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A FORMALISM FOR THE RENORMALIZATION PROCEDURE

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A formalism for the renormalization procedure is given.

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0. Introduction

The purpose of this paper is to analyze the procedure of renormalization from the mathematical point of view. Our original motivation came from trying to really understand the paper [4]. This paper uses the so-called Batalin–Vilkovitski formalism [6, 3]. Its main features include:

(1) given a QFT, one constructs a so-called quantum Batalin–Vilkovitski bracket on the space of observables. Using this bracket one writes a Master equation (a.k.a. Maurer–Cartan equation);
(2) every solution to this equation is supposed to produce a deformation of the QFT.

It is the procedure of constructing such a deformation that is called *renormalization* in the current paper.

Unfortunately, the treatment in [4] does not lead to a (mathematically) non-contradictory definition of the Batalin–Vilkovitski bracket or renormalization (due to divergencies). The goal of this paper is to begin filling this gap up.

Before working with the QFT from [4] (i.e. the Poisson sigma model), it makes sense to start with simpler theories and to define the Batalin–Vilkovitski bracket and the renormalization for them. In this paper we do it for the theory of free boson in $\mathbb{R}^{2n}$, $n > 1$. It turns out that the construction generalizes more or less straightforwardly to the situation in [4], which will be a subject of a subsequent paper.

The author hopes that the constructions of this paper will also work in a more general context.

We deal with QFTs via a $\mathcal{D}_X$-module M of observables of the theory (X is the spacetime) and an OPE-product structure on M. So, we start with a definition of an OPE-product. To this end one first has to prescribe possible singularities of these OPEs. We call such a prescription *a system* (a precise definition is given below). Given a system, we have a notion of an OPE-algebra over this system.

We then construct an appropriate system for the free scalar boson Euclidean theory in $\mathbb{R}^{2n}$, $n > 1$, in which case the only possible singularities are of the type: products of squares of Euclidean distances in the denominator. We denote this system by $\langle \mathfrak{i} \rangle$. The cases $\mathbb{R}^2, \mathbb{R}^{2n+1}$ require semi-integer powers or logarithms, which leads to slightly more complicated definitions. For simplicity we only work with $\mathbb{R}^{2n}$, $n > 1$ throughout the paper.

We then show that the Batalin–Vilkovitski bracket arises due to a certain additional structure on the system. We call a system with such a structure *pre-symmetric*. The system $\langle \mathfrak{i} \rangle$ has no natural pre-symmetric structure, nevertheless we construct a differential graded resolution $\langle \mathfrak{R} \rangle \to \langle \mathfrak{i} \rangle$ which is pre-symmetric. Furthermore, any OPE-algebra over $\langle \mathfrak{i} \rangle$ can be lifted to an OPE-algebra over $\langle \mathfrak{R} \rangle$. The building blocks for the system $\langle \mathfrak{R} \rangle$ are certain spaces of generalized functions. The lifting procedure can be interpreted as a regularization (i.e. passage from usual functions with singularities to generalized functions). It seems to be very similar to the well-known Bogoliubov–Parasyuk–Hepp procedure [11]. There is also some affinity with the approach in [2].

It is worth to mention that the homotopy theory implies that, upto homotopy, nothing should depend on the choice of such a lifting. What is not implied by the homotopy theory is that we can always find an "honest" lifting (as opposed to a lifting upto higher homotopies). Furthermore, we expect that the action of Hopf algebras introduced in [8] (see also [2], where a somewhat similar object appears under the name of "the group of renormalizations") should provide us with ("honest", not quasi-) isomorphisms between the different liftings, which also looks slightly different from what we are used to in the homotopy theory.

Next, we treat the renormalization procedure. It turns out that to accomplish such a procedure, one needs certain additional properties of the system. We call a system with these additional properies *symmetric system*. Unfortunately, the system $\langle \mathfrak{R} \rangle$ is only pre-symmetric, and not symmetric. The reason is very simple: the renormalized OPE have more sophisticated singularities. It turns out, though, that there is a formal "symmetrization" procedure, which produces a symmetric system out of a pre-symmetric one. So, starting from $\langle \mathfrak{R} \rangle$, we get a symmetric system $\langle \mathfrak{R}^{\mathbf{symm}} \rangle$ and construct a renormalized OPE in this system.

Morally, the system $\langle \mathfrak{R}^{\mathbf{symm}} \rangle$ is given in terms of a $\mathcal{D}$-module whose solutions are possible singularities of the renormalized OPE. Our last step is to interpret this $\mathcal{D}$-module as a sub-module in the space of real-analytic functions.

Our approach has to be compared with the ones in [1] and [8, 9]. Our feeling is that our approach is less general than the one in [1] (although, I believe, that they become rather close, if one uses the abstract definition of a system (see 2.3.3); the approach in [8, 9] studies a concrete renormalization procedure, nevertheless, it seems that the Connes–Kreimer Hopf algebra is a rather general phenomenon by means of which one can identify different regularizations (= liftings to $\langle \mathfrak{R} \rangle$) of an OPE-algebra, as was mentioned above.

I hope that the tools developed in this paper can help complete the project described in [10] in a mathematically rigorous way. The major thing which is predicted by physicists (i.e. in [10]) and which is lacking in this paper is a construction of a homotopy d-algebra structure on the de Rham complex of the $\mathcal{D}$-module of observables (we only construct a Lie bracket).

The main technical tool that we use in this paper is a $\mathcal{D}$-module structure on the space of observables. The author started to appreciate this structure in the process of reading [5].

In the case of the free boson the module of observables equals $\mathrm{Sym}_{\mathcal{O}_X} \mathcal{D}_X / \mathcal{D}_X \Delta$, where X is the spacetime and Δ is the Laplacian. This module is not free, which prompts using resolutions and homological algebra.

1. Content of the Paper

The paper consists of three parts. In the first part we introduce the notion of system and the structure of an OPE-algebra over a system. We then discuss a naive approach to renormalization, the naiveness being in ignoring all complications stemming from homological algebra. The rest of the paper is devoted to constructing a homotopically correct (= derived) version of this naive construction. In the second part we explain the main steps in our construction with all technicalities omitted. The third part deals with these omitted technicalities.

Part I: Systems, OPE, Naive Renormalization

2. What is an OPE?

Before giving general definition of OPE, we will introduce this notion in the setting of the theory of free boson. The general notion of an OPE will be obtained via a straightforward generalization.

2.1. *Notations*

We are going to consider the Euclidean theory of free boson. Let $Y := \mathbb{R}^{2N}$ be the spacetime. We will prefer to work with the complexification $X = \mathbb{C}^{2N}$ viewed as an affine algebraic variety over $\mathbb{C}$. Fix a positively-definite quadratic form $q : Y \to \mathbb{R}$. Extend it to X and denote the extension by the same letter: $q : X \to \mathbb{C}$.

For a finite set S, let X^S be the algebraic variety which is the product of $\#S$ copies of X. Let $\mathcal{D}_{X^S}$ be the sheaf of algebras of differential operators on X^S. Let $\mathcal{D}\text{-sh}_S$ be the category of $\mathcal{D}_{X^S}$-sheaves, i.e. non-quasi-coherent $\mathcal{D}_{X^S}$-modules. The usage of non-quasi-coherent modules is indispensable in the setting of this paper; on the other hand, since we are not going to use any of subtleties of the theory of $\mathcal{D}$-modules, $\mathcal{D}_{X^S}$ sheaves will not cause any discomfort.

2.2. *Extension of a $\mathcal{D}$-sheaf from a closed subvariety*

The material in this subsection is standard and can be found for example in [5].

Let $i : Y \to Z$ be a closed embedding of algebraic variety and let M be a $\mathcal{D}_Y$-sheaf. Let Y_n be the nth infinitesimal neighborhood of Y in Z. It is well known that M is a crystal, i.e. it naturally defines an $\mathcal{O}_{Y_n}$-sheaf; denote it by N_n. Set $i^\wedge N := \text{liminv}_n N_n$; it is a topological $\mathcal{D}_Z$-module, the topology is $\mathcal{I}_Y$-adically complete, where $\mathcal{I}_Y$ is the ideal of Y. There is a simple explicit formula for $i^\wedge M$:

$$i^\wedge Y \cong i_{.} \operatorname{Hom}_{\mathcal{O}_Y}(i^*\mathcal{D}_Z, M),$$

where $i^*\mathcal{D}_Z$ is the quasi-coherent inverse image of $\mathcal{D}_Z$ viewed as a quasi-coherent $\mathcal{O}_Y$-module via the left multiplication; $i_{.}$ is the sheaf-theoretic extension by zero; the $\mathcal{D}_Z$-action on $i^\wedge Y$ is via the right action on $i^*\mathcal{D}_Z$.

One can prove an analogue of Kashiwara's theorem in this setting: the functor $i^\wedge$ is an equivalence of the following categories: the first category is the category of $\mathcal{D}_Y$-sheaves; the second category is the category of $\mathcal{D}_Z$-sheaves which are sheaf-theoretically supported on Y and are $\mathcal{I}_Y$-complete, the morphisms are continuous morphisms. One of the corollaries is the existence of natural maps $i^\wedge(M) \otimes_{\mathcal{O}_Z} N \to i^\wedge(M \otimes_{\mathcal{O}_Y} i^*N)$: Kashiwara's theorem implies that the right-hand side is the $\mathcal{I}_Y$-adic completion of the left-hand side.

If i, k are consecutive embeddings, then $i^\wedge k^\wedge \cong (ik)^\wedge$.

2.2.1. All our closed embeddings are going to be the embeddings of a generalized diagonal into some X^S. It is convenient to describe them as surjections $p : T \to S$. Each such a surjection produces a closed embedding $i_p : X^S \to X^T$ in the obvious way.

2.2.2. Another feature of the $\mathcal{D}$-modules theory that will be used in this paper is the existence of exterior product functors

$$\boxtimes_{a\in A} : \prod_{a\in A} \mathcal{D}\text{-sh}_{S_a} \to \mathcal{D}\text{-sh}_S,$$

where $S_a, a \in A$ is a finite family of finite sets and $S = \sqcup_{a\in A} S_a$.

The functor $i^\wedge$ is related with the exterior product in the following way. Let $p_a : T_a \to S_a$ be a family of projections. Let $T = \sqcup_{a\in A} T_a$; $S = \sqcup_{a\in A} S_a$; $p : T \to S$;

$p = \sqcup_{a\in A} p_a$. We then have a natural transformation

$$\boxtimes_{a\in A} \prod_{a\in A} i_a^\wedge \to i^\wedge \boxtimes_{a\in A}$$

both functors act from $\prod \mathcal{D}\text{-sh}_{S_a}$ to $\mathcal{D}\text{-sh}_T$.

2.3. *Construction of functors which are necessary to define an OPE*

Let us now take into account a specific feature of our problem: the presence of the quadratic form q which describes the locus of singularities of the corellators. Let S be a finite set and $s \neq t$ be elements in S. Let $q_{st} : X^S \to \mathcal{C}$ be the function $q(X_s - X_t)$, where X_s are the coordinates of a point on the sth component of X^S (X_t are the coordinates on the tth copy of X). Let $\mathcal{D}_{st}$ be the divisor of zeros of q_{st}. Denote by $Z_S := X^S \backslash (\cup_{st} \mathcal{D}_{ST})$. Let $j_S : Z_S \to X^S$ be an open embedding. Set $\mathcal{B}_S := j_{S*}\mathcal{O}_{Z_S}$. $\mathcal{B}_S$ is a $\mathcal{D}_{X^S}$-module.

For a projection $p : T \to S$ set $\mathcal{B}_p := \boxtimes_{s\in S} \mathcal{B}_{p^{-1}s}$; $\mathcal{B}_p$ is a $\mathcal{D}_{X^T}$-module. Set $\mathfrak{i}_p : \mathcal{C}_S \to \mathcal{C}_T$,

$$\mathfrak{i}_p(M) = i^\wedge(M) \otimes_{\mathcal{O}_{X^T}} \mathcal{B}_T. \tag{1}$$

List the properties of these functors. First of all they interact with the exterior products in the same way as $i_p^\wedge$. The behavior under compositions is different. Let

$$R \xrightarrow{q} T \xrightarrow{p} S$$

be consecutive surjections. We then have a natural transformation

$$\mathfrak{as}_{pq} : \mathfrak{i}_{pq} \to \mathfrak{i}_p \mathfrak{i}_q,$$

which is not an isomorphism. Let us construct $\mathfrak{as}_{pq}$. We need an auxiliary module $\mathcal{B}_{p,q} = j_{p,q*}\mathcal{O}_{Z_{p,q}}$, where $j_{p,q} : Z_{p,q} \to X^R$ is an open subvariety defined by

$$Z_{s,t} = X^R \Bigg\backslash \left(\bigcup_{q(s)\neq q(t)} \mathcal{D}_{st} \right).$$

It is clear that $\mathcal{B}_{pq} \cong \mathcal{B}_{p,q} \otimes \mathcal{B}_q$ and that $i_p^* \mathcal{B}_{p,q} \cong \mathcal{B}_p$. Here i_q^* is the inverse image for $\mathcal{O}_{X^S}$-coherent sheaves.

Define $\mathfrak{as}_{p,q}$ as the composition

$$i_{pq}^\wedge(M) \otimes \mathcal{B}_{pq} \cong i_q^\wedge i_p^\wedge(M) \otimes \mathcal{B}_{p,q} \otimes \mathcal{B}_q \to i_q^\wedge(i_p^\wedge(M) \otimes i_q^* \mathcal{B}_{p,q}) \otimes \mathcal{B}_q \cong \mathfrak{i}_q \mathfrak{i}_p(M).$$

2.3.1. *Co-associativity*

The maps $\mathfrak{as}_{p,q}$ have a co-associativity property. Let

$$U \xrightarrow{r} R \xrightarrow{q} T \xrightarrow{p} S$$

be a sequence of finite sets and their surjections. We then have two transformations from $\mathfrak{i}_{pqr} \to \mathfrak{i}_r\mathfrak{i}_q\mathfrak{i}_p$:
the first one is given by

$$\mathfrak{i}_{pqr}\mathfrak{as}_{r,pq} \longrightarrow \mathfrak{i}_r\mathfrak{i}_{pq} \xrightarrow{\mathrm{Id}\times\mathfrak{as}_{q,p}} \mathfrak{i}_r\mathfrak{i}_q\mathfrak{i}_p$$

and the second one is given by

$$\mathfrak{i}_{pqr} \xrightarrow{\mathfrak{as}_{rq,p}} \mathfrak{i}_{qr}\mathfrak{i}_p \xrightarrow{\mathfrak{as}_{r,q}\times\mathrm{Id}} \mathfrak{i}_r\mathfrak{i}_q\mathfrak{i}_p.$$

The co-associativity property says that these two transformations coincide.

2.3.2. The maps $\mathfrak{as}_{pq}$ interact with the exterior products in the following way. Let

$$R_a \xrightarrow{q_a} T_a \xrightarrow{p_a} S_a \tag{2}$$

be a family of finite sets and their surjections. Let $M_a \in \mathcal{C}_{S_a}$, $a \in A$ be arbitrary objects. Let

$$R \xrightarrow{q} T \xrightarrow{p} S$$

be the disjoint union of (2) over A. Let

$$M = \boxtimes_{a\in A} M_a \in \mathcal{C}_S.$$

Then the following diagram is commutative.

$$\begin{array}{ccccccc} \boxtimes_{a\in A}\mathfrak{i}_{p_a q_a} M_a & \longrightarrow & \boxtimes_{a\in A}\mathfrak{i}_{q_a}\mathfrak{i}_{p_a} M_a & \longrightarrow & \mathfrak{i}_q \boxtimes_{a\in A} \mathfrak{i}_{p_a} M_a & \longrightarrow & \mathfrak{i}_q\mathfrak{i}_p M \\ \downarrow & & & & & \nearrow & \\ \mathfrak{i}_{pq} M & & & & & & \end{array} \tag{3}$$

2.3.3. *Abstract definition*

We abstract the properties of the functors $\mathfrak{i}_p$. Assume that for every surjection $p : S \to T$ of finite sets we are given functors $\jmath_p : \mathcal{D}\text{-sh}_T \to \mathcal{D}\text{-sh}_S$ such that

(1) If p is a bijection, then $\jmath_p$ is the equivalence of categories induced by p;
(2) $\jmath_p$ interact with the exterior products in the same way as $\mathfrak{i}_p$; If all p_a are bijections, then the corresponding transformation is the natural one.

(3) Let

$$R \xrightarrow{q} T \xrightarrow{p} S$$

be a sequence of surjections. We then have transformations

$$\mathfrak{as}_{pq} : \jmath_{pq} \to \jmath_q \jmath_p.$$

This transformation is an isomorphism if at least one of p, q is a bijection. If both p, q are bijections, then the map $\mathfrak{as}_{pq}$ is the natural isomorphism of the corresponding equivalences.

(4) The maps $\mathfrak{as}_{pq}$ satisfy the co-associativity property as in (2.3.1).

(5) The maps $\mathfrak{as}_{pq}$ interact with the exterior product in the same way as in (2.3.2).

If all these properties are the case we say that we have *a system.*

The functors $\mathfrak{i}_p$ and their transformations form a system which we denote by $\langle \mathfrak{i} \rangle$.

2.3.4. *Morphisms of systems*

Let $\langle \jmath \rangle$, $\langle \mathfrak{k} \rangle$ be systems. *A morphism of systems* $F : \langle \jmath \rangle \to \langle \mathfrak{k} \rangle$ is a collection of transformations $F_p : \jmath_p \to \mathfrak{k}_p$ which commute with all elements of the structure of system.

2.4. *Definition of OPE*

With these functors and their properties at hand we are ready to define an OPE-algebra.

First of all, we need to fix a $\mathcal{D}_X$-module $\mathcal{M}$ such that its sections are observables of our theory. In the case of free boson, we set $\mathcal{M} = S_{\mathcal{O}_X} N$, where $N = \mathcal{D}_X / \mathcal{D}_X \cdot \Delta$.

As we know from physics, an OPE is a prescription of maps

$$\mathfrak{ope}_S : \mathcal{M}^{\boxtimes_S} \to \mathfrak{i}_{\pi_S}(\mathcal{M}),$$

where $\pi_S : S \to \{1\}$ is the projection onto a one-element set. These maps should be equivariant with respect to bijections $S \to S'$ of finite sets.

Let us formulate the conditions. It is convenient to define maps $\mathfrak{ope}_p$ for an arbitrary surjection $p : S \to T$,

$$\mathfrak{ope}_p : \mathcal{M}^{\boxtimes S} \to \mathfrak{i}_p(\mathcal{M}^{\boxtimes T})$$

as the composition:

$$\mathcal{M}^{\boxtimes S} \xrightarrow{\boxtimes_{t \in T} \mathfrak{ope}_{p^{-1}t}} \boxtimes_{t \in T} \mathfrak{i}_{\pi_{p^{-1}t}}(\mathcal{M}) \to \mathfrak{i}_p(\mathcal{M}^{\boxtimes T}).$$

Now let $R \xrightarrow{q} T \xrightarrow{p} S$ be a sequence of surjections of finite sets. We can define two maps

$$\mathcal{M}^{\boxtimes R} \to \mathfrak{i}_q \mathfrak{i}_p \mathcal{M}^{\boxtimes S}.$$

The first one is induced by the map $\mathfrak{as}_{q,p}$:

$$\mathcal{M}^{\boxtimes R} \xrightarrow{\mathfrak{ope}_{pq}} \mathfrak{i}_{pq}\mathcal{M}^{\boxtimes S} \xrightarrow{\mathfrak{as}_{q,p}} \mathfrak{i}_q\mathfrak{i}_p\mathcal{M}^{\boxtimes S};$$

the second one is defined by:

$$\mathcal{M}^{\boxtimes_R} \xrightarrow{\mathfrak{ope}_q} \mathfrak{i}_q(\mathcal{M}^T) \xrightarrow{\mathfrak{ope}_p} \mathfrak{i}_q\mathfrak{i}_p(\mathcal{M}^{\boxtimes S}).$$

The axiom is that

Axiom 2.1. *These two maps should coincide.*

3. Additional Features

It turns out that the procedure of renormalization depends on an additional structure possessed by the system $\mathfrak{i}_p$, which we are going to introduce. The importance of this structure is not restricted to the renormalization. The author believes that this structure also plays a key role in formulation of the quasiclassical correspondence principle and in the connection between the Hamiltonian and Lagrangian formalism. Thus, let us describe this structure.

3.1. *Preparation*

3.1.1. *The system* $\mathfrak{l}$

Let $\mathfrak{lie}^1$ be the operad which describes Lie algebras with the bracket of degree 1. Let $\mathfrak{L}(S) := \mathfrak{lie}^1(S)^*$ be the linear dual, here S is a finite set. Let $p : S \to T$ be a map of finite sets. Set

$$\mathfrak{L}(p) := \otimes_{t\in T}\mathfrak{L}(p^{-1}t).$$

We then have maps $\mathfrak{L}(p_1)\otimes\mathfrak{L}(p_2) \to \mathfrak{L}(p_1 \sqcup p_2)$ and $\mathfrak{L}(rq) \to \mathfrak{L}(r)\otimes\mathfrak{L}(q)$, where $p_i : S_i \to T_i$; $r : T \to R$; $q : S \to T$ are maps of finite sets.

Now let $p : S \to T$ be a surjection. Set $\mathfrak{l}_p : \mathcal{D}\text{-sh}_{X^T} \to \mathcal{D}\text{-sh}_{X^S}$; $\mathfrak{l}_p = (i_{p*}) \otimes \mathfrak{L}(p)$, where $i_p : X^S \to X^T$ is an embedding determined by p and i_{p*} is the correspondent $\mathcal{D}$-module theoretic direct image. We then have natural maps

$$\mathfrak{l}_{p_1}(M_1) \boxtimes \mathfrak{l}_{p_2}(M_2) \to \mathfrak{l}_{p_1\sqcup p_2}(M_1 \boxtimes M_2)$$

and $\mathfrak{l}(rq) \to \mathfrak{l}(q)\mathfrak{l}(r)$, where $p_i : S_i \to T_i$; $r : T \to R$; $q : S \to T$ are maps of finite sets and $M_i \in \mathcal{D}_{X^{S_i}}$. These maps are induced by the correspondent maps for $\mathfrak{L}$.

Thus, the functors $\mathfrak{l}$ possess the structure which is similar to the one on $\mathfrak{i}$. One sees that all the properties for $\mathfrak{i}$ stated in 2.3.3 remain true upon substituting $\mathfrak{l}$ for $\mathfrak{i}$. In other words, $\mathfrak{l}$ form a system which we denote by $\langle\mathfrak{l}\rangle$.

An OPE-algebra structure over the system $\langle \mathfrak{l} \rangle$ on a $\mathcal{D}_X$-module M is equivalent to a *-Lie structure on $M[-1]$ as defined in [5]. Let us recall the definition.

3.1.2. *Definition of *-Lie algebra structure*

A *-Lie stucture on a $\mathcal{D}_X$-module M is given by an antisymmetric map $b : M \boxtimes M \to i_* M$, where $i : X \to X \times X$ is the diagonal embedding. The bracket b is supposed to satisfy an analogue of Jacobi identity.

3.1.3. *Quasi-isomorphisms of systems*

Let $\langle \mathfrak{i} \rangle, \langle \mathfrak{j} \rangle$ be systems and let $F : \langle \mathfrak{i} \rangle \to \langle \mathfrak{j} \rangle$ be a morphism of systems. F is a quasi-isomorphism if for every free $\mathcal{D}_{X^S}$-module M the induced map $\mathfrak{j}_p(M) \to \mathfrak{i}_p(M)$ is a quasi-isomorphism for every surjection $p : T \to S$.

3.1.4. *Definition of additional structure I*

The most important part of our additional structure can then be described as a choice of quasi-isomorphisms $\langle \mathfrak{N} \rangle \xrightarrow{\sim} \langle \mathfrak{i} \rangle$, $\langle \mathfrak{l} \rangle \xrightarrow{\sim} \langle \mathfrak{m} \rangle$ and a map of systems $\langle \mathfrak{N} \rangle \to \langle \mathfrak{m} \rangle$:

$$\begin{array}{ccc} \langle \mathfrak{i} \rangle & & \langle \mathfrak{l} \rangle \\ \uparrow & & \downarrow \\ \langle \mathfrak{N} \rangle & \longrightarrow & \langle \mathfrak{m} \rangle \end{array}$$

There is even more structure on $\langle \mathfrak{N} \rangle$ which we shall use. This part is of some importance, but not of principal importance, and will be discussed later (see Sec. 4.1.3).

In the rest of Part I we ignore homotopy-theoretical complications and assume that we have a map systems $\langle \mathfrak{i} \rangle \to \langle \mathfrak{l} \rangle$ (this helps to explain the ideas in a simper way). A precise exposition will be given in the subsequent parts of the paper. Let us now discuss a motivation for the introduced additional structure.

