\times B_n$ but a set of relative measure at most ϵ is mapped inside D. Now the first part of the alternative means that the subgroup N is included in the intersection of the parabolic subgroups P_k and as a corollary its orbit in G/Γ is closed. And we assumed the group N has a dense orbit in G/Γ.

So for n large enough, the total mass of points $(t,s)\in B\times B_n$ such that $\Theta_n(t,s)$ does not belong to D does not exceed 2ϵ times the mass of $B\times B_n$. $\square$

Step 2. The probability measure μ is left-invariant by some unipotent subgroup Z.

Proof. We also handle differently the cases according to the behavior of a_n:

Case 1. *a_n is bounded.*

We may assume that a_n is constantly equal to Id and U^+ is restricted to $\{\mathrm{Id}\}$. Hence the set $O_n=\mathrm{Supp}(\mathrm{Id},t_n)$ is a sequence of balls of radius t_n in H^u and the probability measure $\mathbb{P}(\omega,t_n)$ is the Haar measure of H^u restricted to O_n.

Let Z be a line in the center $\mathfrak{h}^u$ along which ϕ is affine and on which ϕ is the identity. Then $\exp_2(Z)$ is a one-parameter subgroup in the center of H^u, and for all $h \in H^u$ with $h = \exp_2(u)$ and $z \in Z$, we have

$$\exp_2(z)h = \exp(\phi(z))\exp(\phi(u)) = \exp(\phi(u) + \phi(z)) = \exp_2(z + u).$$

By the ellipticity hypothesis, the function $D(\exp_2(u+z))$ is equivalent to $D(\exp_2(u))$ when u goes to ∞.

Hence, for a fixed $z \in Z$, the "norm" $D(\exp_2(z)h)$ is equivalent to $D(h)$. This proves that the ratio $\frac{m_{H^u}(\exp_2(z)O_n \cap O_n)}{m_{H^u}(O_n)}$ tends to 1, which means that μ is left-invariant by $\exp_2(Z)$.

Case 2. a_n *is not bounded.*

Then, by construction a_n has a contracting action on U^+. Moreover, $u^+_*\mu$ is the limit of $u^+_*\pi_*(\mathbb{P}((a_n)_*l, t_n))$. And the last one may be rewritten $\pi_*(\mathbb{P}[(a_n)_*(a_n^{-1}u^+a_n)_*l, t_n])$. As $a_n^{-1}u^+a_n$ goes to Id, Lemma 5.2 implies that μ is U^+ invariant. □

Hence we may use all the tools presented: there exists a class $\mathcal{F}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $\mu(X(P,Z))$ is positive. We want to show that $\mathbf{P} = \mathbf{G}$. This is the third and final step:

Step 3. Any class $\mathcal{F}$-subgroup $\mathbf{P}$ such that $\mu(X(P,Z)) > 0$ is the group G.

Proof. We will naturally use Theorem 4.3. Fix a compact C of $X(P,V)$ of positive measure.

Using Lemma 6.2, we get an ε such that for all measurable subset E in C we have:

If $\frac{1}{\theta_m(B)\theta_r(B_n)}\pi_*((\exp_1)_*(\theta_m) \otimes (\exp_2)_*(\theta_r))(E) \leq \varepsilon$,
then $\pi_*((a_n^{-1})_*[\mathbb{P}((a_n)_*l, t_n)])(E) \leq \frac{\mu(C)}{2}$.

Once again we will apply Theorem 4.3 to the function $\Theta_n(t,s) = a_n \exp_1(t)\exp_2(s)$, restricted to $B \times B_n$, to the compact C and the ε just defined. Then there exists a compact D of $F(P,V)$ such that for all neighborhood W_0 of D there exists a neighborhood W of C such that for all n we get the alternative:

- There exists γ_n in Γ such that $\eta(\Theta_n(B \times B_n)\gamma_n) \subset W_0$
- $\theta_m \otimes \theta_r(\{t, s \in B \times B_n$ such that $\Theta_n(t,s)\Gamma/\Gamma \in W\}) < \varepsilon\theta_m \otimes \theta_r(B \times B_n)$.

Now fix any neighborhood W_0 of D and assume that we are in the second case of the previous alternative. Then by construction, we have:

$$\frac{1}{\theta_m \otimes \theta_r(B \times B_n)}\pi_*((\exp_1)_*(\theta_m) \otimes (\exp_2)_*(\theta_r))(a_n^{-1})(W) < \varepsilon.$$

But we have $\pi_*(\mathbb{P}((a_n)_*l, t_n))(W) > \frac{\mu(C)}{2}$ as W contains C and the measures $\pi_*(\mathbb{P}((a_n)_*l, t_n))$ converges to μ. By definition of ε, the previous inequality does not hold for n large enough.

