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A POSITION-BASED TIME-STEPPING ALGORITHM FOR VIBRO-IMPACT PROBLEMS WITH A MOVING SET OF CONSTRAINTS

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We consider a second-order measure differential inclusion describing the dynamics of a mechanical system subjected to time-dependent frictionless unilateral constraints and we assume inelastic collisions when the contraints are saturated. For this model of impact, we propose a time-stepping algorithm formulated at the position level and we establish its convergence to a solution of the Cauchy problem.

Keywords: Vibro-impact problems; multi-constraint case; time-dependent constraints; inelastic shocks; time-stepping scheme.

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1. Introduction

Motivated by the study of discrete mechanical systems submitted to perfect unilateral constraints, we consider in this paper second-order differential inclusions of the form

$$\ddot{u} + N(K(t), u) \ni g(t, u), \tag{1.1}$$

where $K(t)$ is a subset of $\mathbb{R}^d$ characterized by the following geometrical inequalities

$$u \in K(t) \Leftrightarrow f_\alpha(t, u) \geq 0, \quad \alpha \in \{1, \dots, \nu\}$$

with smooth functions f_α and $N(K(t), u)$ is the normal cone to $K(t)$ at u given by

$$N(K(t), u) = \begin{cases} \{0\} & \text{if } u \in \mathrm{Int}(K(t)), \\ \left\{ \displaystyle\sum_{\alpha \in J(t,u)} \lambda_\alpha \nabla_u f_\alpha(t, u), \lambda_\alpha \leq 0 \right\} & \text{if } u \in \partial K(t), \\ \emptyset & \text{otherwise} \end{cases}$$

with $J(t, u) = \{\alpha \in \{1, \dots, \nu\}; f_\alpha(t, u) \leq 0\}$, i.e. $J(t, u)$ is the set of *active constraints* at the point (t, u).

The inclusion (1.1) may describe the motion of a mechanical system subjected to the frictionless unilateral contraints

$$u(t) \in K(t) \quad \forall\, t. \tag{1.2}$$

Indeed, with the definition of $N(K(t), \cdot)$, any solution of (1.1) will satisfy (1.2) and, as long as $u(t) \in \mathrm{Int}(K(t))$, the motion will be described simply by the Ordinary Differential Equation

$$\ddot{u} = g(t, u).$$

Furthermore, if $u(t) \in \mathrm{Int}(K(t))$ for all $t \in (t_0, t_1) \cup (t_1, t_2)$, with $u(t_1) \in \partial K(t_1)$, then

$$\dot{u}(t_1^-) \in -T(K(t_1), u(t_1)), \quad \dot{u}(t_1^+) \in T(K(t_1), u(t_1)) \tag{1.3}$$

with

$$T(K(t), u) = \{v \in \mathbb{R}^d; \partial_t f_\alpha(t, u) + \langle \nabla_u f_\alpha(t, u), v\rangle \geq 0 \text{ for all } \alpha \in J(t, u)\}.$$

It follows that the velocity may be discontinuous at t_1 and the model has to be completed with an impact law. In this paper, we will assume that

$$\dot{u}(t^+) = \mathrm{Proj}(T(K(t), u(t)), \dot{u}(t^-)). \tag{1.4}$$

Observing that $T(K(t), u(t))$ is the set of kinematically admissible right velocities at the instant t, this relation relies on a minimization property of the kinetic energy at impacts and thus seems to be the most physically relevant (see [11] or [13], for a more mathematical justification in the case of time-independent constraints see also [22]).

The adequate framework for the solutions is thus the set of absolutely continuous functions u which derivative $\dot{u}$ belongs to the space of functions of bounded variation. More precisely, for any initial data $(u_0, v_0) \in K(0) \times T(K(0), u_0)$, we will consider the following Cauchy problem:

Problem (P). *Find* $u : [0, \tau] \to \mathbb{R}^d$, *with* $\tau > 0$, *such that*

(P1) u *is absolutely continuous on* $[0, \tau]$, $\dot{u} \in BV(0, \tau; \mathbb{R}^d)$,
(P2) $u(t) \in K(t)$ *for all* $t \in [0, \tau]$,
(P3) *the measure* $\mu = d\dot{u} - g(\cdot, u)dt$ *satisfies the differential inclusion* (1.1) *in the following sense: there exists* ν *scalar measures* λ_α *such that*

$$\begin{cases} d\dot{u} - g(\cdot, u)dt = \displaystyle\sum_{\alpha=1}^{\nu} \lambda_\alpha \nabla_u f_\alpha(\cdot, u) \\ \lambda_\alpha \geq 0, \quad \mathrm{Supp}(\lambda_\alpha) \subset \{t \in [0, T]; f_\alpha(t, u(t)) = 0\} \quad \forall\, \alpha \in \{1, \dots, \nu\}. \end{cases}$$

(P4) $\dot{u}(t^+) = \mathrm{Proj}(T(K(t), u(t)), \dot{u}(t^-))$ *for all* $t \in (0, \tau)$,
(P5) $u(0) = u_0$, $\dot{u}(0^+) = v_0$.

For this problem, existence and approximation of solutions have been studied by several authors in the case of time-independent constraints, i.e. when the functions f_α do not depend on t.

The first results deal with the single-constraint case, i.e. $\nu = 1$: two different types of time-stepping schemes, formulated either at the position level or at the velocity level, have been proposed and their convergence established (see [14, 19, 20, 23] for position-based algorithms or [10, 8, 9, 6, 7] for velocity-based algorithms). These results have been extended to the multi-constraint case, i.e. $\nu \geq 2$, more recently (see [16–18]). In both cases ($\nu = 1$ or $\nu \geq 2$), the very complete study of Ballard ([1]) can be applied if the data are analytical, leading to the uniqueness of a maximal solution. Unfortunately, for less regular data, uniqueness is not true in general and several counter-examples can be found in the literature (see [5, 24] or [1] for instance).

On the contrary, for time-dependent constraints, very few results are available. In [25], Schatzman established an existence result in the case of a single constraint by using a penalty method, which also provides a sequence of approximate solutions. Unfortunately this technique is not well suited for implementation since it transforms the differential inclusion into a very stiff Ordinary Differential Equation, which stiffness is related to the penalty parameter (see [21] for a more detailed discussion). It is then necessary to try to adapt the time-stepping schemes to the time-dependent contraints framework. A first step in this direction has been achieved in a very recent paper ([2]), in which the authors prove the convergence of a generalization of the velocity based algorithms when the sets of admissible positions are defined as a finite intersection of complements of convex sets (i.e. the mappings f_α are assumed to be convex with respect to their second argument). Of course this last assumption is a severe restriction to the applicability of their result and it is important to relax it. The aim of this paper is thus to propose a generalization of the position-based algorithms and to prove their convergence in a more general geometrical setting than in [2]. Let us also mention another extension of [2] where the sets $K(t)$ are replaced by a given Lipschitz and *admissible* set-valued map $t \mapsto C(t)$ (see [3] and Definition 2.13 of *admissible* set-valued maps). We should emphasize that all these results give, as a by-product, global existence results.

So we adopt the same regularity assumptions for the data as in [2] but we will not assume any convexity property for the mappings f_α. More precisely, let $T > 0$, we assume:

(H1) The mappings f_α, $\alpha \in \{1, \dots, \nu\}$, belong to $C^2([0,T] \times \mathbb{R}^d; \mathbb{R})$ and for all $t \in [0,T]$, the set $K(t) = \{u \in \mathbb{R}^d; f_\alpha(t,u) \geq 0, \forall\, \alpha \in \{1, \dots, \nu\}\}$ is not empty.

We define

$$K = \{(t,u) \in [0,T] \times \mathbb{R}^d; u \in K(t)\}$$

and for any $r > 0$, K_r is the neighborhood of K given by

$$K_r = \{(s, y) \in [0, T] \times \mathbb{R}^d; \exists\,(t, u) \in K / |s - t| \le r, \|y - u\| \le r\}.$$

(H2) There exist $r > 0$, $m > 0$ and $M > 0$ such that, for all $(s, y) \in K_r$:

$$m \le \|\nabla_u f_\alpha(s, y)\| \le M, \|\partial_t f_\alpha(s, y)\| \le M,$$
$$\|\partial_t^2 f_\alpha(s, y)\| \le M, \quad \|\partial_t \nabla_u f_\alpha(s, y)\| \le M, \quad \|D_u^2 f_\alpha(s, y)\| \le M.$$

Moreover there exist $\gamma > 0$ and $\rho > 0$ such that, for all $t \in [0, T]$ and for all $u \in K(t)$:

$$\sum_{\alpha \in J_\rho(t,u)} \lambda_\alpha \|\nabla_u f_\alpha(t, u)\| \le \gamma \left\| \sum_{\alpha \in J_\rho(t,u)} \lambda_\alpha \nabla_u f_\alpha(t, u) \right\|$$
$$\forall\, \lambda_\alpha \in \mathbb{R}_+, \ \ \alpha \in J_\rho(t, u),$$

where $J_\rho(t, u)$ is the set of *almost active constraints* at (t, u) defined by

$$J_\rho(t, u) = \{\alpha \in \{1, \ldots, \nu\}; f_\alpha(t, u) \le \rho\}.$$

(H3) The function g is a Caratheodory function from $[0, T] \times \mathbb{R}^d$ with values in $\mathbb{R}^d$, i.e. $g(\cdot, u)$ is measurable on $[0, T]$ for all $u \in \mathbb{R}^d$ and $g(t, \cdot)$ is continuous on $\mathbb{R}^d$ for all $t \in [0, T]$, and there exist $k_g > 0$ and $F \in L^1(0, T; \mathbb{R})$ such that, for almost every $t \in [0, T]$ we have

$$\|g(t, u) - g(t, \tilde{u})\| \le k_g \|u - \tilde{u}\| \quad \forall\,(u, \tilde{u}) \in (\mathbb{R}^d)^2 \text{ s.t. } (t, u) \in K_r, (t, \tilde{u}) \in K_r,$$
$$\|g(t, u)\| \le F(t) \quad \forall\, u \in \mathbb{R}^d, \text{ s.t. } (t, u) \in K_r.$$

Let us emphasize that (H2) is a kind of uniform positive linear independence property for the vectors $(\nabla_q f_\alpha(t, u))_{\alpha \in J(t,u)}$ which implies a uniform prox-regularity property for the sets $K(t)$, $t \in [0, T]$ (see [2] or [4]) but does not imply convexity. In particular this geometrical framework is much more general than the one considered in [1, 16, 17] since it allows us to consider also cases where the active constraints are not linearly independent, i.e. $(\nabla_u f_\alpha(t, u))_{\alpha \in J(t,u)}$ is not linearly independent.

One of the main interesting consequences of assumption (H2) is the following result.