3.2. *Physical meaning*

Physical meaning of the introduced additional structure can be seen from examining the case when $p : S \to \text{pt}$, where $S = \{1, 2\}$ is a two-element set. As a part of our structure, we have a map

$$\mathfrak{i}_p \to i_{p*} \otimes \mathfrak{L}(S).$$

But $\mathfrak{L}(S) = k[1]$, therefore, we simply get a map

$$\mathfrak{i}_p \to i_{p*}. \tag{4}$$

Recall that

$$\mathfrak{i}_p(M) = i^{\wedge}(M) \otimes_{\mathcal{O}_{X^S}} \mathcal{B}_S$$

and one can show that

$$i_{p*}(M) = i^{\wedge}(M) \otimes_{\mathcal{O}_{X^S}} i_{p*}\mathcal{O}_X.$$

Assume for simplicity that the map (4) is induced in a natural way by a degree +1 map

$$\mathcal{B}_2 \to i_{p*}\mathcal{O}_X \tag{5}$$

(we keep in mind the above identifications).

Such a map specifies an extension $\mathcal{C}_S$ fitting into exact sequence:

$$0 \to i_{p*}\mathcal{O}_X \to \mathcal{C}_S \to \mathcal{B}_S.$$

The meaning of $\mathcal{C}_S$ becomes clear, if we come back to the real (versus complex) picture. The global sections of $\mathcal{B}_S$ produce functions on the real part Y^S with singularities on the diagonal. A global section of $\mathcal{C}_S$ then has a meaning of distribution on Y^2, whose restriction onto the complement $Y^2\backslash Y$ is a function from $\mathcal{B}_S$. If we take the space C' of all such distributions, we shall get a slightly larger extensions as the kernel $C' \to \mathcal{B}_S$ consists of all distributions supported on the diagonal, which is larger than $i_{p*}\mathcal{O}_X$. Nevertheless, it turns out that the space of global sections of $\mathcal{C}_S$ can be defined as a subspace of C' (see 10.1).

Set $\mathcal{I}_S : \mathcal{D}_X \to \mathcal{D}_{X^S}$ to be

$$\mathcal{I}_S(M) = i^{\wedge}(M) \otimes_{\mathcal{O}_{X^S}} \mathcal{C}_S.$$

For good M (say flat as $\mathcal{O}_X$-modules), we have an exact sequence

$$0 \to i_{p*}(M) \to \mathcal{I}_p(M) \to \mathfrak{i}_p(M) \to 0.$$

Now let M be an *OPE*-algebra over $\langle i \rangle$. In particular, we have a map

$$M^{\boxtimes S} \to \mathfrak{i}_p(M).$$

We may now interpret the composition

$$M^{\boxtimes S} \to \mathfrak{i}_p(M) \to i_{p*}(M)$$

of this map with the map (4). Assume that M is a complex of free $\mathcal{D}_X$ modules (bounded from above). Then we can lift the OPE-map to a map

$$\mu : M^{\boxtimes S} \to \mathcal{I}_p(M)$$

with a nonzero differential and the desired composition is equal to $d\mu$. The procedure of lifting from $\mathfrak{i}_p$ to $\mathcal{I}_p$ is nothing else but the *regularization of divergences.* The map μ has the meaning of the commutative product in the Batalin–Vilkovitski formalism. Its differential then has a meaning of the Shouten bracket in the same formalism. This simple physical argument suggests that the map $d\mu$ should be a *-Lie bracket of degree +1.

3.3. *Geometrical meaning*

We will hint at the geometric meaning of the additional structure on $\langle \mathfrak{i} \rangle$. Since our intention is just to give a motivation, the arguments will not be rigorous.

Recall that the functors $\mathfrak{i}$ have been constructed using the $\mathcal{D}_{X^S}$-modules $\mathcal{B}_S$, which are defined as sheaves of functions on certain affine varieties Z_S. Therefore, the de Rham complex of $\mathcal{B}_S$ computes the cohomology of Z_S shifted by $\dim Z_S = 2ns$, where $2n = \dim_{\mathbb{C}} X$ and $s = \#S$. Let $\mathcal{B}'_S := \mathfrak{i}_{\pi_{p_S}}(\mathcal{O}_X) = i^{\wedge}_{\pi_{p_S}}(\mathcal{O}_X) \otimes \mathcal{B}_S$, where $\pi_{p_S} : S \to \mathrm{pt}$ is the map onto a point. The de Rham complex of $\mathcal{B}'_S$ computes the cohomology of the intersection of Z_S with a very small neighborhood of the diagonal $X \subset X^S$.

On the other hand, Z_S contains as its real part the space $Z^r_S := Y^S \backslash (\cup_{s \neq t} \Delta_{st})$, where Δ_{st} is the corresponding diagonal. Thus we have a map from the de Rham cohomology of $\mathcal{B}'_S$ to the cohomology of the intersection of Z^r_S with a very small neighborhood of the diagonal $Y \subset Y^S$ in Y^S which can be easily seen to be the same as the cohomology of Z^r_S. It is well known that $H^{(2n-1)(s-1)}(Z^r_S) \cong \mathfrak{L}(S)[1-s]$, where $2n = \dim Y$ and $s = \#S$. The shift on the right-hand side is made in such a way that both sides have degree zero.

Let us slightly change our point of view. Instead of taking the full de Rham complex, let us pick a point $\sigma \in S$ and let $p_\sigma : X^S \to X$ be the projection onto the correponding component. Let $p_{\sigma *}(\mathcal{B}'_S)$ be the fiber-wise de Rham complex shifted by the dimension of the fiber (in this case $H^0 p_{\sigma *}$ is the usual $\mathcal{D}$-module theoretic direct image).

We see that the induced map $Z_S \to X$ is a trivial fibration whose fiber F_S is homotopy equivalent to Z_S and $\dim F_S = \dim Z_S - 2n$. Let V be a small neighborhood of $X \subset X^S$, then

$$H^i(p_{\sigma *}(\mathcal{B}'_S)) \cong \mathcal{O}_X \otimes H^{2n(s-1)+i}(Z_S \cap V)$$

and we have a through map

$$\begin{aligned} H^{1-s}(p_{\sigma *}(\mathcal{B}_S)) &\cong \mathcal{O}_X \otimes H^{(2n-1)(s-1)}(Z_S \cap V) \\ &\to H^{(2n-1)(s-1)}(Z^r_S \cap V) \otimes \mathcal{O}_X \to \mathfrak{L}(s)[1-s] \otimes \mathcal{O}_X. \end{aligned}$$

Since $H^{>1-s}(p_{\sigma *}(\mathcal{B}_S)) = 0$, we have an induced map

$$p_{\sigma *}(\mathcal{B}'_S) \to \mathcal{O}_X \otimes \mathfrak{L}(S).$$

It is well known that this map induces a map $\mathcal{B}'_S \to i_* \mathcal{O}_X \otimes \mathfrak{L}(S)$ in the derived category of $\mathcal{D}_{X^S}$-sheaves. Thus, the top cohomology of the configuration spaces can be interpreted as maps $\mathcal{B}'_S \to i_* \mathcal{O}_X \otimes \mathfrak{L}(S)$. These maps can be extended to maps $\mathfrak{i}_p(M) \to \mathfrak{l}_p(M)$ in the derived category of $\mathcal{D}_{X^T}$-sheaves on X^T for every free $\mathcal{D}_{X^S}$-module M.

Of course, this argument is insufficient for constructing a map of systems (as opposed to a collection of maps of functors $\mathfrak{i}_p \to \mathfrak{l}_p$).

4. Renormalization. "Naive" Version

Here we will sketch a scheme for renormalization ignoring homotopy-theoretical problems. Although this naive scheme is of purely heuristic value, the correct renormalization scheme is in the same relation to the naive one as derived functors are to usual ones.

So, we shall simply assume that we are given a map $\langle \mathfrak{i} \rangle \to \langle \mathfrak{l} \rangle$.

We start with defining the main ingredients.

4.1. **-Lie structure on $M[-1]$*

Thus, we have a morphism of systems $\langle \mathfrak{i} \rangle \to \langle \mathfrak{l} \rangle$. Assume that M is an OPE algebra over $\mathfrak{i}$. Then it is also an OPE algebra over $\mathfrak{l}$, i.e. $M[-1]$ is a *-Lie algebra. Let $\pi : X \to \mathrm{pt}$ be the projection and denote by $\pi_* M$ the direct image of M;

$$\pi_* M := \omega_X \otimes_{\mathcal{D}_X} M.$$

We know that $\mathfrak{g} := \pi_* M[-1]$ is then a DGLA and this DGLA acts on M. Therefore, for every surjection of finite non-empty sets $p : S \to T$ we have a $\mathfrak{g}$-action on

$$\hom(M^{\boxtimes S}, \mathfrak{i}_p(M^{\boxtimes T})).$$

A very important question for us is whether the elements $\mathfrak{ope}_p$ are $\mathfrak{g}$-invariant. It turns out that in general the answer is no. We are going to impose an extra axiom which would guarantee this property.

4.1.1. *Extra axiom which ensures the $\mathfrak{g}$-invariance of $\mathfrak{ope}_p$*

Let $p : S \to T$ be a surjection of finite sets as above. Pick an arbitrary element $t \in T$; add one more element σ to S and let $p_t : S \sqcup \{\sigma\} \to T$ be a map which extends p in such a way that $p_t(\sigma) = t$. (This extra element is needed to take into account the $\mathfrak{g}$-action.) Let $I : S \to S \sqcup \{\sigma\}$ be the inclusion and let $P : X^{S \sqcup \{\sigma\}} \to X^S$ be the natural projection corresponding to I. Let P_* be the corresponding direct image. We are going to define several maps $P_* \mathfrak{i}_{p_t} \to \mathfrak{i}_p[1]$ as follows. Let $s \in S$ be such that $p(s) = t$. Let $P_s : S \sqcup \sigma \to S$ be the map which is identity on S and $P_s \sigma = s$. Then $p_t = p P_s$. We then have the following composition:

$$P_* \mathfrak{i}_{p_t} \to P_* \mathfrak{i}_{P_s} \mathfrak{i}_p \to P_* \mathfrak{l}_{P_s} \mathfrak{i}_p.$$

Note that

$$\mathfrak{l}_{P_s} \cong i_{P_s *} \otimes_{s' \in S} \mathfrak{L}(P_s^{-1} s') \cong i_{P_s *}[1].$$

Thus, we can continue our composition:

$$P_* \mathfrak{l}_{P_s} \mathfrak{i}_p \to P_* (i_{P_s *}[1]) \mathfrak{i}_p \to \mathfrak{i}_p[1],$$

where we used the natural map

$$P_* i_{P_s *} \to \mathrm{Id}_{X^S}.$$

Let

$$A_s : P_* \mathfrak{i}_{p_t} \to \mathfrak{i}_p[1]$$

be the resulting composition.

There is one more way to decompose p_t. Let $Q := p \sqcup \mathrm{Id} : S \sqcup \{\sigma\} \to T \sqcup \{\sigma\}$. Let $R : T \sqcup \{\sigma\} \to T$ be the identity on T and let $R(\sigma) = t$. Then again $p_t = RQ$. Therefore, we have a composition:

$$P_* \mathfrak{i}_{p_t} \to P_* \mathfrak{i}_Q \mathfrak{i}_R.$$

Let $P_T : X^{T \sqcup \{vs\}} \to X^T$ be the natural projection. It is not hard to see that we have an isomorphism

$$P_* \mathfrak{i}_Q \to \mathfrak{i}_P P_{T*}.$$

Thus, we continue as follows:

$$P_* \mathfrak{i}_Q \mathfrak{i}_R \to \mathfrak{i}_p P_{T*} \mathfrak{i}_R \to \mathfrak{i}_p P_{T*} \mathfrak{l}_R \cong \mathfrak{i}_p P_{T*} i_{R*} \to \mathfrak{i}_p[1].$$

Denote the composition of these maps by

$$B_t : P_* \mathfrak{i}_{p_t} \to \mathfrak{i}_p[1].$$

Let $C_t = B_t - \sum_{s \in S, p(s) = t} A_s$. Let us show that the maps C_t determine the action of $\mathfrak{g}$ on $\mathfrak{as}_p$. Let $X \in \mathfrak{g}$. Let $L(X) := X.\mathfrak{ope}_p$; $L : \mathfrak{g} \to \hom(M^{\boxtimes S}, \mathfrak{i}_p(M^{\boxtimes T}))$.

Claim 4.1. *L is equal to the following composition:*

$$\begin{aligned} \mathfrak{g} &\xrightarrow{\oplus_{t\in T} \mathfrak{as}_{p_t}} \oplus_{t \in T} \mathfrak{g} \otimes \hom(M^{\boxtimes S \sqcup \{\sigma\}}, \mathfrak{i}_{p_t}(M^{\boxtimes T})) \\ &\to \oplus_{t \in T} \mathfrak{g} \otimes \hom(P_* M^{S \sqcup \{\sigma\}}, P_* \mathfrak{i}_{p_t}(M^{\boxtimes T})) \\ &\cong \oplus_{t \in T} \mathfrak{g}[1] \otimes \hom(\mathfrak{g}[1] \otimes M^{\boxtimes S}, P_* \mathfrak{i}_{p_t}(M^{\boxtimes T})) \\ &\to \oplus_{t \in T} \hom(M^{\boxtimes S}[1], P_* \mathfrak{i}_{p_t}(M^{\boxtimes T})) \xrightarrow{\oplus C_t} \hom(M^{\boxtimes S}[1], \mathfrak{i}_p(M^{\boxtimes T})[1]). \end{aligned}$$

Therefore, if $C_t = 0$ for all t, then $L = 0$.

Proof. Straightforward. □

4.1.2. Call a system $\langle \mathfrak{i} \rangle$ endowed with a map $\langle \mathfrak{i} \rangle \to \langle \mathfrak{l} \rangle$ *invariant* if all $C_t = 0$.

4.1.3. *Another axiom*

It turns out that to construct a good theory one has to introduce a one more natural axiom on $\langle \mathfrak{i} \rangle$. The importance of this axiom can be fully appreciated only when one passes to a more precise consideration.

Let us describe this axiom. Let $A = \{1, 2\}$ be a two-element set. Let $q : A \to \mathrm{pt}$. Let $p : S \to T$ be a surjection. Let $i : \mathrm{pt} \to A$ be the inclusion in which a unique element pt goes to 1. For an arbitrary injection $j : U \to V$ let $p_j : X^V \to X^U$ be the corresponding projection and $\mathfrak{p}_i : \mathcal{D}\text{-sh}_U \to \mathcal{D}\text{-sh}_V$ be the corresponding $\mathcal{D}$-module-theoretic direct image.

We then construct two maps

$$\mathfrak{p}_{\mathrm{Id}_S \sqcup i} \mathfrak{i}_{p \sqcup q} \to \mathfrak{i}_{p \sqcup \mathrm{Id}_{\mathrm{pt}}}.$$

The first map is as follows:

$$\begin{aligned} M_I : \mathfrak{p}_{\mathrm{Id}_S \sqcup i} \mathfrak{i}_{p \sqcup q} &\to \mathfrak{p}_{\mathrm{Id}_S \sqcup i} \mathfrak{i}_{\mathrm{Id}_S \sqcup q} \mathfrak{i}_{p \sqcup \mathrm{Id}_{\mathrm{pt}}} \\ &\to \mathfrak{p}_{\mathrm{Id}_S \sqcup i} \delta_{\mathrm{Id}_S \sqcup q} \mathfrak{i}_{p \sqcup \mathrm{Id}_{\mathrm{pt}}} \cong \mathfrak{i}_{p \sqcup \mathrm{Id}_{\mathrm{pt}}} \end{aligned}$$

and the second one is:

$$\begin{aligned} M_{II} : \mathfrak{p}_{\mathrm{Id}_S \sqcup i} \mathfrak{i}_{p \sqcup q} &\to \mathfrak{p}_{\mathrm{Id}_S \sqcup i} \mathfrak{i}_{p \sqcup \mathrm{Id}_{\mathrm{pt}}} \mathfrak{i}_{\mathrm{Id}_S \sqcup q} \\ &\to \mathfrak{p}_{\mathrm{Id}_S \sqcup i} \mathfrak{i}_{p \sqcup \mathrm{Id}_{\mathrm{pt}}} \delta_{\mathrm{Id}_S \sqcup q} \cong \mathfrak{i}_{p \sqcup \mathrm{Id}_{\mathrm{pt}}} \\ &\to \mathfrak{i}_{p \sqcup \mathrm{Id}_{\mathrm{pt}}} \mathfrak{p}_{\mathrm{Id}_S \sqcup i} \delta_{\mathrm{Id}_S \sqcup q}, \end{aligned}$$

where we have used a natural isomorphism

$$\mathfrak{p}_{\mathrm{Id}_S \sqcup i} \mathfrak{i}_{p \sqcup \mathrm{Id}_{\mathrm{pt}}} \cong \mathfrak{i}_{p \sqcup \mathrm{Id}_{\mathrm{pt}}} \mathfrak{p}_{\mathrm{Id}_S \sqcup i}.$$

Call a system $\langle \mathfrak{i} \rangle$ endowed with a map $\langle \mathfrak{i} \rangle \to \langle \mathfrak{l} \rangle$ to be *pre-symmetric* if $M_I = M_{II}$ for all p.

Finally, call a system *symmetric*, if it is both pre-symmetric and invariant.

4.1.4. What is the situation with the system $\langle \mathfrak{R} \rangle$ that we are going to construct? It turns out, that upto homotopies, it is pre-symmetric, but not symmetric. Pre-symmetricity is the additional structure on $\langle \mathfrak{R} \rangle$ which was mentioned in (3.1.4).

The above reasoning suggests that renormalization is only possible in symmetric (or, at least, invariant systems). Therefore, a procedure of "fixing" $\langle \mathfrak{R} \rangle$ (which we call "symmetrization") is needed to perform a renormalization. We shall discuss a naive version of such a symmetrization after a more detailed explanation how renormalization goes on in a symmetric system.

4.2. *Renormalization in a symmetric system*

As was mentioned, the system $\langle \mathfrak{R} \rangle$ that we will construct in the example of free boson is not symmetric. Nevertheless, to appreciate the importance of symmetricity, we will explain in the next section that were $\langle\mathfrak{R}\rangle$ symmetric, the renormalization of any OPE-algebra over $\langle\mathfrak{R}\rangle$ could be defined in a very simple fashion.

Let M be an OPE-algebra in a *symmetric* system $\langle\mathfrak{i}\rangle$. Then, by virtue of the map $\mathfrak{i}\to\mathfrak{l}$, M is also an OPE-algebra in $\mathfrak{l}$, i.e. $M[-1]$ is a *-Lie algebra. Let $\pi_p : X\to \mathrm{pt}$ be the projection onto a point. Then $\pi_{p*}M[1]$ is a DGLA. Let λ be a formal variable (the "interaction constant"). Pick a Maurer–Cartan element

$$\mathfrak{S}\in\lambda(\pi_{p*}M[-1])[[\lambda]]^1=\lambda\pi_{p*}M[[\lambda]];\quad d\mathfrak{S}+1/2[\mathfrak{S},\mathfrak{S}]=0.$$

This equation is called *quantum Master equation.* Using $\mathfrak{S}$ we can perturb the differential on M; let $M' := (M[[\lambda]], d+[\mathfrak{S},\cdot])$ be the corresponding differential graded $\mathcal{D}_{X[[\lambda]]}$-module.

The renormalization is the procedure of constructing a $\mathbb{C}[[\lambda]]$-linear OPE structure over $\langle\mathfrak{i}\rangle$ on M'. In our setting this procedure is trivial. Indeed, since $M' = M[[\lambda]]$ as graded objects; the OPE structure on M gives rise to the maps

$$\mathfrak{ope}'_p : (M')^{\boxtimes_{\mathbb{C}[[\lambda]]}S}\to\mathfrak{i}_p((M')^{\boxtimes_{\mathbb{C}[[\lambda]]}T}).$$

The $\mathfrak{l}$-invariance of $\mathfrak{i}$ and Claim 4.1 imply that these maps are compatible with the differential on M'. Thus, $\mathfrak{ope}'_p$ do define the renormalized OPE on M'.

4.3. *An idea how to fix non-invariance of $\langle\mathfrak{i}\rangle$: Symmetrization*

Let us try to define a system $\mathfrak{i}^{\mathrm{symm}}$ endowed with a map $\langle\mathfrak{i}\rangle\to\langle\mathfrak{i}^{\mathrm{symm}}\rangle$ such that in $\mathfrak{i}^{\mathrm{symm}}$ all $C_t=0$. Then our OPE-algebra M in $\langle\mathfrak{i}\rangle$ determines an OPE-algebra in $\langle\mathfrak{i}^{\mathrm{symm}}\rangle$ and the renormalization of this algebra goes the way as was described above.

The obvious way to define $\langle\mathfrak{i}^{\mathrm{symm}}\rangle$ is to simply put

$$\mathfrak{i}_p^{\mathrm{symm}} := \mathfrak{i}_p/\mathrm{Span}\langle \mathrm{Im} C_t\rangle_{t\in T}.$$

One checks that the structure of system on $\mathfrak{i}$ is naturally transferred onto $\mathfrak{i}_p^{\mathrm{symm}}$.

4.4. *Summary*

Let us first summarize what we have done.

We start with a system $\langle\mathfrak{i}\rangle$ which is quasi-isomorphic to the original system $\langle\mathfrak{i}\rangle$ and is endowed with a map $\langle\mathfrak{i}\rangle\to\langle\mathfrak{l}\rangle$. We then construct a symmetric system $\langle\mathfrak{i}^{\mathrm{symm}}\rangle$ which fits into the diagram $\langle\mathfrak{i}\rangle\to\langle\mathfrak{i}^{\mathrm{symm}}\rangle\to\langle\mathfrak{l}\rangle$. Thereafter, having an OPE algebra M over $\mathfrak{i}$, we observe that $\pi_{p*}M[-1]$ is a DGLA and we pick a Maurer–Cartan element $\mathfrak{S}\in\lambda\pi_{p*}M[[\lambda]][1]$. We then define the $\mathcal{D}_{X[[\lambda]]}$-module M' and define an OPE structure on M' over $\mathfrak{i}^{\mathrm{symm}}$.

What has to be done for this scheme to really work?

Problem 1. We need to construct $\mathfrak{i}$ with the specified properties.

Problem 2. We have an OPE algebra M over $\langle \mathfrak{i} \rangle$ and a quasi-isomorphism $\langle \mathfrak{R} \rangle \to \langle \mathfrak{i} \rangle$. We need to lift M to an OPE algebra over $\langle \mathfrak{R} \rangle$.

Problem 3. The passage from $\langle \mathfrak{R} \rangle$ to $\langle \mathfrak{R}^{\mathbf{symm}} \rangle$ is not stable under quasi-isomorphism of systems. Thus we need to develop a derived version of the map $\langle \mathfrak{R} \rangle \mapsto \langle \mathfrak{R}^{\mathbf{symm}} \rangle$.

Problem 4. After all, we get a renormalized OPE-algebra in an abstract system $\langle \mathfrak{R}^{\mathbf{symm}} \rangle$. To give a physical meaning to this system, we have to find a construction which transforms this OPE-algebra into OPE-products in terms of series of real-analytic functions on Y^S.

4.5. *Plan for the future exposition*

The rest of the paper is devoted to solving these problems. As this involves a lot of technicalities, we shall first retell the content of the paper omitting them. Then the detailed exposition, with proofs, will follow.

First, we shall formulate the list of properties that the system $\langle \mathfrak{R} \rangle$, to be constructed, should possess. These properties form a homotopical variant of the definition of the structure of pre-symmetric system. Every system possessing these properties will be called pre-symmetric (this should not lead to confusion with the naive definition of pre-symmetricity).

Secondly, we shall show how the renormalization can be carried over for OPE-algebras over a pre-symmetric system $\langle \mathfrak{R} \rangle$ (including a construction for symmetrization of $\langle \mathfrak{R} \rangle$ and a construction of the renormalized OPE-algebra over the symmetrized system). These steps constitute a homotopically correct version of the above outlined naive approach. Thereafter, we construct a pre-symmetric system $\langle \mathfrak{R} \rangle$ which is a resolution of the system $\langle \mathfrak{i} \rangle$.

To renormalize an OPE-algebra over $\langle \mathfrak{i} \rangle$ one has to be able to lift it to an OPE-algebra over $\langle \mathfrak{R} \rangle$ so that the lifting be compatible with the quasi-isomorphism of systems $\langle \mathfrak{R} \rangle \to \langle \mathfrak{i} \rangle$. This happens to be a variant of the celebrated Bogoliubov–Parasyuk theorem, saying that such a lifting is always possible. An analogous theorem can be shown by a homotopy-theoretical nonsense, using the quasi-isomorphicity of the map $\langle \mathfrak{R} \rangle \to \langle \mathfrak{i} \rangle$; but for this to work one has to replace the stucture of OPE-algebra upto higher homotopies. Let us stress that Bogoliubov–Parasyuk theorem produces a lifting of usual OPE-algebras, which is a stronger statement. Homotopical approach, on the other hand, provides for a homotopical equivalence of two different liftings. These homotopy-theoretic questions will be discussed in a subsequent paper.

Finally, we solve Problem 4.