Now we use Lemma 5.3. Consider the representation ρ of G in the K module V associated to $\mathbf{P}$ via Chevalley's theorem. And restrict it to a representation of H in V. Let Λ be the discrete set $\eta(h_0\Gamma)$. We want to show that one of these points is invariant under the action of the group N. But this is a direct application of Lemma 5.3: the sets $\rho(a_n B \times B_n)\eta(\gamma_n) = \eta(\Theta_n(B \times B_n))$ are included in W_0 hence bounded. The conclusion Lemma 5.3 being violated, the hypothesis is not fulfilled: one of the points γ_n is N-invariant.

We have done most of the work. Let us conclude, using notations and results of Sec. 4: N is included in $\gamma_n^{-1}N_1(P)\gamma_n$. So the projection of $N_1(P)$ in G/Γ contains a translate of the projection of N. But the latter is dense and the first one is closed: $N_1(P)$ projects onto G/Γ hence is Zariski-dense in G. We conclude that $N_1(P) = G$. That means that $\mathbf{P}$ is a normal subgroup of $\mathbf{G}$, so is equal to $\mathbf{G}$ by simplicity. □

To conclude the proof of Proposition 6.1, note that the rigidity Theorem 4.1 implies that μ is invariant under some finite index subgroup P of G. As G is a simply connected group, G itself is the unique finite index subgroup of G. Eventually μ is G-invariant so is the Haar probability measure on G/Γ. □

6.2. *Equidistribution of spheres*

We need a last step before proving Theorem 2.3: that is a proposition very similar to Proposition 6.1 but more adapted to Cartan decomposition in the group H^{ss}. Recall that, at the beginning of Sec. 6, we defined the Cartan decomposition $H^{ss} = CDA^+C$. The following proposition holds (compare with [19, Corollary 1.2]):

Proposition 6.3. *Let (G, H, Γ) be a triple under study equipped with a size function D elliptic on H^u. Let (h_n) be a sequence in H^{ss}, t_n a sequence of positive numbers going to $+\infty$ and μ a probability measure on C absolutely continuous with respect to the Haar probability measure on C. We assume that for some $\varepsilon > 0$ and for all c in the support of μ, we have $D(h_n c) \leq (1-\varepsilon)t_n$. Let N be the smallest normal subgroup of H such that the projection of the support of $\mathbb{P}((a_n)_*\mu, t_n)$ is bounded in H/N. Assume that ΓN is dense in G.*

Then the projection of probability measures $\mathbb{P}((a_n)_*\mu, t_n)$ in G/Γ becomes equidistributed:

$$\lim_{n\to\infty} \pi_*(\mathbb{P}((a_n)_*\mu, t_n)) = m_{G/\Gamma},$$

that is, for every function ϕ continuous with compact support on G/Γ, we have:

$$\int_H \phi(h\Gamma/\Gamma)d\mathbb{P}((a_n)_*\mu, t_n) \xrightarrow{n\to\infty} \int_{G/\Gamma} \phi dm_{G/\Gamma}.$$

Proof. We will prove that any weak limit of this sequence of probability measure is the Haar measure $m_{G/\Gamma}$.

First of all, we may assume that h_n is an element of A^+. Indeed, using Cartan decomposition, we write $h_n = c_n^1 d_n a_n c_n^2$, and, up to an extraction, the three sequences c_n^1, c_n^2 and d_n converge to respectively c^1, c^2 and d. Now, let μ' be the pushforward of μ under c^2: $\mu'(c^2A) = \mu(A)$. Lemma 5.2 guarantees that the equidistribution of $\pi_*(\mathbb{P}((h_n)_*\mu, t_n))$ is equivalent to the one of $\pi_*(\mathbb{P}((a_n)_*\mu', t_n))$. And by construction, N is also the smallest normal subgroup such that the projection of $\mathrm{Supp}((a_n)_*\mu', t_n)$ is bounded in H/N.