Lemma 1.1. *There exist $\tau \in (0, r]$, $\theta \in (0, r]$, $\kappa > 0$ and $\delta > 0$ such that, for all $t \in [0, T]$ and for all $u \in K(t)$, there exists a unit vector $v(t, u)$ such that, for all $s \in [t - \tau, t + \tau] \cap [0, T]$ and for all $y \in B(u, \theta)$, we have*:

$$\langle \nabla f_\alpha(s, y), v(t, u) \rangle \ge \delta \quad \forall\, \alpha \in J_\kappa(s, y).$$

The proof of the technical lemma can be found in Lemma 5.1 of [4].

Now let us describe our time-discretization algorithm: let $h \in (0, r]$ be a given time-step, we define $t_n = nh$ for all $n \geq 0$ and

- $U^{-1} = u_0 - hv_0$, $U^0 = u_0$,
- for all $n \in \{0, \ldots, \lfloor \frac{T}{h} \rfloor - 1\}$,

$$G^n = \frac{1}{h} \int_{t_n}^{t_{n+1}} g(s, U^n) ds \tag{1.5}$$

and

$$W^n = 2U^n - U^{n-1} + h^2 G^n, \quad U^{n+1} \in \underset{z \in K(t_{n+1})}{\text{Argmin}} \|W^n - z\|. \tag{1.6}$$

We may observe that this scheme coincides with the one proposed in [16] when the constraints do not depend on time and are convex and it is a natural generalization of the position-based algorithms introduced for the first time in [14]. Furthermore, it is also closely related to the algorithm proposed in [2]. Indeed, let us define the discrete velocities as

$$V^n = \frac{U^{n+1} - U^n}{h} \quad \forall\, n \in \left\{-1, \ldots, N(h) := \left\lfloor \frac{T}{h} \right\rfloor - 1\right\}.$$

If we replace $K(t_{n+1})$ by its convex approximation given by

$$\begin{aligned}&\tilde{K}(t_{n+1}, U^n)\\ &\quad = \{q \in \mathbb{R}^d; f_\alpha(t_{n+1}, U^n) + \langle \nabla_q f_\alpha(t_{n+1}, U^n), q - U^n \rangle \geq 0 \ \forall\, \alpha \in \{1, \ldots, \nu\}\},\end{aligned}$$

then (1.6) is replaced by

$$U^{n+1} = \text{Proj}(\tilde{K}(t_{n+1}, U^n), 2U^n - U^{n-1} + h^2 G^n)$$

which is equivalent to

$$U^{n+1} = U^n + hV^n \tag{1.7}$$

with

$$V^n = \text{Proj}(K_h(t_{n+1}, U^n), V^{n-1} + hG^n) \tag{1.8}$$

and

$$\begin{aligned}&K_h(t_{n+1}, U^n)\\ &\quad = \{v \in \mathbb{R}^d; f_\alpha(t_{n+1}, U^n) + h\langle \nabla_q f_\alpha(t_{n+1}, U^n), v \rangle \geq 0 \ \forall\, \alpha \in \{1, \ldots, \nu\}\}.\end{aligned}$$

This is exactly the scheme introduced by Bernicot and Lefebvre-Lepot in [2]. From a numerical point of view, it is clear that the algorithm defined by (1.7) and (1.8) is much more easy to handle than the one defined by (1.6) but (1.7) and (1.8) does not ensure the feasibility of the approximate positions if the functions f_α, $\alpha \in \{1, \ldots, \nu\}$, are not convex with respect to their second argument while we always have $U^n \in K(t_n)$ for all $nh \in [0, T]$ with (1.5) and (1.6).

As a consequence, the convergence proof that is given in the following sections will allow us to extend the results of [14, 19, 16] to the more general setting associated to the assumption (H2) and to extend the result of [2] to the case of non-convex functions f_α. As usual in the multi-constraint case, we cannot expect to prove that the limit satisfies the prescribed impact law (1.4) without introducing some further geometrical assumptions on the active constraints along the limit trajectory. Indeed, the model problem of a material point moving in an angular domain of $\mathbb{R}^2$ shows that continuity on data is lost if the active constraints create obtuse angles (see [15] for a detailed computation). Hence it appears that a necessary condition to ensure continuity on data is given by

$$\langle \nabla_u f_\alpha(t, u(t)), \nabla_u f_\beta(t, u(t)) \rangle \leq 0 \quad \forall\, (\alpha, \beta) \in J(t, u(t))^2,\ \ \alpha \neq \beta,\ \ \forall\, t \in [0, T]$$

and it has been established in [15] that it is also a sufficient condition when the constraints do not depend on time. It is straightforward to extend this result to the smoothly time-dependent framework considered here.

So we define the sequence of approximate solutions $(u_h)_{h>0}$ by a linear interpolation of the U^ns, i.e.

$$u_h(t) = U^n + (t - nh)\frac{U^{n+1} - U^n}{h} \quad \forall\, t \in [nh, (n+1)h],\ \ \forall\, n \in \left\{0, \dots, \left\lfloor \frac{T}{h} \right\rfloor - 1\right\},$$

$$u_h(t) = U^{\lfloor T/h \rfloor} + \left(t - \left\lfloor \frac{T}{h} \right\rfloor h\right) V^{\lfloor T/h \rfloor - 1}\ \forall\, t \in \left[\left\lfloor \frac{T}{h} \right\rfloor h, T\right]$$

and we prove

Theorem 1.1. *Let us assume that* (H1)–(H3) *hold. Let* $(u_0, v_0) \in K(0) \times T(K(0), u_0)$. *Then, possibly extracting a subsequence still denoted* $(u_h)_{h>0}$, *the approximate solutions converge uniformly on* $[0, T]$ *to a limit* u *which satisfies properties* (P1)–(P3). *Furthermore, if*

(H4) $\langle \nabla_u f_\alpha(t, u(t)), \nabla_u f_\beta(t, u(t)) \rangle \leq 0\ \forall\, (\alpha, \beta) \in J(t, u(t))^2, \alpha \neq \beta, \forall\, t \in [0, T]$

then u *also satisfies properties* (P4) *and* (P5) *and is a solution of problem* (P) *on* $[0, T]$.

Let us emphasize that since uniqueness is not true in general for such problems, we cannot expect the convergence of the whole sequence of approximate solutions.

The rest of the paper is organized as follows. In Sec. 2, we establish some *a priori* estimates for the discrete velocities and accelerations. Then, in Sec. 3, we pass to the limit by using Ascoli's and Helly's theorem and we prove that the limit motion is feasible and satisfies properties (P1)–(P3). Finally, assuming that (H4) holds, we prove in Sec. 4 that the initial data and the impact law are satisfied at the limit.

2. *A Priori* Estimates

We prove first two preliminary lemmas.

Lemma 2.1. *For all* $h \in \left(0, \min\left(r, \frac{T}{2}\right)\right)$ *and for all* $n \in \{0, \ldots, N(h)-1\}$, $N(h) := \lfloor \frac{T}{h} \rfloor$, *we have*

$$V^{n-1} - V^n + hG^n \in N(K(t_{n+1}), U^{n+1}). \tag{2.1}$$

Furthermore, if $h\|V^n\| \leq r$, *we get*

$$\partial_t f_\alpha(t_{n+1}, U^{n+1}) + \langle \nabla_q f_\alpha(t_{n+1}, U^{n+1}), V^n \rangle \leq \frac{Mh}{2}(1 + \|V^n\|)^2 \quad \forall\, \alpha \in J(t_{n+1}, U^{n+1}). \tag{2.2}$$

Proof. Let $h \in \left(0, \min\left(r, \frac{T}{2}\right)\right)$ and $n \in \{0, \ldots, N(h) - 1\}$. By definition of the scheme, for all $z \in K(t_{n+1})$ we have

$$\begin{aligned} \|W^n - U^{n+1}\|^2 &\leq \|W^n - z\|^2 \\ &= \|W^n - U^{n+1}\|^2 + 2\langle W^n - U^{n+1}, U^{n+1} - z\rangle + \|U^{n+1} - z\|^2. \end{aligned}$$

Since $W^n - U^{n+1} = h(V^{n-1} - V^n + hG^n)$, it follows that

$$\langle V^{n-1} - V^n + hG^n, z - U^{n+1}\rangle \leq \frac{1}{2}\|U^{n+1} - z\|^2 \quad \forall\, z \in K(t_{n+1}). \tag{2.3}$$

If $U^{n+1} \in \mathrm{Int}(K(t_{n+1}))$, we immediately get $V^{n-1} - V^n + hG^n = 0$ and the announced result holds. Otherwise, $J(t_{n+1}, U^{n+1}) \neq \emptyset$ and we may define

$$T^0(K(t_{n+1}), U^{n+1}) = \{w \in \mathbb{R}^d; \langle \nabla_u f_\alpha(t_{n+1}, U^{n+1}), w\rangle \geq 0 \;\forall\, \alpha \in J(t_{n+1}, U^{n+1})\}$$

and

$$\tilde{T}^0(K(t_{n+1}), U^{n+1}) = \{w \in \mathbb{R}^d; \langle \nabla_u f_\alpha(t_{n+1}, U^{n+1}), w\rangle > 0 \;\forall\, \alpha \in J(t_{n+1}, U^{n+1})\}.$$

Let $\tilde{w} \in \tilde{T}^0(K(t_{n+1}), U^{n+1})$. The C^2-regularity of the mappings f_α, $\alpha = 1, \ldots, \nu$, implies that the smooth curve $\varphi : s \mapsto U^{n+1} + s\tilde{w}$ satisfies $\varphi(s) \in K(t_{n+1})$ for all s in a right neighborhood of 0. Thus, by choosing $z = \varphi(s)$ in (2.3) and letting s goes to 0, we obtain

$$\langle V^{n-1} - V^n + hG^n, \tilde{w}\rangle \leq 0.$$

But $\tilde{T}^0(K(t_{n+1}), U^{n+1})$ is dense in $T^0(K(t_{n+1}), U^{n+1})$. Indeed, with Lemma 1.1, we know that there exists a unit vector $v(t_{n+1}, U^{n+1}) \in \tilde{T}^0(K(t_{n+1}), U^{n+1})$. Thus, for all $w \in T^0(K(t_{n+1}), U^{n+1})$ the sequence $(w_k)_{k\in\mathbb{N}^*}$ defined by $w_k = w + \frac{1}{k}v(t_{n+1}, U^{n+1})$ for all $k \geq 1$ converges to w and satisfies $w_k \in \tilde{T}^0(K(t_{n+1}), U^{n+1})$ for all $k \geq 1$. Hence we have

$$\langle V^{n-1} - V^n + hG^n, w\rangle \leq 0 \quad \forall\, w \in T^0(K(t_{n+1}), U^{n+1})$$

and we may obtain (2.1) by using Farkas's lemma.