The exposition will be organized in such a way that the most difficult technical moments will be omitted at the "first reading", which is Part II, and will be discussed at the "second reading" (i.e. the concluding Part III).

Part II: Exposition Without Technicalities

We shall pass from a naive approach to the realistic one, in which the naive definitions sketched above will be replaced with appropriate homotopically correct versions.

Our plan is as follows. In the following section we give a homotopically correct definition of pre-symmetric system.

Next we show how, having an OPE-algebra in a pre-symmetric system, one can renormalize it.

Next we have to show these definitions work in the example of free scalar boson. The major part of the required work is done in Part III, in this part we only sketch the main steps which are:

(1) We have to construct a pre-symmetric system $\langle\mathcal{R}\rangle$ which maps quasi-isomorphically to the system $\langle\mathfrak{i}\rangle$;

(2) We have to show that every OPE-algebra over $\langle\mathfrak{i}\rangle$ lifts to an OPE-algebra over $\langle\mathcal{R}\rangle$.

Having done this we can apply the symmetrization and renormalization procedures.

(3) And finally, we need to be able to interpret the renormalized OPE in the symmetrized system in terms of expansions whose coefficients are real-analytic functions on Y^n without diagonals.

So, let us follow our plan.

5. Pre-Symmetric Systems

In this section we shall give a homotopy version of the notion of pre-symmetric system (see 4.1.3 for naive version).

The plan is as follows. We shall give two slightly different (and slightly non-equivalent) definitions of a homotopy analog of a pre-symmetric system. Any pre-symmetric system in the sense of the first definition will naturally produce a pre-symmetric system in the sense of the second definition. The first definition is given in terms of functors $\mathcal{R}_p, \delta_p$, in the second definition we replace the functors δ_p with functors of direct image with respect to all projections $X^S \to X^T$. We will see that the second definition looks more natural. Moreover, the second definition encloses all the structures needed for symmetrization and renormalization. So, we consider the second definition as a more basic one. On the other hand, to define a pre-symmetric system in the example of free boson, we shall use the first definition.

We start with formulation of the first definition. First of all, we need to provide for a homotopy-theoretical analog of a map $\langle\mathfrak{i}\rangle \to \langle\mathfrak{l}\rangle$. This will be achieved via replacement of $\langle\mathfrak{l}\rangle$ with a quasi-isomorphic system $\langle\mathfrak{l}\rangle \xrightarrow{\sim} \langle\mathfrak{m}\rangle$. We shall give the definition of such an $\langle\mathfrak{m}\rangle$. A part of a structure of pre-symmetric system on a system $\langle\mathcal{R}\rangle$ will then be a map $\langle\mathcal{R}\rangle \to \langle\mathfrak{m}\rangle$. As was mentioned in 4.1.3, to be pre-symmetric, the system $\langle\mathcal{R}\rangle$ should have additional properties. We will give their homotopical versions. This will accomplish the first definition of a pre-symmetric system. Finally, we formulate the second definition (which is essentially a paraphrasing of the first definition in terms of direct image functors with respect to projections), it will then follow automatically that every pre-symmetric system in the sense of the first definition gives rise to a pre-symmetric system in the sense of the second definition.

5.1. *A homotopy version of the map* $\langle \mathfrak{i} \rangle \to \langle \mathfrak{l} \rangle$

As was explained above, the first step we need to do is to endow the system $\langle \mathfrak{i} \rangle$ with a map of systems $\langle \mathfrak{i} \rangle \to \langle \mathfrak{l} \rangle$. We shall do it in a homotopical sense, i.e. we shall construct systems $\langle \mathcal{R} \rangle$ and $\langle \mathfrak{m} \rangle$ fitting into the following commutative diagram:

$$\begin{array}{ccc} \langle \mathfrak{i} \rangle & & \langle \mathfrak{l} \rangle \\ \sim \uparrow & & \downarrow \sim \\ \langle \mathcal{R} \rangle & \longrightarrow & \langle \mathfrak{m} \rangle \end{array}$$

The vertical arrows should be quasi-isomorphisms.

Let us first define the system $\langle \mathfrak{m} \rangle$.

5.1.1. *The system* $\langle \mathfrak{m} \rangle$

Let us define the complex $\mathfrak{m}_p$ centered in strictly negative degrees by setting

$$\mathfrak{m}_p^{-n} = \oplus \delta_{p_1} \delta_{p_2} \cdots \delta_{p_n}, \tag{6}$$

where the direct sum is taken over all diagrams

$$S \xrightarrow{p_1} S/e_1 \xrightarrow{p_2} S/e_2 \xrightarrow{p_3} \cdots \xrightarrow{p_{n-1}} S/e_{n-1} \xrightarrow{p_n} T, \tag{7}$$

where $\omega > e_1 > e_2 > \cdots > e_n > e$, where e is the equivalence relation induced by p and p_i are natural projections. The differential is given by the alternated sum $d = D_1 - D_2 + \cdots + (-1)^n D_{n-1}$, where

$$D_i : \delta_{p_1} \delta_{p_2} \cdots \delta_{p_n} \to \delta_{p_1} \delta_{p_2} \cdots \delta_{p_{i-1}} \delta_{p_{i+1} p_i} \delta_{p_{i+2}} \cdots \delta_{p_n}$$

is induced by the isomorphism

$$\delta_{p_i} \delta_{p_{i+1}} \to \delta_{p_{i+1} p_i}.$$

The maps $\mathfrak{as}_{q,r} : \mathfrak{m}_{rq} \to \mathfrak{m}_q \mathfrak{m}_r$ are defined in the following natural way. Let $p = rq$. Let f be the equivalence relation on S determined by q and e be the equivalence relation determined by p so that $f > e$. One can assume that $S \xrightarrow{q} S/f \xrightarrow{r} S/e$.

The map $\mathfrak{as}_{q,r}$ restricted to

$$\delta_{p_1} \delta_{p_2} \cdots \delta_{p_n}$$

as in (6), (7), vanish unless there exists a k such that $e_k = f$, in which case it isomorphically maps this term into

$$(\delta_{p_1}\delta_{p_2}\cdots\delta_{p_k})(\delta_{p_{k+1}}\cdots\delta_{p_n}).$$

The factorization maps

$$\mathrm{fact}\colon \boxtimes_{a\in A}\,\mathfrak{m}_{p_a}(M_a)\to\mathfrak{m}_p(\boxtimes_a M_a)$$

are given by a "shuffle product". Here is the construction.

Fix direct summands of $\mathfrak{m}_{p_a}$:

$$\delta_{p_{1a}}\delta_{p_{2a}}\cdots\delta_{p_{n_a a}},$$

where $p_{ia}\colon S_{ia}\to S_{i+1a}$, and define the restriction of the factorization map onto them.

Define *a shuffle* as a sequence

$$\alpha := (a_1, a_2, \ldots, a_N),$$

where

– $a_k \in A$;

– a_k enters into the sequence $a_1, a_2, \ldots, a_N$ exactly n_k times.

Given such a shuffle, let $\alpha_k(a)$ be the number of times a enters into the subsequence $a_1, a_2, \ldots, a_k$.

Let

$$S_k^\alpha := \bigsqcup_{a\in A} S_{\alpha_k(a)a}.$$

Define the map

$$p_k^\alpha\colon S_k^\alpha\to S_{k+1}^\alpha$$

as

$$p_{\alpha_k(a)a}\bigsqcup_{a'\neq a}\mathrm{Id}_{S_{\alpha_k(a')a'}}.$$

We then have a natural map

$$\begin{aligned}\mathrm{fact}(\alpha)\colon &\boxtimes_{a\in A}\,\delta_{p_{1a}}\delta_{p_{2a}}\cdots\delta_{p_{n_a a}}(M_a)\\ &\xrightarrow{\sim}\delta_{p_1^\alpha}\delta_{p_2^\alpha}\cdots\delta_{p_N^\alpha}(\boxtimes_{a\in A}M_a)\to\mathfrak{m}_p(\boxtimes_a M_a).\end{aligned}$$

Set the restriction of the map fact onto

$$\boxtimes_{a\in A}\delta_{p_{1a}}\delta_{p_{2a}}\cdots\delta_{p_{n_a a}}(M_a)$$

to be equal to

$$\sum_\alpha(-1)^{\mathrm{sign}(\alpha)}\mathrm{fact}(\alpha),$$

where $\mathrm{sign}(\alpha)$ is the sign of the shuffle.

Denote by

$$l_p^{\mathfrak{m}} : \mathfrak{m}_p \to \delta_p[1]$$

the natural projection.

Then a map of systems $\langle \mathcal{R} \rangle \to \langle \mathfrak{m} \rangle$ is uniquely determined by the knowledge of compositions

$$l_p : \mathcal{R}_p \to \mathfrak{m}_p \to \mathfrak{l}_p. \tag{8}$$

In the sequel we will work with these maps rather than with the system $\langle \mathfrak{m} \rangle$.

5.1.2. *A quasi-isomorphism* $\langle \mathfrak{l} \rangle \to \langle \mathfrak{m} \rangle$

As a part of our program, we have to define a quasi-isomorphism $\langle \mathfrak{l} \rangle \to \langle \mathfrak{m} \rangle$. As it will not be used in the future, we shall give a very brief description.

It is not hard to see that the cohomology of any complex $\mathfrak{m}_p$ is concentrated in its lowest degree (i.e. $\#T - \#S$, where $p : S \to T$); and it is not hard to see that this cohomology is isomorphic to $\mathfrak{l}_p$, whence the maps $\mathfrak{l}_p \to \mathfrak{m}_p$. The axioms for a map of systems can be easily checked.

5.1.3. *First definition of pre-symmetric system*

As a part of the structure of a pre-symmetric system (in the sense of the first definition) we should include maps (8) which provide for a homotopy-theoretical substitute for a map of systems $\langle \mathfrak{i} \rangle \to \langle \mathfrak{l} \rangle$. To complete the definition we should add a structure which is a homotopical analog of properties 4.1.3. After we formulate this structure, we will formulate the axioms which should be satisfied by the elements of the structure. This will complete the first definition of a pre-symmetric structure.

We shall start with the most natural piece of structure. Let $\phi : S \to T$ and $g : A \to B$ be surjections. Then we should have a natural map

$$\delta_{\mathrm{Id}_S \sqcup \phi} \mathcal{R}_{\phi \sqcup \mathrm{Id}_B} \to \mathcal{R}_{\phi \sqcup \mathrm{Id}_A} \delta_{\mathrm{Id}_T \sqcup g}. \tag{9}$$

Such a natural map also exists if one replaces $\langle \mathcal{R} \rangle$ with $\langle \mathfrak{i} \rangle$.

Indeed:

$$\delta_{\mathrm{Id}_S \sqcup \phi} \mathfrak{i}_{\phi \sqcup \mathrm{Id}_B}(M) \cong i^{\wedge}_{\phi \sqcup g}(M) \otimes_{\mathcal{O}_{X^{S \sqcup A}}} (\mathcal{B}_\phi \boxtimes i_{g*} \mathcal{O}_{X^B}),$$

whereas

$$\mathfrak{i}_{\phi \sqcup \mathrm{Id}_A} \delta_{\mathrm{Id}_T \sqcup g}(M) \cong \mathcal{B}_{\phi \sqcup \mathrm{Id}_A} \otimes_{\mathcal{O}_{X^{S \sqcup A}}} i^{\wedge}_{\phi \sqcup \mathrm{Id}_A}((\mathcal{O}_{X^T} \boxtimes i_{g*} \mathcal{O}_{X_B}) \otimes_{\mathcal{O}_{X^{T \sqcup A}}} i^{\wedge}_{\mathrm{Id}_T \sqcup g}(M))$$

and we see that the right-hand side in (9) is the completion of the left-hand side, whence the desired map.

The corresponding map for $\mathcal{R}$ is constructed following the same principles.

The next piece of structure is more subtle and is given by a family of maps

$$\mathcal{R}_{\phi\sqcup g} \to \mathcal{R}_{\phi\sqcup \mathrm{Id}_A}\delta_{\mathrm{Id}_T\sqcup g},$$

where $\phi : S \to T$ and $g : A \to B$ are arbitrary surjections. The comparison of this additional structure with the naive structure will be given after we list the axioms satisfied by $\mathfrak{l}_p$ and $s(\phi, g)$. A pre-symmetric structure in the sense of the first definition is then a collection of maps $\mathfrak{l}_p$ and $s(\phi, g)$ satisfying the axioms formulated below.

5.2. *Axioms of the pre-symmetric system (in the sense of the first definition)*

5.2.1. *Properties of the maps* $\mathfrak{l}_p$

The properties of $\langle \mathfrak{l}_p\rangle$ we are going to simply express the fact that the collection of maps $\mathfrak{l}_p$ should define a map of systems $\langle R\rangle \to \langle \mathfrak{m}\rangle$.

Property 1. If p is a bijection, then $\mathfrak{l}_p = 0$.

Property 2. Let $f_i : S_i \to T_i$ be nontrivial surjections. Then the composition

$$\mathcal{R}_{f_1}(M_1) \boxtimes \mathcal{R}_{f_2}(M_2) \to \mathcal{R}_{f_1\sqcup f_2}(M_1 \boxtimes M_2) \xrightarrow{\mathfrak{l}_{f_1}\boxtimes \mathfrak{l}_{f_2}} \delta_{f_1\sqcup f_2}(M_1 \boxtimes M_2)$$

is zero.

If f_1 is a bijection, then the above composition equals

$$\mathcal{R}_{f_1}(M_1) \boxtimes \mathcal{R}_{f_2}(M_2) \xrightarrow{a\boxtimes \mathfrak{l}_{f_2}} \delta_{f_1}(M_1) \boxtimes \delta_{f_2}(M_2) \to \delta_{f_1\sqcup f_2}(M_1 \boxtimes M_2),$$

where we used the isomorphism $a : \mathcal{R}_{f_1} \to \delta_{f_1}$ for a bijective f_1.

Property 3. Define the differential $d\mathfrak{l}_p$.

Let $p : S \to T$ and let e be the equivalence relation on S determined by e. Let e_1 be a strictly finer nontrivial equivalence relation. Set $p_1 : S \to S/e_1$, $p_2 : S/e_1 \to T$ to be the natural projections so that $p = p_2p_1$. Set

$$\mathfrak{l}(e_1) : \mathcal{R}_p \to \mathcal{R}_{p_1}\mathcal{R}_{p_2} \xrightarrow{\mathfrak{l}_{p_1},\mathfrak{l}_{p_2}} \delta_{p_1}\delta_{p_2} \cong \delta_p.$$

We then have

$$dl_p + \sum_{e_1} \mathfrak{l}(e_1) = 0,$$

where the sum is taken over all nontrivial equivalence relations on S which are strictly finer than e.

5.2.2. *Properties of maps* $s(\phi, g)$

Property 1. The following diagram is commutative:

$$\begin{array}{ccc} \mathcal{R}_{\phi\sqcup g\sqcup h} & \xrightarrow{s(\phi,g\sqcup h)} & \mathcal{R}_{\phi\sqcup g\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h} \\ & \searrow^{s(\phi\sqcup g,h)} & \downarrow{\scriptstyle s(\phi,g)} \\ & & \mathcal{R}_{\phi\sqcup \mathrm{Id}\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g\sqcup h} \end{array}$$

Property 2. Assume that ϕ is not bijective. Then the composition

$$\mathcal{R}_{\phi\sqcup g} \xrightarrow{s(\phi,g)} \mathcal{R}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \xrightarrow{\iota_{\phi\sqcup\mathrm{Id}}} \delta_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$$

equals

$$\mathcal{R}_{\phi\sqcup g} \xrightarrow{\iota_{\phi\sqcup g}} \delta_{\phi\sqcup g}.$$

If ϕ is bijective and g is not, then the above composition vanishes.

If both ϕ and g are bijections, then the above composition equals the natural identification of the right and left-hand sides.

Property 3. Let $g = g_2 g_1$, where g_1, g_2 are surjections. Introduce a map

$$\begin{aligned} K(\phi_1, \phi_2, g_1, g_2) : \mathcal{R}_{\phi_2\phi_1\sqcup g_2 g_1} &\to \mathcal{R}_{\phi_1\sqcup g_1}\mathcal{R}_{\phi_2\sqcup g_2} \\ &\xrightarrow{s(\phi_1,g_1),s(\phi_2,g_2)} \mathcal{R}_{\phi_1\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{R}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2} \xrightarrow{(9)} \mathcal{R}_{\phi_1\sqcup\mathrm{Id}}\mathcal{R}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2 g_1}. \end{aligned}$$

The property then says: The map

$$\mathcal{R}_{\phi_2\phi_1\sqcup g} \xrightarrow{s(\phi_2\phi_1,g)} \mathcal{R}_{\phi_2\phi_1\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \to \mathcal{R}_{\phi_1\sqcup\mathrm{Id}}\mathcal{R}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$$

is equal to

$$\sum_{g_2 g_1 = g} K(\phi_1, \phi_2, g_1, g_2),$$

where the sum is taken over all diagrams

$$A \xrightarrow{g_1} A/e_1 \xrightarrow{g_2} B, \tag{10}$$

where e_1 is an arbitrary equivalence relation on A such that g passes through A/e, and g_1, g_2 are the natural surjections.

Property 4. The following diagram is commutative:

$$\begin{array}{ccc} \mathcal{R}_{f_1\sqcup g_1}(M_1) \boxtimes \mathcal{R}_{f_2\sqcup g_2}(M_2) & \xrightarrow{s(\phi_1,g_1)\boxtimes s(\phi_2,g_2)} & \mathcal{R}_{f_1\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}(M_1) \boxtimes \mathcal{R}_{f_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2}(M_2) \\ \downarrow & & \downarrow \\ \mathcal{R}_{f_1\sqcup f_2\sqcup g_1\sqcup g_2}(M_1 \boxtimes M_2) & \xrightarrow{s(f_1\sqcup f_2,g_1\sqcup g_2)} & \mathcal{R}_{f_1\sqcup f_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1\sqcup g_2}(M_1 \boxtimes M_2). \end{array}$$

Property 5. Denote

$$s(g_1,\phi,g_2)\colon \mathcal{R}_{\phi\sqcup g} \to \mathcal{R}_{\mathrm{Id}\sqcup g_1}\mathcal{R}_{\phi\sqcup g_2} \xrightarrow{\mathfrak{l}_{\mathrm{Id}\sqcup g_1}} \delta_{\mathrm{Id}\sqcup g_1}\mathcal{R}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2} \xrightarrow{(9)} \mathcal{R}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g};$$

$$s(\phi,g_1,g_2)\colon \mathcal{R}_{\phi\sqcup g} \to \mathcal{R}_{\phi\sqcup g_1}\mathcal{R}_{\mathrm{Id}\sqcup g_2} \xrightarrow{\mathfrak{l}_{\mathrm{Id}\sqcup g_2}} \mathcal{R}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathfrak{g}_1}\delta_{\mathrm{Id}\sqcup g_2} \to \mathcal{R}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}.$$

The property asserts that

$$ds(\phi,g) = \sum_{g=g_2g_1} (s(g_1,\phi,g_2) - s(\phi,g_1,g_2)),$$

where the sum is taken over the same set as in (10).

5.2.3. *Comment on the meaning of* $s(\phi, g)$

To see this meaning consider a special $g\colon A \to \mathrm{pt}$, where $A = \{1,2\}$, and $\phi\colon S \to T$ is a surjection. Calculate the differential $ds(\phi, g)$.

It is equal to the difference $A - B$ of two maps, where

$$A\colon \mathcal{R}_{\phi\sqcup g} \to \mathcal{R}_{\phi\sqcup\mathrm{Id}}\mathcal{R}_{\mathrm{Id}\sqcup g} \xrightarrow{\mathfrak{l}_{\mathrm{Id}\sqcup g}} \mathcal{R}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$$

and

$$B\colon \mathcal{R}_{\phi\sqcup g} \to \mathcal{R}_{\mathrm{Id}\sqcup g}\mathcal{R}_{\phi\sqcup\mathrm{Id}} \to \delta_{\mathrm{Id}\sqcup g}\mathcal{R}_{\phi\sqcup\mathrm{Id}} \to \mathcal{R}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}.$$

Thus, the maps $s(\phi, g)$ provide for the difference $A - B$ to be homotopy equivalent to zero (upto higher homotopies).

Let $j\colon S \sqcup \{1\} \to S \sqcup \{1,2\}$ be the obvious inclusion. Composing $A - B$ with $\mathfrak{p}_j$, we see that $\mathfrak{p}_j A = M_I$, $\mathfrak{p}_j B = M_{II}$ as in Sec. 4.1.3. Thus the maps $s(f, g)$ are responsible for a homotopy analog of pre-symmetricity of $\langle\mathcal{R}\rangle$.

In the next subsection the above described structure will be reformulated in terms of functors of direct image with respect to projections. This will constitute a basis for further exposition.

5.3. *Reformulation in terms of direct images with respect to projections: Second definition of a pre-symmetric system*

Recall that the main ingredient in the renormalization procedure is an element of p_*M, where $p\colon X \to \mathrm{pt}$ is a projection. Thus we have to incorporate into our picture direct images with respect to projections. Let $i\colon S \to T$ be an injection. It induces a projection $p_i\colon X^T \to X^S$. Let $\mathfrak{p}_i\colon \mathcal{D}\text{-sh}_{X^T} \to \mathcal{D}\text{-sh}_{X^S}$ be the corresponding $\mathcal{D}$-module theoretic direct image. We want to incorporate it into our picture and to describe the maps which can be defined on superpositions of various $\mathcal{R}_p$ and $\mathfrak{p}_i$. These maps will be derived from the maps $\mathfrak{l}_p$ and $s(f, g)$. Note that the direct images with respect to injections are not applied, they are only used to *produce* maps between different iterations of $\mathcal{R}_p$ and $\mathfrak{p}_i$.

Thus, we shall now describe these maps and their properties.

5.3.1. The map we shall describe here is somewhat similar to (9).

Let $q : S \to T$ be an surjection and U be a finite set. Consider the following commutative diagram

$$\begin{array}{ccc} S \sqcup U & \overset{p}{\twoheadrightarrow} & T \sqcup U \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ S & \overset{q}{\twoheadrightarrow} & T \end{array} \tag{11}$$

where $p = q \sqcup \mathrm{Id}$, and i, j are the natural injections. Then we have an isomorphism

$$\mathfrak{p}_i \mathcal{R}_p \to \mathcal{R}_q \mathfrak{p}_j. \tag{12}$$

One can see that such an isomorphism is naturally defined, if we replace $\mathcal{R}$ with $\mathfrak{i}$.

5.3.2. Using the maps $\mathfrak{l}_p : \mathcal{R}_p \to \delta_p$, we can do the following.

Consider a commutative triangle

$$\begin{array}{ccc} S & \overset{p}{\twoheadrightarrow} & T \\ {\scriptstyle i}\uparrow & \overset{j}{\nearrow} & \\ R & & \end{array}$$

in which i, j are injections and p is a proper surjection. We then have a degree $+1$ map

$$L(i, p) : \mathfrak{p}_i \mathcal{R}_p \to \mathfrak{p}_j$$

given by

$$\mathfrak{p}_i \mathcal{R}_p \to \mathfrak{p}_i \delta_p \cong \mathfrak{p}_j.$$

5.3.3. Let us now "translate" $s(f, g)$ into our new language. Consider a commutative square

$$\begin{array}{ccc} R & \overset{p}{\twoheadrightarrow} & T \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ S & \overset{q}{\twoheadrightarrow} & P \end{array} \tag{13}$$

in which i, j are injections and p, q are proper surjections. Let $T_1 = T \backslash T_2$ be the subset of all $t \in T$ such that $p^{-1} t \cap i(S)$ consists of ≥ 2 elements.

Call such a square *suitable* if the following is satisfied:

$p^{-1}(T_1) \subset i(S)$, i.e.

$$\#(p^{-1} t \cap i(S)) \geq 2 \Rightarrow p^{-1}(t) \subset i(S).$$

We then have a degree zero map

$$A(i,p,j,q)\colon \mathfrak{p}_i\mathcal{R}_p \to \mathcal{R}_q\mathfrak{p}_j.$$

Construction: Decompose $T = T^1 \sqcup T^2$, where T^1 consists of all $t \in T$ such that $p^{-1}t \subset i(S)$ (so that $T_1 \subset T^1$). Set

$$R^n = p^{-1}T^n, \quad S^n = i^{-1}R^n,$$

etc., so that our suitable square splits into a disjoint sum of two squares:

$$\begin{array}{ccc} R^n & \xrightarrow{p^n} & T^n \\ {\scriptstyle i^n}\uparrow & & \uparrow{\scriptstyle j^n} \\ S^n & \xrightarrow{q^n} & P^n \end{array}$$

where $n = 1, 2$. It follows from the definitions that i^1, q^2 are bijections so that we may assume $S^1 = R^1$, $S^2 = P^2$, $i^1 = \mathrm{Id}, q^2 = \mathrm{Id}$.

