Confluentes Mathematici, Vol. 2, No. 1 (2010) 135–157

DOI: 10.1142/S1793744210000144

HILBERT SPACE-VALUED INTEGRAL OF OPERATOR-VALUED FUNCTIONS

VOLODYMYR TESKO

Institute of Mathematics, National Academy of Sciences of Ukraine
3 Tereshchenkivska Street, Kiev, 01601, Ukraine
tesko@imath.kiev.ua

Received 6 December 2009

In this paper we construct and study an integral of operator-valued functions with respect to Hilbert space-valued measures generated by a resolution of identity. Our integral generalizes the Itô stochastic integral with respect to normal martingales and the Itô integral on a Fock space.

Keywords: Itô integral; normal martingale; resolution of identity; Fock space.

AMS Subject Classification: 46G99, 47B15, 60H05

1. Introduction

It is well known that the Itô integral of adapted square integrable functions plays a fundamental role in the classical stochastic calculus. In the case of complex-valued integrands such integral can be interpreted (roughly speaking) as an ordinary spectral integral applied to a certain vector from an L^2-space. The reason for this is that a square integrable martingale (integrator) can be regarded as a resolution of identity applied to the above-mentioned vector. At the same time, it is considerably more difficult to establish a relation between the Itô and spectral integrals for L^2-valued integrands, since there is a problem in finding an explicit expression for the corresponding spectral integral (see, for example, [5] and reference therein). In this context a natural problem arises, — *to give a suitable definition of "spectral integral" which will generalize the Itô stochastic integral.*

In this paper we introduce a notion of such "spectral integral" and show that it generalizes the classical Itô stochastic integral with respect to normal martingales and the Itô integral on a Fock space. Such point of view enables us to treat all these integrals in one framework.

Let us give a short exposition of our constructions. First of all let us recall the definition of an ordinary square integrable martingale and its interpretation using Hilbert space theory, see e.g. [15, 5, 13]. Throughout this paper, $(\Omega, \mathcal{A}, P)$ is a complete probability space endowed with a right continuous filtration

$\{\mathcal{A}_t\}_{t\in\mathbb{R}_+}$, i.e. with a family $\{\mathcal{A}_t\}_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{A}_t \subset \mathcal{A}$ such that $\mathcal{A}_s \subset \mathcal{A}_t$ if $s \le t$ and $\mathcal{A}_t = \bigcap_{s>t} \mathcal{A}_s$ for all $t \in \mathbb{R}_+$. Furthermore, we assume that $\mathcal{A}$ coincides with the smallest σ-algebra generated by $\bigcup_{t\in\mathbb{R}_+} \mathcal{A}_t$ and $\mathcal{A}_0$ contains all the P-null sets of $\mathcal{A}$.

Since $\mathcal{A}_t \subset \mathcal{A}$, the Hilbert space $L^2(\Omega, \mathcal{A}_t, P)$ is a subspace of $L^2(\Omega, \mathcal{A}, P)$. Denote by E_t the corresponding orthogonal projection in the space $L^2(\Omega, \mathcal{A}, P)$ onto $L^2(\Omega, \mathcal{A}_t, P)$. It can be shown that a projection-valued function $\mathbb{R}_+ \ni t \mapsto E_t$ is a resolution of identity in the space $L^2(\Omega, \mathcal{A}, P)$ and, for each $t \in \mathbb{R}_+$,

$$E_t F = \mathbb{E}[F|\mathcal{A}_t], \quad F \in L^2(\Omega, \mathcal{A}, P),$$

where $\mathbb{E}[\cdot|\mathcal{A}_t]$ denotes a conditional expectation with respect to the σ-algebra $\mathcal{A}_t$. Note that E_t is right continuous (instead of the usual for functional analysis left continuous) since the filtration $\{\mathcal{A}_t\}_{t\in\mathbb{R}_+}$ is right continuous. In what follows, we will consider only right-continuous resolutions of identity.

By definition (see, e.g. [18, 19]) a function $M\colon \mathbb{R}_+ \to L^2(\Omega, \mathcal{A}, P)$ is called a *square integrable martingale* with respect to the filtration $\{\mathcal{A}_t\}_{t\in\mathbb{R}_+}$ if $E_s M_t = M_s$ for all $s \in [0, t]$. Such martingale M is said to be *closed* by a function $M_\infty \in L^2(\Omega, \mathcal{A}, P)$ if $E_t M_\infty = M_t$ for all $t \in \mathbb{R}_+$.

Assume that a square integrable martingale M is closed by $M_\infty \in L^2(\Omega, \mathcal{A}, P)$. Then the Itô stochastic integral $\int_{\mathbb{R}_+} F(t)dM_t$ of a complex-valued function $F\colon \mathbb{R}_+ \to \mathbb{C}$ with respect to M can be interpreted as a spectral integral $\int_{\mathbb{R}_+} F(t)dE_t$ applied to M_∞. That is,

$$\int_{\mathbb{R}_+} F(t)dM_t = \left(\int_{\mathbb{R}_+} F(t)dE_t\right) M_\infty.$$

In general case, when F is a "fine" $L^2(\Omega, \mathcal{A}, P)$-valued function, the Itô integral is well defined but it is impossible to realize the latter equality, because it is not clear what the symbol $\int_{\mathbb{R}_+} F(t)dE_t$ means. In view of this it is natural to ask — *in what sense the expression* $\big(\int_{\mathbb{R}_+} F(t)dE_t\big) M_\infty$ *can be understood in general case*?

A key point for the answer is the observation that the conditional expectation $\mathbb{E}[\cdot|\mathcal{A}_t]$ is a resolution of identity in the Hilbert space $L^2(\Omega, \mathcal{A}, P)$ and that a function $F\colon \mathbb{R}_+ \to L^2(\Omega, \mathcal{A}, P)$ can be naturally viewed as an operator-valued function $\mathbb{R}_+ \ni t \mapsto A_F(t)$ whose values are operators $A_F(t)$ of multiplication by the function $F(t)$ in the space $L^2(\Omega, \mathcal{A}, P)$. This suggests us to consider the above-mentioned question in a more comprehensive sense — *in what sense the expression*

$$\left(\int_{\mathbb{R}_+} A(t)dE_t\right) M_\infty \quad \textit{or, equivalently,} \quad \int_{\mathbb{R}_+} A(t)dM_t,$$

can be understood? Here $E\colon \mathbb{R}_+ \to \mathcal{H}$ is a resolution of identity in a Hilbert space $\mathcal{H}$, $\mathbb{R}_+ \ni t \mapsto A(t)$ is an operator-valued function whose values are linear operators in $\mathcal{H}$, $M_t := E_t M_\infty$ is an abstract martingale and $M_\infty \in \mathcal{H}$.

In the next section we give the answer to this question for a certain class of operator-valued functions $\mathbb{R}_+ \ni t \mapsto A(t)$. The corresponding integral

$$\int_{\mathbb{R}_+} A(t)dM_t \tag{1.1}$$

we define as an element of the Hilbert space $\mathcal{H}$ and call it a *Hilbert space-valued stochastic integral* (or simply *H-stochastic integral*). We stress that formally the construction of H-stochastic integral is similar to the one of the Itô integral. Since the H-stochastic integral integrates with respect to a Hilbert space-valued measure and the integrand is, generally speaking, non-bounded operator-valued function one has to analyze the mutual dependencies of the integrand and the integrator. Therefore, by analogy with the classical Itô integration theory in order to give a correct definition of integral (1.1) we introduce a suitable notion of adaptedness between the integrand A and the integrator M.

We illustrate our abstract constructions with a few examples. Thus, in Sec. 3, we show that the Itô stochastic integral is the particular case of the H-stochastic integral. Namely, we assume that $\mathcal{H} := L^2(\Omega, \mathcal{A}, P)$ and take a normal martingale $N: \mathbb{R}_+ \to L^2(\Omega, \mathcal{A}, P)$ as an $L^2(\Omega, \mathcal{A}, P)$-valued martingale (by definition the process $N = \{N_t\}_{t \in \mathbb{R}_+}$ is a normal martingale if N and $t \mapsto N_t^2 - t$ are both martingales). For a given adapted square integrable function $F \in L^2(\mathbb{R}_+ \times \Omega, dt \times P)$, let $\mathbb{R}_+ \ni t \mapsto A_F(t)$ be the corresponding operator-valued function whose values are operators $A_F(t)$ of multiplication by the function $F(t) = F(t, \cdot) \in L^2(\Omega, \mathcal{A}, P)$ in the space $L^2(\Omega, \mathcal{A}, P)$, i.e.

$$L^2(\Omega, \mathcal{A}, P) \supset \mathrm{Dom}(A_F(t)) \ni G \mapsto A_F(t)G := F(t)G \in L^2(\Omega, \mathcal{A}, P).$$

Then the H-stochastic integral of A_F coincides with the Itô integral $\int_{\mathbb{R}_+} F(t)dN_t$ of F. That is,

$$\int_{\mathbb{R}_+} A_F(t)dN_t = \int_{\mathbb{R}_+} F(t)dN_t.$$

In Secs. 4–6 using the notion of the H-stochastic integral, we give the definition and establish properties of an Itô integral on a symmetric Fock space $\mathcal{F}(H)$ over a real Hilbert space H. In the special important case when $H = L^2(\mathbb{R}_+, dt)$ we obtain the well-known Itô integral on the Fock space $\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}_+, dt))$. This integral is the pre-image (under the corresponding probabilistic interpretation of $\mathcal{F}$) of the classical Itô stochastic integral with respect to normal martingales which possess the Chaotic Representation Property. Observe that the Itô integral on the Fock space $\mathcal{F}$ is the useful tool in the quantum stochastic calculus, see e.g. [4,2] for more details.

Most of the basic results of this paper were announced and partially proved in brief preliminary note [17].

2. A Construction of H-Stochastic Integral

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $(\cdot,\cdot)_{\mathcal{H}}$ and the norm $\|\cdot\|_{\mathcal{H}}$, $\mathcal{L}(\mathcal{H})$ be a space of all bounded linear operators in $\mathcal{H}$. Suppose that $E\colon \mathbb{R}_+ \to \mathcal{L}(\mathcal{H})$ is a right-continuous resolution of identity in $\mathcal{H}$, i.e. $\{E_t\}_{t\in\mathbb{R}_+}$ is a right-continuous and increasing family of projections, such that $E_\infty := \lim_{t\to\infty} E_t = 1$.