In order to prove (2.2) we observe that, for all $\alpha \in J(t_{n+1}, U^{n+1})$, we have $0 = f_\alpha(t_{n+1}, U^{n+1}) \le f_\alpha(t_n, U^n)$ and thus

$$\begin{aligned} 0 &\le f_\alpha(t_n, U^n) - f_\alpha(t_{n+1}, U^{n+1}) \\ &= -h \int_0^1 (\partial_t f_\alpha(t_{n+1} - sh, U^{n+1} - shV^n) \\ &\quad + \langle \nabla_q f_\alpha(t_{n+1} - sh, U^{n+1} - shV^n), V^n \rangle ds. \end{aligned}$$

It follows that

$$\begin{aligned} &\partial_t f_\alpha(t_{n+1}, U^{n+1}) + \langle \nabla_u f_\alpha(t_{n+1}, U^{n+1}), V^n \rangle \\ &\quad \le \int_0^1 (\partial_t f_\alpha(t_{n+1}, U^{n+1}) - \partial_t f_\alpha(t_{n+1} - sh, U^{n+1} - shV^n))ds \\ &\qquad + \int_0^1 \langle \nabla_u f_\alpha(t_{n+1}, U^{n+1}) - \nabla_u f_\alpha(t_{n+1} - sh, U^{n+1} - shV^n), V^n \rangle ds \\ &\quad \le \frac{Mh}{2}(1 + \|V^n\|)^2. \end{aligned}$$

□

We can reformulate (2.1) as follows: for all $h \in big(0, \min\left(r, \frac{T}{2}\right)\right)$ and for all $n \in \{0, \ldots, N(h) - 1\}$ there exists a family of non-negative real numbers $(\lambda_\alpha^n)_{\alpha \in \{1, \ldots, \nu\}}$ such that $\lambda_\alpha^n = 0$ for all $\alpha \notin J(t_{n+1}, U^{n+1})$ and

$$V^n - V^{n-1} - hG^n = \sum_{\alpha=1}^{\nu} \lambda_\alpha^n \nabla_u f_\alpha(t_{n+1}, U^{n+1}). \tag{2.4}$$

This relation is the discrete analogous of property (P3) and $\frac{V^n - V^{n-1}}{h} - G^n$ can be interpreted as a discrete reaction force at t_n.

Let us assume from now on that $h \in (0, h^*]$ with

$$h^* = \min\left(r, \frac{T}{2}, \frac{\kappa}{M\left(1 + \frac{2M}{\delta}\right)}, \frac{1}{\left(1 + \frac{2M}{\delta}\right)^2}, \frac{r\delta}{2M}\right).$$

Lemma 2.2. *Let $h \in (0, h^*]$. For all $n \in \{0, \ldots, N(h) - 1\}$, we have*

$$\|V^n\| \le 2\|V^{n-1}\| + 2h\|G^n\| + \frac{2M}{\delta}. \tag{2.5}$$

Proof. Let $h \in (0, h^*]$ and $n \in \{0, \ldots, N(h) - 1\}$. We define $w = \frac{2M}{\delta} v(t_n, U^n)$, where $v(t_n, U^n)$ is the unit vector defined at Lemma 1.1 for $(t, u) = (t_n, U^n)$. Then $U^n + hw \in K(t_{n+1})$. Indeed, for all $\alpha \notin J_\kappa(t_n, U^n)$, we have

$$\begin{aligned} f_\alpha(t_{n+1}, U^n + hw) &= f_\alpha(t_n, U^n) + h \int_0^1 (\partial_t f_\alpha(t_n + sh, U^n + shw) \\ &\quad + \langle \nabla_u f_\alpha(t_n + sh, U^n + shw), w \rangle)ds \\ &\ge \kappa - hM(1 + \|w\|) \ge \kappa - hM\left(1 + \frac{2M}{\delta}\right) \ge 0 \end{aligned}$$

and for all $\alpha \in J_\kappa(t_n, U^n)$ we have

$$\begin{aligned}
f_\alpha(t_{n+1}, U^n + hw) &= f_\alpha(t_n, U^n) + h(\partial_t f_\alpha(t_n, U^n) + \langle \nabla_q f_\alpha(t_n, U^n), w\rangle) \\
&\quad + h \int_0^1 (\partial_t f_\alpha(t_n + sh, U^n + shw) - \partial_t f_\alpha(t_n, U^n)|)ds \\
&\quad + h \int_0^1 \langle \nabla_u f_\alpha(t_n + sh, U^n + shw) - \nabla_u f_\alpha(t_n, U^n), w\rangle)ds \\
&\geq h(-M + \delta\|w\|) - h^2 M(1 + \|w\|^2) \\
&\geq h\left(M - hM\left(1 + \left(\frac{2M}{\delta}\right)^2\right)\right) \\
&\geq 0.
\end{aligned}$$

By definition of U^{n+1} it follows that

$$\|2U^n - U^{n-1} + h^2 G^n - U^{n+1}\| \leq \|2U^n - U^{n-1} + h^2 G^n - U^n - hw\|$$

and thus

$$\|V^{n-1} - V^n + hG^n\| \leq \|V^{n-1} - w + hG^n\|$$

which yields the conclusion. $\square$

Now we will prove a global uniform estimate for the discrete velocities.

Proposition 2.1. *There exist $h_1 \in (0, h^*]$ and $C > 0$ such that*

$$\|V^n\| \leq C \quad \forall\, n \in \{0, \ldots, N(h)\}, \quad \forall\, h \in (0, h_1]. \tag{2.6}$$

Proof. Let us define two real sequences $(C_k)_{k\in\mathbb{N}}$ and $(\tau_k)_{k\in\mathbb{N}^*}$ by

$$\begin{aligned}
&C_0 = \|v_0\| + 1, \\
&C_k = C_{k-1} + \frac{4M}{\delta} + \|F\|_{L^1(0,T;\mathbb{R}^d)} = C_0 + k\left(\frac{4M}{\delta} + \|F\|_{L^1(0,T;\mathbb{R}^d)}\right) \forall\, k \geq 1
\end{aligned}$$

and

$$\tau_k = \frac{\min(\tau, \theta)}{2C_k} = \frac{\min(\tau, \theta)}{2C_0 + 2k\left(\frac{4M}{\delta} + \|F\|_{L^1(0,T;\mathbb{R}^d)}\right)} \quad \forall\, k \geq 1.$$

It is clear that $\sum_{k\geq 1} \tau_k$ is a divergent sum, thus there exists $k_0 \geq 1$ such that $\sum_{k=1}^{k_0} \tau_k > T$. The main idea of the proof is to show that there exists $h_1 \in (0, h^*]$

such that, for all $h \in (0, h_1]$, there exists a finite family of real numbers $(\tau_k^h)_{1 \le k \le k_0^h}$ such that $\tau_0^h = 0 < \tau_1^h < \cdots < \tau_{k_0^h}^h = T$ with $1 \le k_0^h \le k_0$ and

$$\|V^n\| \le C_k \quad \forall\, n \in \{0, \ldots, N(h) - 1\} \text{ s.t. } nh \in [\tau_{k-1}^h, \tau_k^h), \quad \forall\, k \in \{1, \ldots, k_0^h\}.$$

The conclusion of the proof will follow with the choice $C = C_{k_0}$.

We define

$$\tilde{C} = 2C + 2\|F\|_{L^1(0,T;\mathbb{R}^d)} + \frac{2M}{\delta} \quad \text{and}$$
$$h_1 = \min\left(h^*, \frac{\tau}{3}, \frac{\theta}{3\tilde{C}}, \frac{r}{2\tilde{C}}, \frac{1}{1+\tilde{C}}, \frac{\tau_{k_0}}{2}\right).$$

Let $h \in (0, h_1]$. We will obtain a global uniform estimate for the velocities by an induction argument.

Since $(t_0, U^0) = (0, q_0) \in K$, we define $w^0 = \frac{2M}{\delta} v(t_0, U^0)$. First we observe that $\|V^{-1}\| = \|v_0\| \le C_0 \le C$, which implies, with Lemma 2.2, that $\|V^0\| \le \tilde{C}$. Hence $(t_1, U^1) \in B(t_0, \tau) \times B(U^0, \theta)$ since $0 < h \le h_1$. With Lemma 2.1 we infer that $w^0 - V^0 \in T^0(K(t_1), U^1)$. Indeed, for all $\alpha \in J(t_1, U^1)$

$$\begin{aligned}\langle \nabla_u f_\alpha(t_1, U^1), w^0 - V^0 \rangle &\ge \langle \nabla_u f_\alpha(t_1, U^1), w^0 \rangle + \partial_t f_\alpha(t_1, U^1) - \frac{Mh}{2}(1 + \|V^0\|) \\ &\ge \delta \|w^0\| - M - \frac{Mh}{2}(1 + \tilde{C}) \ge \frac{M}{2}.\end{aligned}$$

It follows that

$$\langle (V^{-1} - w^0) - (V^0 - w^0) + hG^0, w^0 - V^0 \rangle \le 0.$$

Thus

$$\|V^0 - w^0\| \le \|V^{-1} - w^0\| + h\|G^0\|$$

and

$$\|V^0\| \le \|V^{-1}\| + \frac{4M}{\delta} + h\|G^0\| \le C_1 \le C.$$

We may reproduce the same computations and prove that

$$\|V^n - w^0\| \le \|V^{-1} - w^0\| + h \sum_{l=0}^{n} \|G^l\| \quad \forall\, n \in \{0, \ldots, N(h) - 1\} \text{ s.t. } nh \in [0, \tau_1].$$

Indeed, let us assume that $n \in \{1, \ldots, N(h) - 1\}$ such that $nh \in [0, \tau_1]$ and that

$$\|V^k - w^0\| \le \|V^{-1} - w^0\| + h \sum_{l=0}^{k} \|G^l\| \quad \forall\, k \in \{0, \ldots, n - 1\}.$$

Then $\|V^k\| \leq C_1$ for all $k \in \{0, \ldots, n-1\}$ and, with Lemma 2.2, we infer that $\|V^n\| \leq \tilde{C}$. Thus $(t_{n+1}, U^{n+1}) \in B(t_0, \tau) \times B(U^0, \theta)$ since $0 < h \leq h_1$ and, with Lemma 2.1 $w^0 - V^n \in T^0(K(t_{n+1}), U^{n+1})$. Thus

$$\langle (V^{n-1} - w^0) - (V^n - w^0) + hG^n, w^0 - V^n \rangle \leq 0$$

and

$$\|V^n - w^0\| \leq \|V^{n-1} - w^0\| + h\|G^n\| \leq \|V^{-1} - w^0\| + h \sum_{l=0}^{n} \|G^l\|.$$

Hence

$$\|V^n\| \leq \|V^0\| + \frac{4M}{\delta} + h \sum_{l=0}^{n} \|G^l\| \leq C_1.$$

Now let $\tau_0^h = 0$ and $n_1^h \in \mathbb{N}$ such that $n_1^h h \leq \min(\tau_1, T) < n_1^h h + h$. If $n_1^h < N(h) - 1$, we define $\tau_1^h = (n_1^h + 1)h$, otherwise $\tau_1^h = T$. If $n_1^h < N(h) - 1$, we have $\tau_1^h - \tau_0^h = \tau_1^h \geq \tau_1$. Moreover, $T > \tau_1^h$, so $k_0 > 1$ and $(t_{n_1^h}, U^{n_1^h}) \in K$ and $\|V^{n_1^h}\| \leq C_1 \leq C$.