So, we have the following diagram:

$$\begin{array}{ccc} S^1 \sqcup R^2 & \xrightarrow{p^1\sqcup p^2} & T^1 \sqcup T^2 \\ {\scriptstyle \mathrm{Id}\sqcup i^2}\uparrow & & \uparrow{\scriptstyle j^1\sqcup j^2} \\ S^1 \sqcup S^2 & \xrightarrow{q^1\sqcup \mathrm{Id}} & P^1 \sqcup S^2 \end{array} \tag{14}$$

The desired map is then defined as follows:

$$\begin{aligned} \mathfrak{p}_{i^1\sqcup i^2}\mathcal{R}_{p^1\sqcup p^2} &\to \mathfrak{p}_{\mathrm{Id}\sqcup i^2}\mathcal{R}_{p^1\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup p^2} \to \mathcal{R}_{p^1\sqcup\mathrm{Id}}\mathfrak{p}_{\mathrm{Id}\sqcup i^2}\delta_{\mathrm{Id}\sqcup p^2} \\ &\to \mathcal{R}_{p^1\sqcup \mathrm{Id}}\mathfrak{p}_{\mathrm{Id}\sqcup p^2 i^2} = \mathcal{R}_q\mathfrak{p}_j. \end{aligned}$$

5.3.4. *Properties*

The above defined maps have the following properties, easily derived from the ones of the maps $\mathfrak{l}_p, s(\phi, g)$. We shall now list them.

(1) Let

$$\begin{array}{ccc} R & \xrightarrow{p} & T \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ S & \xrightarrow{q} & P \end{array}$$

be a suitable square and $q = q_2q_1$, where q_1, q_2 are surjections.

Define the set $X(q_1, q_2)$ of isomorphism classes of commutative diagrams

$$\begin{array}{ccccc} R & \xrightarrow{p_1} & U & \xrightarrow{p_2} & T \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j'} & & \uparrow{\scriptstyle j} \\ S & \xrightarrow{q_1} & V & \xrightarrow{q_2} & P \end{array}$$

We will refer to such a diagram as (p_1, p_2, j'). Both squares in every such a diagram are automatically suitable. Therefore, every element $x := (p_1, p_2, j') \in X(q_1, q_2)$ determines a map

$$m_x : \mathfrak{p}_i \mathcal{R}_p \to \mathfrak{p}_i \mathcal{R}_{p_1} \mathcal{R}_{p_2} \to \mathcal{R}_{q_1} \mathfrak{p}_{j'} \mathcal{R}_{p_2} \to \mathcal{R}_{q_1} \mathcal{R}_{q_2} \mathfrak{p}_j.$$

Then the composition

$$\mathfrak{p}_i \mathcal{R}_p \to \mathcal{R}_q \mathfrak{p}_j \to \mathcal{R}_{q_1} \mathcal{R}_{q_2} \mathfrak{p}_j$$

equals

$$\sum_{x \in X(q_1, q_2)} m_x$$

(2) Consider the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{p} & T \\ {\scriptstyle i_2}\uparrow & & \uparrow{\scriptstyle j_2} \\ S_1 & \xrightarrow{q} & P_1 \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle j_1} \\ S & \xrightarrow{r} & P \end{array}$$

in which both small squares are suitable. Then the large square is also suitable and the following maps coincide:

$$\mathfrak{p}_{i_2 i_1} \mathcal{R}_p \to \mathcal{R}_r \mathfrak{p}_{j_2 j_1}$$

and

$$\mathfrak{p}_{i_2 i_1} \mathcal{R}_p \to \mathfrak{p}_{i_1} \mathfrak{p}_{i_2} \mathcal{R}_p \to \mathfrak{p}_{i_1} \mathcal{R}_q \mathfrak{p}_{j_2} \to \mathcal{R}_r \mathfrak{p}_{j_1} \mathfrak{p}_{j_2} \to \mathcal{R}_r \mathfrak{p}_{j_2 j_1}.$$

(3) Consider the following commutative diagram:

$$\begin{array}{ccc} R & \overset{p}{\twoheadrightarrow} & T \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ S & \overset{q}{\twoheadrightarrow} & P \\ {\scriptstyle k}\uparrow & \nearrow{\scriptstyle l} & \\ Q & & \end{array}$$

where the upper square is suitable. Then the following maps coincide:

$$\mathfrak{p}_{ik} \mathcal{R}_p \to \mathfrak{p}_k \mathfrak{p}_i \mathcal{R}_p \to \mathfrak{p}_k \mathcal{R}_q \mathfrak{p}_j \to \mathfrak{p}_{qk} \mathfrak{p}_j = \mathfrak{p}_l$$

and

$$\mathfrak{p}_{ik} \mathcal{R}_p \to \mathfrak{p}_{pik} = \mathfrak{p}_l.$$

(4) Let

$$\begin{array}{ccc} R & \xrightarrow{p} & T \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ S & \xrightarrow{q} & X \end{array}$$

and

$$\begin{array}{ccc} R_1 & \xrightarrow{p_1} & T_1 \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle j_1} \\ S_1 & \xrightarrow{q_1} & X_1 \end{array}$$

be suitable squares and let $s : S \to S_1$, $r : R \to R_1$, $t : T \to T_1$, $x : X \to X_1$ be bijections fitting the two squares into a commutative cube. Then the map $A(i, p, j, q)$ can be expressed in terms of $A(i_1, p_1, j_1, q_1)$ in the following natural way:

$$\begin{aligned} \mathfrak{p}_i \mathcal{R}_p &\cong \mathfrak{p}_s \mathfrak{p}_{i_1} \mathfrak{p}_{r^{-1}} \mathfrak{p}_r \mathcal{R}_{p_1} \mathfrak{p}_{t_1^{-1}} \cong \mathfrak{p}_s \mathfrak{p}_{i_1} \mathcal{R}_{p_1} \mathfrak{p}_{t_1^{-1}} \xrightarrow{A(i_1, p_1, j_1, q_1)} \mathfrak{p}_s \mathcal{R}_{q_1} \mathfrak{p}_{j_1} \mathfrak{p}_{t_1^{-1}} \\ &\cong \mathfrak{p}_s \mathcal{R}_{q_1} \mathfrak{p}_{x^{-1}} \mathfrak{p}_x \mathfrak{p}_{j_1} \mathfrak{p}_{t_1^{-1}} \mathcal{R}_q \mathfrak{p}_j. \end{aligned}$$

(5) Let (i_k, p_k, j_k, q_k), $k \in K$ be a collection of suitable squares. Let $i_k : S_k \to R_k$; let M_k be a collection of $\mathcal{D}_{X^{S_k}}$-sheaves. Let $i = \sqcup_{k \in K} i_k$, $p = \sqcup_{k \in K} p_k$, $j = \sqcup_{k \in K} j_k$, $q = \sqcup_{k \in K} q_k$, and $M = \boxtimes_{k \in K} M_k$. Then the square i, p, j, q is also suitable and the following compositions coincide:

$$\boxtimes_{k \in K} \mathfrak{p}_{i_k} \mathcal{R}_{p_k}(M_k) \to \boxtimes_{k \in K} R_{q_k} \mathfrak{p}_{j_k}(M_k) \to \mathcal{R}_q \mathfrak{p}_j(M)$$

and

$$\boxtimes_{k \in K} \mathfrak{p}_{i_k} \mathcal{R}_{p_k}(M_k) \to \mathfrak{p}_i \mathcal{R}_p(M) \to \mathcal{R}_q \mathfrak{p}_j(M).$$

(6) Let $i_k : S_k \to R_k$, $k \in K$ be injections and $p_k : R_k \to T_k$, $k \in K$ be surjections such that $j_k := p_k i_k$ are injections. Let M_k be $\mathcal{D}_{X^{T_k}}$-modules. Let i, j, p, M be disjoint unions of the respective objects.

Assume that at least two of the maps p_k are proper surjections. Then the composition

$$\boxtimes_{k \in K} \mathfrak{p}_{i_k} \mathcal{R}_{p_k}(M_k) \to \mathfrak{p}_i \mathcal{R}_p(\boxtimes_k M_k) \to \mathfrak{p}_j(M)$$

vanishes.

If only one of the surjections p_k is proper, say p_κ, $\kappa \in K$, then the above composition equals

$$\begin{aligned} &\boxtimes_{k \in K} \mathfrak{p}_{i_k} \mathcal{R}_{p_k}(M_k) = \mathfrak{p}_{i_\kappa} \mathcal{R}_{p_\kappa}(M_\kappa) \boxtimes_{k \in K \setminus \{\kappa\}} \mathfrak{p}_{i_k} \mathcal{R}_{p_k}(M_k) \\ &\quad \xrightarrow{L(i_\kappa, p_\kappa)} \mathfrak{p}_{j_\kappa}(M_\kappa) \boxtimes_{k \in K \setminus \{\kappa\}} \mathfrak{p}_{i_k} \mathcal{R}_{p_k}(M_k) \to \boxtimes_{k \in K} \mathfrak{p}_{j_k}(M_k) \to \mathfrak{p}_j(M). \end{aligned}$$

(7) The diagram (11) is suitable, and the corresponding map $A(i, p, j, q)$ is the isomorphism (12).

5.3.5. *Differentials*

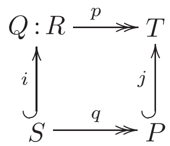
The differential of the map $L(i,p)$ is computed as follows. Consider the set of all equivalence classes of decompositions $p = p_2p_1$, where p_1, p_2 are surjections and p_1i is injection. We then have a map

$$l(p_1,p_2)\colon \mathfrak{p}_i\mathcal{R}_p \to \mathfrak{p}_i\mathcal{R}_{p_1}\mathcal{R}_{p_2} \to \mathfrak{p}_{p_1i}R_{p_2} \to \mathfrak{p}_{p_2p_1i} = \mathfrak{p}_{pi}.$$

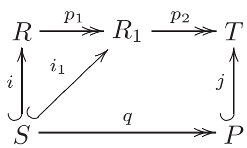
We then have

$$dL(i,p) + \sum_{(p_1,p_2)} l(p_1,p_2) = 0.$$

(2) Let

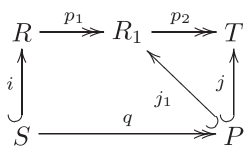


be a suitable square. Define two sets $L(Q)$ and $R(Q)$ as follows. The set $L(Q)$ is the set of all isomorphism classes of diagrams:



such that $p = p_1p_2$. It is clear that the internal commutative square in this diagram is also suitable.

Define the set $R(Q)$ as the set of isomorphisms classes of diagrams



where $p = p_1p_2$. The internal square in such a diagram is always suitable as well.

Every element $l := (p_1, p_2, i_1) \in L(Q)$ determines a map

$$f_l\colon \mathfrak{p}_i\mathcal{R}_p \to \mathfrak{p}_i\mathcal{R}_{p_1}\mathcal{R}_{p_2} \to \mathfrak{p}_{i_1}\mathcal{R}_{p_2} \to \mathcal{R}_q\mathfrak{p}_j.$$

Every element $r = (p_1, p_2, j_1) \in R(Q)$ determines a map

$$g_r\colon \mathfrak{p}_i\mathcal{R}_p \to \mathfrak{p}_i\mathcal{R}_{p_1}\mathcal{R}_{p_2} \to \mathcal{R}_q\mathfrak{p}_{j_1}\mathcal{R}_{p_2} \to \mathcal{R}_q\mathfrak{p}_j.$$

We then have

$$dA(i,p,j,q) = \sum_{l\in L(Q)} f_l - \sum_{r\in R(Q)} g_r = 0.$$

This completes the list of properties.

5.3.6. *Second definition of a pre-symmetric system*

Call a system $\langle\mathcal{R}\rangle$ endowed with the above specified maps having the above properties *a pre-symmetric system* (in the sense of the second definition). As we will mainly use pre-symmetric systems in the sense of the second definition, we shall simply refer to them as pre-symmetric.

6. Renormalization in Pre-Symmetric Systems

We are going to describe the renormalization procedure for algebras over pre-symmetric systems. The plan is as follows.

First of all given an algebra M over a pre-symmetric system, we show that the direct image p_*M has an L_∞-structure, (here $p: X \to \mathrm{pt}$). Next we have to show how, given a solution to the Master equation, one can deform the algebra M. As in the naive approach, we see that to be able to renormalize, one needs an extra structure on our system, and we define this structure (it is called symmetric). Next, we show how the renormalization goes in symmetric systems, and finally, we discuss a procedure by means of which, given a pre-symmetric system one can produce a symmetric system (we call this procedure *symmetrization*). So, the renormalization of an algebra over a pre-symmetric system includes:

(1) symmetrization of the system so that we get an OPE-algebra over a symmetric system;
(2) renormalization in the symmetric system.

6.0.1. *An L_∞-structure on $p_*M[1]$, where M is an OPE-algebra over $\langle\mathcal{R}\rangle$*

Let M be an OPE-algebra over $\langle\mathcal{R}\rangle$. We are going to introduce an L_∞ structure on p_*M, where $p: X \to \mathrm{pt}$ is the projection Let S be a finite set and $i_S: \emptyset \to S$ be an embedding. Let $\mathfrak{p}_S := \mathfrak{p}_{i_S}$. It is clear that $\mathfrak{p}_{\mathrm{pt}} = p_*$ and that

$$\mathfrak{p}_S(M^{\boxtimes S}) \cong (p_*M)^{\otimes S}.$$

Finally, set $p_S: S \to \mathrm{pt}$.

Define a degree $+1$ map

$$C_S: (p_*M)^{\otimes s} \to p_*M$$

as the composition:

$$(p_*M)^{\otimes s} \cong \mathfrak{p}_S(M^{\boxtimes S}) \xrightarrow{\mathfrak{ope}_S} \mathfrak{p}_S\mathfrak{i}_{p_S}(M) \xrightarrow{L(i_S,p_S)} \mathfrak{p}_{i_{\mathrm{pt}}}M \cong p_*M.$$

Claim 6.1. *The maps C_S endow $p_*M[1]$ with an L_∞-structure.*

Proof. The key ingredient in the proof is

Lemma 6.2. *Let $q: S \to T$ be a surjection such that one can decompose $S = S_1 \sqcup S_2$, $T = T_1 \sqcup T_2$, $q = q_1 \sqcup q_2$, where $q_i: S_i \to T_i$, $i = 1, 2$ are both non-bijective* *surjections. Then the composition*

$$\mathfrak{p}_S(M^{\boxtimes S}) \to \mathfrak{p}_S\mathfrak{i}_q(M^{\boxtimes T}) \to \mathfrak{p}_T(M^{\boxtimes T})$$

vanishes.

Proof. Let $A = \{1,2\}$. Then the above composition equals:

$$\mathfrak{p}_S(M^{\boxtimes S}) \cong \mathfrak{p}_{i_{S_1}\sqcup i_{S_2}}(M^{\boxtimes S_1} \boxtimes M^{\boxtimes S_2}) \to \mathfrak{p}_{i_{S_1}\sqcup i_{S_2}}(\mathfrak{i}_{q_1}(M^{T_1}) \boxtimes \mathfrak{i}_{q_2}(M^{T_2}))$$
$$\to \mathfrak{p}_S\mathfrak{i}_q(M^{\boxtimes T_1} \boxtimes M^{\boxtimes T_2}) \to \mathfrak{p}_T(M^{\boxtimes T}).$$

Here $i_{S_1} : \emptyset \to S_1$, $i_2 : \emptyset \to S_2$.

The composition of the last two arrows vanishes by Property 6 in the previous subsection. □

The Claim now follows directly from the formula of the differential of $L(i,p)$. □

6.0.2. *Action of the DGLA $p_*M[1]$ on M*

Define the maps

$$A_S : (p_*M)^{\otimes S} \otimes M \to M$$

as follows. Let $S_0 = S \sqcup \mathrm{pt}$. Let $k : \mathrm{pt} \to S_0$ be the natural embedding. Let $p_{S_0} : S_0 \to \mathrm{pt}$.

We then set

$$A_S : (p_*M)^{\otimes S} \otimes M \cong \mathfrak{p}_k(M^{\boxtimes S_0}) \xrightarrow{\mathfrak{ope}_{S_0}} \mathfrak{p}_k\mathfrak{i}_{p_{S_0}}(M) \xrightarrow{L(k,p_{S_0})} M.$$

It is not hard to see that the collection of maps A_S determines an L_∞-action of $p_*M[1]$ on M.

6.1. *Symmetric systems*

Pre-symmetric systems do not fit for renormalization. The reason is more or less the same as in the naive approach, but let us reformulate it in terms of direct images with respect to projections.

Let $p : S \to \mathrm{pt}$ and pick an element $s \in S$.

Let $S' := S \sqcup \{s\}$. Let $t \in S'$. Define $p_t : S \to S'$ as follows:

$$p_t(r) = r$$

if $r \neq s$;

$$p_t(s) = t.$$

Let $p_s : S \to \{a, s\}$, where a is an abstract element, $a \neq s$, by setting

$$p_s(t) = a$$

if $t \neq s$; $p_s(s) = s$.

Let $q : S' \to \mathrm{pt}$ and $r : \{a, s\} \to \mathrm{pt}$. Let $i : S' \to S$, $j : \{s\} \to \{a, s\}$ be natural embeddings.

We then have several maps

$$\mathfrak{p}_i \mathcal{R}_p \to \mathcal{R}_q.$$

(a) Let $t \in S'$. Set

$$L_t : \mathfrak{p}_i \mathcal{R}_p \to \mathfrak{p}_i \mathcal{R}_{p_t} \mathcal{R}_q \to \mathcal{R}_q;$$

Set

$$R : \mathfrak{p}_i \mathcal{R}_p \to \mathfrak{p}_i \mathcal{R}_{p_s} \mathcal{R}_r \to \mathcal{R}_q \mathfrak{p}_j \mathcal{R}_r \to \mathcal{R}_q.$$

Then luck of symmetricity manifests itself in the fact that the difference

$$R - \sum_{t \in S'} L_t$$

is not homotopic to 0.

We thus need to add extra homotopies which would take care about it. It turns out that this can be accomplished in a very symple way:

Call a system $\langle \mathcal{R} \rangle$ *symmetric* if the maps $A(i, p, j, q)$ are defined for *all* commutative squares

$$\begin{array}{ccc} R & \xrightarrow{p} & T \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ S & \xrightarrow{q} & P \end{array}$$

where p, q are both non-bijective surjections (not necessarily suitable). The properties remain the same as for pre-symmetric system except that we drop the suitability condition everywhere.

We shall demonstrate how the renormalization goes in symmetric systems.

Let now $\mathfrak{D} \in \lambda p_* M^0[[\lambda]]$ be a MC element. For a finite set T set

$$\mathfrak{D}_T := \mathfrak{D}^{\boxtimes T} \in \mathfrak{p}_T \lambda^{|T|} M^{\boxtimes T}[[\lambda]].$$

Let $i : R \to S$ be an injection. Let $T = S \backslash i(R)$. We then have a map

$$M^{\boxtimes R} \to \mathfrak{p}_i \lambda^{|T|} M^{\boxtimes S}[[\lambda]]$$

defined by:

$$M^{\boxtimes R} \xrightarrow{\otimes \mathfrak{D}_T} M^{\boxtimes R} \otimes \mathfrak{p}_T \lambda^{|T|} M^{\boxtimes T}[[\lambda]] \cong \mathfrak{p}_i \lambda^{|T|} M^{\boxtimes S}[[\lambda]].$$

6.1.1. Let $i : S \to R$ be an injection and $q : R \to T$ be a surjection such that $p := qi$ is a surjection.

We then have a map

$$\mathfrak{ope}(q,i) : M^{\boxtimes S} \to \mathfrak{p}_i M^{\boxtimes R} \to \mathfrak{p}_i \mathcal{R}_q M^{\boxtimes T} \to \mathcal{R}_p M^{\boxtimes T}.$$

Set

$$\mathfrak{ope}_p^r = \sum \mathfrak{ope}(q,i), \tag{15}$$

where the sum is taken over all isomorphism classes of decompositions $p = qi$. Let $M^r := M[[\lambda]], d_{\mathfrak{D}}$, where $d_{\mathfrak{D}}$ is the differential twisted by $\mathfrak{D}$. Then $(M', \mathfrak{ope}^r)$ is the renormalized OPE-algebra.

Note that the sum (15) is infinite but it converges in the λ-adic topology.

6.2. *Symmetrization*

Finally, we need a method on how, given a pre-symmetric system, one gets a symmetric system.

The idea is as follows. Let $f : S \to T$ be a map of finite sets. Construct a category $\mathcal{B}_{\mathbf{presymm}}(f)$ whose objects are compositions $\mathfrak{p}_i \mathcal{R}_{p_1} \mathcal{R}_{p_2} \cdots \mathcal{R}_{p_n}$, where i is injective, p_k are surjective, all the maps are composable and

$$p_n p_{n-1} \cdots p_1 i = f.$$

The morphisms are all possible morphisms one can get using the axioms of pre-symmetric system. Given a pre-symmetric system $\langle \mathcal{R} \rangle$ and a $\mathfrak{D}_{X^T}$-sheaf N, the application

$$\mathfrak{p}_i \mathcal{R}_{p_1} \mathcal{R}_{p_2} \cdots \mathcal{R}_{p_n} \mapsto \mathfrak{p}_i \mathcal{R}_{p_1} \mathcal{R}_{p_2} \cdots \mathcal{R}_{p_n} N$$

produces a functor

$$\mathfrak{r}^f(N) : \mathcal{B}_{\mathbf{presymm}}(f) \to \mathcal{D}\text{-sh}_{X^T}.$$

Let $\mathcal{B}_{\mathbf{symm}}(f)$ be the same thing, but we use axioms of a symmetric system. We then have a tautological functor $R : \mathcal{B}_{\mathbf{presymm}}(f) \to \mathcal{B}_{\mathbf{symm}}(f)$. One can construct a bifunctor

$$B : \mathcal{B}^{\mathrm{op}}_{\mathbf{presymm}}(f) \times \mathcal{B}_{\mathbf{symm}}(f) \to \text{complexes},$$

where $B(X,Y) = \hom_{\mathcal{B}_{\mathbf{symm}}(f)}(X,Y)$.

Set

$$\mathfrak{r}^f_{\mathbf{symm}}(N) : \mathcal{B}_{\mathbf{symm}}(f) \to \text{complexes}$$

to be

$$\mathfrak{r}^f(N)\otimes_{\mathcal{B}_{\mathbf{presymm}}(f)} B.$$

Remark. Let R^{-1} be a functor from:

the category of functors $\mathcal{B}_{\mathbf{symm}}(f)\to$ complexes
to
the category of functors $\mathcal{B}_{\mathbf{presymm}}(f)\to$ complexes

which is the pre-composition with R. One can show that R^{-1} has a left adjoint $R_!$ and that $\mathfrak{r}^f_{\mathbf{symm}}(N)=R_!\mathfrak{r}^f(N)$.

We can now construct a system $\langle\mathcal{R}^{\mathbf{symm}}\rangle$ which is a symmetrization of $\mathcal{R}$ by setting $\mathcal{R}^{\mathbf{symm}}_p(N)=\mathfrak{r}^p(N)(\mathcal{R}_p)$. We have to say that the introduction of a structure of system on the collection of functors $\langle\mathcal{R}^{\mathbf{symm}}_p\rangle$ is not at all a consequence of a general nonsense. It turns out that in order to define such a structure one has to use certain specific features of the categories $\mathcal{B}_{\mathbf{presymm}},\mathcal{B}_{\mathbf{symm}}$.

We also have a natural map $\langle\mathcal{R}\rangle\to\langle\mathcal{R}^{\mathbf{symm}}\rangle$. Therefore, given an OPE-algebra over $\langle\mathcal{R}\rangle$, we can transform it into an OPE-algebra over $\langle\mathcal{R}^{\mathbf{symm}}\rangle$ and then renormalize it.

We shall now give a more explicit construction of $\langle\mathcal{R}^{\mathbf{symm}}\rangle$. In fact, the resulting system $\langle\mathcal{R}^{\mathbf{symm}}\rangle$ is isomorphic to the above described one. This follows from a more detailed study of the categories $\mathcal{B}_{\mathbf{presymm}},\mathcal{B}_{\mathbf{symm}}$ which id done in 18.4.

7. Explicit Construction of $\langle\mathcal{R}^{\mathbf{symm}}\rangle$

7.1. *Main objects*

7.1.1. *Groupoid* C'_f

Let $f:S\to T$ be a surjection. Define a groupoid C'_f whose objects are diagrams

$$S\xrightarrow{i}U\xrightarrow{p}T,$$

where i is injective, p is surjective, and $pi=f$. Isomorphisms are morphisms of these diagrams inducing identities on S,T.

7.1.2. *Groupoid* C_f

Let $(i,p)\in C_f$. Call p *i-super-surjective* if for every $t\in T$, the pre-image $p^{-1}t$ either:

contains at least two elements from $i(S)$
or
consists of one element from $i(S)$.

Let C_f be the full sub-groupoid of C'_f consisting of all pairs (i,p), where p is i-super-surjective.