Moreover, up to another extraction, we assume that a_n is simplified. Consider now the opposite parabolic subgroup P^- to U^+ in H^{ss} and U^- the expanded unipotent subgroup:

$$U^- = \left\{ h \in H^{ss} \text{ such that } \lim_{n\to+\infty} a_n h a_n^{-1} = e \right\}.$$

Every neighborhood of an element c in C, contains a neighborhood which is homeomorphic to a neighborhood of Id in $P^- \times U^+$ via the application $(p^-, u^+) \mapsto p^- u^+ c$. We may split the support of μ in such sets (up to a negligible set), or in other words, we assume μ to be supported inside an open set homeomorphic to an open set $\Omega^- \times \Omega^+$ in $P^- \times U^+$. We furthermore assume that both Ω^- and Ω^+ are product set of the form $\prod_{\nu\in S} \Omega_\nu$. Moreover, at the Archimedean places, we may "thicken" a little bit μ to construct a measure absolutely continuous with respect to $m_{H^{ss}}$: let λ be a probability measure on a sufficiently small neighborhood O of Id in U_∞^- (the Archimedean part of U^-) absolutely continuous with respect to the Haar measure on U_∞^-. Then $\lambda \otimes \mu$ is absolutely continuous with respect to the Haar measure on H^{ss} (see [19, p. 15]).

Looking at the action of a_n on U^- and using Lemma 5.1 it is clear that for every function f continuous with compact support in G/Γ, the integrals of f for the both measures $\pi_*(\mathbb{P}((a_n)_*\lambda \otimes \mu, t_n))$ and $\pi_*(\mathbb{P}((a_n)_*\mu, t_n))$ are equivalent as n go to ∞:

$$\begin{aligned}
&\left| \int_{G/\Gamma} f d\pi_*(\mathbb{P}((a_n)_*\lambda \otimes \mu, t_n)) - \int_{U^+\times H^u} f(x) d\pi_*(\mathbb{P}((a_n)_*\mu, t_n)) \right| \\
&\quad \leq \int_{U^-} \left| \int_H f(x) d\pi_*((\mathbb{P}((a_n o a_n^{-1}) a_n)_*\mu, t_n) - (\mathbb{P}((a_n)_*\mu, t_n)))(x) \right| d\lambda(o) \\
&\quad \xrightarrow{n\to\infty} 0.
\end{aligned} \tag{6.1}$$

The limit is obtained using $a_n o a_n^{-1} \xrightarrow{n\to\infty} \mathrm{Id}$, Lemma 5.1 and the dominated convergence theorem.

We work now with $\lambda \otimes \mu$. Remark that, at non-Archimedean places, we do not have to modify μ, as maximal compact subgroups are also open.

Now, using [19, Proposition 6.1], we may decompose this probability measure $\lambda \otimes \mu$ in the product $\Omega^- \times \Omega^+$: there are a probability measure ν^- on Ω^- and for almost all x in Ω^-, a probability measure ν_x^+ on Ω^+ such that:

- ν^- and all the ν_ω^+ are absolutely continuous with respect to the Haar measure on P^- and U^+ respectively.

- for all ϕ continuous with compact support in H^{ss}, we have

$$\int_{H^{ss}} \phi d(\lambda \otimes \mu) = \int_{\Omega^-} \int_{\Omega^+} \phi(xy) d\nu_x^+(y) d\nu^-(x).$$

Consider now a function f continuous with compact support in G/Γ. We have:

$$\int_{G/\Gamma} f d\pi_*(\mathbb{P}((a_n)_*\lambda \otimes \mu, t_n)) = \int_{\Omega^-} \int_{\Omega^+ \times H^u} f(y\Gamma/\Gamma) d\mathbb{P}((a_n x)_* \nu_x^+, t_n)(y) d\nu^-(x).$$

So the last difficulty that remains is to compare the two probability measures $\mathbb{P}((a_n x)_* \nu_x^+, t_n)$ and $\mathbb{P}((a_n)_* \nu_x^+, t_n)$: if we prove that they are sufficiently close, then we may use Proposition 6.1 to conclude that the limit is the Haar probability measure $m_{G/\Gamma}$. But under conjugacy by a_n, the elements in P^- remains bounded. So, if we choose the support of λ small enough, Lemma 5.1 ensures that the two measures $\mathbb{P}((a_n x)_* \nu_x^+, t_n) = \mathbb{P}((a_n x a_n^{-1})_* (a_n)_* \nu_x^+, t_n)$ and $\mathbb{P}((a_n)_* \nu_x^+, t_n)$ are arbitrarily closed.