Let us assume now that $n_1^h < N(h) - 1$. We define $w^1 = \frac{2M}{\delta} v(t_{n_1^h}, U^{n_1^h})$ and we can prove again by induction that, for all $n \in \{0, \ldots, N(h) - 1\}$ such that $nh \in [\tau_1^h, \tau_1^h + \tau_2]$:

$$\|V^n - w^1\| \leq \|V^{n_1^h} - w^1\| + h \sum_{l=n_1^h+1}^{n} \|G^l\|$$

and

$$\|V^n\| \leq \|V^{n_1^h}\| + \frac{4M}{\delta} + h \sum_{l=n_1^h+1}^{n} \|G^l\| \leq C_2 \leq C.$$

We define $n_2^h \in \mathbb{N}$ such that $n_2^h h \leq \min(\tau_1^h + \tau_2, T) < n_2^h h + h$ and $\tau_2^h = n_2^h h$ if $n_2^h < N(h) - 1$, otherwise $\tau_2^h = T$. We can check immediately that, if $n_2^h < N(h) - 1$, we have $\tau_2^h - \tau_1^h \geq \tau_2$ and $k_0 > 2$. Finally we complete the proof with a finite induction argument. $\square$

Let us come now to the estimate of the discrete accelerations.

Proposition 2.2. *There exists $C' > 0$ such that, for all $h \in (0, h_1]$, we have*

$$\sum_{n=1}^{N(h)-1} \|V^n - V^{n-1}\| \leq C'.$$

Proof. Let $h \in (0, h_1]$. We again use the decomposition of the interval $[0, T]$ with the subintervals $[\tau_k^h, \tau_{k+1}^h]$, $k \in \{0, \ldots, k_0^h - 1\}$, which have been defined in the previous proposition. We recall that, for all $k \in \{0, \ldots, k_0^h - 1\}$ and for all $n \in \{0, \ldots, N(h) - 1\}$ such that $nh \in [\tau_k^h, \tau_{k+1}^h]$ we have $(t_{n+1}, U^{n+1}) \in B(t_{n_k^h}, \tau) \times B(U^{n_k^h}, \theta)$. Furthermore, with the definition of $w^k = \frac{2M}{\delta} v(t_{n_k^h}, U^{n_k^h})$ we also have

$$\langle \nabla_u f_\alpha(t_{n+1}, U^{n+1}), w - V^n \rangle \geq \frac{M}{2}$$

and thus $\bar{B}(w^k - V^n, \frac{1}{2}) \subset T^0(K(t_{n+1}), U^{n+1})$.

Using [12] we infer that, for all $z \in \mathbb{R}^d$:

$$\begin{aligned}
&\|z - \mathrm{Proj}(T^0(t_{n+1}, U^{n+1}), z)\| \\
&\qquad \leq \|z - w^k + V^n\|^2 - \|\mathrm{Proj}(T^0(t_{n+1}, U^{n+1}), z) - w^k + V^n\|^2.
\end{aligned}$$

Then we apply this estimate with $z = V^{n-1} - V^n + hG^n$: using Lemma 2.1 we know that $z \in N(K(t_{n+1}), U^{n+1})$ and since $N(K(t_{n+1}), U^{n+1})$ and $T^0(K(t_{n+1}), U^{n+1})$ are convex polar cones, we get

$$\begin{aligned}
&\|V^{n-1} - V^n\| \\
&\quad \leq h\|G^n\| + \|(V^{n-1} - V^n + hG^n) - \mathrm{Proj}(T^0(t_{n+1}, U^{n+1}), V^{n-1} - V^n + hG^n)\| \\
&\quad \leq h\|G^n\| + (\|V^{n-1} - w^k + hG^n\|^2 - \|V^n - w^k\|^2) \\
&\quad \leq h\|G^n\| + (\|V^{n-1} - w^k\|^2 - \|V^n - w^k\|^2 + h^2\|G^n\|^2 + 2h\langle G^n, V^{n-1} - w^k\rangle) \\
&\quad \leq \left(1 + \|F\|_{L^1(0,T;\mathbb{R}^d)} + 2C + \frac{4M}{\delta}\right) h\|G^n\| + (\|V^{n-1} - w^k\|^2 - \|V^n - w^k\|^2).
\end{aligned}$$

We add all these inequalities for $n = n_k^h, \ldots, n_{k+1}^h - 1$ and for $k = 0, \ldots, k_0^h - 2$ and for $n = n_{k_0^h}^h, \ldots, N(h) - 1$ if $k = k_0^h - 1$. We get

$$\begin{aligned}
\sum_{n=0}^{N(h)-1} \|V^{n-1} - V^n\| &= \sum_{k=0}^{k_0^h-2} \sum_{n=n_k^h}^{n_{k+1}^h-1} \|V^{n-1} - V^n\| + \sum_{n=n_{k_0^h-1}^h}^{N(h)-1} \|V^{n-1} - V^n\| \\
&\leq \left(1 + \|F\|_{L^1(0,T;\mathbb{R}^d)} + 2C + \frac{4M}{\delta}\right) \sum_{n=0}^{N(h)-1} h\|G^n\| \\
&\quad + \sum_{k=0}^{k_0^h-2} (\|V^{n_k^h-1} - w^k\|^2 - \|V^{n_{k+1}^h-1} - w^k\|^2) \\
&\quad + (\|V^{n_{k_0^h-1}^h-1} - w^{k_0^h-1}\|^2 - \|V^{N(h)-1} - w^{k_0^h-1}\|^2) \\
&\leq \left(1 + \|F\|_{L^1(0,T;\mathbb{R}^d)} + 2C + \frac{4M}{\delta}\right) \|F\|_{L^1(0,T;\mathbb{R}^d)} \\
&\quad + 2k_0^h \left(C + \frac{2M}{\delta}\right)^2.
\end{aligned}$$

Recalling that $k_0^h \leq k_0$ for all $h \in (0, h_1]$, we may conclude. □

3. Convergence of the Approximate Solutions $(u_h)_{h^* \geq h > 0}$

Now we can pass to the limit in the same way as in [17]. We recall that the approximate solutions are defined as

$$u_h(t) = \begin{cases} U^n + (t - nh)\frac{U^{n+1} - U^n}{h} & \forall\, t \in [nh, (n+1)h],\ \forall\, n \in \{0, \ldots, N(h) - 1\}, \\ U^{N(h)} + (t - N(h)h)V^{N(h)-1} & \forall\, t \in [N(h)h, T] \end{cases}$$

and we let

$$v_h(t) = \begin{cases} V^n = \frac{U^{n+1} - U^n}{h} & \forall\, t \in [nh, (n+1)h), \quad \forall\, n \in \{0, \ldots, N(h) - 1\}, \\ = V^{N(h)-1} & \forall\, t \in [N(h)h, T] \end{cases}$$

for all $h \in (0, h_1]$.

From Propositions 2.1 and 2.2 we know that the sequence $(u_h)_{h_1 \geq h > 0}$ is uniformly Lipschitz continuous and that $(v_h)_{h_1 \geq h > 0}$ is uniformly bounded in $L^\infty(0, T; \mathbb{R}^d)$ and in $BV(0, T; \mathbb{R}^d)$. Thus, applying Ascoli's and Helly's theorem, we can extract a subsequence, still denoted $(u_h)_{h_1 \geq h > 0}$ and $(v_h)_{h_1 \geq h > 0}$, and there exist $u \in C^0([0, T]; \mathbb{R}^d)$ and $v \in BV(0, T; \mathbb{R}^d)$ such that

$$\begin{aligned} u_h &\to u \quad && \text{strongly in } C^0([0, T]; \mathbb{R}^d), \\ v_h &\to v && \text{pointwise in } [0, T], \\ dv_h &\rightharpoonup dv && \text{weakly* in } \mathcal{M}^1(0, T; \mathbb{R}^d). \end{aligned} \tag{3.1}$$

Furthermore, the definitions of u_h and v_h imply that

$$u_h(t) = u_0 + \int_0^t v_h(s)ds \quad \forall\, t \in [0, T],\ \ \forall\, h \in (0, h_1].$$

We can pass to the limit by using Lebesgue's theorem and we obtain

$$u(t) = u_0 + \int_0^t v(s)ds \quad \forall\, t \in [0, T]. \tag{3.2}$$

Hence u is C-Lipschitz continuous on $[0, T]$ and $\dot{u} = v \in BV(0, T; \mathbb{R}^d)$. Moreover, we can check easily that

Lemma 3.1. *For all $t \in [0, T], u(t) \in K(t)$.*

Proof. This is a direct consequence of the feasibility of the approximate positions. Indeed, for all $t \in [0, T]$ and for all $h \in (0, h_1]$ there exists $n \in \{0, \ldots, N(h)\}$ such that $t \in [nh, (n+1)h)$. Then we can use a Taylor's expansion to estimate from below $f_\alpha(u(t))$, $\alpha \in \{1, \ldots, \nu\}$. More precisely, for all $\alpha \in \{1, \ldots, \nu\}$,

$$\begin{aligned} f_\alpha(t, u(t)) &= f_\alpha(nh, u_h(nh)) \\ &\quad + \int_0^1 (\partial_t f_\alpha(nh + s(t - nh), u_h(nh) + s(u(t) - u_h(nh)))(t - nh)ds \end{aligned}$$

$$+\int_0^1 \langle \nabla_u f_\alpha(nh+s(t-nh), u_h(nh)$$
$$+s(u(t)-u_h(nh))), u(t)-u(nh)\rangle)ds.$$

But

$$\|u(t)-u_h(nh)\| \le \|u(t)-u(nh)\| + \|u(nh)-u_h(nh)\|$$
$$\le C(t-nh) + \|u-u_h\|_{C^0([0,T];\mathbb{R}^d)}.$$

Using the uniform convergence of $(u_h)_{h_1 \ge h > 0}$ to u on $[0,T]$, we infer that there exists $h_2 \in (0,h_1]$ such that

$$Ch + \|u-u_h\|_{C^0([0,T];\mathbb{R}^d)} \le r \quad \forall\, h \in (0,h_2].$$

It follows that, for all $h \in (0,h_2]$:

$$f_\alpha(t,u(t)) \ge f_\alpha(t_n,U^n) - M(h + \|u(t)-u_h(nh)\|)$$
$$\ge -M((1+C)h + \|u-u_h\|_{C^0([0,T];\mathbb{R}^d)}),$$

which allows us to conclude. □

Next we prove that the limit trajectory satisfies property (P3). With the definition of u_h and v_h, the Stieltjes measure $\ddot{u}_h = d\dot{u}_h = dv_h$ is a sum of Dirac's measures:

$$\ddot{u}_h(t) = \sum_{n=1}^{N(h)-1} (V^n - V^{n-1})\delta(t-nh)$$

and we define

$$G_h(t) = \sum_{n=1}^{N(h)-1} hG^n\delta(t-nh)$$
$$+ \sum_{n=1}^{N(h)-1}\sum_{\alpha=1}^{\nu} \lambda_\alpha^n(\nabla_u f_\alpha(t_{n+1},U^{n+1}) - \nabla_u f_\alpha(t_n,u(t_n)))\delta(t-nh),$$
$$\lambda_{\alpha,h}(t) = \sum_{n=1}^{N(h)-1} \lambda_\alpha^n\delta(t-nh) \quad \forall\, \alpha \in \{1,\dots,\nu\}.$$

Then relation (2.4) can be rewritten as

$$dv_h = \sum_{\alpha=1}^{\nu} \lambda_{\alpha,h}\nabla_u f_\alpha(\cdot,u) + G_h \tag{3.3}$$

and we have to pass to the limit in the above relation.