A slight generalization of the notion of square integrable martingale is the following.

Definition 2.1. A function $M\colon \mathbb{R}_+ \to \mathcal{H}$, $t \mapsto M_t$, is called an *$\mathcal{H}$-valued martingale* with respect to E if $E_s M_t = M_s$ for all $t \in \mathbb{R}_+$ and all $s \in [0,t]$.

It is clear that an ordinary square integrable martingale $M\colon \mathbb{R}_+ \to L^2(\Omega,\mathcal{A},P)$ with respect to the filtration $\{\mathcal{A}_t\}_{t\in\mathbb{R}_+}$ is an $L^2(\Omega,\mathcal{A},P)$-valued martingale with respect to the resolution of identity $E_t = \mathbb{E}[\cdot|\mathcal{A}_t]$. By analogy with the classical probability, the $\mathcal{H}$-valued martingale M will be called *closed* by a vector $M_\infty \in \mathcal{H}$ if $M_t := E_t M_\infty$, $t \in \mathbb{R}_+$.

Sometimes it will be convenient for us to regard the $\mathcal{H}$-valued martingale $M\colon \mathbb{R}_+ \to \mathcal{H}$ as an $\mathcal{H}$-valued measure $\mathcal{B}(\mathbb{R}_+) \ni \alpha \mapsto M(\alpha) \in \mathcal{H}$ on the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$. To this end, for any interval $(s,t] \subset \mathbb{R}_+$, we set

$$M((s,t]) := M_t - M_s, \quad M(\{0\}) := M_0, \quad M(\emptyset) := 0,$$

and then extend this definition to all Borel subsets of $\mathbb{R}_+$. Using such $\mathcal{H}$-valued measure, we construct a non-negative Borel measure

$$\mathcal{B}(\mathbb{R}_+) \ni \alpha \mapsto \mu(\alpha) := \|M(\alpha)\|^2_{\mathcal{H}} \in \mathbb{R}_+.$$

The purpose of this section is to define and study a *Hilbert space-valued stochastic integral* (or *H-stochastic integral*)

$$\int_{\mathbb{R}_+} A(t)dM_t \tag{2.1}$$

for a certain class of functions $\mathbb{R}_+ \ni t \mapsto A(t)$ whose values are linear operators in $\mathcal{H}$. We give a step-by-step construction of this integral, beginning with the simplest class of operator-valued functions. Since in the special case (see above) we want to obtain a generalization of the classical Itô integral, we have to introduce a suitable notion of adaptedness between $A(t)$ and M_t such that on one hand gives a natural generalization of the usual notion of adaptedness in the classical stochastic calculus, and on the other hand, allows us to obtain an analogue of the Itô isometry property and as a result to extend integral (2.1) from simple to a wider class of operator-valued functions.

Let us introduce the required class of simple functions. For each point $t \in \mathbb{R}_+$, we denote by

$$\mathcal{H}_M(t) := \operatorname{span}\{M_{s_2} - M_{s_1} | (s_1,s_2] \subset (t,\infty)\} \subset \mathcal{H}$$

the linear span of the set $\{M_{s_2} - M_{s_1} | (s_1, s_2] \subset (t, \infty)\}$, and

$$\mathcal{L}_M(t) = \mathcal{L}(\mathcal{H}_M(t), \mathcal{H})$$

will represent the set of linear operators in $A: \mathcal{H} \to \mathcal{H}$ such that continuously act from $\mathcal{H}_M(t)$ to $\mathcal{H}$. We stress that the set $\mathcal{L}_M(t)$ consists of all linear (bounded or non-bounded) operators $A: \mathcal{H} \to \mathcal{H}$ such that

$$\|A\|_{\mathcal{L}_M(t)} := \sup \left\{ \frac{\|Ag\|_{\mathcal{H}}}{\|g\|_{\mathcal{H}}} \middle| g \in \mathcal{H}_M(t), g \neq 0 \right\} < \infty.$$

Definition 2.2. A linear operator A in $\mathcal{H}$ will be called M_t-measurable if

(i) $A \in \mathcal{L}_M(t)$ and $\|A\|_{\mathcal{L}_M(t)} = \|A\|_{\mathcal{L}_M(s)}$ for all $s \in [t, \infty)$.
(ii) A is partially commuting with the resolution of identity E. More precisely,

$$AE_s g = E_s A g, \quad g \in \mathcal{H}_M(t), \quad s \in [t, \infty).$$

Evidently, if an operator A in $\mathcal{H}$ is M_t-measurable for some $t \in \mathbb{R}_+$ then A is M_s-measurable for all $s \in [t, \infty)$.

Definition 2.3. We say that a family $\{A(t)\}_{t \in \mathbb{R}_+}$ of linear operators in $\mathcal{H}$ is an M-adapted operator-valued function if, for every $t \in \mathbb{R}_+$, the operator $A(t)$ is M_t-measurable.

An M-adapted operator-valued function $\mathbb{R}_+ \ni t \mapsto A(t)$ will be called simple if there exists a partition $0 = t_0 < t_1 < \cdots < t_n < \infty$ of $\mathbb{R}_+$ such that

$$A(t) = \sum_{k=0}^{n-1} A_k \varkappa_{(t_k, t_{k+1}]}(t), \quad t \in \mathbb{R}_+, \tag{2.2}$$

where $\varkappa_\alpha(\cdot)$ is the characteristic function of a Borel set $\alpha \in \mathcal{B}(\mathbb{R}_+)$. It is clear that each operator A_k in (2.2) is M_{t_k}-measurable, since $A(t)$ is the M_t-measurable operator for every $t \in \mathbb{R}_+$.

Denote by $S = S(M)$ the space of all simple M-adapted operator-valued functions on $\mathbb{R}_+$. For each $A \in S$ of kind (2.2), we introduce a quasinorm

$$\|A\|_{S_2} := \left(\int_{\mathbb{R}_+} \|A(t)\|^2_{\mathcal{L}_M(t)} d\mu(t) \right)^{\frac{1}{2}} := \left(\sum_{k=0}^{n-1} \|A_k\|^2_{\mathcal{L}_M(t_k)} \mu((t_k, t_{k+1}]) \right)^{\frac{1}{2}}, \tag{2.3}$$

where, as above, μ is a Borel measure on $\mathbb{R}_+$ defined by $\mu(\alpha) := \|M(\alpha)\|^2_{\mathcal{H}}$. Since a simple operator-valued function A is M-adapted, we conclude that, for each $t \in \mathbb{R}_+$,

$$\|A(t)\|_{\mathcal{L}_M(t)} = \|A(t)\|_{\mathcal{L}_M(s)}, \quad s \in [t, \infty).$$

Due to the latter equality and finite additivity of the measure μ, definition (2.3) is correct, i.e. it does not depend on the choice of representation A in S.

Let us give the definition of an H-stochastic integral.

Definition 2.4. For $A \in S$ of kind (2.2), the H-*stochastic integral* with respect to M is defined as an element of $\mathcal{H}$ given by

$$\int_{\mathbb{R}_+} A(t)dM_t := \sum_{k=0}^{n-1} A_k(M_{t_{k+1}} - M_{t_k}). \tag{2.4}$$

Of course, definition (2.4) does not depend on the choice of representation of the simple function A in the space S. Further, let $0 \le s \le \infty$ be fixed. Clearly, if A belongs to S then $A\varkappa_{[0,s]}$ also belongs to S. Hence, for every $A \in S$, we can define an indefinite H-stochastic integral by

$$\int_0^s A(t)dM_t := \int_{\mathbb{R}_+} A(t)\varkappa_{[0,s]}(t)dM_t.$$

The following statement gives a description of the properties of our integral.

Theorem 2.1. *Let $0 \le s \le \infty$ be fixed. For all constants $a, b \in \mathbb{C}$ and for all simple operator-valued functions $A, B \in S$, we have*

$$\int_0^s (aA(t) + bB(t))dM_t = a\int_0^s A(t)dM_t + b\int_0^s B(t)dM_t, \tag{2.5}$$

$$\left\| \int_0^s A(t)dM_t \right\|_{\mathcal{H}}^2 \le \int_0^s \|A(t)\|^2_{\mathcal{L}_M(t)} d\mu(t). \tag{2.6}$$

Moreover, the indefinite integral $\int_0^s A(t)dM_t$ of $A \in S$ is an $\mathcal{H}$-valued martingale with respect to resolution of identity E. That is, for any $u \in [0, s]$,

$$E_u\left(\int_0^s A(t)dM_t\right) = \int_0^u A(t)dM_t. \tag{2.7}$$

Proof. Equalities (2.5) and (2.7) are obvious. Let us check inequality (2.6).

Taking into account the definition of $\int_0^s A(t)dM_t$, it will be enough to prove (2.6) replacing s with ∞. Let $A \in S$ be of the form

$$A(t) = \sum_{k=0}^{n-1} A_k \varkappa_{\Delta_k}(t), \quad \Delta_k := (t_k, t_{k+1}].$$

Since M is the H-valued martingale with respect to E, we see that $M(\Delta_k) = E(\Delta_k)M_{t_{k+1}}$, where $E(\Delta_k) := E_{t_{k+1}} - E_{t_k}$. Using Definition 2.2 and the properties of resolution of identity E we get

$$\begin{aligned}\left\| \int_{\mathbb{R}_+} A(t)dM_t \right\|_{\mathcal{H}}^2 &= \left(\int_{\mathbb{R}_+} A(t)dM_t, \int_{\mathbb{R}_+} A(t)dM_t \right)_{\mathcal{H}} \\ &= \sum_{k,m=0}^{n-1} (A_k M(\Delta_k), A_m M(\Delta_m))_{\mathcal{H}} \\ &= \sum_{k,m=0}^{n-1} (A_k E(\Delta_k)M_{t_{k+1}}, A_m E(\Delta_m)M_{t_{m+1}})_{\mathcal{H}}\end{aligned}$$

$$= \sum_{k,m=0}^{n-1} (E(\Delta_k)A_k E(\Delta_k)M_{t_{k+1}}, E(\Delta_m)A_m E(\Delta_m)M_{t_{m+1}})_{\mathcal{H}}$$

$$= \sum_{k=0}^{n-1} (A_k E(\Delta_k)M_{t_{k+1}}, A_k E(\Delta_k)M_{t_{k+1}})_{\mathcal{H}} = \sum_{k=0}^{n-1} \|A_k M(\Delta_k)\|^2_{\mathcal{H}}$$

$$\leq \sum_{k=0}^{n-1} \|A_k\|^2_{\mathcal{L}_M(t_k)} \|M(\Delta_k)\|^2_{\mathcal{H}} = \sum_{k=0}^{n-1} \|A_k\|^2_{\mathcal{L}_M(t_k)} \mu(\Delta_k)$$

$$= \int_{\mathbb{R}_+} \|A(t)\|^2_{\mathcal{L}_M(t)} d\mu(t).$$

□

Inequality (2.6) enables us to extend the H-stochastic integral to operator-valued functions $\mathbb{R}_+ \ni t \mapsto A(t)$ which are not necessarily simple. Namely, denote by $S_2 = S_2(M)$ *a Banach space associated with the quasinorm* $\|\cdot\|_{S_2}$. For its construction, at first it is necessary to pass from S to the factor space $\dot{S} := S/\{A \in S | \|A\|_{S_2} = 0\}$ and then to take the completion of $\dot{S}$. It is not difficult to understand that elements of the space S_2 are equivalence classes of operator-valued functions on $\mathbb{R}_+$ whose values are linear operators in the space $\mathcal{H}$. These classes are completely characterized by their following properties:

- An operator-valued function $\mathbb{R}_+ \ni t \mapsto A(t)$ belongs to some equivalence class from S_2 if and only if, for μ-almost all $t \in \mathbb{R}_+$, $A(t)$ is an M_t-measurable function and there exists a sequence $(A_n)_{n=0}^{\infty}$ of simple functions $A_n \in S$ such that

$$\int_{\mathbb{R}_+} \|A(t) - A_n(t)\|^2_{\mathcal{L}_M(t)} d\mu(t) \to 0 \quad \text{as } n \to \infty. \tag{2.8}$$

- Operator-valued functions $\mathbb{R}_+ \ni t \mapsto A(t)$ and $\mathbb{R}_+ \ni t \mapsto B(t)$ belong to the same equivalence class from S_2 if and only if, for μ-almost all $t \in \mathbb{R}_+$,

$$A(t)g = B(t)g, \quad g \in \mathcal{H}_M(t).$$

By convention, in what follows we will not make the distinctions between the equivalence class and operator-valued function from this class.

Definition 2.5. An operator-valued function $\mathbb{R}_+ \ni t \mapsto A(t)$ is said to be *H-stochastic integrable* with respect to M if A belongs to the space S_2.

Let A be an H-stochastic integrable function with respect to M and $(A_n)_{n=0}^{\infty}$ be a sequence of the simple functions $A_n \in S$ such that (2.8) holds. Due to inequality (2.6) a limit

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} A_n(t) dM_t$$

exists in $\mathcal{H}$ and does not depend on the choice of the sequence $(A_n)_{n=0}^{\infty} \subset S$ satisfying (2.8). We denote such limit by

$$\int_{\mathbb{R}_+} A(t)dM_t := \lim_{n\to\infty} \int_{\mathbb{R}_+} A_n(t)dM_t \in \mathcal{H}$$

and call it the *H-stochastic integral of A with respect to M*. It is clear that if A and B belong to the same equivalence class from S_2, then

$$\int_{\mathbb{R}_+} A(t)dM_t = \int_{\mathbb{R}_+} B(t)dM_t.$$

Note that for all H-stochastic integrable functions the assertions of Theorem 2.1 hold. To prove this fact, one should write (2.6) and other expression from Theorem 2.1 for approximating functions from S and then pass to the limit. We omit the details, note only that due to (2.6) the limit transition is always possible.

To clarify slightly the definition of H-stochastic integral, we consider a simple example which shows that an ordinary spectral integral applied to a certain vector from $\mathcal{H}$ can be thought of as the H-stochastic integral.

Example 2.1. Let $M: \mathbb{R}_+ \to \mathcal{H}$ be an $\mathcal{H}$-valued martingale closed by a certain vector $M_\infty \in \mathcal{H}$. Suppose $f: \mathbb{R}_+ \to \mathbb{C}$ is a square integrable function with respect to μ, that is $f \in L^2(\mathbb{R}_+, \mu)$. In the space $\mathcal{H}$ we consider a family $\{A_f(t)\}_{t\in\mathbb{R}_+}$ of the operators of multiplication by the complex number $f(t)$, i.e.

$$A_f(t)g := f(t)g, \quad g \in \mathcal{H}.$$

It is clear that $A_f \in S_2$ and

$$\int_{\mathbb{R}_+} A_f(t)dM_t = \int_{\mathbb{R}_+} f(t)dM_t = \left(\int_{\mathbb{R}_+} f(t)dE_t\right) M_\infty.$$

Note one simple property of the integral introduced above. Let U be some unitary operator acting from $\mathcal{H}$ onto another complex Hilbert space $\mathcal{K}$. Then

$$G: \mathbb{R}_+ \to \mathcal{K}, \quad t \mapsto G_t := UM_t,$$

is a $\mathcal{K}$-valued martingale with respect to the resolution of identity

$$X: \mathbb{R}_+ \to \mathcal{L}(\mathcal{K}), \quad t \mapsto X_t := UE_tU^{-1}.$$

Proposition 2.1. *If an operator-valued function $\mathbb{R}_+ \ni t \mapsto A(t)$ is H-stochastic integrable with respect to M, then the operator-valued function $\mathbb{R}_+ \ni t \mapsto UA(t)U^{-1}$ is H-stochastic integrable with respect to G and*

$$U\left(\int_{\mathbb{R}_+} A(t)dM_t\right) = \int_{\mathbb{R}_+} UA(t)U^{-1}dG_t \in \mathcal{K}.$$

Proof. These properties are immediate if A is a simple adapted function, and a proof of the general case is not difficult. ◻

3. The Itô Stochastic Integral as a Particular Case of the H-Stochastic Integral

Before stating the results let us recall the definition of the *Itô stochastic integral*. In our presentation of stochastic integration we will restrict ourselves to normal martingales. Note that this family of martingales includes Brownian motion, the compensated Poisson process and the Azéma martingales as particular cases, see e.g. [8, 6, 13, 11, 3].

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a right continuous filtration $\{\mathcal{A}_t\}_{t\in\mathbb{R}_+}$. Suppose that $\mathcal{A}$ coincides with the smallest σ-algebra generated by $\bigcup_{t\in\mathbb{R}_+} \mathcal{A}_t$ and $\mathcal{A}_0$ contains all the P-null sets of $\mathcal{A}$. In addition, assume that $\mathcal{A}_0$ is trivial, that is every $\alpha \in \mathcal{A}_0$ has probability 0 or probability 1.

By definition a process $N = \{N_t\}_{t\in\mathbb{R}_+}$ is a *normal martingale* on $(\Omega, \mathcal{A}, P)$ with respect to $\{\mathcal{A}_t\}_{t\in\mathbb{R}_+}$ if $\{N_t\}_{t\in\mathbb{R}_+}$ and $\{N_t^2 - t\}_{t\in\mathbb{R}_+}$ are martingales with respect to $\{\mathcal{A}_t\}_{t\in\mathbb{R}_+}$. In other words, N is a *normal martingale* if $N_t \in L^2(\Omega, \mathcal{A}_t, P)$ for all $t \in \mathbb{R}_+$ and

$$\mathbb{E}[N_t - N_s|\mathcal{A}_s] = 0, \quad \mathbb{E}[(N_t - N_s)^2|\mathcal{A}_s] = t - s \tag{3.1}$$

for all $s, t \in \mathbb{R}_+$ such that $s \leq t$. In what follows, without loss of generality we assume that $N_0 = 0$.

Let us introduce the space of functions for which the Itô integral is defined. We will denote by $L^2(\mathbb{R}_+ \times \Omega)$ the space of all $\mathcal{B}(\mathbb{R}_+) \times \mathcal{A}$-measurable functions (classes of functions) $F: \mathbb{R}_+ \times \Omega \to \mathbb{C}$ such that

$$\int_{\mathbb{R}_+} \int_\Omega |F(t,\omega)|^2 dP(\omega) dt = \int_{\mathbb{R}_+} \|F(t)\|^2_{L^2(\Omega,\mathcal{A},P)} dt < \infty,$$

and $L_a^2(\mathbb{R}_+ \times \Omega)$ will present the subspace of $\mathcal{A}$-adapted functions. Recall, a function $F \in L^2(\mathbb{R}_+ \times \Omega)$ is called $\mathcal{A}$-adapted if $F(t, \cdot)$ is $\mathcal{A}_t$-measurable for almost all $t \in \mathbb{R}_+$, that is $F(t, \cdot) = \mathbb{E}[F(t, \cdot)|\mathcal{A}_t]$ for almost all $t \in \mathbb{R}_+$.

Assume that $F(t) = F(t, \omega)$ is a simple function from the space $L_a^2(\mathbb{R}_+ \times \Omega)$, i.e. there exists a partition $0 = t_0 < t_1 < \cdots < t_n < \infty$ of $\mathbb{R}_+$ such that

$$F(\cdot) = \sum_{k=0}^{n-1} F_k \varkappa_{(t_k, t_{k+1}]}(\cdot) \in L_a^2(\mathbb{R}_+ \times \Omega).$$

The *Itô integral* of such function F with respect to N is defined by

$$\int_{\mathbb{R}_+} F(t) dN_t := \sum_{k=0}^{n-1} F_k (N_{t_k} - N_{t_{k-1}}) \in L^2(\Omega, \mathcal{A}, P)$$

and has the Itô isometry property

$$\left\| \int_{\mathbb{R}_+} F(t) dN_t \right\|^2_{L^2(\Omega,\mathcal{A},P)} = \int_{\mathbb{R}_+} \|F(t)\|^2_{L^2(\Omega,\mathcal{A},P)} dt.$$

Since the set $L^2_{a,s}(\mathbb{R}_+ \times \Omega)$ of all simple functions from $L^2_a(\mathbb{R}_+ \times \Omega)$ is dense in the space $L^2_a(\mathbb{R}_+ \times \Omega)$ (with respect to the topology of $L^2(\mathbb{R}_+ \times \Omega)$), extending the mapping

$$L^2_a(\mathbb{R}_+ \times \Omega) \supset L^2_{a,s}(\mathbb{R}_+ \times \Omega) \ni F \mapsto \int_{\mathbb{R}_+} F(t)dN_t \in L^2(\Omega, \mathcal{A}, P)$$

by continuity, we obtain a definition of the Itô integral on $L^2_a(\mathbb{R}_+ \times \Omega)$. In what follows, we keep the same notation $\int_{\mathbb{R}_+} F(t)dN_t$ for the extension.