Fix $\epsilon > 0$ and choose the support O of λ such that we have: for all $x \in O$, all n

$$\left| \int_{\Omega^+ \times H^u} f(y\Gamma/\Gamma) d\mathbb{P}((a_n x)_* \nu_x^+, t_n)(y) - \int_{\Omega^+ \times H^u} f(y\Gamma/\Gamma) d\mathbb{P}((a_n)_* \nu_x^+, t_n)(y) \right| \leq \epsilon.$$

Then, we have:

$$\left| \int_{G/\Gamma} f d\pi_*(\mathbb{P}((a_n)_*\lambda \otimes \mu, t_n)) - \int_{\Omega^-} \int_{\Omega^+ \times H^u} f(y\Gamma/\Gamma) d\mathbb{P}((a_n)_* \nu_x^+, t_n)(y) d\nu^-(x) \right| \leq \epsilon.$$

Now, Proposition 6.1 states that for all x, we have the limit:

$$\int_{\Omega^+ \times H^u} f(y\Gamma/\Gamma) d\mathbb{P}((a_n)_* \nu_x^+, t_n)(y) \xrightarrow{n\to\infty} \int_{G/\Gamma} f dm_{G/\Gamma}.$$

We conclude applying the dominated convergence theorem:

$$\left| \int_{G/\Gamma} f d\pi_*(\mathbb{P}((a_n)_*\lambda \otimes \mu, t_n)) - \int_{G/\Gamma} f dm_{G/\Gamma} \right| \leq \epsilon.$$

So the previous inequality together with (6.1) leads to (for n large enough):

$$\left| \int_{G/\Gamma} f d\pi_*(\mathbb{P}((a_n)_*\mu, t_n)) - \int_{G/\Gamma} f dm_{G/\Gamma} \right| \leq 2\epsilon.$$

As this is true for arbitrary ϵ, we have finally obtained the desired result:

$$\int_{G/\Gamma} f d\pi_*(\mathbb{P}((a_n)_*\mu, t_n)) \xrightarrow{n\to\infty} \int_{G/\Gamma} f dm_{G/\Gamma}.$$

The proposition is proven. □

Thanks to this proposition, we are able to define a subset of large relative volume in H such that, basically, as soon as the support of $\mathbb{P}(h_* m_C, t)$ hits this subset, the projection of this measure in G/Γ is closed to the Haar probability measure:

Corollary 1. *Let (G, H, Γ) be a triple under study, together with a size function D elliptic on H^u. Assume that every dominant normal subgroup of H has a dense orbit in G/Γ. Fix $\varepsilon > 0$, f a continuous function with compact support in G/Γ, and O some open subset in C. Then there is a finite number of non-dominant normal subgroups $N_1, \dots, N_k$ of H, a compact subset B in H such that*:

For h in H, $O' \subset C$ containing O with μ the probability measure on O' proportional to the Haar measure on C and $t > 0$ verifying for all o in O, $D(go) \leq \frac{t}{1+\varepsilon}$, we have:

If the support of $\mathbb{P}(h_*\mu, t)$ is not included in any BN_i, then

$$\left| \int_{G/\Gamma} f d\pi_*(\mathbb{P}(h_*\mu, t)) - \int_{G/\Gamma} f dm_{G/\Gamma} \right| \leq \varepsilon.$$

Proof. Take the N_i's to be the maximal normal non-dominant subgroups. They are in finite number. Assume they do not verify the corollary. Then we construct a sequence h_n, t_n, O_n such that the supports of $\mathbb{P}((h_n)_*\mu_n, t_n)$ are not included in any compact neighborhood of a non-dominant normal subgroup and the difference of integrals is always greater than ε:

$$\left| \int_{G/\Gamma} f d\pi_*(\mathbb{P}((h_n)_*\mu_n, t_n)) - \int_{G/\Gamma} f dm_{G/\Gamma} \right| > \varepsilon. \tag{6.2}$$

Up to an extraction, we may assume that μ_n converges to a measure which is equal to the probability measure μ_∞ on an open O'_∞ containing O and proportional to the Haar measure of C.

These supports are yet included in a compact neighborhood of some normal subgroup N which has to be dominant. By assumption, N has a dense projection in G/Γ. So we may apply the above proposition to this sequence: the projection $\pi_*(\mathbb{P}((h_n)_*\mu_\infty, t_n))$ converges to the Haar probability measure $m_{G/\Gamma}$. But, using Lemma 5.2, letting n go to infinity, the measure $\pi_*(\mathbb{P}((h_n)_*\mu_\infty, t_n))$ is arbitrarily closed to $\pi_*(\mathbb{P}((h_n)_*\mu_n, t_n))$. This contradicts 6.2. □

6.3. *Equidistribution of balls*

At last we are able to conclude the proof of equidistribution of balls. Fix a function f continuous with compact support in G/Γ. Fix $\varepsilon > 0$. Let $\eta > 0$ be such that $\frac{m_H(H_{(1+\eta)t})}{m_H(H_t)} \leq 1 + \varepsilon$ for all t.