First we observe that

Lemma 3.2. *For all $\alpha \in \{1, \dots, \nu\}$ and for all $h \in (0, h_1]$ we have*

$$\sum_{n=1}^{N(h)-1} |\lambda_\alpha^n| \le \frac{\gamma}{m}(TV(v_h) + \|F\|_{L^1(0,T;\mathbb{R}^d)}).$$

Proof. Let $\alpha \in \{1, \dots, \nu\}$ and $n \in \{1, \dots, N(h)-1\}$. With relation (2.4) we have

$$\left\| \sum_{\beta=1}^{\nu} \lambda_\beta^n \nabla_u f_\beta(t_{n+1}, U^{n+1}) \right\| \le \|V^n - V^{n-1}\| + h\|G^n\|$$

and $\lambda_\beta^n \ge 0$ for all $\beta \in \{1, \dots, \nu\}$ with $\lambda_\beta^n = 0$ if $\beta \notin J(t_{n+1}, U^{n+1})$. Thus, using assumption (H2), we get

$$\begin{aligned}
\left\| \sum_{\beta=1}^{\nu} \lambda_\beta^n \nabla_u f_\alpha(t_{n+1}, U^{n+1}) \right\| &= \left\| \sum_{\beta \in J(t_{n+1}, U^{n+1})} \lambda_\beta^n \nabla_u f_\alpha(t_{n+1}, U^{n+1}) \right\| \\
&\ge \frac{1}{\gamma} \sum_{\beta \in J(t_{n+1}, U^{n+1})} \lambda_\beta^n \|\nabla_u f_\alpha(t_{n+1}, U^{n+1})\| \\
&= \frac{1}{\gamma} \sum_{\beta=1}^{\nu} \lambda_\beta^n \|\nabla_q f_\alpha(t_{n+1}, U^{n+1})\| \\
&\ge \frac{m}{\gamma} \sum_{\beta=1}^{\nu} \lambda_\beta^n \ge \frac{m}{\gamma} \lambda_\alpha^n.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{n=1}^{N(h)-1} |\lambda_\alpha^n| &= \sum_{n=1}^{N(h)-1} \lambda_\alpha^n \\
&\le \frac{\gamma}{m} \sum_{n=1}^{N(h)-1} (\|V^n - V^{n-1}\| + h\|G^n\|) \\
&\le \frac{\gamma}{m}(TV(v_h) + \|F\|_{L^1(0,T;\mathbb{R})}).
\end{aligned}$$ □

Reminding the uniform estimate of v_h in $BV(0, T; \mathbb{R}^d)$ obtained at Proposition 2.2, we infer that the scalar measures $\lambda_{\alpha,h}$, $\alpha \in \{1, \dots, \nu\}$, are uniformly bounded in $\mathcal{M}^1(0, T; \mathbb{R})$. Thus, possibly extracting another subsequence, there exist non-negative scalar measures λ_α, such that for all $\alpha \in \{1, \dots, \nu\}$:

$$\lambda_{\alpha,h} \rightharpoonup \lambda_\alpha \quad \text{weakly* in } \mathcal{M}^1(0, T; \mathbb{R}).$$

It remains to pass to the limit in the last term of the right-hand side of (3.3).

Lemma 3.3. *The sequence $(G_h)_{h^* \geq h > 0}$ converges weakly to $g(\cdot, u)dt$ in $\mathcal{M}^1(0, T; \mathbb{R}^d)$, where $g(\cdot, u)dt$ is the measure of density $g(\cdot, u)$ with respect to Lebesgue's measure on $[0, T]$.*

Proof. Let $\phi \in C^0([0, T]; \mathbb{R}^d)$. By definition of G_h we have

$$
\begin{aligned}
&\langle G_h, \phi \rangle_{\mathcal{M}^1(0,T;\mathbb{R}^d), C^0([0,T];\mathbb{R}^d)} \\
&\quad = \sum_{n=1}^{N(h)-1} h \langle G^n, \phi(nh) \rangle \\
&\qquad + \sum_{n=1}^{N(h)-1} \sum_{\alpha=1}^{\nu} \lambda_\alpha^n \langle \nabla_u f_\alpha(t_{n+1}, U^{n+1}) - \nabla_u f_\alpha(t_n, u(t_n)), \phi(nh) \rangle \\
&\quad = \sum_{n=1}^{N(h)-1} \int_{t_n}^{t_{n+1}} \langle g(s, U^n), \phi(nh) \rangle ds \\
&\qquad + \sum_{n=1}^{N(h)} \sum_{\alpha=1}^{\nu} \lambda_\alpha^n \langle \nabla_u f_\alpha(t_{n+1}, U^{n+1}) - \nabla_u f_\alpha(t_n, u(t_n)), \phi(nh) \rangle \\
&\quad = \int_0^T \langle g(s, u(s)), \phi(s) \rangle ds - \int_{N(h)h}^T \langle g(s, u(s)), \phi(s) \rangle ds \\
&\qquad + \sum_{n=1}^{N(h)-1} \int_{t_n}^{t_{n+1}} \langle g(s, U^n) - g(s, u(s)), \phi(s) \rangle ds \\
&\qquad + \sum_{n=1}^{N(h)-1} \int_{t_n}^{t_{n+1}} \langle g(s, U^n), \phi(nh) - \phi(s) \rangle dt \\
&\qquad + \sum_{n=1}^{N(h)-1} \sum_{\alpha=1}^{\nu} \lambda_\alpha^n \langle \nabla_u f_\alpha(t_{n+1}, U^{n+1}) - \nabla_u f_\alpha(t_n, u(t_n)), \phi(nh) \rangle.
\end{aligned}
$$

But, for all $n \in \{1, \ldots, N(h) - 1\}$, we have $(t_n, u(t_n)) \in K$ and

$$
\begin{aligned}
\|U^{n+1} - u(t_n)\| &\leq \|U^{n+1} - U^n\| + \|u_h(t_n) - u(t_n)\| \\
&\leq Ch + \|u - u_h\|_{C^0([0,T];\mathbb{R}^d)}.
\end{aligned}
$$

As in Lemma 3.1 we define $h_2 \in (0, h_1]$ such that

$$
Ch + \|u - u_h\|_{C^0([0,T];\mathbb{R}^d)} \leq r \quad \forall\, h \in (0, h_2].
$$

It follows that, for all $h \in (0, h_2]$, we have

$$\left\| \sum_{n=1}^{N(h)-1} \sum_{\alpha=1}^{\nu} \lambda_\alpha^n \langle \nabla_u f_\alpha(t_{n+1}, U^{n+1}) - \nabla_u f_\alpha(t_n, u(t_n)), \phi(nh) \rangle \right\|$$

$$\begin{aligned}
&\leq \sum_{n=1}^{N(h)-1} \sum_{\alpha=1}^{\nu} \lambda_\alpha^n M(h + \|U^{n+1} - u(nh)\|)\|\phi(nh)\| \\
&\leq \sum_{n=1}^{N(h)-1} \sum_{\alpha=1}^{\nu} \lambda_\alpha^n M((C+1)h + \|u - u_h\|_{C^0([0,T];\mathbb{R}^d)})\|\phi\|_{C^0([0,T];\mathbb{R}^d)} \\
&\leq M\nu((C+1)h + \|u - u_h\|_{C^0([0,T];\mathbb{R}^d)}) \\
&\quad \times \|\phi\|_{C^0([0,T];\mathbb{R}^d)} \frac{\gamma}{m} \left(TV(v_h) + \|F\|_{L^1(0,T;\mathbb{R}^d)} \right).
\end{aligned}$$

Moreover,

$$\left| \sum_{n=1}^{N(h)-1} \int_{t_n}^{t_{n+1}} \langle g(s, U^n), \phi(nh) - \phi(s) \rangle ds \right|$$

$$\leq \sum_{n=1}^{N(h)-1} \int_{t_n}^{t_{n+1}} \|g(s, U^n)\| \|\phi(nh) - \phi(s)\| ds \leq \omega_\phi(h) \|F\|_{L^1(0,T;\mathbb{R}^d)},$$

where ω_ϕ denotes the modulus of continuity of ϕ. Furthermore, using assumption (H3) we have

$$\left| \sum_{n=1}^{N(h)-1} \int_{t_n}^{t_{n+1}} \langle g(s, U^n) - g(s, u(s)), \phi(s) \rangle ds \right|$$

$$\begin{aligned}
&\leq \sum_{n=1}^{N(h)-1} \int_{t_n}^{t_{n+1}} k_g \|U^n - u(s)\| \|\phi(s)\| ds \\
&\leq k_g (Ch + \|u - u_h\|_{C^0([0,T];\mathbb{R}^d)}) \int_0^T \|\phi(s)\| ds
\end{aligned}$$

for all $h \in (0, h_2]$, since

$$\begin{aligned}
\|U^n - u(s)\| &\leq \|u_h(nh) - u_h(s)\| + \|u_h(s) - u(s)\| \\
&\leq Ch + \|u - u_h\|_{C^0([0,T];\mathbb{R}^d)} \\
&\leq r.
\end{aligned}$$

Finally

$$\left| \int_{N(h)h}^{T} \langle g(s, u(s)), \phi(s) \rangle ds \right| \leq \|\phi\|_{C^0([0,T];\mathbb{R}^d)} \int_{N(h)h}^{T} F(s) ds$$

and we can pass to the limit as h tends to zero to get the announced result. □

Hence we can pass to the limit in (3.3) and we get

$$d\dot{u} = dv = \sum_{\alpha=1}^{\nu} \lambda_\alpha \nabla_u f_\alpha(\cdot, u) + g(\cdot, u)dt.$$