Let us show that the Itô integral can be interpreted as the H-stochastic integral. To this end, we set $\mathcal{H} := L^2(\Omega, \mathcal{A}, P)$ and consider in this space a resolution of identity

$$E\colon \mathbb{R}_+ \to \mathcal{L}(L^2(\Omega, \mathcal{A}, P)), \quad t \mapsto E_t := \mathbb{E}[\cdot|\mathcal{A}_t],$$

generated by the filtration $\{\mathcal{A}_t\}_{t\in\mathbb{R}_+}$. It is easy to see that the normal martingale N is the $L^2(\Omega, \mathcal{A}, P)$-valued martingale with respect to $E_t := \mathbb{E}[\cdot|\mathcal{A}_t]$ and

$$\mu([0,t]) = \|N_t\|^2_{L^2(\Omega,\mathcal{A},P)} = \mathbb{E}[N_t^2] = \mathbb{E}[N_t^2|\mathcal{A}_0] = t$$

is the ordinary Lebesgue measure on $\mathbb{R}_+$.

Let us prove that in the context of this section N_t-measurability is equivalent to the usual $\mathcal{A}_t$-measurability. More precisely, the following result holds (the proof being similar to the one in [17] is omitted).

Lemma 3.1. *Let $t \in \mathbb{R}_+$. For a given function $F \in L^2(\Omega, \mathcal{A}, P)$ the operator A_F of multiplication by F in the space $L^2(\Omega, \mathcal{A}, P)$, i.e.*

$$L^2(\Omega, \mathcal{A}, P) \supset \mathrm{Dom}(A) \ni G \mapsto A_F G := FG \in L^2(\Omega, \mathcal{A}, P),$$
$$\mathrm{Dom}(A_F) := \{G \in L^2(\Omega, \mathcal{A}, P) | FG \in L^2(\Omega, \mathcal{A}, P)\},$$

is N_t-measurable if and only if the function F is $\mathcal{A}_t$-measurable.

Moreover, if $F \in L^2(\Omega, \mathcal{A}, P)$ is a $\mathcal{A}_t$-measurable function, then

$$\|A_F\|_{\mathcal{L}_N(s)} = \|F\|_{L^2(\Omega,\mathcal{A},P)}, \quad s \in [t, \infty).$$

This allows us to establish the following result.

Theorem 3.1. *For given $F \in L^2(\mathbb{R}_+ \times \Omega)$ the family $\{A_F(t)\}_{t\in\mathbb{R}_+}$ of the operators $A_F(t)$ of multiplication by $F(t) \in L^2(\Omega, \mathcal{A}, P)$ in the space $L^2(\Omega, \mathcal{A}, P)$,*

$$L^2(\Omega, \mathcal{A}, P) \supset \mathrm{Dom}(A_F(t)) \ni G \mapsto A_F(t)G := F(t)G \in L^2(\Omega, \mathcal{A}, P),$$

is H-stochastic integrable with respect to the normal martingale N (i.e. A_F belongs to $S_2 = S_2(N)$) if and only if F belongs to the space $L^2_a(\mathbb{R}_+ \times \Omega)$.

Proof. This fact is an immediate consequence of Lemma 3.1 and the definitions of the spaces $L^2_a(\mathbb{R}_+ \times \Omega)$ and S_2. □

The next theorem shows that the Itô stochastic integral with respect to the normal martingale N can be interpreted as the H-stochastic integral.

Theorem 3.2. *Let $F \in L_a^2(\mathbb{R}_+ \times \Omega)$ and $\{A_F(t)\}_{t\in\mathbb{R}_+}$ be the corresponding family of the operators $A_F(t)$ of multiplication by $F(t)$ in the space $L^2(\Omega, \mathcal{A}, P)$. Then*

$$\int_{\mathbb{R}_+} A_F(t)dN_t = \int_{\mathbb{R}_+} F(t)dN_t.$$

Proof. In view of Theorem 3.1, Lemma 3.1 and the definitions of the integrals, it is sufficient to prove the statement only for simple functions $F \in L_a^2(\mathbb{R}_+ \times \Omega)$. But in this case it can be easily proved directly. □

4. An Itô Integral on a Fock Space

Let $\mathcal{F}(H)$ be a *symmetric Fock space* over a complex separable Hilbert space H, that is

$$\mathcal{F}(H) := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} H^{\odot n} n!,$$

where $\odot$ stands for the symmetric tensor product ($\otimes$ is the ordinary tensor product). Thus, $\mathcal{F}(H)$ is a complex Hilbert space of sequences $f = (f_n)_{n=0}^{\infty}$ such that each f_n belongs to $H^{\odot n}$ ($H^{\odot 0} := \mathbb{C}$) and

$$\|f\|^2_{\mathcal{F}(H)} = |f_0|^2 + \sum_{n=1}^{\infty} \|f_n\|^2_{H^{\odot n}} n! < \infty.$$

Let us fix a resolution of identity $\mathbb{R}_+ \ni t \mapsto P_t \in \mathcal{L}(H)$ in the space H. For each $t \in \mathbb{R}_+$, we define the second quantization of P_t by the formula

$$\operatorname{Exp} P_t := I \oplus \bigoplus_{n=1}^{\infty} P_t^{\otimes n},$$

i.e.

$$\operatorname{Exp} P_t f := (f_0, P_t f_1, \ldots, P_t^{\otimes n} f_n, \ldots) \in \mathcal{F}(H) \tag{4.1}$$

for all $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}(H)$. It is clear that $\operatorname{Exp} P \colon \mathbb{R}_+ \to \mathcal{F}(H)$ is a resolution of identity in $\mathcal{F}(H)$.

Suppose that $M \colon \mathbb{R}_+ \to \mathcal{F}(H)$ is an $\mathcal{F}(H)$-valued martingale with respect to $E_t := \operatorname{Exp} P_t$ of the form

$$M_t = (\underbrace{0, \ldots, 0}_{\text{k times}}, M_k(t), 0, 0, \ldots) \in \mathcal{F}(H), \tag{4.2}$$

where $k \in \mathbb{N}$ is fixed. In what follows, we fix such martingale M.

In this section, starting with the notion of the H-stochastic integral, we define an Itô integral (on the Fock space $\mathcal{F}(H)$) with respect to such martingale M and derive its properties. More exactly, we give a meaning to the expression

$$\int_{\mathbb{R}_+} f(t)dM_t \tag{4.3}$$

for a certain class of $\mathcal{F}(H)$-valued functions $f \colon \mathbb{R}_+ \to \mathcal{F}(H)$.

First, let us describe a class of functions $f\colon \mathbb{R}_+ \to \mathcal{F}(H)$ for which expression (4.3) will be defined. In the Fock space $\mathcal{F}(H)$, for a fixed vector $f \in \mathcal{F}(H)$, we consider the operator A_f of a Wick multiplication by f, i.e.

$$\mathcal{F}(H) \supset \mathrm{Dom}(A_f) \ni g \mapsto A_f g := f \Diamond g \in \mathcal{F}(H),$$
$$\mathrm{Dom}(A_f) := \{g \in \mathcal{F}(H) | f \Diamond g \in \mathcal{F}(H)\}.$$

Recall, for given $f = (f_n)_{n=0}^{\infty}$ and $g = (g_n)_{n=0}^{\infty}$ from $\mathcal{F}(H)$ the *Wick product* $f \Diamond g$ is defined by

$$f \Diamond g := \left(\sum_{m=0}^{n} f_m \odot g_{n-m} \right)_{n=0}^{\infty} \tag{4.4}$$

provided the latter sequence belongs to $\mathcal{F}(H)$. It is easily seen that $\mathrm{Dom}(A_f)$ is dense in $\mathcal{F}(H)$, since $\mathcal{F}_{\mathrm{fin}}(H) \subset \mathrm{Dom}(A_f)$ and $\mathcal{F}_{\mathrm{fin}}(H)$ is a dense subset of $\mathcal{F}(H)$. Here $\mathcal{F}_{\mathrm{fin}}(H)$ denotes the set of all finite sequences $(f_n)_{n=0}^{\infty}$ from $\mathcal{F}(H)$.

The following statement underlies the definition of a class of functions for which integral (4.3) will be defined.

Lemma 4.1. *For given $t \in \mathbb{R}_+$ and $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}(H)$ an operator A_f of Wick multiplication by f in the space $\mathcal{F}(H)$ is M_t-measurable if and only if*

$$Exp\, P_t f = f. \tag{4.5}$$

Moreover, if $f \in \mathcal{F}(H)$ has property (4.5), then

$$\|A_f\|_{\mathcal{L}_M(s)} = \|f\|_{\mathcal{F}(H)}, \quad s \in [t, \infty). \tag{4.6}$$

Proof. Suppose that equality (4.5) holds for some $t \in \mathbb{R}_+$ and $f = (f)_{n=0}^{\infty} \in \mathcal{F}(H)$. Let us prove that the corresponding operator A_f of Wick multiplication by f is M_t-measurable. Clearly, for this we need to verify hypotheses of Definition 2.2.

First, let us check that $A_f \in \mathcal{L}_M(t)$, i.e. there exists a constant $C > 0$ such that

$$\|A_f g\|_{\mathcal{F}(H)} \leq C \|g\|_{\mathcal{F}(H)}$$

for all $g \in \mathcal{H}_M(t) := \mathrm{span}\{M_{s_2} - M_{s_1} | (s_1, s_2] \subset (t, \infty)\} \subset \mathcal{F}(H)$.