There is a neighborhood O of Id in C such that for all $h \in H^{ss}$ we have $D(ho) \leq \sqrt{1+\eta} D(h)$. And we may choose O such that C is a disjoint union of translates of O (up to a negligible set): there exist $c_1, \dots, c_s$ such that $c_i O \cap c_j O$

has measure 0 and the union $\cup_1^s c_i O$ is of full measure in C. Note μ_O the restriction of the probability Haar measure of C to O.

Let $\tilde{H}_t$ be the union over $c \in CD$, $a \in A^+$, and $1 \leq i \leq s$ with $D(cac_i) \leq t$, of the support of $m((cac_i)_*\mu_O, (1+\eta)t)$. Thanks to the Cartan decomposition, up to a negligible set, $\tilde{H}_t$ contains H_t, is contained in $H_{(1+\eta)t}$ and the restriction of m_H to $\tilde{H}_t$ may be written:

$$(m_H)_{|\tilde{H}_t} = \sum_1^s \int_{c\in CD, a\in A, D(cac_i)\leq t} m((cac_i)_*\mu_O, (1+\eta)t).$$

Let E_t be the union of the supports of measures $m((ca)_*\mu_{ca}, (1+\eta)t)$ which are completely included in $B\bigcup_1^k N_i$ (the sets constructed in the above corollary). As none of the N_i's are dominant, for t large enough, the relative mass of E_t in $\tilde{H}_t$ is less than ε and the symmetric difference between H_t and $\tilde{H}_t \backslash E_t$ is almost negligible:

$$\frac{m_H(H_t \Delta (\tilde{H}_t \backslash E_t))}{m_H(H_t)} \leq 2\varepsilon.$$

Corollary 1 implies that for all $a \in A$, $c \in CD$ and $1 \leq i \leq s$, if the support $\mathrm{Supp}(m((cac_i)_*\mu_O, (1+\eta)t))$ is not included in E_t, then its projection is pretty well distributed:

$$\left| \int_H f\, dm((cac_i)_*\mu_O, (1+\eta)t) - m((cac_i)_*\mu_O, (1+\eta)t)(H) \int_{G/\Gamma} f\, dm_{G/\Gamma} \right| \leq \varepsilon.$$

Integrating all these approximation over c, a and c_i leads to:

$$\left| \frac{1}{m_H(\tilde{H}_t \backslash E_t)} \int_{\tilde{H}_t \backslash E_t} f d\pi_*(m_H) - \int_{G/\Gamma} f dm_{G/\Gamma} \right| \leq \varepsilon.$$

So going back to the desired integral, we get (for t large enough):

$$\left| \frac{1}{m_H(H_t)} \int_{H_t} f d\pi_*(m_H) - \int_{G/\Gamma} f dm_{G/\Gamma} \right| \leq (1 + 4\max(|f|))\varepsilon.$$

As ϵ is arbitrarily small, we get the desired result:

$$\frac{1}{m_H(H_t)} \int_{H_t} f d\pi_*(m_H) \xrightarrow{t\to\infty} \int_{G/\Gamma} f dm_{G/\Gamma}.$$

This concludes the proof of Theorem 2.3.

7. Applications

We conclude this text by some explanations on the applications described in the Introduction.

7.1. *In dimension* 2

Recall the framework: we consider the group $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{Q}_p)$ for p a prime number, and the lattice $\Gamma = \mathrm{SL}(2,\mathbb{Z}[\frac{1}{p}])$. We fix here (for sake of simplicity) the standard Euclidean norm $|.|_\infty$ on the matrix algebra $\mathcal{M}(2,\mathbb{R})$ and the max-norm $|\cdot|_p$ on $\mathcal{M}(2,\mathbb{Q}_p)$. For a point v in $\mathbb{R}^2$, we note also $|v|_\infty$ the norm of the matrix whose first column is v and the second one is 0. We define similarly the norm of a point in $\mathbb{Q}_p^2$. We choose a Haar measure $m = m_\infty \otimes m_p$ on G.

The first result was:

Application 1.1. Let O be a bounded open subset of $\mathrm{SL}(2,\mathbb{Q}_p)$. Note Γ_T^O the set of elements $\gamma \in \Gamma$ such that $|\gamma|_\infty \leq T$ and $\gamma \in O$ as an element of $\mathrm{SL}(2,\mathbb{Q}_p)$. Let v be a point of the plane $\mathbb{R}^2 \backslash \{0\}$ with coordinates independent over $\mathbb{Q}$.