7.1.3. *Functors* $\mathcal{M}(i,p)$, $\mathcal{M}_f$

For an object (i,p) in C_f, set $\mathcal{M}(i,p) := \mathfrak{p}_i\mathcal{R}_p$. It is clear that $\mathcal{M}(,)$ is a functor from C_f to the category of functors from the category of $\mathcal{D}_{X^T}$-sheaves to the category of $\mathcal{D}_{X^S}$-sheaves. Set

$$\mathcal{M}_f := \mathrm{limdir}_{C_f}\mathcal{M}(i,p).$$

Denote by $I(i,p)\colon \mathcal{M}(i,p) \to \mathcal{M}_f$ the natural map. It is clear that $I(i,p)$ passes through $\mathcal{M}(i,p)_{\mathrm{Aut}_{C_f}(i,p)}$. Furthermore, we have an isomorphism

$$\oplus\mathcal{M}(i,p)_{\mathrm{Aut}_{C_f}(i,p)} \to \mathcal{M}_f, \tag{16}$$

where the sum is taken over an arbitrary set of representatives of isomorphism classes of C_f.

7.2. *Differential*

The symmetrized resolution $\mathcal{R}_f^{\mathbf{symm}}$ is given by the functor $\mathcal{M}_f$ as in (16), on which a new differential is introduced. This differential is of the form $d + L + R$, where d is the differential on $\mathcal{M}_f$, and degree $+1$ endomorphisms $L, R\colon \mathcal{M}_f \to \mathcal{M}_f$ shall be defined below.

7.2.1. *Map* $L\colon \mathcal{M}_f \to \mathcal{M}_f$

7.2.2. *Set* $E_L(i,p)$

Let

$$S \xrightarrow{i} U \xrightarrow{p} T$$

be an object in C_f. Define a finite set $E_L(i,p)$ whose elements are equivalence relations e on U such that

(1) p passes through U/e;
(2) the composition

$$S \xrightarrow{i} U \longrightarrow U/e$$

is injective.

Let $\pi_e : U \to U/e$ be the natural projection, let $p_e : U/e \to T$ be the map induced by p, and $i_e = \pi_e i$.

It turns out that $(i_e, \pi_e) \in C_f$. Indeed, $\pi_e^{-1}(t)$ is the quotient of $p^{-1}t$ by e and elements of $i(S)$ are e-non-equivalent, which implies the super-surjectivity.

7.2.3. *The map* L

Define a map $L_e : \mathcal{M}(i,p) \to \mathcal{M}(i_e, p_e)$ as follows:

$$\mathfrak{p}_i\mathcal{R}_p \to \mathfrak{p}_i\mathcal{R}_{\pi_e}\mathcal{R}_{p_e} \xrightarrow{L(i,\pi_e)} \mathfrak{p}_{i_e}\mathcal{R}_{p_e}.$$

Define a map $L(i,p)\colon \mathcal{M}(i,p) \to \mathcal{M}_f$ by setting

$$L(i,p) = \sum_{e \in E_L(i,p)} I(i_e, p_e) L_e.$$

It is easy to see that the collection of maps $L(i,p)$ descends to a map $L\colon \mathcal{M}_f \to \mathcal{M}_f$.

7.3. *Map* $R\colon \mathcal{M}_f \to \mathcal{M}_f$

7.3.1. *Set* $E_R(i,p)$

Let

$$S \xrightarrow{i} U \xrightarrow{p} T$$

be an object in C_f. Define a finite set $E_R(i,p)$ whose elements are equivalence relations e on U such that

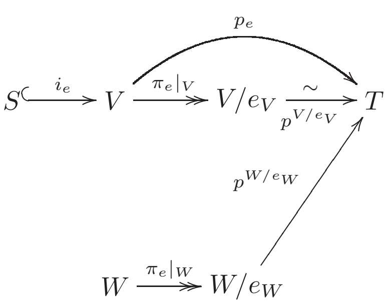
(1) p passes through U/e;
(2) The restriction of e on S coincides with the equivalence relation on S determined by f.

Let $\pi_e\colon U \to U/e$. Let $T_e := \mathrm{Im}(\pi_e i)$ and $V := V_e := \pi_e^{-1} T_e$ and $W := W_e := U \backslash U_e$. Let e_V (respectively e_W) be the restriction of e on V (respectively W).

It is clear that

(1) $i(S) \subset V$;
(2) The map $p^{V/e_V}\colon V/e_V \to T$ induced by p is bijective.

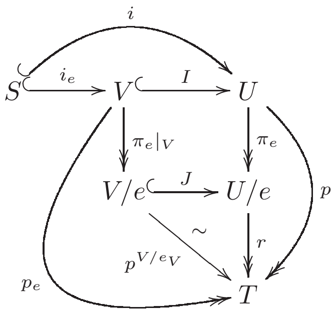
So, we have a diagram:



Elements of $E_R(i,p)$ can be equivalently defined as collections (W, e_W), where $W \subset U$, $W \cap i(S) = \emptyset$, and e_W is an equivalence relation on W such that $p|_W$ passes through e_W. Indeed, let $V := U \backslash W$ and let e_V be induced on V by $p|_V$. Set $e := e_V \sqcup e_W$. This establishes a 1-1 correspondence between different descriptions of $E_R(i,p)$.

Let us check that $(i_e, p_e) \in C_f$. Indeed, for every $t \in T$, $p_e^{-1} t = p^{-1} t \cap V$. Since $V \supset i(S)$, we have: if $p^{-1} t \cap i(S)$ has at least two elements, then so does $p_e^{-1} t$; otherwise $p^{-1} t$ consists of exactly one element from $i(S)$ and $p_e^{-1} t = p^{-1} t$.

We will now define a map $R_e : \mathcal{M}(i, p) \to \mathcal{M}(i_e, p_e)$. To this end we shall consider a diagram:



We then observe that the square $(I, \pi_e, J, \pi_e|_V)$ is clearly suitable. We can therefore define $R_e : M(i, p) \to M(i_e, p_e)$ via the following chain of maps:

$$R_e : \mathfrak{p}_i \mathcal{R}_p \cong \mathfrak{p}_{i_e} \mathfrak{p}_I \mathcal{R}_{rp_e} \to \mathfrak{p}_{i_e} \mathfrak{p}_I \mathcal{R}_{p_e} \mathcal{R}_r \to \mathfrak{p}_{i_e} \mathcal{R}_{\pi_e|_V} \mathfrak{p}_J \mathcal{R}_r$$
$$\to \mathfrak{p}_{i_e} \mathcal{R}_{\pi_e|_V} \mathfrak{p}_{p^{V/e_V}} \cong \mathfrak{p}_{i_e} \mathcal{R}_{p_e}.$$

We then define

$$R = \sum_{e \in R(i,p)} I(i_e, p_e) R_e.$$

7.3.2. *Definition of the differential*

We define the differential on $\langle \mathcal{R}^{\mathbf{symm}} \rangle$ as a sum $d + L + R$.

7.4. *Asymptotic decomposition maps* $\mathfrak{as}_{f_1,f_2} : \mathcal{R}^{\mathbf{symm}}_{f_2 f_1} \to \mathcal{R}^{\mathbf{symm}}_{f_1} \mathcal{R}^{\mathbf{symm}}_{f_2}$

Suppose we have a chain of surjections

$$S \overset{f_1}{\twoheadrightarrow} R \overset{f_2}{\twoheadrightarrow} T,$$

so that $f = f_2 f_1$.

Let

$$S \overset{i}{\hookrightarrow} U \overset{p}{\twoheadrightarrow} T$$

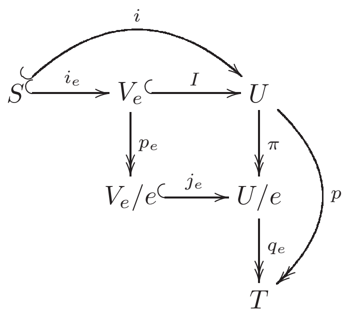
be in C_f. The map $\mathcal{I}(i, p) : \mathcal{M}(i, p) \to \mathcal{M}_f$ determines a similar map $\mathcal{I}(i, p) \to \mathcal{R}^{\mathbf{symm}}_f$. In order to construct the map $\mathfrak{as}_{f,g}$ we will first define maps

$$\mathfrak{as}(i, p, f, g) : \mathcal{M}(i, p) \to \mathcal{M}_{f_1} \mathcal{M}_{f_2}.$$

Define the set $E(i, p, f_1, f_2)$ whose elements are equivalence relations e on U such that

(1) p passes through U/e
(2) The restriction $e|_S$ coincides with the equivalence relation on S determined by f_1.

Let $V_e \subset U$ be the set of all elements which are equivalent (with respect to e) to elements of S. Let $W_e = U \backslash V_e$. Let $i_e : S \to V_e$; $p_e : V_e \to V_e/e$, $j_e : V_e/e \to U/e$, $q_e : U/e \to T$ be the map induced by p. We then have the following commutative diagram:



It is easy to check that the square (I, π, j_e, p_e) is suitable. This allows us to define a map

$$\mathfrak{as}(i, p, e) : \mathcal{M}(i, p) \to \mathcal{M}(i_e, p_e)\mathcal{M}(j_e, q_e)$$

as follows:

$$\begin{aligned}\mathcal{M}(i, p) &\cong \mathfrak{p}_i \mathcal{R}_p \to \mathfrak{p}_{i_e}\mathfrak{p}_I \mathcal{R}_{q_e\pi} \to \mathfrak{p}_{i_e}\mathfrak{p}_I \mathcal{R}_\pi \mathcal{R}_{q_e} \\ &\to \mathfrak{p}_{i_e}\mathcal{R}_{p_e}\mathfrak{p}_{j_e}\mathcal{R}_{q_e} = \mathcal{M}(i_e, p_e)\mathcal{M}(j_e, q_e).\end{aligned}$$

Let

$$\mathfrak{as}(i, p, f_1, f_2) : \mathcal{M}(i, p) \to \mathcal{M}_{f_1}\mathcal{M}_{f_2}$$

be given by the formula:

$$\mathfrak{as}(i, p, f_1, f_2) = \sum_{e \in E(i,p,f_1,f_2)} \mathcal{I}_{(j_e,q_e)}\mathcal{I}_{(i_e,p_e)}\mathfrak{as}(i, p, e).$$

This completes the definition of the map $\mathfrak{as}_{f_1,f_2}$.

7.5. *Factorization maps*

Let $f_a : S_a \to T_a$, $a \in A$ be a family of surjections.

Let $(i_a, p_a) \in C_{f_a}$, $a \in A$, be a family of objects. Let $i = \sqcup_{a\in A} i_a$, $p = \sqcup_{a\in A} p_a$, $f = \sqcup_{a\in A} f_a$.

Let $M_a \in \mathcal{D}\text{-sh}_{X^{T^a}}$. Let $M := \boxtimes_{a\in A} M_a$.

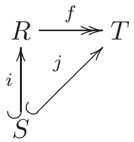
We then have a natural map

$$\boxtimes_a \mathfrak{p}_{i_a} \mathcal{R}_{p_a}(M_a) \to \mathfrak{p}_i \mathcal{R}_p(M),$$

induced by the factorization maps for $\langle \mathcal{R} \rangle$. These maps give rise to the factorization maps in $\langle \mathcal{R}^{\mathbf{symm}} \rangle$.

7.6. *Maps* $L(i, f) : \mathfrak{p}_i \mathcal{R}^{\mathbf{symm}}_f \to \mathfrak{p}_j$

Let



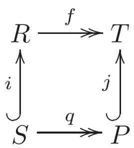
be a commutative diagram. The map $L(i, p) : \mathfrak{p}_i \mathcal{R}^{\mathbf{symm}}_f \to \mathfrak{p}_j$ is then defined via maps

$$\mathfrak{p}_i \mathcal{M}(k, p) \cong \mathfrak{p}_{ki} \mathcal{R}_p \xrightarrow{L(ki,p)} \mathfrak{p}_j,$$

where $pk = f$.

7.7. *The maps* $A(i, p, j, q) : \mathfrak{p}_i \mathcal{R}^{\mathbf{symm}}_p \to \mathcal{R}^{\mathbf{symm}}_q \mathfrak{p}_j$

Let

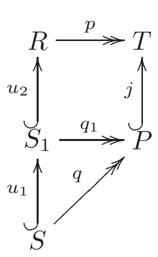


be a commutative diagram. The maps

$$A(i, f, j, q) : \mathfrak{p}_i \mathcal{R}^{\mathbf{symm}}_f \to \mathcal{R}^{\mathbf{symm}}_q \mathfrak{p}_j$$

are defined as follows.

Let $(k, p) \in C_f$. Let $u = ki$. One can show that there exists a unique, upto an isomorphism decomposition $u = u_2 u_1$ into a product of two injections such that in the diagram



uniquely, upto an isomorphism, constructed, given a decomposition $u = u_2u_1$, the square

$$(u_2, p, j, q_1)$$

is suitable, and in the pair

$$(u_1, q_1),$$

the map q_1 is super-surjective.

The map $A(i, f, j, q)$ goes as follows:

$$\mathfrak{p}_i\mathcal{M}(k, p) \cong \mathfrak{p}_{ki}\mathcal{R}_p \cong \mathfrak{p}_{u_1}\mathfrak{p}_{u_2}\mathcal{R}_p \xrightarrow{A(u_2,p,j,q_1)} \mathfrak{p}_{u_1}\mathcal{R}_{q_1}\mathfrak{p}_j \cong \mathcal{M}(u_1, q_1)\mathfrak{p}_j.$$

8. Constructing the System $\langle\mathcal{R}\rangle$ with the Above Explained Properties

8.1. *Step 1. Spaces of generalized functions C_S*

Our motivation comes from the construction in 3.2. In the case when $p: S \to \text{pt}$, where S has two elements, this construction suggests that one can replace $\mathfrak{i}_p$ with a complex $0 \to i_{p*} \to \mathcal{I}_p \to 0$, where we put $\mathcal{I}_p$ in degree 0. Denote this complex by $\mathcal{R}_p$. On the one hand, we have a map $\mathcal{R}_p \to \mathfrak{i}_p$, so that the induced map $\mathcal{R}_p(M) \to \mathfrak{i}_p(M)$ is a quasi-isomorphism for good M's; on the other hand we have a map $\mathcal{R}_p \to i_{p*}\mathcal{O}_X$ of degree $+1$. Thus, $\mathcal{R}_p$ has all the desired properties.

Let us try to expand this construction to an arbitrary case. It is natural to start with constructing certain spaces of generalized functions $\mathcal{C}_S$ on X^S so that each $\mathcal{C}_S$ is a sub-$\mathcal{D}_{X^S}$ submodule of the space of complex-valued generalized functions on Y^S with compact support. In pursuit of making $\mathcal{C}_S$ as small as possible we construct $\mathcal{C}_S$ in such a way that they are holonomic $\mathcal{D}_{X^S}$-modules; their structure is as follows. Let D be a generalized diagonal in X^S and let $\mathcal{C}_{S,[D]}$ be the maximal submodule supported on D. This defines a filtration on $\mathcal{C}_S$ whose terms are labeled by the ordered set of generalized diagonals in X^S. The associated graded term

$$\mathcal{C}_{S,[D]}/\text{span}_{E\subsetneq D}\mathcal{C}_{S,[E]} \cong i_{D*}\mathcal{B}_D,$$

where $\mathcal{B}_D$ is the $\mathcal{D}_D$-module of all meromorphic functions with singularities along hyper-surfaces $q(X_i - X_j) = 0$, where $X_i|_D \neq X_j|_D$.

Construction of such $\mathcal{C}_S$ is done by means of certain analytical considerations. Some of them a very similar to standard methods of regularization of divergent integrals. The detailed exposition is in Secs. 10.2–11.

8.2. *Step 2. Functors $\mathcal{I}_p$ and their properties*

Next we construct the functors $\mathcal{I}_p$ out of $\mathcal{C}_S$ in the same way as $\mathfrak{i}_p$ was constructed out of $\mathcal{B}_S$: let $p: S \to T$ be a surjection of finite sets; set

$$\mathcal{C}_p := \boxtimes_{t\in T}\mathcal{C}_{p^{-1}t}.$$

Define $\mathcal{I}_p : \mathcal{D}_{X^T} \to \mathcal{D}_{X^S}$ by

$$\mathcal{I}_p(M) = i_p^\wedge(M) \otimes_{\mathcal{O}_{X^S}} \mathcal{C}_p.$$

We then have natural maps $\mathcal{I}_p \to \mathfrak{i}_p$. We then ask ourselves whether $\mathcal{I}_p$ form a system. The answer is no. It probably could be yes if $\mathcal{C}_S$ would be a bit larger subspace of generalized functions, because we have a technique of asymptotic decomposition of generalized functions due to Bernstein (unpublished). But there are examples in which we see that already for the set $S = \{1, 2, 3\}$ consisting of three elements there are functions $f \in C_S$, whose asymptotic decomposition near the diagonal $X^1 = X^2$ requires introduction of such functions as $\log(X_1 - X_3)$. For example, let $Y = \mathrm{Re}^4$ and take

$$f(X^1, X^2, X^3) = \frac{1}{|X^1 - X^3|^2 |X^2 - X^3|^2}.$$

This is a locally L^1-function, therefore, it determines a generalized function. Let us investigate its asymptotic as X^1 approaches X^2. According to Bernstein, we should consider the following expression:

$$u(\lambda) = \int \frac{g(X_1, X_1 + (X_1 - X_2)/\lambda, X_3)}{|X^1 - X^3|^2 |X^2 - X^3|^2} d^4X^1 d^4X^2 d^4X^3,$$

where g is a compactly supported smooth function and λ is a small positive parameter. Our goal is to find an asymptotic for $a(\lambda)$. Let $x = X_1$, $a = X_2 - X_1$, $b = X_3 - X_1$. Let $G(x, a, b) := g(x, x + a, x + b)$. We then have

$$u(\lambda) = \int \frac{G(x, a/\lambda, b)}{|b|^2 |b + a|^2} d^4b d^4a d^4x.$$

One can show that

$$u(\lambda) = C \int G(x, a/\lambda, 0) \ln(|a|^2) d^4a d^4x + v(\lambda),$$

where $v(\lambda)$ is bounded as $\lambda \to +0$, and C is a constant.

This means that

$$\begin{aligned} v(\lambda) = \int & g(X_1, X_1 + (X_1 - X_2)/\lambda, X_3) \\ & \times \left\{ \frac{1}{|X^1 - X^3|^2 |X^2 - X^3|^2} - C \ln(|X_1 - X_2|^2) \delta(X^1 - X^3) \right\} \\ & \times d^4X^1 d^4X^2 d^4X^3 \end{aligned}$$

is bounded as $\lambda \to +0$. This demonstrates that, at least, we have to include $\ln(|X_1 - X_2|^2)$ into our picture to get an asymptotic decomposition of

$$\frac{1}{|X^1 - X^3|^2 |X^2 - X^3|^2}.$$

The geometrical meaning of this phenomenon is that the cohomology of the complex variety which is the complement in $\mathbb{C}^4 \times \mathbb{C}^4$ to the set of complex zeroes of $|Z_1 - Z_2|^2 = 0$ differs from the cohomology of the real part, which is $\mathbb{R}^4 \times \mathbb{R}^4$ minus the diagonal. We need to add functions which would kill the de-Rham cocycles which are nontrivial on the complexification but become trivial upon restriction to the real part.

Nevertheless, we have maps

$$\mathcal{I}_{pq} \to \mathcal{I}_q \mathfrak{i}_p \tag{17}$$

for all surjections p, q.

For certain p, q we also have maps

$$\mathcal{I}_{pq} \to \mathcal{I}_q \mathcal{I}_p. \tag{18}$$

Namely, this happens if

$$q = q_1 \sqcup \mathrm{Id} : S_1 \sqcup S_2 \to R_1 \sqcup S_2$$

and

$$p = \mathrm{Id} \sqcup p_1 : R_1 \sqcup S_2 \to R_1 \sqcup T_2,$$

or if p, q can be brought to this form via conjugations by bijections. This circumstance will play an important role in the future steps, but now let us concentrate only on the maps $\mathcal{I}_{pq} \to \mathcal{I}_q \mathfrak{i}_p$. They have associativity properties similar to those of $\mathfrak{i}$ and they nicely behave with respect to $\boxtimes$. They are compatible with the corresponding maps $\mathfrak{i}_{pq} \to \mathfrak{i}_q \mathfrak{i}_p$.

There is an additional feature stemming from the fact that the submodule $\mathcal{C}_{S,\Delta} \subset \mathcal{C}_S$, where $\Delta \subset X^S$ is a generalized diagonal, is isomorphic to $i_{\Delta *}\mathcal{C}_\Delta$.

Let p be a surjection. Denote $\delta_p := i_{p*}$. We then have a natural map

$$\delta_p \mathcal{I}_q \to \mathcal{I}_{qp}, \tag{19}$$

whenever surjections p, q are composable. These maps behave nicely with respect to the other parts of the structure.

8.2.1. *Iterations of functors $\mathcal{I}$ and $\mathfrak{i}$*

We will work with all possible functors of the form

$$\mathfrak{j}^1_{p_1} \mathfrak{j}^2_{p_2} \cdots \mathfrak{j}^n_{p_n},$$

where $p_i : S_i \to S_{i+1}$ are surjections and $\mathfrak{j}^s_{p_s}$ is either $\mathfrak{i}_{p_s}$ or $\mathcal{I}_{p_s}$. Fix a surjection $p : S \to T$ and consider the class $\mathbf{Zebra}_p$ of all such compositions with $p_n p_{n-1} \cdots p_1 = p$ (in particular, $S_1 = S$, $S_{n+1} = T$). The asymptotic decomposition maps (17) and their compositions produce maps between objects of $\mathbf{Zebra}_p$ (warning: we exclude the maps (18)). For example, we can construct a map $\mathcal{I}_{qrp} \to \mathcal{I}_p \mathfrak{i}_r \mathfrak{i}_q$ as a composition:

$$\mathcal{I}_{qrp} \to \mathcal{I}_p \mathfrak{i}_{qr} \to \mathcal{I}_p \mathfrak{i}_r \mathfrak{i}_q.$$

We can also take another composition:

$$\mathcal{I}_{qrp} \to \mathcal{I}_{rp} \mathfrak{i}_q \to \mathcal{I}_p \mathfrak{i}_r \mathfrak{i}_q.$$

The associativity property implies that these compositions are equal.

On the other hand, there is no way to construct a map $\mathcal{I}_{qrp} \to \mathfrak{i}_p \mathcal{I}_q \mathfrak{i}_r$.

Thus, $\mathbf{Zebra}_p$ is naturally a category. Furthermore, it turns out that, because of the associativity properties, there is at most one arrow between different arrows, i.e. $\mathbf{Zebra}_p$ is equivalent to a poset which will be denoted by $\mathbf{Zebra}(p)$. Let us describe it. First of all, each isomorphism class in $\mathbf{Zebra}_p$ does not even form a set because of the indeterminacy in the choice of intermediate sets S_i. This can be easily resolved by demanding each S_i to be S/e_i, where e_i is an equivalence relation on S. More precisely, let e be the equivalence relation on S determined by $p : S \to T$, T being identified with S/e. Let $\mathbf{Eq}_e$ be the poset of all equivalence relations on S which are finer than e. Let us write $e_1 > e_2$ if e_1 is finer than e_2. Denote by ω the trivial (the finest) equivalence relation on S. An element of $\mathbf{Zebra}(p)$ is then a pair $F, \{\mathfrak{j}^s\}$, where $F = (\omega = e_1 > \cdots e_{n+1} = e)$ is a proper flag of equivalence relations and $\{j^s\}_{s=1}^n$ is a sequence of symbols $\mathfrak{i}$ or $\mathcal{I}$. It is convenient to visualize an object of zebra as a subdivision of a large segment into n small subsegments; the equivalence relations are associated with the nodes (e_s is associated with the sth node from the left) and $\mathfrak{j}^s$ determines one of two colors of the small segment between the sth and the $(s+1)$th node.

To such data we associate the functor

$$[F, \{\mathfrak{j}^s\}] = \mathfrak{j}^1_{p_1} \mathfrak{j}^2_{p_2} \cdots \mathfrak{j}^n_{p_n},$$

where $p_i : S/e_i \to S/e_{i+1}$ is the natural projection. Let us describe the order (we assume that an arrow $X \to Y$ exists iff $X \leq Y$). We say that $X < Y$ if

(1) the flag of Y is a refinement of the flag of X. Thus, each small segment of the flag of X is then subdivided into even smaller segments (call them microscopic) of the flag of Y.
(2) If a small segment of the flag of X is colored into the color "$\mathfrak{i}$", then all its microscopic subsegments are also colored into "$\mathfrak{i}$". If a small segment is colored into "$\mathcal{I}$", then the color of its leftmost microscopic segment may be arbitrary, but the colors of its remaining microscopic segments must by "$\mathfrak{i}$". The detailed exposition can be found in Sec. 14.

8.3. Step 3. OPE-algebras over the collection of functors $\mathcal{I}_p$. The functors $\mathfrak{P}_p$

Albeit the functors $\mathcal{I}_p$ do not form a system, it is still possible to make a meaningful definition of an OPE-algebra over a collection of functors $\mathcal{I}_p$, which we will now do.