Finally we prove that

Lemma 3.4. *For all $\alpha \in \{1, \ldots, \nu\}$ we have*

$$\mathrm{Supp}(\lambda_\alpha) \subset \{t \in [0, T]; f_\alpha(t, u(t)) = 0\}.$$

Proof. Let $\alpha \in \{1, \ldots, \nu\}$ and $\phi \in C^0([0, T]; \mathbb{R})$ such that $\phi \not\equiv 0$ and

$$\mathrm{Supp}(\phi) \subset [0, T] \backslash \{t \in [0, T]; f_\alpha(t, u(t)) = 0\} = \{t \in [0, T]; f_\alpha(t, u(t)) > 0\}.$$

Using the continuity of the mappings f_α, $\alpha \in \{1, \ldots, \nu\}$, we obtain that, for all $t \in \mathrm{Supp}(\phi)$ there exists $r_t \in (0, r)$ such that

$$f_\alpha(s, y) \geq \frac{1}{2} f_\alpha(t, u(t)) > 0 \quad \forall\, s \in [t - r_t, t + r_t] \cap [0, T], \quad \forall\, y \in \bar{B}(u(t), r_t).$$

Then

$$\mathrm{Supp}(\phi) \subset \bigcup_{t \in \mathrm{Supp}(\phi)} \left(t - \frac{r_t}{4(C+1)}, t + \frac{r_t}{4(C+1)} \right)$$

and, since $\mathrm{Supp}(\phi)$ is a compact subset of $\mathbb{R}$, there exists a finite family $(t^i)_{1 \leq i \leq p}$ of points of $\mathrm{Supp}(\phi)$ such that

$$\mathrm{Supp}(\phi) \subset \bigcup_{i=1}^{p} \left(t^i - \frac{r_{t^i}}{4(C+1)}, t^i + \frac{r_{t^i}}{4(C+1)} \right).$$

Let $\tilde{r} = \min_{1 \leq i \leq p} \frac{r_{t^i}}{4(C+1)}$ and $h_1^* \in \left(0, \min\left(h_1, \frac{\tilde{r}}{4(C+1)}\right)\right]$ such that

$$\|u - u_h\|_{C^0([0,T];\mathbb{R}^d)} \leq \frac{\tilde{r}}{4} \quad \forall\, h \in (0, h_1^*].$$

Then, by definition of $\lambda_{\alpha,h}$ we have

$$\langle \lambda_{\alpha,h}, \phi \rangle_{\mathcal{M}^1(0,T;\mathbb{R}), C^0([0,T];\mathbb{R})} = \sum_{n=1}^{N(h)-1} \lambda_\alpha^n \phi(nh) \quad \forall\, h \in (0, h_1].$$

But, for all $nh \in \mathrm{Supp}(\phi)$ there exists $t^i \in \{t^1, \ldots, t^p\}$ such that $nh \in \big(t^i - \frac{r_{t^i}}{4(C+1)}, t^i + \frac{r_{t^i}}{4(C+1)}\big)$. It follows that, for all $h \in (0, h_1^*]$, we have

$$|(n+1)h - t^i| < h + \frac{r_{t^i}}{4(C+1)} \leq \frac{r_{t^i}}{2(C+1)} < r_{t^i}$$

and

$$\begin{aligned}\|U^{n+1} - u(t^i)\| &\le \|u_h(t_{n+1}) - u(t_{n+1})\| + \|u(t_{n+1}) - u(t^i)\| \\ &\le \|u - u_h\|_{C^0([0,T];\mathbb{R}^d)} + C|(n+1)h - t^i| \\ &\le \frac{\tilde{r}}{4} + C\frac{r_{t^i}}{2(C+1)} \\ &< r_{t^i}.\end{aligned}$$

Thus $f_\alpha(t_{n+1}, U^{n+1}) > 0$ and $\lambda_\alpha^n = 0$ for all $nh \in \mathrm{Supp}(\phi)$. We infer that

$$\langle \lambda_{\alpha,h}, \phi \rangle_{\mathcal{M}^1(0,T;\mathbb{R}), C^0([0,T];\mathbb{R})} = \sum_{n=1}^{N(h)-1} \lambda_\alpha^n \phi(nh) = 0 \quad \forall\, h \in (0, h_1^*]$$

which allows us to conclude. □

4. Transmission of the Velocity at Impacts

In this section we prove that the limit trajectory satisfies the impact law (P4) and the initial data (P5).

First we observe that the impact law is satisfied at any instant $t \in (0,T)$ such that $J(t, u(t)) = \emptyset$. Indeed, by continuity of the mappings f_α, $\alpha \in \{1, \dots, \nu\}$, we may define $r_t \in (0, \min(r, t, T-t))$ such that, for all $\alpha \in \{1, \dots, \nu\}$ we have

$$f_\alpha(s, y) \ge \frac{1}{2} f_\alpha(t, u(t)) > 0 \quad \forall\, s \in [t - r_t, t + r_t], \;\; \forall\, y \in \bar{B}(u(t), r_t)$$

and we define $h_t \in \left(0, \min\left(h_1, \frac{r_t}{4(C+1)}\right)\right]$ such that $\|u - u_h\|_{C^0([0,T];\mathbb{R}^d)} \le \frac{r_t}{4}$ for all $h \in (0, h_t]$. Then, for all $\tilde{r} \in (0, r_t]$ and for all $h \in (0, h_t]$, we define

$$n_- = \left\lfloor \frac{t - \frac{\tilde{r}}{4(C+1)}}{h} \right\rfloor + 1, \quad n_+ = \left\lfloor \frac{t + \frac{\tilde{r}}{4(C+1)}}{h} \right\rfloor.$$

It follows that

$$\begin{aligned} 2h < (n_- - 1)h \le t - \frac{\tilde{r}}{4(C+1)} < n_- h < \cdots < n_+ h \\ \le t + \frac{\tilde{r}}{4(C+1)} < (n_+ + 1)h < T - 2h \end{aligned}$$

and

$$V^{n_- - 1} = v_h\left(t - \frac{\tilde{r}}{4(C+1)}\right), \quad V^{n_+} = v_h\left(t + \frac{\tilde{r}}{4(C+1)}\right).$$

With relation (2.4) we get

$$V^{n_+} - V^{n_- - 1} = \sum_{n=n_-}^{n_+} hG^n + \sum_{n=n_-}^{n_+} \sum_{\alpha=1}^{\nu} \lambda_\alpha^n \nabla_u f_\alpha(t_{n+1}, U^{n+1}).$$

But, for all $n \in \{n_-, \dots, n_+\}$ we have $t_n = nh \in \left[t - \frac{\tilde{r}}{4(C+1)}, t + \frac{\tilde{r}}{4(C+1)}\right]$ and

$$|t_{n+1} - t| \le \frac{\tilde{r}}{4(C+1)} + h \le \frac{r_t}{2(C+1)} < r_t,$$
$$\begin{aligned}\|U^{n+1} - u(t)\| &\le \|U^{n+1} - u_h(t)\| + \|u_h(t) - u(t)\| \\ &\le C|t_{n+1} - t| + \|u - u_h\|_{C^0([0,T];\mathbb{R}^d)} < r_t.\end{aligned}$$

It follows that $f_\alpha(t_{n+1}, U^{n+1}) > 0$ and $\lambda_\alpha^n = 0$ for all $\alpha \in \{1, \dots, \nu\}$ and for all $n \in \{n_-, \dots, n_+\}$. Thus

$$\begin{aligned}&\left\| v_h\left(t + \frac{\tilde{r}}{4(C+1)}\right) - v_h\left(t - \frac{\tilde{r}}{4(C+1)}\right)\right\| \\ &\quad = \left\| \sum_{n=n_-}^{n_+} hG^n \right\| \le \int_{t_{n_-}}^{t_{n_+ + 1}} F(s)ds \le \int_{t - \frac{\tilde{r}}{4(C+1)}}^{t + \frac{\tilde{r}}{4(C+1)} + h} F(s)ds.\end{aligned}$$

We can pass to the limit as h tends to zero, then as r tends to zero and we get

$$\|v(t^-) - v(t^+)\| \le 0,$$

i.e. $v(t^-) = \dot{u}(t^-) = \dot{u}(t^+) = v(t^+)$.

Now let us consider $t \in (0, T)$ such that $J(t, u(t)) \neq \emptyset$. If $J(t, u(t)) = \{1, \dots, \nu\}$, we let $r_t = \frac{1}{2}\min(r, t, T - t)$. Otherwise, using again the continuity of the mappings f_α, $\alpha \in \{1, \dots, \nu\}$, we define $r_t \in (0, \min(r, t, T - t))$ such that, for all $\alpha \in \{1, \dots, \nu\} \backslash J(t, u(t))$ we have

$$f_\alpha(s, y) \ge \frac{1}{2} f_\alpha(t, u(t)) > 0 \quad \forall\, s \in [t - r_t, t + r_t],\ \ \forall\, y \in \bar{B}(u(t), r_t).$$

Then, using the uniform convergence of $(u_h)_{h_1 \ge h > 0}$ to u on $[0, T]$, we define $h_t \in \left(0, \min\left(h_1, \frac{r_t}{4(C+1)}\right)\right]$ such that $\|u - u_h\|_{C^0([0,T];\mathbb{R}^d)} \le \frac{r_t}{4}$ for all $h \in (0, h_t]$. It follows that, for all $h \in (0, h_t]$ and for all $nh \in \left[t - \frac{r_t}{4(C+1)}, t + \frac{r_t}{4(C+1)}\right]$ we have $J(t_{n+1}, U^{n+1}) \subset J(t, u(t))$. Indeed, let $h \in (0, h_t]$ and $nh \in \left[t - \frac{r_t}{4(C+1)}, t + \frac{r_t}{4(C+1)}\right]$. We have

$$|t_{n+1} - t| \le \frac{r_t}{4(C+1)} + h \le \frac{r_t}{2(C+1)} < r_t,$$
$$\begin{aligned}\|U^{n+1} - u(t)\| &\le \|U^{n+1} - u_h(t)\| + \|u_h(t) - u(t)\| \\ &\le C|t_{n+1} - t| + \|u - u_h\|_{C^0([0,T];\mathbb{R}^d)} < r_t\end{aligned}$$

and we infer that

$$f_\alpha(t_{n+1}, U^{n+1}) > 0 \quad \forall\, \alpha \notin J(t, u(t)).$$

Then we split $J(t, u(t))$ as $J(t, u(t)) = J_1(t, u(t)) \cup J_2(t, u(t))$ with

$$J_1(t, u(t)) = \Big\{\alpha \in J(t, u(t)); \exists\, r_\alpha \in (0, r_t], \exists\, h_\alpha \in (0, h_t]/\forall\, h \in (0, h_\alpha], \\ \forall\, nh \in \left[t - \frac{r_\alpha}{4(C+1)}, t + \frac{r_\alpha}{4(C+1)}\right] \cap [0, T], f_\alpha(t_{n+1}, U^{n+1}) > 0\Big\} \tag{4.1}$$

and

$$J_2(t, u(t)) = \Big\{\alpha \in J(t, u(t));\ \forall\, r_\alpha \in (0, r_t],\ \forall\, h_\alpha \in (0, h_t], \exists\, h \in (0, h_\alpha], \\ \exists\, nh \in \left[t - \frac{r_\alpha}{4(C+1)}, t + \frac{r_\alpha}{4(C+1)}\right] \cap [0, T]/f_\alpha(t_{n+1}, U^{n+1}) \le 0\Big\}. \tag{4.2}$$

Since $J_1(t, u(t))$ is a finite set, we may define $\tilde{r}_t = \min_{\alpha \in J_1(t,u(t))} r_\alpha$, $\tilde{h}_t = \min_{\alpha \in J_1(t,u(t))} h_\alpha$ if $J_1(t, u(t)) \neq \emptyset$, and $\tilde{r}_t = r_t$ and $\tilde{h}_t = h_t$ if $J_1(t, u(t)) = \emptyset$.