Fix $g \in \mathcal{H}_M(t)$. From the definition of the space $\mathcal{H}_M(t)$ and (4.2), we conclude that

$$g = (\underbrace{0, \ldots, 0}_{\text{k times}}, g_k, 0, 0, \ldots). \tag{4.7}$$

Therefore, according to (4.4) we have

$$A_f g = f \Diamond g = (\underbrace{0, \ldots, 0}_{\text{k times}}, f_0 \odot g_k, f_1 \odot g_k, \ldots).$$

Taking into account that $P_t^{\otimes k} g_k = 0$ (because $g \in \mathcal{H}_M(t)$ and M is the $\mathcal{F}(H)$-valued martingale with respect to $\operatorname{Exp} P$) and $P_t^{\otimes n} f_n = f_n$ for all $n \in \mathbb{N} \cup \{0\}$, we obtain

$$\|f_n \odot g_k\|^2_{H^{\odot n+k}} = \frac{n!k!}{(n+k)!} \|f_n\|^2_{H^{\odot n}} \|g_k\|^2_{H^{\odot k}}.$$

Hence

$$\begin{aligned} \|A_f g\|^2_{\mathcal{F}(H)} = \|f \Diamond g\|^2_{\mathcal{F}(H)} &= \sum_{n=0}^{\infty} \|f_n \odot g_k\|^2_{H^{\odot n+k}} (n+k)! \\ &= \sum_{n=0}^{\infty} \|f_n\|^2_{H^{\odot n}} \|g_k\|^2_{H^{\odot k}} n!k! = \|f\|^2_{\mathcal{F}(H)} \|g\|^2_{\mathcal{F}(H)}. \end{aligned} \tag{4.8}$$

So, we can assert that $A_f \in \mathcal{L}_M(t)$ and equality (4.6) holds.

Let us check that A_f is partially commuting with $\operatorname{Exp} P$, i.e.

$$A_f \operatorname{Exp} P_s g = \operatorname{Exp} P_s A_f g, \quad g \in \mathcal{H}_M(t), \quad s \in [t, \infty). \tag{4.9}$$

Using (4.1) and (4.7) we get

$$\begin{aligned} A_f \operatorname{Exp} P_s g &= f \Diamond \operatorname{Exp} P_s g \\ &= (\underbrace{0, \ldots, 0}_{\text{k times}}, f_0 \odot P_s^{\otimes k} g_k, \ldots, f_n \odot P_s^{\otimes k} g_k, \ldots). \end{aligned} \tag{4.10}$$

On the other hand, we have

$$\begin{aligned} \operatorname{Exp} P_s A_f g &= \operatorname{Exp} P_s (f \Diamond g) \\ &= (\underbrace{0, \ldots, 0}_{\text{k times}}, f_0 \odot P_s^{\otimes k} g_k, \ldots, P_s^{\otimes n} f_n \odot P_s^{\otimes k} g_k, \ldots). \end{aligned} \tag{4.11}$$

Since $f = \operatorname{Exp} P_t f$ and $\operatorname{Exp} P_s \operatorname{Exp} P_t = \operatorname{Exp} P_t$ for all $s \in [t, \infty)$, we conclude that $P_s^{\otimes n} f_n = f_n$ for all $n \in \mathbb{N}$ and all $s \in [t, \infty)$. Hence, from (4.10) and (4.11) it follows (4.9).

So, the first part of the lemma is proved. Let us prove the converse statement: if for given $f \in \mathcal{F}(H)$ the corresponding operator A_f of Wick multiplication by f in the space $\mathcal{F}(H)$ is M_t-measurable then (4.5) holds.

Since A_f is the M_t-measurable, we see that for any $s \in [t, \infty)$

$$A \operatorname{Exp} P_s g = \operatorname{Exp} P_s A g, \quad g \in \mathcal{H}_M(t).$$

Let $s \in (t, \infty)$ and $(s_1, s_2] \subset (t, s]$ be fixed. We take

$$g := M_{s_2} - M_{s_1} = (\underbrace{0, \ldots, 0}_{\text{k times}}, g_k, 0, 0, \ldots) \in \mathcal{H}_N(t),$$

where $g_k := M_k(s_2) - M_k(s_1)$. Then, from the one hand,

$$A E_s g = A g = f \Diamond g = (\underbrace{0, \ldots, 0}_{\text{k times}}, f_0 \odot g_k, f_1 \odot g_k, \ldots),$$

and on the other hand, taking into account that $P_s^{\otimes k} g_k = g_k$, we get

$$\begin{aligned} E_s A g = E_s(f \Diamond g) &= (\underbrace{0, \dots, 0}_{\text{k times}}, f_0 \odot P_s^{\otimes k} g_k, \dots, P_s^{\otimes n} f_n \odot P_s^{\otimes k} g_k, \dots) \\ &= (\underbrace{0, \dots, 0}_{\text{k times}}, f_0 \odot g_k, \dots, P_s^{\otimes n} f_n \odot g_k, \dots). \end{aligned}$$

Hence, $f_n \odot g_k = P_s^{\otimes n} f_n \odot g_k$ for each $n \in \mathbb{N}$ and all $s \in (t, \infty)$. Since resolution of identity (4.1) is right continuous, the latter equality still holds for $s = t$, and therefore equality (4.5) takes place. □

Now we are ready to introduce a class of $\mathcal{F}(H)$-valued functions for which integral (4.3) will be defined. Denote by $L^2(\mathbb{R}_+, \mu; \mathcal{F}(H)) := L^2(\mathbb{R}_+ \to \mathcal{F}(H), \mu)$ the Hilbert space of $\mathcal{F}(H)$-valued functions

$$f : \mathbb{R}_+ \to \mathcal{F}(H), \quad \|f\|^2_{L^2(\mathbb{R}_+, \mu; \mathcal{F}(H))} := \int_{\mathbb{R}_+} \|f(t)\|^2_{\mathcal{F}(H)} d\mu(t) < \infty,$$

with the corresponding scalar product, where as above $\mu(\alpha) := \|M(\alpha)\|^2_{\mathcal{F}(H)}$.

Definition 4.1. A function $f(\cdot) = (f_n(\cdot))_{n=0}^{\infty} \in L^2(\mathbb{R}_+, \mu; \mathcal{F}(H))$ is said to be *adapted* with respect to $\operatorname{Exp} P$ if for μ-almost all $t \in \mathbb{R}_+$

$$\operatorname{Exp} P_t f(t) = f(t).$$

We will use the notation $L_a^2(\mathbb{R}_+, \mu; \mathcal{F}(H))$ for the corresponding subspace of $L^2(\mathbb{R}_+, \mu; \mathcal{F}(H))$ formed by adapted functions.

Let $\mathcal{E} = \mathcal{E}(\operatorname{Exp} P)$ denote the class of simple adapted functions with respect to the resolution of identity $\operatorname{Exp} P$. That is, a function f belongs to $\mathcal{E}$ if it belongs to $L_a^2(\mathbb{R}_+, \mu; \mathcal{F}(H))$ and can be written as

$$f(t) = \sum_{k=0}^{n-1} f_{(k)} \varkappa_{(t_k, t_{k+1}]}(t) \in \mathcal{F}(H) \tag{4.12}$$

for μ-almost all $t \in \mathbb{R}_+$, where $0 = t_0 < t_1 < \cdots < t_n < \infty$ is a partition of $\mathbb{R}_+$. We denote by $\mathcal{E}_2 = \mathcal{E}_2(\operatorname{Exp} P)$ the closure of the set $\mathcal{E}$ in the space $L^2(\mathbb{R}_+, \mu; \mathcal{F}(H))$. Observe that if the structures of the elements of the Hilbert space H and the orthogonal projection P_t are not known, then it is difficult to characterize the space $\mathcal{E}_2$ (clear only that $\mathcal{E}_2 \subset L_a^2(\mathbb{R}_+, \mu; \mathcal{F}(H))$). In a special case when $H = L^2(\mathbb{R}_+, dt)$ and P_t is the resolution of identity of the operator of multiplication by t in the space $L^2(\mathbb{R}_+, dt)$, the space $\mathcal{E}_2$ can be analyzed in a deeper way, in particular, it can be shown that $\mathcal{E}_2 = L_a^2(\mathbb{R}_+, \mu; \mathcal{F}(H))$, see the next section for more details.

Let us derive the connection between the spaces $\mathcal{E}_2(\operatorname{Exp} P)$ and $S_2(M)$.

Proposition 4.1. *A function $f : \mathbb{R}_+ \to \mathcal{F}(H)$ belongs to the space $\mathcal{E}_2(Exp\, P)$ if and only if the corresponding operator-valued function $\mathbb{R}_+ \ni t \mapsto A_f(t)$ whose* *values are operators of Wick multiplication by $f(t)$ in the Fock space $\mathcal{F}(H)$ belongs to the space $S_2(M)$.*

Proof. This fact is an immediate consequence of Lemma 4.1 and the definitions of the spaces $\mathcal{E}_2(\operatorname{Exp} P)$ and $S_2(M)$. □

Now we are ready to give a definition of Itô integral on the Fock space $\mathcal{F}(H)$.

Definition 4.2. An Itô integral of $f \in \mathcal{E}_2$ with respect to M is defined by

$$\int_{\mathbb{R}_+} f(t) dM_t := \int_{\mathbb{R}_+} A_f(t) dM_t \in \mathcal{F}(H),$$

where $A_f(t)$ is the operator of Wick multiplication by $f(t)$ in the space $\mathcal{F}(H)$.

Clearly, for a simple function $f \in \mathcal{E}$ of kind (4.12), we have

$$\int_{\mathbb{R}_+} f(t) dM_t = \sum_{k=0}^{n-1} f_{(k)} \Diamond (M_{t_{k+1}} - M_{t_k}) \in \mathcal{F}(H).$$

Theorem 4.1. *Let $f, g \in \mathcal{E}_2$ and $a, b \in \mathbb{C}$ then*

$$\int_{\mathbb{R}_+} (af(t) + bg(t)) dM_t = a \int_{\mathbb{R}_+} f(t) dM_t + b \int_{\mathbb{R}_+} g(t) dM_t$$

and

$$\left\| \int_{\mathbb{R}_+} f(t) dM_t \right\|^2_{\mathcal{F}(H)} = \int_{\mathbb{R}_+} \|f(t)\|^2_{\mathcal{F}(H)} d\mu(t). \tag{4.13}$$

Proof. It is sufficient to prove the statement for simple adapted functions. The first assertion is obvious. Concerning the second one, we use the technique of the proof of Theorem 2.1 and equality (4.8). Thus, for a simple function $f \in \mathcal{E}$ of type (4.12) we obtain

$$\begin{aligned}
\left\| \int_{\mathbb{R}_+} f(t) dM_t \right\|^2_{\mathcal{F}(H)} &= \left\| \sum_{k=0}^{n-1} f_{(k)} \Diamond (M_{t_{k+1}} - M_{t_k}) \right\|^2_{\mathcal{F}(H)} \\
&= \sum_{k=0}^{n-1} \| f_{(k)} \Diamond (M_{t_{k+1}} - M_{t_k}) \|^2_{\mathcal{F}(H)} \\
&= \sum_{k=0}^{n-1} \| f_{(k)} \|^2_{\mathcal{F}(H)} \| M_{t_{k+1}} - M_{t_k} \|^2_{\mathcal{F}(H)} \\
&= \sum_{k=0}^{n-1} \| f_{(k)} \|^2_{\mathcal{F}(H)} \mu((t_k, t_{k+1}]) \\
&= \int_{\mathbb{R}_+} \|f(t)\|^2_{\mathcal{F}(H)} d\mu(t).
\end{aligned}$$

□

Remark 4.1. Note as a consequence of the previous theorem that the integral $\int_{\mathbb{R}_+} f(t)dM_t$ is well defined linear isometry operator from a subspace $\mathcal{E}_2$ of $L^2(\mathbb{R}_+, \mu; \mathcal{F}(H))$ into $\mathcal{F}(H)$.