Let M be a $\mathcal{D}_X$-module. An OPE-structure over a collection of $\mathcal{I}_p$ is a collection of maps

$$\mathfrak{ope}_{p_S} : M^{\boxtimes S} \to \mathcal{I}_{p_S}(M),$$

where $p_S : S \to \mathrm{pt}$, with certain properties. To formulate them, we first form maps

$$\mathfrak{ope}_p : M^{\boxtimes S} \to \mathcal{I}_p(M^{\boxtimes T})$$

for an arbitrary surjection $p : S \to T$, in the same way as it was done in the definition of an OPE-algebra over a system.

The natural maps $\mathcal{I}_p \to \mathfrak{i}_p$ give rise to maps

$$\mathfrak{ope}^{\mathfrak{i}}_p : M^{\boxtimes S} \to \mathfrak{i}_p(M^{\boxtimes T}).$$

Let $p = p_n p_{n-1} \cdots p_1$, where $p_i : S_i \to S_{i+1}$ and $\mathfrak{j}^1, \mathfrak{j}^2, \ldots, \mathfrak{j}^n$ be as above. We can construct maps

$$M^{\boxtimes S} \to \mathfrak{j}^1_{p_1} \cdots \mathfrak{j}^n_{p_n} M^{\boxtimes T}$$

as follows:

$$M^{\boxtimes S} \xrightarrow{\mathfrak{ope}^{\mathfrak{j}^1}_{p_1}} \mathfrak{j}^1_{p_1} M^{\boxtimes S_2} \xrightarrow{\mathfrak{ope}^{\mathfrak{j}^2}_{p_2}} \mathfrak{j}^1_{p_1} \mathfrak{j}^2_{p_2} M^{\boxtimes S_3} \cdots .$$

Thus for every object $X \in \mathbf{Zebra}_p$, we have a map

$$\mathfrak{ope}_X : M^{\boxtimes S} \to X(M^{\boxtimes T}).$$

Let $u : X \to Y$ be an arrow in $\mathbf{Zebra}_p$. We then have a composition

$$\mathfrak{ope}_X \circ u(M^{\boxtimes T}) : M^{\boxtimes S} \to Y(M^{\boxtimes T}).$$

We demand that this composition be equal to $\mathfrak{ope}_Y$. If this is the case, then we say that the maps $\mathfrak{ope}_{p_S}$ define an OPE-algebra structure on M over the collection $\mathcal{I}$.

We can now do the following. Set

$$\mathfrak{P}_p(M) = \mathrm{liminv}_{X \in \mathbf{Zebra}_p} X(M^{\boxtimes T}).$$

Then the above axiom implies that the maps $\mathfrak{ope}_X$ produce a map

$$\mathfrak{ope}^{\mathfrak{P}}_p : M^{\boxtimes S} \to \mathfrak{P}_p(M^{\boxtimes T}).$$

It is not hard to see that the functors $\mathfrak{P}_p$ form a system. Indeed: let $p = rs$. Then $\mathfrak{P}_s\mathfrak{P}_r$ can be realized as an inverse limit of $X(M^{\boxtimes T})$ over a full subcategory (= subset with an induced order) of $\mathbf{Zebra}(p)$ formed by all Xs whose flags contain the equivalence relation on S determined by r, whence a map

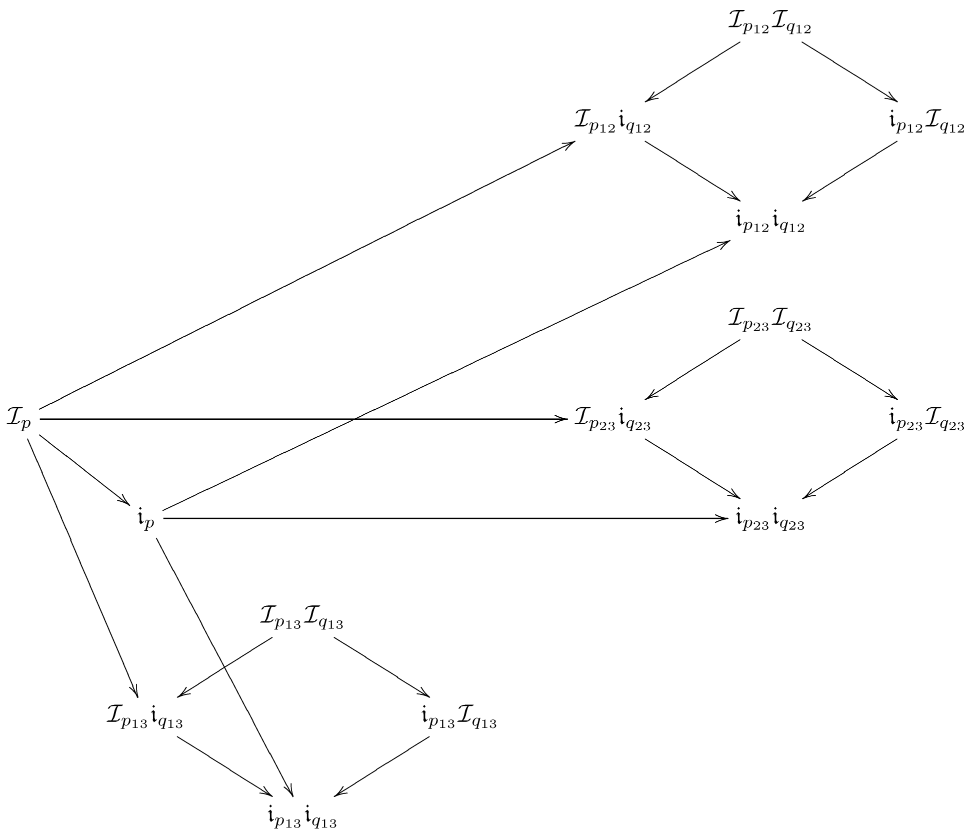
$$\mathfrak{P}_p \to \mathfrak{P}_s\mathfrak{P}_r.$$

8.3.1. *Example*

Let $S = \{1, 2, 3\}$ and $p : S \to \mathrm{pt}$. We have the following equivalence relations on S:

(a) the finest one ω;
(b) the relations e_{ij}, $i \neq j$, $i, j \in \{1, 2, 3\}$, in which $i \sim j$, and the remaining element is only equivalent to itself;
(c) the coarsest relation α in which all elements are equivalent.

Let $S_{ij} := S/e_{ij}$. Let $p_{ij} : S \to S/e_{ij}$ and $q_{ij} : S/e_{ij} \to \mathrm{pt}$. Then $\mathfrak{P}_p$ is the inverse limit of the following diagram:



This diagram is co-final to the sub-diagram:

$$\begin{array}{ccccc} & \nearrow & \mathcal{I}_{p_{12}}\mathfrak{i}_{q_{12}} & \xleftarrow[1]{} & \mathcal{I}_{p_{12}}\mathcal{I}_{q_{12}} \\ \mathcal{I}_p & \longrightarrow & \mathcal{I}_{p_{23}}\mathfrak{i}_{q_{23}} & \xleftarrow[2]{} & \mathcal{I}_{p_{23}}\mathcal{I}_{q_{23}} \\ & \searrow & \mathcal{I}_{p_{13}}\mathfrak{i}_{q_{13}} & \xleftarrow[3]{} & \mathcal{I}_{p_{13}}\mathfrak{i}_{q_{13}} \end{array} \tag{20}$$

We see that $\mathfrak{P}_p$ is an extension of $\mathcal{I}_p$ by the kernels of the arrows $1, 2, 3$, which are $\mathcal{I}_{p_{ij}}\mathfrak{i}_{q_{ij}*}$, where $i \neq j$, $i, j = 1, 2, 3$.

8.3.2. The features of functoriality of the collection of functors $\mathfrak{P}_p$ are inherited from those of the collection $\delta_p, \mathfrak{i}_p, \mathcal{I}_p$. The most important ones are the following ones:

(1) the structure of system on the collection of functors $\mathfrak{P}_p$;
(2) maps $\mathfrak{P}_p\delta_q\mathfrak{P}_r \to \mathfrak{P}_{rqp}$, where p, q, r are surjections and q is *not* a bijection.

Let us sketch the definition. First of all, such a map is uniquely defined by prescribing all compositions

$$f_X : \mathfrak{P}_p\delta_q\mathfrak{P}_r \to \mathfrak{P}_{rqp} \to X,$$

where X runs through the set of all elements in **Zebra**(p).

Let $R := rqp$; $R : S \to T$; let $Q = qp$. Let e (respectively e_q, respectively e_p) be the equivalence relation determined by R (respectively Q, respectively p). It follows that

$$\omega \geq e_p > e_q \geq e,$$

where ω is the trivial equivalence relation on S. Without loss of generality, we may assume that $p : S \to S/e_p$, $q : S/e_p \to S/e_q$, $r : S/e_q \to S/e$ are the natural projections.

Now let X be given by a flag

$$(\omega = f_1 > f_2 > \cdots f_{n+1} = e)$$

and a coloring $\mathfrak{j}^1, \mathfrak{j}^2, \ldots, \mathfrak{j}^n$.

The map f_X is then specified by the following conditions:

(1) $f_X = 0$ unless there exists a k such that $f_k = e_p > e_q \geq f_{k+1}$ and $\mathfrak{j}_k = \mathcal{I}_k$.
(2) Assume that such a k exists. Let $\rho : S/f_{k+1} \to T$ be the natural projection. Let $\sigma : S/e_q \to S/f_{k+1}$ so that $r = \rho\sigma$ and $\sigma q : S/f_k \to S/f_{k+1}$ is the natural projection. Define elements $X_r \in$ **Zebra**(r) and $X_\pi \in$ **Zebra**(π) as follows:

    X_r is given by the flag $\omega > f_1 > \cdots > f_k = e_r$, and the coloring $(\mathfrak{j}^1, \mathfrak{j}^2, \ldots, \mathfrak{j}^{k-1})$;

X_p is given by the flag

$$\omega_{S/f_k} \geq f_{k+1}/f_k > f_{k+2}/f_k > \cdots > e/f_k,$$

of equivalence relations on S/f_k. It follows that X decomposes as $X = X_p\mathcal{I}_{\sigma q}X_\rho$.

The map f_X then goes as follows:

$$\mathfrak{P}_p\delta_q\mathfrak{P}_r \to \mathfrak{P}_p\delta_q\mathfrak{P}_\sigma\mathfrak{P}_\rho \to X_p\delta_q\mathcal{I}_\sigma X_\rho \to X_p\mathcal{I}_{\sigma q}X_\rho = X.$$

8.3.3. *Example*

Let us come back to our example $S = \{1, 2, 3\}$ and $p : S \to \mathrm{pt}$. We know that $\mathfrak{P}_p$ is the inverse limit of the diagram (20). Let us describe the map

$$\delta_{p_{12}}\mathfrak{P}_{q_{12}} \to \mathfrak{P}_q.$$

First of all, $\mathfrak{P}_{q_{12}} \to \mathcal{I}_{q_{12}}$ is an isomorphism.

We then have maps

$$\delta_{p_{12}}\mathcal{I}_{q_{12}} \to \mathcal{I}_{p_{12}}\mathcal{I}_{q_{12}}$$

and

$$\delta_{p_{12}}\mathcal{I}_{q_{12}} \to \mathcal{I}_q.$$

The diagram

$$\begin{array}{ccc} \delta_{p_{12}}\mathcal{I}_{q_{12}} & \longrightarrow & \mathcal{I}_{p_{12}}\mathcal{I}_{q_{12}} \\ \downarrow & & \downarrow \\ \mathcal{I}_p & \longrightarrow & \mathcal{I}_{p_{12}}\mathfrak{i}_{q_{12}} \end{array}$$

turns out to be commutative (this is hidden behind the words "these maps behave well with respect to the other elements of the structure" after (19)). Furthermore, the compositions

$$\delta_{p_{12}}\mathcal{I}_{q_{12}} \to \mathcal{I}_q \to \mathcal{I}_{p_{23}}\mathfrak{i}_{q_{23}}, \mathcal{I}_{p_{13}}\mathfrak{i}_{q_{13}}$$

as well as

$$\delta_{p_{12}}\mathcal{I}_{q_{12}} \to \mathcal{I}_{p_{12}}\mathcal{I}_{q_{12}} \to \mathfrak{i}_{p_{12}}\mathcal{I}_{q_{12}}$$

all vanish, whence the desired map $\delta_{p_{12}}\mathfrak{P}_{q_{12}} \to \mathfrak{P}_p$.

Consider now the map $\mathfrak{P}_{p_{12}}\delta_{q_{12}} \to \mathfrak{P}_p$. Again, we have an isomorphism

$$\mathfrak{P}_{p_{12}} \to \mathcal{I}_{p_{12}}.$$

We also have a map

$$\mathcal{I}_{p_{12}}\delta_{q_{12}} \to \mathcal{I}_{p_{12}}\mathcal{I}_{q_{12}},$$

the composition

$$\mathcal{I}_{p_{12}}\delta_{q_{12}} \to \mathcal{I}_{p_{12}}\mathcal{I}_{q_{12}} \to \mathcal{I}_{p_{12}}\mathfrak{i}_{q_{12}}$$

being zero. Furthermore, the sequence

$$0 \to \mathcal{I}_{p_{12}}\delta_{q_{12}} \to \mathcal{I}_{p_{12}}\mathcal{I}_{q_{12}} \to \mathcal{I}_{p_{12}}\mathfrak{i}_{q_{12}} \to 0$$

is exact. Therefore, the map

$$\mathcal{I}_{p_{12}}\delta_{q_{12}} \to \mathfrak{P}_p$$

realizes an embedding of the kernel of the arrow 1 in (20) into $\mathfrak{P}_p$.

Describe the map $\delta_p \to \mathfrak{P}_p$. It is given by the inclusion $\delta_p \to \mathcal{I}_p$; since the composition of this map with every arrow coming out of $\mathcal{I}_p$ vanishes, this is a well-defined map. This map can also be described as a composition:

$$\delta_p \cong \delta_{p_{12}}\delta_{q_{12}} \to \delta_{p_{12}}\mathcal{I}_{q_{12}} \to \mathfrak{P}_p.$$

Finally, the map

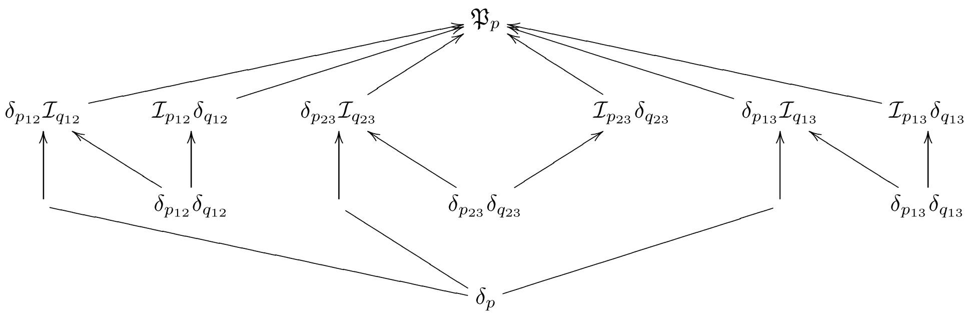
$$\delta_{p_{12}}\delta_{q_{12}} \to \mathfrak{P}_p$$

is given by

$$\delta_{p_{12}}\delta_{q_{12}} \to \mathcal{I}_{p_{12}}\mathcal{I}_{q_{12}}$$

and is different from the previous one!

The maps that we considered fit into a commutative diagram



This diagram specifies a map from the direct limit of its three lowest floors to $\mathfrak{P}_p$. It turns out that this map is an inclusion whose cokernel is isomorphic to $\mathfrak{i}_p$ via the natural map $\mathfrak{P}_p \to \mathcal{I}_p \to \mathfrak{i}_p$.

This implies that $\mathfrak{P}_p$ has a three-term filtration (the two lowest floors are combined) whose successive quotients are

(1) $\delta_p \oplus_{i<j} \delta_{p_{ij}}\delta_{q_{ij}}$;

(2) $\oplus_{i<j}\delta_{p_{ij}}\mathfrak{i}_{q_{ij}} \oplus \mathfrak{i}_{p_{ij}}\delta_{q_{ij}}$
(3) $\mathfrak{i}_p$.

8.3.4. *Filtration on* $\mathfrak{P}$

The filtration on functors I_p define a filtration on $\mathfrak{P}_p$. See Secs. 15.1–15.2.3 for its description. Its successive quotients are direct sums of the terms of the form

$$\mathfrak{i}_{p_1}\delta_{q_1}\mathfrak{i}_{p_2}\delta_{q_2}\cdots\delta_{q_n}\mathfrak{i}_{p_{n+1}}$$

with fixed n. Here $p_{n+1}q_np_n\cdots q_1p_1 = p$; all p's and q's are surjective and all q's are not bijective.

8.4. *Resolution* $\mathcal{R}$

We are now ready to define the desired resolution. The starting point is the maps $\mathfrak{P}_p \to \mathfrak{i}_p$, which are surjections. Our goal is to kill the kernel, which turns out to be spanned by the images of all maps

$$\mathfrak{P}_a\delta_b\mathfrak{P}_c \to \mathfrak{P}_p,$$

where $cba = p$.

Thus, it makes sense to assign

$$\mathcal{R}_p^0 := \mathfrak{P}_p$$

and

$$\mathcal{R}_p^{-1} := \oplus\mathfrak{P}_a\delta_b\mathfrak{P}_c,$$

where the direct sum is taken over all sequences

$$S \xrightarrow{c} S/e_1 \xrightarrow{b} S/e_2 \xrightarrow{a} T,$$

where $e_1 > e_2 > e$ are equivalence relations on S, e is determined by p, and a, b, c are natural projections. The differential is given by the above described maps $\mathfrak{P}_c\delta_b\mathfrak{P}_a$.

The nth term $\mathcal{R}_p^{-n}$ is given by the direct sum of the terms

$$\mathfrak{P}_{p_1}\delta_{q_1}\mathfrak{P}_{p_2}\delta_{q_2}\cdots\delta_{q_n}\mathfrak{P}_{p_{n+1}},$$

where the sum is taken over all diagrams of the form

$$S \xrightarrow{p_1} S/e_1 \xrightarrow{q_1} S/f_1 \xrightarrow{p_2} S/e_2 \xrightarrow{q_2} S/f_2 \xrightarrow{p_3} \cdots \xrightarrow{q_n} S/f_n \xrightarrow{p_{n+1}} T,$$

where

$$e_1 > f_1 \geq e_2 > f_2 \geq \cdots > f_n \geq e,$$

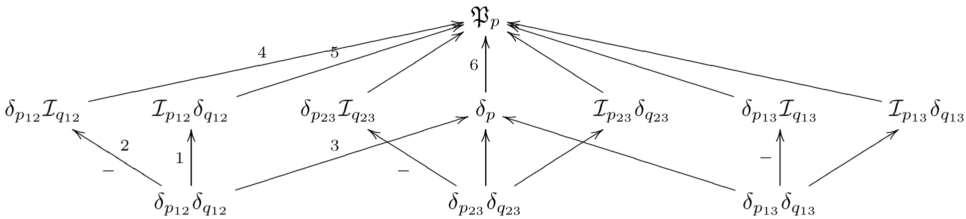
and p_i, q_j are all natural projections. The differential $d : \mathcal{R}_p^{-n} \to \mathcal{R}_p^{-n+1}$ is given by the alternated sum of maps induced by

(a) $\mathfrak{P}_{p_i}\delta_{q_i}\mathfrak{P}_{p_{i+1}} \to \mathfrak{P}_{p_{i+1}q_ip_i}$ and
(b) $\delta_{q_i}\mathfrak{P}_{p_{i+1}}\delta_{q_{i+1}} \to \delta_{q_{i+1}p_{i+1}q_1}$, which are nonzero iff $p_{i+1} = \mathrm{Id}$, in which case they are natural isomorphisms.

One then has to check that $d^2 = 0$ and to define on $\mathcal{R}_p$ a structure of system. For all this we refer the reader to Sec. 15.5.

8.4.1. *Example*

Let $S = \{1, 2, 3\}$. Then the complex $\mathcal{R}_p$ is depicted as follows:



where all the arrows are the natural maps; the arrows marked with — are taken with the negative sign. Let us check that $d^2 = 0$. It suffices to check that

$$d^2|_{\delta_{p_{12}}\delta_{q_{12}}} : \delta_{p_{12}}\delta_{q_{12}} \to \mathfrak{P}_p$$

is zero. This reduces to checking that the compositions

$$Ad^2|_{\delta_{p_{12}}\delta_{q_{12}}} : \delta_{p_{12}}\delta_{q_{12}} \to \mathfrak{P}_p \xrightarrow{A} \mathcal{I}_p;$$

$$B_{ij}d^2|_{\delta_{p_{12}}\delta_{q_{12}}} : \delta_{p_{12}}\delta_{q_{12}} \to \mathfrak{P}_p \xrightarrow{B_{ij}} \mathcal{I}_{p_{ij}}\mathcal{I}_{q_{ij}}$$

do all vanish. Let us so do.

Ad^2. We have: $A42 = A63$; $A51 = 0$. Hence $Ad^2 = A42 - A63 + A51 = 0$.

$B_{ij}d^2$. If $\{i, j\} \neq \{1, 2\}$, then all three maps

$$B_{ij}42 = B_{ij}63 = B_{ij}51 = 0.$$

Consider now the remaining case $B_{12}d^2$. We then have: $B_{12}42 = B_{12}51$ and $B_{12}63 = 0$, which implies that $B_{12}d^2 = 0$.

8.5. *The system $\langle\mathfrak{m}\rangle$ and the map $\langle\mathcal{R}\rangle \to \langle\mathfrak{m}\rangle$*

Recall that the whole purpose of constructing $\langle\mathcal{R}\rangle$ was to establish a link between the systems $\langle\mathfrak{i}\rangle$ and $\langle\mathfrak{l}\rangle$. Unfortunately, there is no direct map $\langle\mathcal{R}\rangle \to \langle\mathfrak{l}\rangle$.

Instead, we shall construct a map $m : \langle \mathcal{R} \rangle \to \langle \mathfrak{m} \rangle$ satisfying the properties described in Sec. 5.1.

Define a map $m_p : \mathcal{R}_p \to \mathfrak{m}_p$ by the following conditions.

(1) m_p vanishes on all terms

$$\mathcal{R}_{p_1} \delta_{q_1} \mathcal{R}_{p_2} \delta_{q_2} \cdots \delta_{q_n} \mathcal{R}_{p_{n+1}}$$

where at least on $p_i \neq \mathrm{Id}$. Otherwise, m_p is the identical embedding onto the term

$$\delta_{q_1} \delta_{q_2} \cdots \delta_{q_n}$$

of $\mathfrak{m}_p$.

Denote by

$$l_p : \mathcal{R}_p \to \mathfrak{m}_p \to \delta_p[1]$$

the natural composition.

8.6. *The additional structure induced by the maps* (17)

Recall that the collection of maps $\langle \mathcal{I}_p \rangle$ has a functoriality (17) which we have never used. It turns out that this additional functoriality yields an additional structure on the system $\langle \mathcal{R} \rangle$.

To obtain this additional structure one has to first understand the additional structure on the system $\langle \mathfrak{P} \rangle$ produced by these functors. Consider some examples.

Let $A = \{1, 2\}$ be a two-element set and let $g : A \to \mathrm{pt}$. Let $f : S \to T$ be a surjection. Let

$$f \sqcup g : S \sqcup A \to T \sqcup \mathrm{pt}$$

be a disjoint union.

We may define two maps

$$n_1, n_2 : \mathfrak{P}_{f \sqcup g} \to \mathcal{I}_{f \sqcup \mathrm{Id}_A} \mathcal{I}_{\mathrm{Id}_T \sqcup g}.$$

The map n_1 is just the natural projection onto a member of $\mathbf{Zebra}_{f \sqcup g}$. The map n_2 is the composition

$$\mathfrak{P}_{f \sqcup g} \to \mathcal{I}_{f \sqcup g} \to \mathcal{I}_{f \sqcup \mathrm{Id}_A} \mathcal{I}_{\mathrm{Id}_T \sqcup g},$$

where we first apply the natural projection and then the map (17).

It follows that the compositions of n_1, n_2 with the map

$$\lambda : \mathcal{I}_{f \sqcup \mathrm{Id}_A} \mathcal{I}_{\mathrm{Id}_T \sqcup g} \to \mathcal{I}_{f \sqcup \mathrm{Id}_A} \mathfrak{i}_{\mathrm{Id}_T \sqcup g}$$

do coincide, therefore the difference $n_2 - n_1$ determines a map to the kernel of λ, i.e. a map

$$\xi(f,g)\colon \mathfrak{P}_{f\sqcup g} \to \mathcal{I}_{f\sqcup \mathrm{Id}_A}\delta_{\mathrm{Id}_T\sqcup g}.$$

This is only true because of the special form of g.

For a general $g\colon A \to B$ the kernel of λ is spanned by the images of all maps

$$\mathcal{I}_{f\sqcup \mathrm{Id}_A}\delta_{u_1}\mathcal{I}_{u_2} \to \mathcal{I}_{f\sqcup \mathrm{Id}_A}\mathcal{I}_{\mathrm{Id}_T\sqcup g},$$

where u_1, u_2 are surjections, u_1 is not a bijection, and

$$u_2 u_1 = \mathrm{Id}_T \sqcup g.$$

So that the structure of $n_1 - n_2$ becomes more complicated.