Now let $\tilde{r} \in (0, \tilde{r}_t]$ and $h \in (0, \tilde{h}_t]$. We define as previously

$$n_- = \left\lfloor \frac{t - \frac{\tilde{r}}{4(C+1)}}{h} \right\rfloor + 1, \quad n_+ = \left\lfloor \frac{t + \frac{\tilde{r}}{4(C+1)}}{h} \right\rfloor$$

which implies that

$$2h < (n_- - 1)h \le t - \frac{\tilde{r}}{4(C+1)} < n_- h < \cdots < n_+ h \le t + \frac{\tilde{r}}{4(C+1)} \\ < (n_+ + 1)h < T - 2h$$

and

$$V^{n_- - 1} = v_h\left(t - \frac{\tilde{r}}{4(C+1)}\right), \quad V^{n_+} = v_h\left(t + \frac{\tilde{r}}{4(C+1)}\right).$$

Thus

$$V^{n_+} - V^{n_- - 1} = \sum_{n=n_-}^{n_+} hG^n + \sum_{n=n_-}^{n_+} \sum_{\alpha=1}^{\nu} \lambda_\alpha^n \nabla_u f_\alpha(t_{n+1}, U^{n+1}).$$

But, for all $n \in \{n_-, \ldots, n_+\}$, we have $t_n = nh \in \left[t - \frac{\tilde{r}}{4(C+1)}, t + \frac{\tilde{r}}{4(C+1)}\right]$. Hence $J(t_{n+1}, U^{n+1}) \subset J(t, u(t))$ and $\alpha \notin J(t_{n+1}, U^{n+1})$ if $\alpha \in J_1(t, u(t))$, so

$$V^{n_+} - V^{n_- - 1} = \sum_{n=n_-}^{n_+} hG^n + \sum_{\alpha \in J_2(t,u(t))} \sum_{n=n_-}^{n_+} \lambda_\alpha^n \nabla_u f_\alpha(t_{n+1}, U^{n+1}). \tag{4.3}$$

If $J_2(t, u(t)) = \emptyset$ we may conclude as previously that $\dot{u}(t^+) = \dot{u}(t^-)$.

On the other hand, we have $u(s) \in K(s)$ for all $s \in [0, T]$, so $\dot{u}(t^+) \in T(K(t), u(t))$. We infer that $\dot{u}(t^+) = \dot{u}(t^-) \in T(K(t), u(t))$ and thus $\dot{u}(t^+) = \dot{u}(t^-) = \mathrm{Proj}(T(K(t), u(t)), \dot{u}(t^-))$.

Otherwise, if $J_2(t, u(t)) \neq \emptyset$, we rewrite (4.3) as follows:

$$
\begin{aligned}
&v_h\left(t + \frac{\tilde{r}}{4(C+1)}\right) - v_h\left(t - \frac{\tilde{r}}{4(C+1)}\right) \\
&\quad = \sum_{\alpha \in J_2(t,u(t))} \left(\sum_{n=n_-}^{n_+} \lambda_\alpha^n \right) \nabla_u f_\alpha(t, u(t)) + \sum_{n=n_-}^{n_+} hG^n \\
&\qquad + \sum_{\alpha \in J_2(t,u(t))} \sum_{n=n_-}^{n_+} \lambda_\alpha^n (\nabla_u f_\alpha(t_{n+1}, U^{n+1}) - \nabla_u f_\alpha(t, u(t))). \qquad (4.4)
\end{aligned}
$$

We may deduce that

Lemma 4.1. *We have*

$$
v(t^+) - v(t^-) \in \sum_{\alpha \in J_2(t,u(t))} \mathbb{R}^+ \nabla_u f_\alpha(t, u(t)).
$$

Proof. We can estimate the last two terms of (4.4) as follows:

$$
\left\| \sum_{n=n_-}^{n_+} hG^n \right\| \leq \int_{t_{n_-}}^{t_{n_+}+h} F(s)ds \leq \int_{t-\frac{\tilde{r}}{4(C+1)}}^{t+\frac{\tilde{r}}{4(C+1)}+h} F(s)ds
$$

and, using Lemma 3.2

$$
\begin{aligned}
&\left\| \sum_{\alpha \in J_2(t,u(t))} \sum_{n=n_-}^{n_+} \lambda_\alpha^n (\nabla_u f_\alpha(t_{n+1}, U^{n+1}) - \nabla_u f_\alpha(t, u(t))) \right\| \\
&\quad \leq \sum_{\alpha \in J_2(t,u(t))} \sum_{n=n_-}^{n_+} \lambda_\alpha^n \|\nabla_u f_\alpha(t_{n+1}, U^{n+1}) - \nabla_u f_\alpha(t, u(t))\| \\
&\quad \leq \sum_{\alpha \in J_2(t,u(t))} \sum_{n=n_-}^{n_+} \lambda_\alpha^n M(|t_{n+1} - t| + \|U^{n+1} - u(t)\|) \\
&\quad \leq \sum_{\alpha \in J_2(t,u(t))} \sum_{n=n_-}^{n_+} \lambda_\alpha^n M\left(\left(h + \frac{\tilde{r}}{4(C+1)}\right)(C+1) + \|u - u_h\|_{C^0([0,T];\mathbb{R}^d)}\right) \\
&\quad \leq M\left(\left(h + \frac{\tilde{r}}{4(C+1)}\right)(C+1) + \|u - u_h\|_{C^0([0,T];\mathbb{R}^d)}\right) \\
&\qquad \times \frac{\nu\gamma}{m}(TV(v_h) + \|F\|_{L^1(0,T;\mathbb{R}^d)}).
\end{aligned}
$$

Reminding the uniform estimate of $TV(v_h)$ obtained at Proposition 2.2, we infer that

$$\lim_{\tilde{r}\to 0^+}\lim_{h\to 0^+}\left\| v_h\left(t+\frac{\tilde{r}}{4(C+1)}\right)-v_h\left(t-\frac{\tilde{r}}{4(C+1)}\right)\right.$$
$$\left. -\sum_{\alpha\in J_2(t,u(t))}\left(\sum_{n=n_-}^{n_+}\lambda_\alpha^n\right)\nabla_u f_\alpha(t,u(t))\right\| = 0. \tag{4.5}$$

Finally we infer from assumption (H2) that $\mathcal{C} := \sum_{\alpha\in J_2(t,u(t))}\mathbb{R}^+\nabla_u f_\alpha(t,u(t))$ is a closed subset of $\mathbb{R}^d$. Indeed, let $(x_n)_{n\in\mathbb{N}}$, with $x_n = \sum_{\alpha\in J_2(t,u(t))} x_{\alpha,n}\nabla_u f_\alpha(t,u(t))$ for all $n\in\mathbb{N}$, be a sequence of $\mathcal{C}$. With assumption (H2) we have

$$\begin{aligned} m\sum_{\alpha\in J_2(t,u(t))} x_{\alpha,n} &\le \sum_{\alpha\in J_2(t,u(t))} x_{\alpha,n}\|\nabla_u f_\alpha(t,u(t))\| \\ &\le \gamma\left\|\sum_{\alpha\in J_2(t,u(t))} x_{\alpha,n}\nabla_u f_\alpha(t,u(t))\right\| \end{aligned}$$

for all $n\in\mathbb{N}$. Hence, if $(x_n)_{n\in\mathbb{N}}$ converges to x_* in $\mathbb{R}^d$, the sequence $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded and all the non-negative real sequences $(x_{\alpha,n})_{n\in\mathbb{N}}$, $\alpha\in J_2(t,u(t))$, are bounded. Possibly extracting a subsequence, still denoted $(x_n)_{n\in\mathbb{N}}$, we may infer that there exist non-negative real numbers $x_{\alpha,*}$ such that

$$x_{\alpha,n}\xrightarrow[n\to+\infty]{} x_{\alpha,*}\quad \forall\,\alpha\in J_2(t,u(t)).$$

Then we get

$$\begin{aligned}\left\|x_n-\sum_{\alpha\in J_2(t,u(t))} x_{\alpha,*}\nabla_u f_\alpha(t,u(t))\right\| &\le \sum_{\alpha\in J_2(t,u(t))} |x_{\alpha,n}-x_{\alpha,*}|\|\nabla_u f_\alpha(t,u(t))\| \\ &\le M\sum_{\alpha\in J_2(t,u(t))} |x_{\alpha,n}-x_{\alpha,*}|\quad \forall n\in\mathbb{N}\end{aligned}$$

and we obtain at the limit $x_* = \sum_{\alpha\in J_2(t,u(t))} x_{\alpha,*}\nabla_u f_\alpha(t,u(t))\in\mathcal{C}$. Hence, using (4.5) and passing to the limit as h tends to zero, then as r tends to zero in (4.4), we obtain the announced result. □

Now we will prove that

Proposition 4.1. *For all $\alpha\in J_2(t,u(t))$ we have*

$$\partial_t f_\alpha(t,u(t)) + \langle\nabla_u f_\alpha(t,u(t)),\dot{u}(t^+)\rangle = 0.$$

Proof. Since we already know that $\dot{u}(t^+) \in T(K(t), u(t))$, we only need to prove that

$$\partial_t f_\alpha(t, u(t)) + \langle \nabla_u f_\alpha(t, u(t)), \dot{u}(t^+) \rangle \leq 0 \quad \forall\, \alpha \in J_2(t, u(t)).$$