5. Itô Integral on the Fock Space $\mathcal{F}(\mathbf{L^2}(\mathbb{R}_+))$

Let $L^2(\mathbb{R}_+) := L^2(\mathbb{R}_+, dt)$ be a complex L^2-space with respect to the Lebesgue measure $dt = dm(t)$. The aim of this section is to show that the definition of the Itô integral on the Fock space $\mathcal{F} := \mathcal{F}(L^2(\mathbb{R}_+))$ proposed in [4] (see also [2]) is the particular case of the one proposed in the previous section. Note that the Fock space $\mathcal{F}$ is closely related to the chaos expansion in probability theory (see, e.g. [13, 9, 11] and also Sec. 6 below) and from a physics point of view it describes a field of bosonic particles, like photons. We shall always identify (in the natural way) the space $L^2(\mathbb{R}_+)^{\odot n}$ with the space $L^2_{\text{sym}}(\mathbb{R}^n_+)$ of all complex-valued symmetric functions from $L^2(\mathbb{R}^n_+)$. Now

$$\begin{aligned}\|f_n\|^2_{L^2(\mathbb{R}_+)^{\odot n}} &= \int_{\mathbb{R}^n_+} |f_n(t_1, \ldots, t_n)|^2 dt_1 \cdots dt_n \\ &= n! \int_0^\infty \int_0^{t_n} \cdots \left(\int_0^{t_2} |f_n(t_1, \ldots, t_n)|^2 dt_1 \right) \cdots dt_{n-1} dt_n\end{aligned}$$

for all $f_n \in L^2(\mathbb{R}_+)^{\odot n} \cong L^2_{\text{sym}}(\mathbb{R}^n_+)$.

Recall the definition of the Itô integral on the Fock space $\mathcal{F}$ proposed in [4] (see also [2]). First of all a function $\mathbb{R}_+ \ni t \mapsto f(t) = (f_n(t))_{n=0}^\infty \in \mathcal{F}$ is called *Itô integrable* if f belongs to $L^2(\mathbb{R}_+, dt; \mathcal{F})$ and, for almost all $t \in \mathbb{R}_+$,

$$f(t) = (f_0(t), \varkappa_{[0,t]} f_1(t), \ldots, \varkappa_{[0,t]^n} f_n(t), \ldots).$$

In other words, in our terminology, a function $f \in L^2(\mathbb{R}_+, dt; \mathcal{F})$ is Itô integrable if and only if f is an adapted function with respect to the resolution of identity $\operatorname{Exp} \mathcal{X} \colon \mathbb{R}_+ \to \mathcal{F},\ t \mapsto \operatorname{Exp} \mathcal{X}_t$, constructed according to (4.1) using the resolution of identity

$$\mathbb{R}_+ \ni t \mapsto \mathcal{X}_t f := \varkappa_{[0,t]} f \in L^2(\mathbb{R}_+), \quad f \in L^2(\mathbb{R}_+),$$

of the operator of multiplication by t in the space $L^2(\mathbb{R}_+)$. Following the notations of the previous section we will denote by $L^2_a(\mathbb{R}_+, dt; \mathcal{F})$ the set of all adapted functions with respect to $\operatorname{Exp} \mathcal{X}$.

Observe here that each component $f_n(t_1, \ldots, t_n; t)$ of a function $f(\cdot) = (f_n(\cdot))_{n=0}^\infty \in L^2(\mathbb{R}_+, dt; \mathcal{F})$ belongs to the space $L^2_{\text{sym}}(\mathbb{R}^n_+) \otimes L^2(\mathbb{R}_+)$. It means that f_n belongs to $L^2(\mathbb{R}^{n+1}_+)$ and it is a symmetric function in the first n variables, i.e. for m-almost all $t \in \mathbb{R}_+$ and for $m^{\otimes n}$-almost all $(t_1, \ldots, t_n) \in \mathbb{R}^n_+$,

$$f_n(t_1, \ldots, t_n; t) = \frac{1}{n!} \sum_\sigma f_n(t_{\sigma(1)}, \ldots, t_{\sigma(n)}; t),$$

where σ running over all permutations of $\{1, \ldots, n\}$.

Definition 5.1. The Itô integral $\mathbb{I}(f)$ of $f \in L^2_a(\mathbb{R}_+, dt; \mathcal{F})$ is defined as the unique linear isometric mapping

$$\mathbb{I}\colon L^2(\mathbb{R}_+, dt; \mathcal{F}) \to \mathcal{F}, \quad \mathrm{Dom}(\mathbb{I}) := L^2_a(\mathbb{R}_+, dt; \mathcal{F}), \tag{5.1}$$

such that

$$\begin{aligned}\mathbb{I}(g\varkappa_{(s_1,s_2]}) &= g\Diamond(0, \varkappa_{(s_1,s_2]}, 0, 0, \ldots) \\ &= (0, g_0 \odot \varkappa_{(s_1,s_2]}, \ldots, g_n \odot \varkappa_{(s_1,s_2]}, \ldots) \in \mathcal{F}\end{aligned} \tag{5.2}$$

for any $(s_1, s_2] \subset \mathbb{R}_+$ and any $g = (g_n)_{n=0}^\infty \in \mathcal{F}$ such that $\mathrm{Exp}\,\mathcal{X}_{s_1} g = g$.

We stress that a set of all functions $\mathbb{R}_+ \ni t \mapsto f(t) := g\varkappa_{(s_1,s_2]}(t) \in \mathcal{F}$ such that $(s_1, s_2] \subset \mathbb{R}_+$ and $g = \mathrm{Exp}\,\mathcal{X}_{s_1} g \in \mathcal{F}$ is a total in the space $L^2_a(\mathbb{R}_+, dt; \mathcal{F})$. The isometry of $\mathbb{I}$ means that

$$\|\mathbb{I}(f)\|^2_{\mathcal{F}} = \int_{\mathbb{R}_+} \|f(t)\|^2_{\mathcal{F}} dt, \quad f \in L^2_a(\mathbb{R}_+, dt; \mathcal{F}). \tag{5.3}$$

Let us show that the integral $\mathbb{I}$ is a particular case of the integral introduced in the previous section. To this end, note that

$$Z\colon \mathbb{R}_+ \to \mathcal{F}, \quad t \mapsto Z_t := (0, \varkappa_{[0,t]}, 0, 0, \ldots),$$

is an $\mathcal{F}$-valued martingale with respect to $\mathrm{Exp}\,\mathcal{X}$ and

$$\mu([0,t]) := \|Z_t\|^2_{\mathcal{F}} = \|\varkappa_{[0,t]}\|^2_{L^2(\mathbb{R}_+)} = m([0,t]) = t$$

is the Lebesgue measure on $\mathbb{R}_+$.

Theorem 5.1. *We have* $\mathcal{E}_2(Exp\,\mathcal{X}) = L^2_a(\mathbb{R}_+, dt; \mathcal{F})$ *and*

$$\mathbb{I}(f) = \int_{\mathbb{R}_+} f(t) dZ_t, \quad f \in L^2_a(\mathbb{R}_+, dt; \mathcal{F}).$$

Proof. Since the set $\mathcal{E}(\mathrm{Exp}\,\mathcal{X})$ of all simple adapted functions with respect to the resolution of identity $\mathrm{Exp}\,\mathcal{X}$ is dense in the spaces $\mathcal{E}_2(\mathrm{Exp}\,\mathcal{X})$ and $L^2_a(\mathbb{R}_+, dt; \mathcal{F})$, in view of the definitions of the latter spaces, we conclude that $\mathcal{E}_2(\mathrm{Exp}\,\mathcal{X}) = L^2_a(\mathbb{R}_+, dt; \mathcal{F})$.

The second assertion of the theorem directly follows from the definitions of the integrals. □

Now one can develop an abstract Itô calculus on the Fock space $\mathcal{F}$. In particular, one can construct iterated Itô integrals and the corresponding Fock space chaotic expansion, to prove an analog of Clark–Ocone formula and so on. We do not discuss this in details (see, e.g. [4,2]), but we recall here the expression of the Itô integral $\mathbb{I}$ in terms of the Fock space structure. Since this fact is easily proven, for the reader's convenience we also present a proof.

Theorem 5.2. *Let $f(\cdot) = (f_n(\cdot))_{n=0}^\infty$ belongs to $L_a^2(\mathbb{R}_+, dt; \mathcal{F})$. Then*

$$\mathbb{I}(f) = \int_{\mathbb{R}_+} f(t) dZ_t = (0, \hat{f}_1, \dots, \hat{f}_n, \dots), \tag{5.4}$$

where each $\hat{f}_n(t_1, \dots, t_n)$ is the symmetrization of $f_{n-1}(t_1, \dots, t_{n-1}; t)$ with respect to n variables, i.e. in view of the fact that $f_{n-1}(t_1, \dots, t_{n-1}; t)$ is symmetric in the first $n-1$ variables,

$$\hat{f}_n(t_1, \dots, t_n) := \frac{1}{n} \sum_{k=1}^n f_{n-1}(t_1, \dots, \not{t_k}, \dots, t_n; t_k).$$

Proof. According to (5.3) and Theorem 3.2 from [1], for all $f \in L_a^2(\mathbb{R}_+, dt; \mathcal{F})$, we have

$$\|\mathbb{I}(f)\|_{\mathcal{F}}^2 = \int_{\mathbb{R}_+} \|f(t)\|_{\mathcal{F}}^2 dt \quad \text{and} \quad \|(0, \hat{f}_1, \dots, \hat{f}_n, \dots)\|_{\mathcal{F}}^2 = \int_{\mathbb{R}_+} \|f(t)\|_{\mathcal{F}}^2 dt.$$

In other words, the linear mappings

$$f \mapsto \mathbb{I}(f) \quad \text{and} \quad f \mapsto (0, \hat{f}_1, \dots, \hat{f}_n, \dots)$$

from $L_a^2(\mathbb{R}_+, dt; \mathcal{F})$ to $\mathcal{F}$ are continuous (more exactly, isometric). Therefore, it is sufficient to check (5.4) for functions $\mathbb{R}_+ \ni t \mapsto f(t) := g\varkappa_{(s_1,s_2]}(t) \in \mathcal{F}$ such that $(s_1, s_2] \subset \mathbb{R}_+$, and $g = \operatorname{Exp} \mathcal{X}_{s_1} g$.