Nevertheless, one can define maps

$$\xi(f,g)\colon \mathfrak{P}_{f\sqcup g} \to \mathcal{I}_{f\sqcup \mathrm{Id}_A}\delta_{\mathrm{Id}_T\sqcup g}$$

for an arbitrary g by means of the following inductive process. Let $|g| = |A| - |B|$. Since g is a surjection, $|g| \geq 0$. If g is a bijection, we then have a natural isomorphism

$$\mathfrak{P}_{f\sqcup g} \to \mathfrak{P}_{f\sqcup \mathrm{Id}_A}\delta_{\mathrm{Id}_T\sqcup g},$$

because $\mathrm{Id}_T \sqcup g$ is a bijection.

Set $\xi(f,g)$ to be the composition of this isomorphism with the natural map

$$\mathfrak{P}_{f\sqcup \mathrm{Id}_A}\delta_{\mathrm{Id}_T\sqcup g} \to \mathcal{I}_{f\sqcup \mathrm{Id}_A}\delta_{\mathrm{Id}_T\sqcup g}.$$

Let us now assume that $\xi(f,g)$ is defined for all g with $|g| < N$. Define it for all g with $|g| = N$. Let e be an equivalence relation on A induced by g. Let $\omega_S > \epsilon \geq e$, let $h_\epsilon\colon A \to A/\epsilon$ and $k_\epsilon\colon S/\epsilon \to B$ so that $k_\epsilon h_\epsilon = g$.

Define a map

$$C(\epsilon)\colon \mathfrak{P}_{f\sqcup g} \to \mathfrak{P}_{f\sqcup k_\epsilon}\mathfrak{P}_{\mathrm{Id}_T\sqcup \mathrm{Id}_{A/\epsilon}} \to \mathcal{I}_{f\sqcup \mathrm{Id}_A}\delta_{\mathrm{Id}_T\sqcup k_\epsilon}\mathcal{I}_{\mathrm{Id}_T\sqcup \mathrm{Id}_{A/\epsilon}} \to \mathcal{I}_{f\sqcup \mathrm{Id}_A}\mathcal{I}_{\mathrm{Id}_T\sqcup g},$$

set

$$\xi'(f,g) := -\sum_{\epsilon} C(\epsilon).$$

If $\xi'(f,g)$ passes through $\mathcal{I}_{f\sqcup \mathrm{Id}_A}\delta_{\mathrm{Id}_T\sqcup g}$, it determines a map

$$\mathfrak{P}_{f\sqcup g} \to \mathcal{I}_{f\sqcup \mathrm{Id}_A}\delta_{\mathrm{Id}_T\sqcup g},$$

which we assign to be $\xi(f,g)$. It can be checked that if this rule was obeyed when $\xi(f,g)$ was defined for all g with $|g| < N$, then $C'(f,g)$ passes through $\mathcal{I}_{f\sqcup \mathrm{Id}_A}\delta_{\mathrm{Id}_T\sqcup g}$ and gives rise to the map $\xi(f,g)$.

On the next step the maps $\xi(f,g)$ are lifted to maps

$$c(f,g):\mathfrak{P}_{f\sqcup g}\to\mathfrak{P}_{f\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g},$$

which in turn produce maps

$$s(f,g):\mathcal{R}_{f\sqcup g}\to\mathcal{R}_{f\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$$

with a nonzero differential, which is described in (17.10).

8.6.1. Thus, we described a construction of a pre-symmetric (upto homotopies) system $\langle\mathcal{R}\rangle$.

9. Realization of the System $\langle\mathcal{R}^{\mathbf{symm}}\rangle$ in the Spaces of Real-Analytic Functions

Our answer to the renormalization problem l is given in terms of a system $\langle\mathcal{R}^{\mathbf{symm}}\rangle$. To be able to get a physically meaningful answer we need an OPE expansion in terms of series of real-analytic functions on the Y^S minus all generalized diagonals.

The nicest possible way to do it includes constructing a system which is explicitly linked to the spaces of real-analytic functions on Y^S minus all generalized diagonals and constructing a map from $\langle\mathcal{R}^{\mathbf{symm}}\rangle$ to this system. Unfortunately, we do not know how to realize this project. The problem is that arbitrary real-analytic functions do not have a good asymptotic expansion in a neighborhood of generalized diagonals, therefore, we cannot form a system based on such spaces.

Let us describe a palliative measure we take instead.

First of all, we shall work with spaces of global sections rather than with sheaves. So, whenever we use a notation for a sheaf, it will actually mean the space of global sections. If our sheaf is a $\mathcal{D}_{X^S}$-module, then its space of global sections is a module over the space of global sections of $\mathcal{D}_{X^S}$. Whenever we say "a $\mathcal{D}_{X^S}$-module", we actually mean "a module over the space of global sections of $\mathcal{D}_{X^S}$".

Let $Y\subset Y^S$ be the main diagonal. We pick a vector field which contracts everything to Y and take analytic functions on Y^S minus the complement to all generalized diagonals which are generalized eigenvalues of this field.

Denote this space spanned by such functions by $\mathcal{A}^\circ{}_S$. This space has a grading given by the generalized eigenvalue. Let $\mathcal{A}^\circ{}_S^{\geq N}$ be the span of all elements whose generalized eigenvalue is $\geq N$.

Then the spaces $\mathcal{A}^\circ{}_S^{\leq N}:=\mathcal{A}^\circ{}_S/\mathcal{A}^\circ{}_S^{\geq N}$ do not depend on a choice of particular vector field.

Let $p:S\to T$ be a projection. We define a functor $\mathcal{A}^\circ{}_p$ from the category of $\mathcal{D}_{X^T}$-modules to the category of $\mathcal{D}_{X^S}$-modules by the formula

$$\mathcal{A}^\circ{}_p(M)=\mathrm{liminv}_N i_p^\wedge(M)\otimes_{\mathcal{O}_{X^S}}\mathcal{A}^\circ{}_S^{\leq N}.$$

These functors do not form a system. Nevertheless, given $a \in \mathcal{A}^\circ{}_p(M)$, $b \in \mathcal{A}^\circ{}_{p_1}\mathcal{A}^\circ{}_{p_2}(M)$, where $p = p_2 p_1$, one can say whether b is an asymptotic decomposition of a or not. The problem is that not every a has such a decomposition.

We define a functor

$$\Gamma^{\circ\circ}(p_1, p_2) \subset \mathcal{A}^\circ{}_p \oplus \mathcal{A}^\circ{}_{p_1}\mathcal{A}^\circ{}_{p_2}$$

so that $\Gamma^{\circ\circ}(p_1, p_2)(M)$ consists of all pairs (a, b) such that b is an asymptotic decomposition of a. In other words, instead of a map $\mathcal{A}^\circ{}_p \to \mathcal{A}^\circ{}_{p_1}\mathcal{A}^\circ{}_{p_2}$ we have a "correspondence" given by $\Gamma^{\circ\circ}(p_1, p_2)$.

Next, we construct maps $\int_p : \mathcal{R}_p^{\mathbf{symm}} \to \mathcal{A}^\circ{}_p$. We then show that these maps are compatible with the correspondences $\Gamma^{\circ\circ}(p_1, p_2)$ as follows:

Let

$$\int_{p_1, p_2} : \mathcal{R}_p^{\mathbf{symm}} \to \mathcal{R}_{p_1}^{\mathbf{symm}}\mathcal{R}_{p_2}^{\mathbf{symm}} \xrightarrow{\int_{p_1} \otimes \int_{p_2}} \mathcal{A}^\circ{}_{p_1}\mathcal{A}^\circ{}_{p_2}.$$

We then show that

$$\int_p \oplus \int_{p_1, p_2} : \mathcal{R}_p^{\mathbf{symm}} \to \mathcal{A}^\circ{}_p \oplus \mathcal{A}^\circ{}_{p_1}\mathcal{A}^\circ{}_{p_2}$$

passes through $\Gamma^{\circ\circ}(p_1, p_2)$.

This construction provides us with an OPE product on M in terms of series of real-analytic functions on Y^S.

The construction of the maps $\int_p$ resembles the construction of the maps of Sec. 8.6, which is based on the maps (17). The construction of $\int_p$ is based on the existence of asymptotic decompositions of generalized functions from $\mathcal{C}_S$ near generalized diagonals. Namely, let $p_S : S \to \mathrm{pt}$, and let $p_S = p_2 p_1$ be a decomposition. We construct maps

$$\mathcal{C}_S \to \mathcal{A}_{p_1}\mathcal{I}_{p_2},$$

where $\mathcal{A}_{p_1}$ is constructed in the same way as $\mathcal{A}^\circ{}_{p_1}$ but generalized functions which are non-singular on the complement to generalized diagonals and are generalized eigenvectors of the vector field which shrinks everything to the main diagonal, are used.

Part III: Technicalities

In the concluding part of the paper we give constructions and proof required for everything in the previous part to work. This includes

(1) constructing the system $\langle \mathcal{R} \rangle$ and endowing it with a pre-symmetric structure;
(2) Bogoliubov–Parasyuk lifting theorem;

(3) more details on the symmetrization procedure and on the renormalization in symmetric systems. To this end we need to develop certain machinery ("pseudo-tensor bodies");
(4) real-analytic interpretation of the symmetric system that we obtain from $\langle\mathcal{R}\rangle$.

10. Constructing the System $\langle\mathcal{R}\rangle$

10.0.1. Let $Y = \mathcal{R}^N$, where N is a fixed natural even number. We fix the coordinates $x^1, x^2, \ldots, x^N$ on Y. For $x \in Y$ we set $q(x) = \sum_{i=1}^N (x^i)^2$. Also we take the standard orientation on Y.

10.0.2. Let S be a finite set. Let Y^S be the space of functions $S \to Y$. Let $[n] = \{1, 2, \ldots, n\}$, then $Y^{[n]} \cong Y^n$. Since Z is even-dimensional, the orientation on Y produces canonically an orientation on Y^S. Thus, Y^S will be assumed to have an orientation.

Let e be an equivalence relation on S. Denote by $\Delta_e \subset Y^S$ the corresponding generalized diagonal consisting of points $y : S \to Y$ such that $s \sim_e t \Rightarrow y(s) = y(t)$.

Let $f : S \to T$ be a map of finite sets. We have an induced map $f^\# : Y^T \to Y^S$. Let $p_e : S \to S/e$. Then $\Delta_e = \mathrm{Im} p_e$. If f is surjective, then f identifies Δ_e with $Y^{S/e}$. We will use this identification.

For two equivalence relations e_1 and e_2 on a finite set S we write $e_1 \leq e_2$ iff $s \sim_{e_2} t \Rightarrow s \sim_{e_1} t$. We have $e_1 \leq e_2$ iff $\Delta_{e_1} \subseteq \Delta_{e_2}$. Denote by α the least equivalence relation (i.e. every two points are equivalent) and by ω the greatest equivalence relation (i.e. every two distinct points are *not* equivalent). Let $s, t \in S$ be distinct elements.

Let $T \subset S$. Denote by e_T the equivalence relation in which two distinct elements are equivalent iff both of them are in T. For example, $\omega = e_\emptyset$; $\alpha = e_S$. Set $\Delta_T := \Delta_{e_T}$; $\Delta_{st} := \Delta_{\{s;t\}}$.

10.0.3. Denote

$$U_S = Y^S - \bigcup_{e \neq \omega} \Delta_e.$$

Obviously, a point $y : S \to Y$ is in U_S iff the map y is injective.

10.0.4. Let $s, t \in S$ be distinct elements. Denote by $q_{st} : Y^S \to \mathcal{R}$ the function defined according to the rule

$$q_{st}(y) = q(y(s) - y(t)), \tag{21}$$

where $y : S \to Y$ is a point in Y^S and q is the standard quadratic form on Y. Of course, the set of zeros of $q_{st}(y)$ is Δ_{st}.

10.0.5. Denote by B_S the space of functions $U_S \to \mathcal{C}$ which can be expressed as a ratio $P(Y)/Q(Y)$, where P is an arbitrary polynomial and Q is a product of non-negative integer powers of q_{st} for arbitrary s, t.

10.0.6. As usual, we denote by $\mathfrak{D}_{Y^S}$ the space of compactly supported top forms on Y and by $\mathfrak{D}'_{Y^S}$ the space of distributions on $\mathfrak{D}_{Y^S}$. Any smooth function on Y^S will be regarded as a distribution in the usual way (recall that the orientation on Y produces a canonical orientation on Y^S).

10.1. *Quasi-polynomial distributions*

We have a diagonal action of the group $\mathcal{R}^N$ on Y^S by translations. This induces an action of the abelian N-dimensional Lie algebra $\mathfrak{t}_N$ on $\mathfrak{D}_{Y^S}$; $\mathfrak{D}'_{Y^S}$. Call a distribution f quasi-polynomial if there exists an M such that $t^M f = 0$. Let $\mathfrak{P}_{Y^S}$ be the subspace of all quasi-polynomial distributions.

We have natural continuous maps

$$T_{S_1S_2} : \mathfrak{D}'_{Y^{S^1}} \otimes \mathfrak{D}'_{Y^{S^2}} \to \mathfrak{D}'_{Y^{S^1 \sqcup S^2}}$$

which induce maps:

$$T_{S_1S_2} : \mathfrak{P}_{Y^{S^1}} \otimes \mathfrak{P}_{Y^{S^2}} \to \mathfrak{P}_{Y^{S^1 \sqcup S^2}}. \tag{22}$$

10.2. *Definition of subspaces $C_S \in \mathfrak{P}_{Y^S}$*

We define these subspaces recursively.

(1) If S is empty or has only one element, we set $C_S := \mathfrak{P}_{Y^S}$.
(2) Suppose, we have already defined $C_S \subset \mathfrak{P}_{Y^S}$ for all S with at most m elements. For an S with $m+1$ elements, we say that a quasi-polynomial distribution f on Y^S is in C_S iff for any partition $S = S_1 \sqcup S_2$, there exists an integer M such that

$$\left(\prod_{s_1 \in S_1; s_2 \in S_2} q_{s_1 s_2}^M \right) f \in T_{S_1S_2}(C_{S_1} \otimes C_{S_2}), \tag{23}$$

where $T_{S_1S_2}$ is as in (22) and $q_{s_1s_2}$ is as in (21).

10.3. *Example*

Let $S = \{1, 2\}$. We will also use the symbol [2] for $\{1, 2\}$. For $y : [2] \to Y$ we write $y_1 := y(1)$ and $y_2 := y(2)$. Then $f \in C_S$ iff f is quasi-polynomial and there exists an M such that $q(y_1 - y_2)^M f = P(y_1, y_2)$, where P is a polynomial.

Define a map $\pi : C_S \to B_S$ by $\pi f = P/q^M$. It is clear that this map is well-defined and that $\operatorname{Ker}\pi$ consists of all functions $f \in C_S$ supported on the diagonal. Denote $C_{S,\Delta} := \operatorname{Ker}\pi$. We are going to describe this space.

Let $s = (y, y)$ be a point on the diagonal. Then on any relatively compact neighborhood U of s, any distribution supported on the diagonal is of the form

$$f = \sum_\mu f_\mu(y_1)\delta^\mu(y_2 - y_1), \tag{24}$$

where the sum is taken over a finite set of multi-indices μ and f_μ are distributions on Y.

Suppose that $f \in C_{S,\Delta}$. Then f is quasi-polynomial, $t^{M'} f = 0$ for some M'. Therefore, $t^{M'} f_\mu = 0$ for all μ meaning that each f_μ is a polynomial of degree less than M'. This immediately implies that (24) is true everywhere for some polynomials f_μ. Conversely, if all f_μ are polynomials, then $f \in C_{S,\Delta}$.

The map π defines an injection $C_S/C_{S,\Delta} \to B_S$. Let us show that this is in fact a bijection. This means that for any integer $M > 0$ and any polynomial $P(y_1, y_2)$ there exists a distribution F such that $Fq(y_1 - y_2)^M = P$. It is sufficient to do it for $P = 1$. Let us construct such an F.

10.3.1. To this end, take an $f dy_1 dy_2 \in \mathfrak{D}_{Y^S}$, where dy is the standard volume form on $Y = R^N$ and consider the expression

$$Z(s, f) = \int_{Y^2} f(y_1, y_2) q(y_1 - y_2)^s dy_1 dy_2.$$

Claim 10.1. *This integral uniformly converges on any strip* $\operatorname{Re} s > K$, *where* $K > -N/2$.

Proof. To show it, change the variables $y = y_1$, $z = y_2 - y_1$, and $g(y, z) = f(y, y + z)$. Then

$$Z(s, f) = \int_{Y^2} g(y, z) q(z)^s dy dz.$$

Let $S^{N-1} \subset Y$ be the unit sphere $q(y) = 1$. Let $\alpha : R_+ \times S^{N-1} \to Y$ be the map: $\alpha(r, n) = rn$. Let dn be the measure on S^{n-1} determined by q. Then

$$Z(s, f) = \int_0^\infty r^{2s+N-1} h_f(r) dr,$$

where

$$h_f(r) = \int_{Y \times S^{N-1}} g(y, rn) dy dn.$$

Whence the statement. $\square$

10.3.2. Since $Z(s,f)$ is (up to a shift) the Mellin transform of h_f, we know that $Z(s,f)$ has a meromorphic continuation to the whole complex plane, the poles can only occur at $s = -(N+k)/2$, $k = 0, 1, 2, \ldots$ and are of at most first order. Denote

$$U_M(f) = \mathrm{res}_{s=-M} \frac{Z(s,f)}{s+M}.$$

Claim 10.2. *U_M is a distribution.*

Proof. Set $s' = 2s + N - 1$; $M' = -2M + N - 1$ (M' corresponds to $s = M$). Integration by parts yields:

$$Z(s,f) = \frac{(-1)^P}{(s'+1)(s'+2)\cdots(s'+P)} \int_0^\infty \left(\frac{d^P}{dr^P} h_f(r)\right) r^{s'+P} dr,$$

whenever $s' + P > 0$ and $s' \neq -1, -2, \ldots, -P+1$. Choose P large enough so that $M' + P > 1$. Set

$$l(s', r) = \frac{(-1)^P}{(s'+1)(s'+2)\cdots(s'+P)} r^{s'+P}$$

and

$$\lambda(r) = \mathrm{res}_{s'=M'} l(s', r) = r(M'+P) C_{M',P},$$

where $C_{M',P}$ is a constant. Thus,

$$U_{M'}(f) = C_{M',P} \int_0^\infty r^{M'+P} \frac{d^P}{dr^P} h_f(r) dr.$$

It is clear that the function h_f is smooth and rapidly decreasing as $r \to \infty$. Furthermore, $f \mapsto h_f$ is a continuous map from $\mathfrak{D}_{Y^S}$ to the space of rapidly decreasing infinitely differentiable functions on $[0, \infty]$, in which the topology is given by the family of seminorms

$$\|h\|_{K,L} = \max_r r^L |h^{(K)}(r)|.$$

Since the map $f \mapsto h_f$ is continuous, so is U_M, whence the statement. □

Claim 10.3. (1) $q^M(Y_1 - Y_2)U_M = 1$
(2) $\mathfrak{t}.U_M = 0$. (*for* $\mathfrak{t}$ *see Sec.* 10.1).

Proof. (1)

$$U_M(q^M(Y_1 - Y_2)f) = \mathrm{res}_{s=-M} \frac{Z(s, q^M f)}{s+M} = \mathrm{res}_{s=-M} \frac{Z(s+M, f)}{s+M}$$
$$= \mathrm{res}_{s=0} \frac{Z(s,f)}{s} = Z(0,f) = \int f dy_1 dy_2.$$

(2) Obvious. □

Corollary 10.4. *$U_M \in C_S$; $\pi(U_M) = 1/q^M$. Therefore, π is surjective.*

Thus, we have an exact sequence:

$$0 \to C_{S,\Delta} \to C_S \to B_S \to 0. \tag{25}$$

We see that this is an extension of $\mathfrak{D}_S$-modules. From our description of $C_{S,\Delta}$, it follows that $C_{S,\Delta} = i_* \mathcal{O}_\Delta$, where $i : \mathcal{D} \to Y^2$ is the diagonal embedding. One can show that this extension does not split. One can construct a similar extension when N is odd, in which case it splits; the reason is that the Green function for the Laplace operator requires extraction a square root.

11. Study of C_S

11.1. *Action of differential operators*

Denote by $\mathfrak{D}_{Y^S}$ the algebra of polynomial differential operators on Y^S, it is clear that each $\mathfrak{P}_{Y^S}$ is a $\mathfrak{D}_{Y^S}$-module.

Claim 11.1. *Each C_S is a $\mathfrak{D}_{Y^S}$-submodule of $\mathfrak{P}_{Y^S}$.*

Proof. This is obvious when S has 0 or 1 element. For an arbitrary S the proof can be easily done by induction. Indeed, we only need to check that for any $f \in C_S$ and any polynomial differential operator D, Df satisfies (23). It suffices to consider only operators of zeroth and first order. If the order of D is zero, the statement is immediate. Assume that the order of D is 1 and $D1 = 0$. Let

$$Q_{S_1 S_2} = \prod_{s_1 \in S_1; s_2 \in S_2} q_{s_1 s_2}$$

and $fQ^M \in T_{S_1 S_2}(C_{S_1} \otimes C_{S_2})$. It is immediate that the space on the right-hand side is a $\mathfrak{D}_{Y^S}$-submodule of $\mathfrak{P}_{Y^S}$. We then have

$$Q^{M+1} Df = D(Q^{M+1} f) - (M+1) Q^M (DQ) f \in T_{S_1 S_2}(C_{S_1} \otimes C_{S_2}). \qquad \square$$

11.2. *Map $\pi : C_S \to B_S$ and its surjectivity*

11.2.1. Let $f \in C_S$.

Claim 11.2. *There exists a natural number M such that*

$$\left(\prod_{\{s,t\} \subset S} q_{st} \right)^M f = P, \tag{26}$$

where P is a polynomial and the product is taken over all 2-element subsets of S.

Proof. This is obvious when S is empty or has only one element. For general S the argument follows from (23) by induction on the number of elements in S. $\square$

Write

$$\pi(f) = P\Big/\left(\prod_{s\neq t} q_s t\right)^M .$$

It is clear that $\pi(f)$ depends only on f and that $\pi : C_S \to B_S$ is a $\mathfrak{D}_{Y^S}$-module map.

Proposition 11.3. *The map π is surjective.*

11.3. *Proof of Proposition 11.3*

It is sufficient to construct for every integer $M > 0$ an $F \in C_S$ such that

$$F\left(\prod_{\{s,t\}\subset S} q_{st}\right)^M = 1.$$

This is what we are going to do.

11.3.1. For convenience, denote by $P := P_2(S)$ the set of all 2-element subsets of S; for $T = \{s,t\} \in P$ write $q_T = q_{st}$. Denote $\mathcal{U} = \mathbb{C}^P$; for $s \in U$, write

$$q^s := \prod_{T\in P} q_T^{s_T}.$$

It is clear that for every $s \in \mathcal{U}$, $q^s : Y^S \to \mathbb{C}$ is an analytic function on U_S.

11.3.2. Denote by dy the standard volume form on Y; set

$$\Omega := \prod_{s\in S} dy_s.$$

Note that the product does not depend on the order of multiples. Let $f\Omega \in \mathfrak{D}_{Y^S}$. Write

$$Z(f,s) = \int_{Y^S} (f q^s \Omega).$$

This integral converges if $\operatorname{Re} s_T > 0$ for every $T \in P$.

11.3.3.

Claim 11.4. *For any f, Z extends to a meromorphic function on $\mathcal{U}$. It can only have poles of the first order along the divisors of the form*

$$D(R,n) := \left\{\left(\sum_{T\subset R} 2s_T\right) + n = 0\right\},$$

where $R \subset S$ *is a subset with at least* 2 *elements*; T *is an arbitrary* 2*-element subset of* R; $n > (\#R-1)(N-1)$ *is a positive integer.*

Proof. Let $\mathfrak{FM}$ be the real Fulton–MacPherson compactification of U_S so that we have a surjection $P : \mathfrak{FM} \to Y^S$. Denote $V = P^{-1}U_S$. We know that P identifies V and U_S. The complement $\mathfrak{FM}\backslash V$ can be represented as $\mathfrak{FM}\backslash V = \cup_{R\subset S}\mathfrak{FM}_R$, where $\#R > 1$ and each $\mathfrak{FM}_R$ is a smooth subvariety of codimension 1; $P(\mathfrak{FM}_R) = \mathcal{D}_R$, where $\mathcal{D}_R$ is the diagonal given by the equivalence relation e_R on S in which $x \sim_{e_R} y$ and $x \neq y$ iff $x, y \in R$.