Let $\alpha \in J_2(t, u(t))$ and $\tilde{r} \in (0, \tilde{r}_t]$. Using the definition of $J_2(t, u(t))$ (see (4.1) and (4.2)), we may define a subsequence $(h_i)_{i \in \mathbb{N}}$ strictly decreasing to zero such that, for all $i \in \mathbb{N}$ we have $h_i \in (0, \tilde{h}_t]$ and there exists $nh_i \in \left[t - \frac{\tilde{r}}{4(C+1)}, t + \frac{\tilde{r}}{4(C+1)}\right]$ such that $f_\alpha(t_{n+1}, U^{n+1}) \leq 0$, i.e. $\alpha \in J(t_{n+1}, U^{n+1})$. We define

$$n_i = \max\left\{ n \in \mathbb{N}; nh_i \in \left[t - \frac{\tilde{r}}{4(C+1)}, t + \frac{\tilde{r}}{4(C+1)}\right] \text{ and } \alpha \in J(t_{n+1}, U^{n+1}) \right\}.$$

With Lemma 2.1 we have

$$\begin{aligned} \partial_t f_\alpha(t_{n_i+1}, U^{n_i+1}) + \langle \nabla_u f_\alpha(t_{n_i+1}, U^{n_i+1}), V^{n_i} \rangle &\leq \frac{Mh_i}{2}(1 + \|V^{n_i}\|)^2 \\ &\leq \frac{Mh_i}{2}(1 + C)^2. \end{aligned}$$

It follows that

$$\begin{aligned} &\partial_t f_\alpha(t, u(t)) + \langle \nabla_u f_\alpha(t, u(t)), V^{n_+} \rangle \\ &\quad \leq \frac{Mh_i}{2}(1 + C)^2 + (\partial_t f_\alpha(t, u(t)) - \partial_t f_\alpha(t_{n_i+1}, U^{n_i+1})) \\ &\qquad + \langle \nabla_u f_\alpha(t, u(t)), V^{n_+} - V^{n_i} \rangle \\ &\qquad + \langle \nabla_u f_\alpha(t, u(t)) - \nabla_u f_\alpha(t_{n_i+1}, U^{n_i+1}), V^{n_i} \rangle. \end{aligned} \tag{4.6}$$

We can estimate the second and fourth terms of the right-hand side of (4.6) as

$$\begin{aligned} &\|\partial_t f_\alpha(t, u(t)) - \partial_t f_\alpha(t_{n_i+1}, U^{n_i+1})\| \\ &\quad \leq M(|t - t_{n_i+1}| + \|U^{n_i+1} - u(t)\|) \\ &\quad \leq M\left(\left(\frac{\tilde{r}}{4(C+1)} + h_i\right)(C+1) + \|u - u_{h_i}\|_{C^0([0,T];\mathbb{R}^d)}\right) \end{aligned}$$

and

$$\begin{aligned} &\|\langle \nabla_u f_\alpha(t, u(t)) - \nabla_u f_\alpha(t_{n_i+1}, U^{n_i+1}), V^{n_i} \rangle\| \\ &\quad \leq M(|t - t_{n_i+1}| + \|U^{n_i+1} - u(t)\|)\|V^{n_i}\| \\ &\quad \leq MC\left(\left(\frac{\tilde{r}}{4(C+1)} + h_i\right)(C+1) + \|u - u_{h_i}\|_{C^0([0,T];\mathbb{R}^d)}\right). \end{aligned}$$

If $n_i = n_+$, the third term of the right-hand side of (4.6) vanishes. Otherwise we rewrite it as follows

$$
\begin{aligned}
&\langle \nabla_u f_\alpha(t, u(t)), V^{n_+} - V^{n_i} \rangle \\
&\quad = \left\langle \nabla_u f_\alpha(t, u(t)), \sum_{n=n_i+1}^{n_+} hG^n \right\rangle \\
&\qquad + \left\langle \nabla_u f_\alpha(t, u(t)), \sum_{n=n_i+1}^{n_+} \sum_{\beta \in J(t_{n+1}, U^{n+1})} \lambda_\beta^n \nabla_u f_\beta(t_{n+1}, U^{n+1}) \right\rangle \\
&\quad \leq M \int_{t-\frac{\tilde{r}}{4(C+1)}}^{t+\frac{\tilde{r}}{4(C+1)}+h_i} F(s) ds \\
&\qquad + \left\langle \nabla_u f_\alpha(t, u(t)), \sum_{n=n_i+1}^{n_+} \sum_{\beta \in J(t_{n+1}, U^{n+1})} \lambda_\beta^n \nabla_u f_\beta(t, u(t)) \right\rangle \\
&\qquad + \left\langle \nabla_u f_\alpha(t, u(t)), \sum_{n=n_i+1}^{n_+} \sum_{\beta \in J(t_{n+1}, U^{n+1})} \lambda_\beta^n (\nabla_u f_\beta(t_{n+1}, U^{n+1}) \right. \\
&\qquad \left. - \nabla_u f_\beta(t, u(t))) \right\rangle .
\end{aligned}
$$

Since $\alpha \notin J(t_{n+1}, U^{n+1})$ for all $n \in \{n_i + 1, \ldots, n_+\}$ by definition of n_i and $J(t_{n+1}, U^{n+1}) \subset J(t, u(t))$, assumption (H4) implies that the second term of the right-hand side of this last inequality is non-positive. Furthermore, the last term can be estimated as

$$
\begin{aligned}
&\left\| \left\langle \nabla_u f_\alpha(t, u(t)), \sum_{n=n_i+1}^{n_+} \sum_{\beta \in J(t_{n+1}, U^{n+1})} \lambda_\beta^n (\nabla_u f_\beta(t_{n+1}, U^{n+1}) - \nabla_u f_\beta(t, u(t))) \right\rangle \right\| \\
&\quad \leq \sum_{n=n_i+1}^{n_+} \sum_{\beta \in J(t_{n+1}, U^{n+1})} \lambda_\beta^n M^2 (|t - t_{n+1}| + \|U^{n+1} - u(t)\|) \\
&\quad \leq M^2 \nu \left(\left(\frac{\tilde{r}}{4(C+1)} + h_i \right) (C+1) + \|u - u_{h_i}\|_{C^0([0,T];\mathbb{R}^d)} \right) \\
&\qquad \times (TV(v_{h_i}) + \|F\|_{L^1(0,T;\mathbb{R}^d)}).
\end{aligned}
$$

Then we can pass to the limit in all the terms of the right-hand side of (4.6), and recalling that $V^{n_+} = v_h\big(t + \frac{\tilde{r}}{4(C+1)}\big)$, we obtain

$$
\begin{aligned}
&\lim_{\tilde{r} \to 0^+} \lim_{h_i \to 0^+} \partial_t f_\alpha(t, u(t)) + \langle \nabla_u f_\alpha(t, u(t)), V^{n_+} \rangle \\
&\quad = \partial_t f_\alpha(t, u(t)) + \langle \nabla_u f_\alpha(t, u(t)), v(t^+) \rangle \leq 0.
\end{aligned}
$$

□

Now we can easily check that

$$\dot{u}(t^+) = \mathrm{Proj}(T(K(t), u(t)), \dot{u}(t^-)).$$

Indeed we already know that $\dot{u}(t^+) \in T(K(t), u(t))$ and that $\dot{u}(t^+) - \dot{u}(t^-) \in \sum_{\alpha \in J_2(t,u(t))} \mathbb{R}^+ \nabla_u f_\alpha(t, u(t))$. Hence there exist non-negative real numbers $\bar{\lambda}_\alpha$, for $\alpha \in J_2(t, u(t))$, such that

$$\dot{u}(t^+) - \dot{u}(t^-) = \sum_{\alpha \in J_2(t,u(t))} \bar{\lambda}_\alpha \nabla_u f_\alpha(t, u(t))$$

and for all $w \in T(K(t), u(t))$

$$\langle \dot{u}(t^-) - \dot{u}(t^+), w - \dot{u}(t^+) \rangle = - \sum_{\alpha \in J_2(t,u(t))} \bar{\lambda}_\alpha \langle \nabla_u f_\alpha(t, u(t)), w - \dot{u}(t^+) \rangle.$$

But, using the previous proposition, for all $w \in T(K(t), u(t))$ and for all $\alpha \in J_2(t, u(t))$, we have

$$\begin{aligned} \langle \nabla_u f_\alpha(t, u(t)), w - \dot{u}(t^+) \rangle &= \left(\partial_t f_\alpha(t, u(t)) + \langle \nabla_u f_\alpha(t, u(t)), w \rangle \right) \\ &\quad - \left(\partial_t f_\alpha(t, u(t)) + \langle \nabla_u f_\alpha(t, u(t)), \dot{u}(t^+) \rangle \right) \\ &= \partial_t f_\alpha(t, u(t)) + \langle \nabla_u f_\alpha(t, u(t)), w \rangle \\ &\geq 0. \end{aligned}$$

Hence

$$\langle \dot{u}(t^-) - \dot{u}(t^+), w - \dot{u}(t^+) \rangle \leq 0 \quad \forall\, w \in T(K(t), u(t))$$

which allows us to conclude since $T(K(t), u(t))$ is a closed convex subset of $\mathbb{R}^d$.

Finally we observe that the limit trajectory satisfies the initial data. Indeed, with (3.2) we have immediately $u(0) = u_0$. Moreover, recalling that $v_0 \in T(K(0), u_0)$ we can prove that

$$\dot{u}(0^+) = v_0 = \mathrm{Proj}(T(K(0), u_0), v_0)$$

by the same kind of computations. Indeed, if $t = t_0 = 0$, we may define $r_{t_0} \in (0, \min(r, T))$ such that

$$J(s, y) \subset J(t_0, u(t_0)) \quad \forall\, s \in [t_0 - r_{t_0}, t_0 + r_{t_0}] \cap [0, T], \ \ \forall\, y \in \bar{B}(u(t_0), r_{t_0})$$

and we define h_{t_0} (respectively, $\tilde{r}_{t_0}$ and $\tilde{h}_{t_0}$ if $J(t_0, u(t_0)) \neq \emptyset$) in the same way as previously. Then, for all $\tilde{r} \in (0, r_{t_0}]$ and for all $h \in (0, h_{t_0}]$ (respectively, for all $\tilde{r} \in (0, \tilde{r}_{t_0}]$ and for all $h \in (0, \tilde{h}_{t_0}]$ if $J(t_0, u(t_0)) \neq \emptyset$) we define

$$n_- = 0, \quad n_+ = \left\lfloor \frac{t_0 + \frac{\tilde{r}}{4(C+1)}}{h} \right\rfloor.$$

We get

$$V^{n_- - 1} = V^{-1} = v_0, \quad V^{n_+} = v_h \left(t_0 + \frac{\tilde{r}}{4(C+1)} \right)$$

and the rest of the computation is straightforward.

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