Fix $(s_1, s_2] \subset \mathbb{R}_+$ and $g = (g_n)_{n=0}^\infty \in \mathcal{F}$ such that $g = \operatorname{Exp} \mathcal{X}_{s_1} g$. For

$$f(t) := g\varkappa_{(s_1,s_2]}(t) = (f_n(t))_{n=0}^\infty, \quad f_n(t) := g_n \varkappa_{(s_1,s_2]}(t),$$

we have

$$\begin{aligned}\int_{\mathbb{R}_+} f(t) dZ(t) &= g \Diamond (Z_{s_2} - Z_{s_1}) \\ &= (0, g_0 \odot \varkappa_{(s_1,s_2]}, \dots, g_{n-1} \odot \varkappa_{(s_1,s_2]}, \dots) = (0, \hat{f}_1, \dots, \hat{f}_n, \dots). \qquad \square\end{aligned}$$

6. Relationship Between the Classical Itô Integral and the Itô Integral on the Fock Space $\mathcal{F}$

The main purpose of this section is to connect the objects of Secs. 3 and 5 with each other. Such results are fairly standard, but we include it here for the sake of completeness.

Without going into details, let us give a brief introduction to the theory of probabilistic interpretations of the Fock space $\mathcal{F}$, see e.g. [13, 9] for more details. As before, let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a right continuous filtration $\{\mathcal{A}_t\}_{t \in \mathbb{R}_+}$, $\mathcal{A}_0$ be the trivial σ-algebra containing all the P-null sets of $\mathcal{A}$ and $\mathcal{A}$ coincides with the smallest σ-algebra generated by $\bigcup_{t \in \mathbb{R}_+} \mathcal{A}_t$. Suppose $N = \{N_t\}_{t \in \mathbb{R}_+}$, $N_0 = 0$, is a *normal martingale* on $(\Omega, \mathcal{A}, P)$ with respect to $\{\mathcal{A}_t\}_{t \in \mathbb{R}_+}$.

It is well known that the mapping (so-called *chaos expansion*)

$$I\colon \mathcal{F} \to L^2(\Omega, \mathcal{A}, P), \quad f = (f_n)_{n=0}^{\infty} \mapsto If := \sum_{n=0}^{\infty} I_n(f_n), \tag{6.1}$$

is well-defined and isometric. Here $I_0(f_0) := f_0$ and for each $n \in \mathbb{N}$

$$I_n(f_n) := n! \int_0^{\infty} \int_0^{t_n} \cdots \left(\int_0^{t_2} f_n(t_1, \ldots, t_n) dN_{t_1} \right) \ldots dN_{t_{n-1}} dN_{t_n}$$

is an n-iterated Itô integral with respect to N.

In what follows we will always assume that the normal martingale N has the *Chaotic Representation Property* (CRP), i.e. the mapping $I : \mathcal{F} \to L^2(\Omega, \mathcal{A}, P)$ is unitary. In this way we get a probabilistic interpretation of the Fock space $\mathcal{F}$. We observe that the Brownian motion, the compensated Poisson process and some Azéma martingales are examples of normal martingales which possess the CRP, see for instance [13, 9, 8, 11, 3].

It is not difficult to show that the mapping I is completely characterized by the following properties:

(i) $I : \mathcal{F} \to L^2(\Omega, \mathcal{A}, P)$ is the unitary operator. That is, I maps the whole space $\mathcal{F}$ onto whole $L^2(\Omega, \mathcal{A}, P)$ and $\|If\|_{L^2(\Omega,\mathcal{A},P)} = \|f\|_{\mathcal{F}}$ for all $f \in \mathcal{F}$.

(ii) $I_0(f_0) = f_0$ for all $f_0 \in \mathbb{C}$.

(iii) For each $n \in \mathbb{N}$ and every disjoint sets $\alpha_1, \ldots, \alpha_n$ from $\mathcal{B}(\mathbb{R}_+)$ of finite Lebesgue measure,

$$I_n(\varkappa_{\alpha_1} \odot \cdots \odot \varkappa_{\alpha_n}) = N(\alpha_1) \cdot \ldots \cdot N(\alpha_n), \tag{6.2}$$

where $\mathcal{B}(\mathbb{R}_+) \ni \alpha \mapsto N(\alpha) \in L^2(\Omega, \mathcal{A}, P)$ is a vector-valued measure generated by the normal martingale N, i.e. we set

$$N((s_1, s_2]) = N_{s_2} - N_{s_1}, \quad N(\{0\}) := N_0 = 0, \quad N(\emptyset) := 0,$$

and extend this definition to all Borel subsets of $\mathbb{R}_+$.

Let us pass to the establishing of the relationship between the objects of Secs. 3 and 5. From property (iii) of the mapping I it immediately follows that the normal martingale N is the I-image of the $\mathcal{F}$-valued martingale

$$Z\colon \mathbb{R}_+ \to \mathcal{F}, \quad t \mapsto Z_t := (0, \varkappa_{[0,t]}, 0, 0, \ldots),$$

i.e. for almost all $t \in \mathbb{R}_+$, $N_t = IZ_t \in L^2(\Omega, \mathcal{A}, P)$. The following step is to show that the resolution of identity

$$E\colon \mathbb{R}_+ \to L^2(\Omega, \mathcal{A}, P), \quad t \mapsto E_t := \mathbb{E}[\cdot|\mathcal{A}_t],$$

is the I-image of the resolution of identity $\operatorname{Exp}\mathcal{X}\colon \mathbb{R}_+ \to \mathcal{F}$, where as before

$$\operatorname{Exp}\mathcal{X}_t := I \oplus \bigoplus_{n=1}^{\infty} \mathcal{X}_t^{\otimes n}, \quad \mathcal{X}_t f = \varkappa_{[0,t]} f, \quad f \in L^2(\mathbb{R}_+).$$

We have the following result.

Lemma 6.1. *For all $t \in \mathbb{R}_+$ we have $I^{-1}\mathbb{E}[\cdot|\mathcal{A}_t]I = Exp\,\mathcal{X}_t$. That is,*

$$I^{-1}\mathbb{E}[If|\mathcal{A}_t] = Exp\,\mathcal{X}_t f = (f_0, \varkappa_{[0,t]}f_1, \ldots, \varkappa_{[0,t]^n}f_n, \ldots)$$

for all $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}$.

Proof. In order to prove the assertion of the lemma it is sufficient to show that

$$\mathbb{E}[I_n(f_n)|\mathcal{A}_t] = I_n(\varkappa_{[0,t]^n}f_n) \tag{6.3}$$

for any $f_n \in L^2(\mathbb{R}_+)^{\odot n}, n \in \mathbb{N}$.

Since functions

$$f_n := \varkappa_{\alpha_1}\odot\cdots\odot\varkappa_{\alpha_n}, \quad \alpha_i \in \mathcal{B}(\mathbb{R}_+), \quad \alpha_i \cap \alpha_j = \emptyset, \quad i \neq j,$$

form a total set in $L^2(\mathbb{R}_+)^{\odot n}$, it is sufficient to check (6.3) for such functions. Using (6.2), (3.1) and the properties of the conditional expectation, we get

$$\begin{aligned}\mathbb{E}[I_n(f_n)|\mathcal{A}_t] &= \mathbb{E}[I_n(\varkappa_{\alpha_1}\odot\cdots\odot\varkappa_{\alpha_n})|\mathcal{A}_t] = \mathbb{E}[N(\alpha_1)\cdot\ldots\cdot N(\alpha_n)|\mathcal{A}_t]\\ &= \mathbb{E}\left[\prod_{i=1}^{n}(N(\alpha_i\cap[0,t]) - N(\alpha_i\cap(t,\infty]))\middle|\mathcal{A}_t\right]\\ &= N(\alpha_1\cap[0,t])\cdot\ldots\cdot N(\alpha_n\cap[0,t])\\ &= I_n(\varkappa_{\alpha_1\cap[0,t]}\odot\cdots\odot\varkappa_{\alpha_n\cap[0,t]}) = I_n(\varkappa_{[0,t]^n}f_n).\end{aligned}$$

□

As an immediate consequence we have.

Corollary 6.1. *For given $t \in \mathbb{R}_+$ a function $F \in L^2(\Omega, \mathcal{A}, P)$ is an $\mathcal{A}_t$-measurable if and only if $f = Exp\,\mathcal{X}_t f$, where $f := I^{-1}F \in \mathcal{F}$.*

The next result explains the relationship between the Wick multiplication $\lozenge$ on $\mathcal{F}$ and the ordinary multiplication on $L^2(\Omega, \mathcal{A}, P)$.