Then we have the following limit, for any function ϕ continuous with compact support in $\mathbb{R}^2 \backslash \{0\}$:

$$\frac{1}{T} \sum_{\Gamma_T^O} \phi(\gamma(v)) \xrightarrow{T\to\infty} \frac{2m_p(O)}{m(G/\Gamma)|v|_\infty} \int_{\mathbb{R}^2} \phi(w) \frac{dw}{|w|_\infty}.$$

Proof. We work here in the product of $\mathbb{R}^2 \backslash \{0\}$ and $\mathrm{SL}(2,\mathbb{Q}_p)$. We see it as the homogeneous space $H\backslash G$ with $H = \mathrm{Stab}(v)$ the stabilizer of v for the linear action of $\mathrm{SL}(2,\mathbb{R})$ on the plane.

Then it is not difficult to see that the hypotheses on the norm are fulfilled and that H has no dominant subgroup except itself. Moreover, the volume of balls are explicitly computed: the ratios of $m_H(H_t g)$ and $m_H(H_t)$ tends to $\frac{1}{|v||w|}$ where $w = g(v)$ [10, Sec. 12.4]. Remark that there is no need to split the parameter space.

It remains to prove that $H.\mathrm{SL}(2,\mathbb{Z}[\frac{1}{p}])$ is dense in G. But it contains $\mathrm{Stab}(v)$. $\mathrm{SL}(2,\mathbb{Z})$ which is by hypothesis dense in $\mathrm{SL}(2,\mathbb{R})$. Now we may use the strong approximation in $\mathrm{SL}(2)$ [18]: the algebraic group $\mathrm{SL}(2)$ is semisimple simply connected, hence the product $\mathrm{SL}(2,\mathbb{R}).\mathrm{SL}(2,\mathbb{Z}[\frac{1}{p}])$ is dense in G. This yields the desired property: $H.\mathrm{SL}(2,\mathbb{Z}[\frac{1}{p}])$ is dense in G.

Now, Theorem 2.1 implies the stated result. □

The second application was the following one. Recall that on the p-adic plane, we normalize the measure such that it gives mass 1 to $\mathbb{Z}_p^2$. The result is that if your beginning point generates the whole plane among the $\mathbb{Q}$-subspaces, then its orbit is dense and you get a distribution result (the function E appearing is the integer part):

Application 1.2. Let (v_∞, v_p) be an element of $(\mathbb{R}^2 \backslash 0) \times (\mathbb{Q}_p^2 \backslash 0)$. Suppose that any $\mathbb{Q}$-subspace V of $\mathbb{Q}^2$ verifying $v_\infty \in V \otimes_{\mathbb{Q}} \mathbb{R}$ and $v_p \in V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is $\mathbb{Q}^2$. Denote Γ_T the set of elements $\gamma \in \Gamma$ with $|\gamma|_\infty \leq T$ and $|\gamma|_p \leq T$.

Then, for all function ϕ continuous with compact support in $(\mathbb{R}^2\backslash 0) \times (\mathbb{Q}_p^2\backslash 0)$, we have the following limit:

$$\frac{1}{Tp^{E(\ln_p(T))}} \sum_{\Gamma_T} \phi(\gamma v_\infty, \gamma v_p)$$

$$\xrightarrow{T\to\infty} \frac{2(p^2-1)}{p^2 m(G/\Gamma)|v_\infty|_\infty |v_p|_p} \int_{\mathbb{R}^2\times\mathbb{Q}_p^2} \phi(v,w)\frac{dvdw}{|w|_\infty |w|_p}.$$

Proof. The proof here is similar to the previous one, the group H being $\mathrm{Stab}(v_\infty)\times \mathrm{Stab}(v_p)$. The hypotheses on the norm are fulfilled, as H is unipotent. The volume ratio limits are easy to compute and left to the reader. You just have to be careful with the normalizations of measures, letting this constant $\frac{p^2-1}{p^2}$ appear.

So it just remains to prove that $H.\mathrm{SL}(2,\mathbb{Z}[\frac{1}{p}])$ is dense in G. The key point is that its closure must be (up to finite index) the $\mathbb{R}\times\mathbb{Q}_p$-points of a $\mathbb{Q}$-subgroup of $\mathrm{SL}(2)$, by Tomanov theorem: it is a closed subset in G/Γ invariant under unipotent subgroups.