Lemma 6.2. *Suppose $t \in \mathbb{R}_+$ and $F \in L^2(\Omega, \mathcal{A}, P)$ is an $\mathcal{A}_t$-adapted function. Then for each $\alpha \in \mathcal{B}([t,\infty))$ the function $FN(\alpha)$ belongs to $L^2(\Omega, \mathcal{A}, P)$ and the I^{-1}-image of $FN(\alpha)$ has the form*

$$I^{-1}(FN(\alpha)) = I^{-1}F\lozenge I^{-1}(N(\alpha)) = I^{-1}F\lozenge Z(\alpha). \tag{6.4}$$

Proof. Using (6.2) and taking into account that for any fixed interval $[a,b] \subset \mathbb{R}_+$ functions $\varkappa_{\alpha_1}\odot\cdots\odot\varkappa_{\alpha_n}$, where each $\alpha_i \in \mathcal{B}([a,b])$ and $\alpha_i \cap \alpha_j = \emptyset$ for $i \neq j$, form a total set in the space $L^2_{\text{sym}}([a,b]^n)$ of all complex-valued symmetric functions from $L^2([a,b]^n)$, we get

$$I_{n+1}(f_n \odot g_1) = I_n(f_n)I_1(g_1)$$

for all $f_n \in L^2(\mathbb{R}_+)^{\odot n}$ and all $g_1 \in L^2(\mathbb{R}_+)$ such that $f_n = \varkappa_{[0,t]^n}f_n$ and $g_1 = \varkappa_{(t,\infty)}g_1$. Hence, for any $f = (f_n)_{n=0}^{\infty} \in \mathcal{F}$ such that $f = \mathrm{Exp}\,\mathcal{X}_t f$ and any $\alpha \in \mathcal{B}([t,\infty))$,

$$I(f\lozenge Z(\alpha)) = If\cdot IZ(\alpha) = If\cdot N(\alpha).$$

From the latter equality and Corollary 6.1, we immediately obtain the assertion of the lemma. □

Corollary 6.2. *Suppose $t \in \mathbb{R}_+$ and $F \in L^2(\Omega, \mathcal{A}, P)$ is an $\mathcal{A}_t$-measurable function. Let A_F be the corresponding operator of multiplication by F in the space $L^2(\Omega, \mathcal{A}, P)$, i.e.*

$$L^2(\Omega, \mathcal{A}, P) \supset \mathrm{Dom}(A_F) \ni G \mapsto A_F G := FG \in L^2(\Omega, \mathcal{A}, P).$$

Then, for all $g \in \mathcal{H}_Z(t) = \mathrm{span}\{Z_{s_2} - Z_{s_1} | (s_1, s_2] \subset (t, \infty)\} \subset \mathcal{F}$, we have

$$I^{-1} A_F I g = I^{-1} F \Diamond g.$$

Proof. Let $F \in L^2(\Omega, \mathcal{A}, P)$ be an $\mathcal{A}_t$-measurable function, A_F be the corresponding operator of multiplication by F in the space $L^2(\Omega, \mathcal{A}, P)$ and $g \in \mathcal{H}_Z(t)$. Using (6.4) we get

$$I^{-1} A_F I g = I^{-1} A_F G = I^{-1}(FG) = I^{-1} F \Diamond I^{-1} G = I^{-1} F \Diamond g,$$

where $G := Ig \in \mathcal{H}_N(t) = \mathrm{span}\{N_{s_2} - N_{s_1} | (s_1, s_2] \subset (t, \infty)\}$. □

Remark 6.1. It should be noticed that in general the I-image of the Wick multiplication $\Diamond$ distinguishes from the ordinary multiplication. To be precise, there exist functions $F, G \in L^2(\Omega, \mathcal{A}, P)$ such that $FG \in L^2(\Omega, \mathcal{A}, P)$ but

$$I^{-1}(FG) \neq I^{-1} F \Diamond I^{-1}(G).$$

Before establishing the relationship between the classical Itô integral and the Itô integral on the Fock space $\mathcal{F}$ we note that the spaces $L^2(\mathbb{R}_+ \times \Omega)$ and $L^2(\mathbb{R}_+, dt; \mathcal{F})$ can be interpreted as tensor products $L^2(\mathbb{R}_+) \otimes L^2(\Omega, \mathcal{A}, P)$ and $L^2(\mathbb{R}_+) \otimes \mathcal{F}$ respectively. Therefore

$$1 \otimes I : L^2(\mathbb{R}_+, dt; \mathcal{F}) \to L^2(\mathbb{R}_+ \times \Omega)$$

is a well-defined unitary operator.

Theorem 6.1. *We have $L^2_a(\mathbb{R}_+ \times \Omega) = (1 \otimes I) L^2_a(\mathbb{R}_+, dt; \mathcal{F})$ and*

$$I\left(\int_{\mathbb{R}_+} f(t) dZ_t\right) = \int_{\mathbb{R}_+} I f(t) dN_t$$

for arbitrary $f \in L^2_a(\mathbb{R}_+, dt; \mathcal{F})$.

Proof. This result follows from Proposition 2.1 combined with Corollaries 6.1 and 6.2. □

Remark 6.2. Since $L^2_a(\mathbb{R}_+ \times \Omega) = (1 \otimes I) L^2_a(\mathbb{R}_+, dt; \mathcal{F})$, for any function $F \in L^2_a(\mathbb{R}_+ \times \Omega)$ there exists a uniquely defined vector $f(\cdot) = (f_n(\cdot))_{n=0}^{\infty} \in L^2_a(\mathbb{R}_+, dt; \mathcal{F})$

such that $F(t) = If(t) = \sum_{n=0}^{\infty} I_n(f_n(t))$ for almost all $t \in \mathbb{R}_+$. In view of (5.4), (6.1) and Theorem 6.1, we get

$$\int_{\mathbb{R}_+} F(t)dN_t = I\left(\int_{\mathbb{R}_+} f(t)dZ_t\right) = \sum_{n=1}^{\infty} I_n(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P).$$

Using this representation we can easily construct a natural extension of the Itô integral, called the Hitsuda–Skorohod integral, whose study has become very fashionable (see, e.g. [7, 16, 14, 11]). Namely, the expression

$$\mathbb{I}_{\mathrm{HS}}(F) := \sum_{n=1}^{\infty} I_n(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P), \quad F \in \mathrm{Dom}(\mathbb{I}_{\mathrm{HS}}),$$

$$\mathrm{Dom}(\mathbb{I}_{\mathrm{HS}}) := \left\{ G \in L^2(\mathbb{R}_+ \times \Omega), G(t) = \sum_{n=0}^{\infty} I_n(g_n(t)) \,\middle|\, \sum_{n=1}^{\infty} I_n(\hat{g}_n) \in L^2(\Omega, \mathcal{A}, P) \right\},$$

is called the *Hitsuda–Skorohod integral* of F. When N is a Brownian motion, it is exactly the integral introduced by Hitsuda [7] and Skorohod [16].

It is clear that $L_a^2(\mathbb{R}_+, dt; \mathcal{F}) \subset \mathrm{Dom}(\mathbb{I}_{\mathrm{HS}})$ and the operator $\mathbb{I}_{\mathrm{HS}}$ restricted to the space $L_a^2(\mathbb{R}_+, dt; \mathcal{F})$ coincides with the Itô integral, i.e.

$$\mathbb{I}_{\mathrm{HS}}(F) = \int_{\mathbb{R}_+} F(t)dN_t, \quad F \in L_a^2(\mathbb{R}_+, dt; \mathcal{F}).$$

Note that the Hitsuda–Skorohod integral can also be defined as the adjoint of the Malliavin derivative, see e.g. [20, 14, 12].

Remark 6.3. In this section we have seen that the Itô integral with respect to normal martingales with CRP can be interpreted as the image of Itô integral (5.1) on the Fock space $\mathcal{F}$. Because of this fact we can treat the stochastic analysis connected with all these martingales in one framework — as the analysis on the Fock space.

In connection with this it is natural to ask: "*is it possible to obtain a similar result for some normal martingales without the CRP?*". Recently it became clear (see surveys [10] and the references therein) that this is possible at least for the case of stochastic integration with respect to Gamma, Pascal and Meixner processes. But in this case it is necessary to use more complicated "extended Fock space" instead of the Fock space $\mathcal{F}$. It can be shown that the corresponding Itô integral on the extended Fock space is the particular case of the H-stochastic integral.

Acknowledgment

I would like to thank Prof. Yu. M. Berezansky for the statement of the question and helpful discussions.

References

1. S. Albeverio, Yu. M. Berezansky and V. Tesko, A generalization of an extended stochastic integral, *Ukr. Mat. Zh.* **59** (2007) 588–617.
2. S. Attal, *Quantum Noises, Open Quantum Systems. II. The Markovian Approach*, Lect. Notes in Math., Vol. 1881 (Springer-Verlag, 2006).
3. S. Attal and A. Belton, The chaotic-representation property for some Azema martingales, *Probab. Th. Relat. Fields* **139** (2007) 543–562.
4. S. Attal and J. M. Lindsay, Quantum stochastic calculus with maximal operator domains, *Ann. Probab.* **32** (2004) 488–529.
5. Yu. M. Berezansky, N. W. Zhernakov and G. F. Us, A spectral approach to quantum stochastic integrals, *Rep. Math. Phys.* **28** (1989) 347–360.
6. C. Dellacherie, B. Maisonneuve and P. A. Meyer, *Probabilites et Potentiel. Ch XVII a XXIV. Processus de Markov (fin), Complements de calcul stochastique* (Paris, 1992).
7. M. Hitsuda, Formula for Brownian partial derivatives, *Second Japan–USSR Symp. Probab.* **2** (1972) 111–114.
8. M. Emery, *On the Azema Martingales*, Lect. Notes in Math., Vol. 1372 (Springer, 1989).
9. M. Emery, On the chaotic representation property for martingales, in *Probability Theory and Mathematical Statistics*, eds. I. A. Ibragimov and A. Yu. Zaitsev (Gordon and Breach, 1996), pp. 155–166.
10. N. A. Kachanovsky and V. A. Tesko, Stochastic integral of Hitsuda–Skorohod type on the extended Fock space, *Ukr. Math. J.* **61** (2009) 733–764.
11. J. Ma, P. Protter and J. S. Martin, Anticipating integrals for a class of martingales, *Bernoulli* **4** (1998) 81–114.
12. P. Malliavin, *Stochastic Analysis* (Springer-Verlag, 1997).
13. P. A. Meyer, *Quantum Probability for Probabilists*, Lect. Notes in Math., Vol. 1538 (Springer-Verlag, 1993).
14. D. Nualart, *The Malliavin Calculus and Related Topics* (Springer-Verlag, 1995).
15. B. Simon, *The* $P(\varphi)_2$ *Euclidian (Quantum) Field Theory* (Princeton Univ. Press, 1974).
16. A. V. Skorohod, On a generalization of the stochastic integral, *Teor. Verojatnost. i Primenen.* **20** (1975) 223–238 (in Russian).
17. V. Tesko, A stochastic integral of operator-valued functions, *Methods Funct. Anal. Topology* **14** (2008) 132–141.
18. I. I. Gikhman and A. V. Skorokhod, *The Theory of Random Processes* (in Russian) (Nauka, 1975), Vol. 3.
19. Ph. E. Protter, *Stochastic Integration and Differential Equations* (Springer-Verlag, 2004).
20. S. Watanabe, *Lecture on Stochastic Differential Equations and Malliavin Calculus* (Springer-Verlag, 1994).