Hence, if either v_∞ or v_p has coordinates independent over $\mathbb{Q}$, the argument in previous application show the density. The only remaining case is when both v_∞ and v_p are stabilized by a $\mathbb{Q}$-unipotent group. But the assumption that v_∞ and v_p "generates" $\mathbb{Q}^2$ is then equivalent to the fact that these two stabilizers are different. Now we may conclude, arguing that two different unipotent subgroups of $\mathrm{SL}(2,\mathbb{Q})$ generate the whole group. Hence the smallest $\mathbb{Q}$-subgroup of $\mathrm{SL}(2,\mathbb{Q})$ such that its real points contains the stabilizer of v_∞ and its p-adic the stabilizer of v_p is $\mathrm{SL}(2)$. And the closure of $H.\mathrm{SL}(2,\mathbb{Z}[\frac{1}{p}])$ is G. □

The two previous examples showed how to profit of both the rigidity of orbit closures in an S-arithmetic setting and algebraic features such as strong approximation in the ambient group G. These arguments are also the core of the next case.

7.2. *In greater dimension*

Recall that we look at the action of $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ on the kth exterior power $\Lambda^k(\mathbb{R}^n)$. And we fix the standard Euclidean norm $|\cdot|$ on $\mathcal{M}(n,\mathbb{R})$. We consider also the standard Euclidean norm on $\Lambda^k(\mathbb{R}^n)$ and m is a Haar measure on $\mathrm{SL}(n,\mathbb{R})$. We want to prove:

Application 1.3. Let v be a nonzero element of $\Lambda^k(\mathbb{R}^n)$ such that its corresponding k-plane of $\mathbb{R}^n$ contains no rational vector. Denote Γ_T the set of elements $\gamma\in\Gamma$ with $|\gamma|\leq T$.

Then we have a positive real constant c (independent of Γ and v) such that for all function ϕ continuous with compact support on $\Lambda^k(\mathbb{R}^n)\backslash\{0\}$:

$$\frac{1}{T^{n^2+k^2-nk-n}}\sum_{\Gamma_T}\phi(\gamma v) \xrightarrow{T\to\infty} \frac{c}{m(G/\Gamma)|v|}\int_{\Lambda^k(\mathbb{R}^n)}\phi(v')\frac{dv'}{|v'|}.$$

Proof. Here we have to be more careful than in previous section. We consider the subgroup $H = \mathrm{Stab}(v)$. It is a conjugate of the group H_0 of the form:

$$H_0 = \begin{pmatrix} \mathrm{SL}(k, \mathbb{R}) & H^u \\ 0 & \mathrm{SL}(n-k, \mathbb{R}) \end{pmatrix} := \begin{pmatrix} H_k & H^u \\ 0 & H_{n-k} \end{pmatrix}.$$

So it is a semidirect product of a semisimple and a unipotent group. Moreover, the quotient $H_0 \backslash G$ identifies with $\Lambda^k(\mathbb{R}^n) \backslash \{0\}$ via the projection associating at an element of $\mathrm{SL}(n, \mathbb{R})$ the exterior product of its k-first lines.

We have to prove the orthogonality property for the norm on $H = gH_0g^{-1}$ ($g \in \mathrm{SL}(n, \mathbb{R})$). The key point is that one may use Iwasawa decomposition to write $g = oan$ where o belongs to $\mathrm{SO}(n)$, a is diagonal and n is upper triangular and unipotent (so is an element of H_0). Hence $H = oaH_0a^{-1}o^{-1}$ and H inherits the decomposition $H^{ss} \rtimes H^u$ from the canonical one of H_0 by conjugation by oa. Now the bi-invariance of the Euclidean norm under $\mathrm{SO}(n)$, and the fact that a normalizes the semisimple part of H_0 and the unipotent one imply that for $h = oah_0a^{-1}o^{-1}$ with the obvious notation:

$$|h|^2 = |ah_0a^{-1}|^2 = |ah_0^{ss}a^{-1}|^2 + |ah_0^u a^{-1}|^2 = |h^{ss}|^2 + |h^u|^2.$$

Now it is clear that H_0 has no dominant subgroup except itself, so the same holds for H. Let us prove that $H\Gamma$ is dense before evaluating the volume ratio limits. A way to see it is to pull back the dynamic on the space of k-frames: choose a family of k vectors in $\mathbb{R}^n$ such that their exterior product is v. Then the hypothesis on v is that the k-plane generated by this family of vector contains no nonzero rational vectors. By a theorem of Dani and Raghavan [3], it implies that the orbit of this family under Γ is dense in the space of k-frames. This in turn implies by projection that the orbit of v under Γ is dense in $H_0 \backslash G$, i.e. that $H\Gamma$ is dense in G.

We have computed the volume ratios to get the limiting density. Precisely, let $w = H_0g' = g^{-1}H(g'g^{-1})$ be a nonzero point in $\Lambda^k(\mathbb{R}^n)$. Then the limiting density at w given by Theorem 2.1 is the ratio:

$$\frac{m_H(H_t(g'g^{-1}))}{m_H(H_t)}.$$

The set $H_t(g'g^{-1})$ is by definition $\{h \in H \text{ such that } |hgg'| \leq t\}$; or the set $\{h_0 \in H_0 \text{ such that } |gh_0g'| \leq t\}$. Hence we have to compute the measure $M_t(g, g') = m_{H_0}(\{h_0 \in H_0 \text{ such that } |gh_0g'| \leq t\})$. The choice of normalization of m_{H_0} has no importance, as we only want to compute ratios. Using the bi-invariance of the norm and the Iwasawa decomposition of g and g'^{-1}, we immediately see that $M_t(g, g') = \frac{1}{|\mathrm{Vol}(g)||\mathrm{Vol}(g')|} M_t(1, 1)$, where $\mathrm{Vol}(g)$ is the determinant of the k first line of g. And, by the definition of the exterior product, the absolute value of this determinant is the Euclidean norm of their exterior product. So we may rewrite $M_t(g, g') = \frac{1}{|v||w|} M_t(1, 1)$. This gives the limiting density.

At this point, we need a last estimation: an equivalent of $M_T(1, 1)$ which gives the renormalization factor $T^{n^2+k^2-nk-n}$. So we want to compute the volume of the set $\{h_0 \in H_0 \text{ such that } |h_0| \leq T\}$ for the standard Haar measure on H_0: the product of the standard Haar measure on the three groups $\mathrm{SL}(k,\mathbb{R})$, $\mathrm{SL}(n-k,\mathbb{R})$ and H^u. Using the estimations of Maucourant [16], we see that the volume of the sphere of radius T in these groups are respectively of order T^{k^2-k-1}, $T^{(n-k)^2-(n-k)-1}$ and $T^{k(n-k)-1}$. So the leading term of the volume of the ball of radius T is of order:

$$\int_{T_1^2+T_2^2+T_3^2\leq T^2} T_1^{k^2-k-1}T_2^{(n-k)^2-(n-k)-1}T_3^{k(n-k)-1}.$$

Hence the leading term is of order:

$$T^{k^2-k+(n-k)^2-(n-k)+n(n-k)} = T^{n^2+k^2-nk-n}.$$

This concludes the proof of Application 1.3. □

We conclude this paper with the S-arithmetic generalization of the previous result. We leave the proof to the reader. All the arguments are in the three previous proofs except an estimation of the volume of the ball of radius T in $\mathrm{SL}(k,\mathbb{Q}_p)$ (p being a prime number). Using Cartan decomposition and some basic combinatorics, we get that the leading term of this volume is $(p^{E(\ln_p(T))})^{k^2-k}$. We fix the max norm in the standard basis on $\mathcal{M}(n,\mathbb{Q}_p)$ and $\Lambda^k(\mathbb{Q}_p^n)$. The group Γ is $\mathrm{SL}(n,\mathbb{Z}[\frac{1}{p}])$, and we note for an element $\gamma\in\Gamma$, $|\gamma|$ the max of its real Euclidean norm and p-adic max norm.

Application 7.1. Let $v=(v_\infty,v_p)$ be a nonzero element of $\Lambda^k(\mathbb{R}^n\times\mathbb{Q}_p^n)$ such that there is no nonzero rational vector belonging to both the real k-planes associated to v_∞ and the p-adic one associated to v_p. Denote Γ_T the set of elements $\gamma\in\Gamma$ with $|\gamma|\leq T$.

Then we have a positive real constant c (independent of Γ and v) such that for all function ϕ continuous with compact support on $\Lambda^k(\mathbb{R}^n)\backslash\{0\}$:

$$\frac{1}{(Tp^{E(\ln_p(T))})^{n^2+k^2-nk-n}}\sum_{\Gamma_T}\phi(\gamma v) \xrightarrow{T\to\infty} \frac{c}{m(G/\Gamma)|v_\infty|_\infty}\int_{\Lambda^k(\mathbb{R}^n\times\mathbb{Q}_p^n)}\phi(v'_\infty,v'_p)\frac{dv'_\infty v'_p}{|v'_\infty|_\infty|v_p|_p}.$$

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