Let $P'(S)$ be the set of non-empty subsets of S. Let $K \subset P'(S)$. Then

$$\mathfrak{FM}_K := \bigcap_{R\in K} \mathfrak{FM}_R \neq \emptyset$$

if and only if for every R_1, R_2 from K, either one of them is inside the other, or they do not intersect. In this case we call K *forest*. Let

$$\mathfrak{FM}_K^o := \mathfrak{FM}_K \Big\backslash \bigcup_{L\supset K, L\neq K} \mathfrak{FM}_L.$$

For every point $x \in \mathfrak{FM}_K^o$, there exists a neighborhood W of x and a nondegenerate system of functions t_R, $R \in K$ (i.e. all dt_R are linearly independent at every point $y \in W$) such that $\mathfrak{FM}_R$ is given by the equation $t_R = 0$.

Claim 11.5. (1) *We have*

$$P^{-1}\Omega = \prod_{R\in K} t_R^{(N-1)(\#R-1)}\omega,$$

where ω *is nondegenerate at* x.
(2)

$$P^{-1}q_{st} = \left(\prod_{\{s,t\}\subseteq R} t_R^2\right) u_{st},$$

where $u_{st}(x) \neq 0$.

Without loss of generality we can assume that

(1) both ω and all u_{st} do not vanish on W;
(2) $\phi := P^{-1}f$ is supported on W.

11.3.4. We have

$$Z(s,f) = \int_{Y^S} \left(\prod_{R\in K} t_R^{2s_R+(\#R-1)(N-1)}\right) F(s,y)\phi\Omega, \tag{27}$$

where $F(s,y)$ is an integer function in s and $s_R = \sum_{T\subset R} s_T$. Therefore, $Z(s,f)$ can only have poles of at most first order along the divisors $D(R,n)$, where $n > (\#R-1)(N-1)$. □

11.3.5. Let $M \in \mathcal{U}$ be such that all M_T are integer. Choose an arbitrary total order $<_P$ on $P_2(S)$ and a point $\epsilon \in \mathcal{U}$ such that

(1) each λ_T is positive real number;
(2)

$$\sum_{T \in P} \lambda_T < 1;$$

(3) for all T,

$$\lambda_T > \sum_{T' < T} \lambda_{T'}.$$

Let $C \subset \mathbb{C}$ be the unit cirle. Then for all $z \in C^{P_2(S)}$, $Z(s, f)$ is regular at $M + \lambda z$. Set

$$U(M, \lambda, f) = \frac{1}{(2\pi i)^{\#P}} \int_{C^P} Z(M + \lambda\epsilon, f) \prod_{T \in P} \frac{d\epsilon_T}{\epsilon_T}.$$

Note that the sign of this integral is well-defined.

It is clear that $U(M, \lambda, f)$ is independent of λ; we set $U(M, <_P, f) := U(M, \lambda, f)$.

Claim 11.6. *$f \mapsto U(M, <_P)$ is a distribution.*

Proof. Let $P : \mathfrak{FM} \to Y^S$, $x \in \mathfrak{FM}$ and a neighborhood W of x be as in the proof of Claim 11.4.

It is sufficient to check that $U(M, <_P)$ is continuous when restricted to a subspace $\mathfrak{D}_W$ of densities f such that $P^{-1} f$ is supported in W.

Let $s'_R := 2s_R + (\#R - 1)(N - 1)$. Let L_R be arbitrary positive integers. Then we can modify (27) as follows:

$$Z(s, f) = \prod_{R \in K} \frac{(-1)^L}{(s'_R + 1)(s'_R + 2) \cdots (s'_R + L_R)} \int_W \times \left(\prod_R t_R^{s'_R + L_R} \partial_{t_R}^{L_R} (\phi(y) F(s, y)) \right) \omega,$$

where we assume that we have extended the set of functions t_R to a coordinate system on W and that ω is the standard density in this coordinate system.

Pick L_R to be large enough. Then it is immediate that

$$U(M, <_P, f) = \int_W \left(\sum_{S \in K} \partial_t^S A_S \right) f\omega,$$

where A_S are smooth functions on W. Therefore, $U(M, >_P)$ is a distribution. ◻

Let $S = S_1 \sqcup S_2$ so that $Y^S = Y^{S_1} \times Y^{S_2}$. Let $f_i \in \mathfrak{D}_{Y^{S^i}}$; define $f_1 \boxtimes f_2 : Y^S \to \mathbb{C}$ by $f_1 \boxtimes f_2(Y_1 \times Y_2) = f_1(Y_1) f_2(Y_2)$, where $Y_i \in Y^{S_i}$. Assume that $M_T \geq 0$ whenever $T = \{s_1, s_2\}$ with $s_1 \in S_1$, $s_2 \in S_2$.

Claim 11.7. *We have*

$$U(M, <_P, f_1 \boxtimes f_2) = U(M|_{S_1}, <_{(P|_{P_2(S_1)})}, f_1) U(M|_{S_2}, <_{(P|_{P_2(S_2)})}, f_2).$$

Proof. Clear

Claim 11.8. (1) $q^L U(M, <_P) = U(M + L, <_P)$;
(2) $q^M(U(M, <_P)) = 1$;
(3) $U(M, <_P) \in C_S$. □

Proof. (1) Clear;
(2) follows from (1);
(3) Note that $\mathfrak{t}.U(M, <_P) = 0$, therefore $U(M, <_P)$ is quasi-polynomial. The property (23) follows by induction from Claim 11.7. □

Thus, we have shown that $\pi : C_S \to B_S$ is surjective.

11.4. *Filtration on C_S*

Let $\mathrm{Diag}^n \subset Y^S$ be the union of generalized diagonals of codimension n. Let $F^n C_S := C_{S,\mathrm{Diag}^n} \subset C_S$ be the submodule consisting of distributions supported on Diag^n. We will study this filtration.

11.4.1. Let $\Delta \subset Y^S$ be a diagonal. Let $i_\Delta : \Delta \to Y^S$ be the corresponding inclusion. Let $\mathfrak{D}_\Delta$ be the algebra of polynomial differential operators on Δ. Let ω_Δ (respectively ω_{Y^S}) be the bundle of top forms on Δ (respectively on Y^S). It is well known that

$$\mathcal{D}_{\Delta \to Y} := \omega_{\mathcal{D}_\Delta} \otimes_{\mathcal{O}_\Delta} \mathcal{D}_{Y^S} \otimes_{\mathcal{O}_{Y^S}} \omega_{Y^S}^{-1}$$

is a right $\mathcal{D}_\Delta$ and a left $\mathcal{D}_{Y^S}$-module. Let M be a left $\mathcal{D}_\Delta$-module. Set

$$i_{\Delta *} M := M \otimes_{\mathcal{D}_{D^e}} \mathcal{D}_{\Delta \to Y}.$$

For example, let $\mathfrak{D}'_{Y^S,\Delta} \subset \mathfrak{D}'_{Y^S}$ be the submodule of distributions F such that

(1) F is supported on Δ
(2) there exists an $M = M(F)$ such that $Fg = 0$ for any smooth function vanishing on Δ at order $\geq M(F)$. (Note that locally on Δ the condition (2) is always

true.) We have a natural isomorphism

$$I_\Delta := I_{\Delta,Y^S} : i_{\Delta *}\mathfrak{D}'_\Delta \xrightarrow{\sim} \mathfrak{D}'_{Y^S,\Delta}.$$

Claim 11.9. (1) *Let $\Delta_1 \subset \Delta_2 \subset \Delta_3$. Consider the composition*

$$i_{\Delta_1\Delta_3 *}\mathcal{C}_{\Delta_1} \cong i_{\Delta_2\Delta_3 *} i_{\Delta_1\Delta_2 *}\mathcal{C}_{\Delta_1} \xrightarrow{I_{\Delta_1\Delta_2}} i_{\Delta_2\Delta_3 *}\mathcal{C}_{\Delta_2} \xrightarrow{I_{\Delta_2\Delta_3}} \mathcal{C}_{\Delta_3}.$$

It is equal to $I_{\mathcal{D}_1\mathcal{D}_3}$.

11.4.2. Let Δ be given by an equivalence relation e on S. We then have an isomorphism $\Delta \cong Y^{S/e}$. Denote $C_{\mathcal{D}} := C_{Y^{S/e}}$.

Proposition 11.10. (1)

$$I_\Delta(i_{\Delta *}C_\Delta) \subset C_{Y^S;\Delta};$$

(2)

$$I_\Delta|_{C_\Delta} : i_{\Delta *}C_\Delta \to C_{Y^S;\Delta}$$

is an isomorphism.

Proof. (1) It suffices to show that $I_\Delta(i_{\Delta *}C_\Delta) \subset C_S$. Let $A \subset i_{\Delta *}C_\Delta$ be the subspace of all elements annihilated by multiplication by any function vanishing on Δ. Let $f \in \mathfrak{D}_{Y^S}$, $a \in C_{\mathcal{D}}$ and $u \in \omega_{\mathcal{D}} \otimes_{\mathcal{O}_{Y^S}} \omega_{Y^S}^{-1}$. Then $uf|_\Delta \in \mathfrak{D}_\Delta$ and $I_\Delta(au)(f) = a(uf|_\Delta)$.

Using this formula and a simple induction, we see that $I_\Delta(A) \subset C_S$. It is also well-known that $i_{\Delta *}C_\Delta$ is generated by A. This completes the proof of (1).

(2) We need the lemma:

Lemma 11.11. *Let $U \subset Y$ be a non-empty open set and assume that $F \in C_S$ vanishes on U. Then $F = 0$.*

Proof of Lemma. The statement is obvious when S is empty or has 1 element. Let us now use induction. Let $S = S_1 \sqcup S_2$. We know that for some M

$$\prod_{s_1\in S_1, s_2\in S_2} q_{s_1s_2}^M F \in T_{S_1S_2}(C_{S_1} \otimes C_{S_2}).$$

There exist non-empty open sets $A_i \in Y^{S_i}$ such that $A_1 \times A_2 \subset U$. Write:

$$\prod_{s_1\in S_1, s_2\in S_2} q_{s_1s_2}^M F \in T_{S_1S_2}\left(\sum_i a_i \otimes b_i\right),$$

where $a_i \in C_{S_1}$, $b_i \in C_{S_2}$ and a_i are linearly independent. By induction assumption, restrictions of a_i onto A_1 are also linearly independent (because if these restrictions are dependent, then the same dependence holds for the whole Y^{S^1}). Therefore, there exist $p_i \in \mathfrak{D}_{A_1}$ such that $a_i(p_j) = \delta_{ij}$. Let $q \in \mathfrak{D}_{A_2}$. We know that $F(p_i \boxtimes q) = 0$. Therefore, $b_i(q) = 0$, since q is arbitrary, b_i vanishes on A_2, hence by induction assumption, $b_i = 0$. Therefore, $\prod_{s_1 \in S_1, s_2 \in S_2} q^M_{s_1 s_2} F = 0$. Therefore, F is supported on $\mathcal{D}_{S_1 S_2} := \cap_{s_i \in S_i} \mathcal{D}_{s_1 s_2}$, hence on

$$E = \cap_{S = S_1 \sqcup S_2} \mathcal{D}_{S_1 S_2}.$$

Show that E is the smallest diagonal $\Delta\alpha$. Indeed it is clear that $\Delta_\alpha \subset E$. If $y \notin \Delta_\alpha$, then there exists a partition $S = S_1 \sqcup S_2$ such that S_1, S_2 are non-empty and $Y_{s_1} \neq Y_{s_2}$ whenever $s_i \in S_i$. Therefore $y \notin \Delta_{S_1 S_2}$, hence not in E.

Thus, F is supported on Δ_α. Since F is quasi-polynomial and vanishes on U, it also vanishes on $U + a$ for all $a \in \Delta_\alpha$. Therefore, F vanishes on a neighborhood of Δ_α. Therefore, $F = 0$.

Proof of Proposition 11.10(2). (1) Choose a relatively compact open set $U \in Y^S$. Then it is well-known that there exists M such that F is annihilated by multiplication by any function vanishing on $\Delta \cap U$ of order $\geq M$. By virtue of the lemma, this implies that F is actually annihilated by multiplication by any function vanishing on Δ of order $\geq M$. (2) It suffices to check that there exists $f \in I_\Delta(i_{\Delta *} C_\Delta)$ such that $F - f$ vanishes on U. It is easy to see that the latter is equivalent to the following: for any polynomial P vanishing on Δ of order $M - 1$, $Pf \in I_\Delta(A)$. This follows easily by induction. □

11.5. Let $\mathfrak{Diag}_n := \mathfrak{Diag}_n(S)$ be the set (not the union!) of all diagonals in Y^S of codimension n. We have a map

$$I_n := \oplus_{\mathcal{D} \in \mathfrak{Diag}_n} I_\Delta \oplus i \colon \oplus_{\mathcal{D} \in \mathfrak{Diag}_n} i_{\mathcal{D}*} C_{\mathcal{D}} \oplus C_{S,\, \mathrm{Diag}_{n+1}} \to C_{S,\, \mathrm{Diag}_n}.$$

Claim 11.12. (1) I_n *is surjective*;
(2) *if* $\sum_{\Delta \in \mathfrak{Diag}_n} f_\Delta + g \in \operatorname{Ker} I_n$, *where* $f_\Delta \in i_{\Delta *}\Delta$ *and* $g \in C_{S, \Delta_{n+1}}$, *then all* f_Δ *are supported on* Δ_{n+1}.

Proof. We need the following lemma.

Lemma 11.13. *Let* $\Delta \in \mathfrak{Diag}(S)_n$. *There exists* $M \colon P_2(S) \to \mathbb{Z}_{\geq 0}$ *such that* $q_\Delta := q^M = 0$ *on any* $\Delta' \in \mathfrak{Diag}_n$, $\Delta' \neq \Delta$ *but* $q^M \neq 0$ *on* Δ.

Proof. Set $M_{st} = 1$ if $\Delta_{st} \supset \Delta$; otherwise set $M_{st} = 0$. □

Proof of Claim 11.12. (1) Let $F \in \mathcal{C}_{S, \Delta_n}$. Then $F(q_\Delta)^m \in C_{S,\Delta}$ for $m \gg 0$. We have an isomorphism $C_{S,\Delta} \cong i_{\Delta *} C_\Delta$. We also have the map $\pi \colon C_\Delta \to B_\Delta$ which induces a map

$$i_*(\pi) \colon i_{\Delta *} C_\Delta \to i_{\Delta *} B_\Delta.$$

In particular $[i_{\Delta *}(\pi)]F(q_\Delta)^m \in i_{\Delta *}B_\Delta$. Since the multiplication onto q_Δ is invertible on $i_{\Delta *}B_\Delta$, there is an element $x \in i_{\Delta *}B_\Delta$ such that

$$x(\mathfrak{q}_\Delta)^m = (i_{\Delta *}(\pi)(F)(q_\Delta)^m).$$

Since π is surjective, so is $i_{\Delta *}\pi$. Pick a pre-image $x' := x'_\Delta$ of x in $i_{\Delta *}C_\Delta$. Then

$$([i_{\Delta *}(\pi)](F(q_\Delta)^M - x'(q_\Delta)^M)) = 0.$$

It then follows that $F - x'$ is supported on the union of all n-dimensional diagonals except $\mathcal{D}$. Since each $x'_\mathcal{D}$ is supported on Δ, we have:

$$F - \sum_\Delta x'_\Delta$$

is supported on Δ_{n+1}.

Proof of (2). Let $\sum f_\Delta + g \in \operatorname{Ker} I_n$. It follows that $(q_\Delta)^m f_\Delta$ is supported on Δ_{n+1} it is easy to check that if $x \in \Delta$ and $(q_\Delta)^m(x) = 0$, then $x \in \Delta_{n+1}$. Therefore, f_Δ is supported on Δ_{n+1}. □

11.5.1.

Corollary 11.14. *The map* $\oplus_{\Delta \in \mathbf{Eq}(S)_n} I_\mathcal{D}$ *induces an isomorphism*:

$$\oplus_{\Delta \in \mathfrak{Diag}_n} i_* B_\mathcal{D} \to C_{S,\mathfrak{Diag}_n}/C_{S,\mathbf{Eq}(S)_{n+1}}.$$

11.5.2. Let $X := \mathbb{C}^N$ be the complexification of Y viewed as an algebraic variety over $\mathbb{C}$. Let $\mathcal{D}_{X^S}$ be the sheaf of differential operators on X^S. Then C_S defines a $\mathcal{D}_{X^S}$-module $\mathcal{C}_S$ in the usual way. The above claim implies that $\mathcal{C}_S$ is a holonomic $\mathcal{D}_{X^S}$-module (because each quotient $C_{S,\mathfrak{Diag}(S)_n}/C_{S,\mathfrak{Diag}(S)_{n+1}}$ determines a holonomic $\mathcal{D}_{X^S}$-module).

11.5.3. Let $\mathfrak{Diag}(S)$ be the set of diagonals in X^S ordered with respect to the inclusion. We denote by the same symbol the corresponding category. We have a functor $\mathcal{D} \mapsto \mathcal{C}_{S,\mathcal{D}}$ from $\mathbf{Eq}(S)$ to the category of $\mathfrak{D}_{X^S}$-modules.

11.5.4. Let I be a small category and $\mathcal{C}$ an abelian k-linear category. Let $F : I \to \mathcal{C}$ be a functor. Let I' be the abelian category of functors $I^{\mathrm{op}} \to \mathrm{vect}$. For $A \in I'$ we can form the Eilenberg–MacLane tensor product $F \otimes_I A \in \mathcal{C}$. We call F *perfect* if the functor $A \mapsto F \otimes_I A$ is exact.

Claim 11.15. *The functor* $\Delta \mapsto \mathcal{C}_{S,\Delta}$ *is perfect.*

Proof. Let $n > 0$ be an integer and $\Delta \in \mathfrak{Diag}(S)$. Set $F_n(e) := (\mathcal{C}_{S,\Delta})_n$. We see that $F_n : \mathfrak{Diag}(S) \to \mathcal{D}_{X^S}$ are subfunctors of our functor $F = F_0$.

It suffices to show that for every n, $G_n := F_n/F_{n+1}$ is perfect.

It follows that

$$G_n(s) = \bigoplus_{t \in \mathfrak{Diag}_n; t \leq s} G_t,$$

where $G_t = i_{t*}\mathcal{B}_{\Delta_t}$. The structure maps are the obvious ones.

We have

$$G_n \otimes_{\mathfrak{Diag}(S)} A = \bigoplus_{t \in \mathfrak{Diag}(S)_n} A(t) \otimes G_t$$

and we see that the functor

$$A \mapsto \mathcal{G}_n \otimes_{\mathfrak{Diag}(S)}$$

is exact. Therefore, G_n is perfect. □

11.5.5. Let $S_a, a \in A$ be a finite family of finite sets. Then we have a $\prod_{a \in A} \mathfrak{Diag}(S_a)$-filtration on $\prod_{a \in A} \mathcal{C}_{S_a}$ viewed as a $\mathcal{D}_{X^{\sqcup_{a \in A} S_a}}$-module. The same agument shows that the corresponding functor from the category $\prod_{a \in A} \mathfrak{Diag}(S_a)$ to the category of $\mathfrak{D}_{X^{\sqcup_{a \in A} S_a}}$-modules is perfect.

11.5.6. We are going to study how the map $I_{\mathcal{D}_1\mathcal{D}_2} : i_{\mathcal{D}_1\mathcal{D}_2*}\mathcal{C}_{\mathcal{D}_1} \to \mathcal{C}_{\mathcal{D}_2}$ is compatible with the filtrations. The answer is very simple: this map induces an isomorphism

$$i_{\mathcal{D}_1\mathcal{D}_2*}\mathcal{C}_{\mathcal{D}_1} \to \mathcal{C}_{\mathcal{D}_2,\mathcal{D}_1}.$$

The filtration on the L.H.S. induced by the filtration on $\mathcal{C}_{\mathcal{D}_1}$ coincides with the filtration induced by the one on $\mathcal{C}_{\mathcal{D}_2,\mathcal{D}_1}$.

12. Asymptotic Maps

12.1. *Construction*

Let $\mathcal{D}_e \subset Y^S$ be a diagonal given by an equivalence relation e on S. Let $p : S \to S/e$ be the canonical projection. Let $S_i := p^{-1}i$, $i \in S/e$. Denote

$$\mathcal{C}_S^e := i_{\mathcal{D}_e Y^S} \wedge (\mathcal{B}_{\mathcal{D}}) \otimes \boxtimes_{i \in S/e} \mathcal{C}_{S_i}.$$

The multiplication by q_{st} is invertible on $\mathcal{C}_S^e$ whenever $p(s) \neq p(t)$. Let Q_e be the product of all such q_{st}.

12.1.1. We are going to construct a map

$$\mathfrak{as}_{S,e} : \mathcal{C}_S \to \mathcal{C}_S^e$$

as follows.

First of all it suffices to define a corresponding map on the level of global sections. Let $F \in C_S$. It follows from the definition that there exists an M such that

$$Q_e^M F \in T((\otimes_{i\in S/e} C_{S_i})),$$

where the tensor product is taken over $\mathbb{C}$. Where T is the natural inclusion

$$(\otimes_{i\in S/e} C_{S_i}) \to C_S,$$

induced by the superposition of maps from (22). On the other hand, we have an obvious map

$$(\otimes_{i\in S/e} C_{S_i}) \to C_S^e.$$

Since the multiplication by Q is invertible on C_S^e, we have a well-defined map $\mathfrak{as}'_{S,e} : C_S \to C_S^e$, which determines the desired map $\mathfrak{as}_S$.

13. Properties of $\mathfrak{as}_S$

13.1. *Compatibility with the filtrations*

13.1.1. *Filtration on C_S^e*

Let $f \geq e$ be an equivalence relation. It can be equivalently described as a set of equivalence relations f_i on S_i. Set

$$(\mathcal{C}_S^e)_f := i^{\wedge}_{\mathcal{D}_e Y^S}(\mathcal{B}_S) \otimes \boxtimes_i \mathcal{C}_{S_i,\Delta_{f_i}} \subset \mathcal{C}_S^e.$$

Thus we have a filtration of $\mathcal{C}_S^e$ indexed by the ordered set $\mathfrak{Diag}(S)^{\geq e}$ of all equivalence relations which are greater than or equal to e. It is clear that this filtration is perfect (i.e. the corresponding functor

$$\mathfrak{Diag}(S)^{\geq e} \to \mathcal{D}_{X^S}\text{-mod}$$

is perfect). We can also consider $\mathcal{C}_S^e$ as an object perfectly filtered by $\mathfrak{Diag}(S)$ such that $(C_S^e)_f = 0$ if f is not greater than or equal to e.

We have an isomorphism

$$\mathrm{Gr}_f \mathcal{C}_S^e = i_{\mathcal{D}_f Y^S *}\{\mathfrak{i}_{\mathcal{D}_e \mathcal{D}_f}{}^{\wedge}(\mathcal{B}_{S/e}) \otimes \boxtimes_i \mathcal{B}_{S_i/f_i}\}$$

if $f \geq 0$ (otherwise the corresponding element is zero).

13.1.2. The map $\mathfrak{as}_{S,e}$ is compatible with the filtrations. Let $f \geq e$. The induced map from $\mathrm{Gr}_f \mathcal{C}_S \cong i_{\mathcal{D}_f Y^S} \mathcal{B}_{S/f}$ to $\mathrm{Gr}_f \mathcal{C}_S^e$ is induced by the asymptotic map

$$\mathcal{B}_S \to i^{\wedge}_{\mathcal{D}_e \mathcal{D}_f}(\mathcal{B}_{S/e}) \otimes \boxtimes_i \mathcal{B}_{S_i/f_i}.$$

14. Formalism $\mathcal{I}, \mathfrak{i}, \delta$

In this section we will define functors $\mathcal{I}, \mathfrak{i}, \delta$. The functors $\mathfrak{i}$ are the same as the ones used to define an OPE (see (1)). The functors δ are the functors of direct image in the theory of Δ-modules.

The functors $\mathcal{I}$ are built from $\mathcal{C}_S$.

These functors will be used to construct a required resolution of the system $\langle \mathfrak{i} \rangle$.

14.1. *Main definitions*

14.1.1. Let $\Delta_e \subset \Delta_f$ be two diagonals in X^S determined by the equivalence relations $e \le f$. Let $p: S/f \to S/e$ be the canonical projection. Let $(S/f)_i := p^{-1}(i)$, $i \in S/e$. Set

$$\mathcal{I}_{\Delta_1\Delta_2}, \mathfrak{i}_{\Delta_1\Delta_2}, \delta_{\Delta_1\Delta_2} : \mathcal{D}\text{-mod}_{X^{\Delta_1}} \to \mathcal{D}\text{-mod}_{X^{\Delta_2}}$$

to be defined by the formulas:

$$\mathcal{I}_{\Delta_2\Delta_1}(M) = i_{\Delta_1\Delta_2}(M)^{\wedge} \otimes \boxtimes_{i\in S/e}\mathcal{C}_{(S/f)_i};$$
$$\mathfrak{i}_{\Delta_2\Delta_1}(M) = i^{\wedge}_{\Delta_1\Delta_2}(M) \otimes \boxtimes_{i\in S/e}\mathcal{B}_{(S/f)_i};$$
$$\delta_{\Delta_2\Delta_1}(M) = i_{\Delta_1\Delta_2*}(M).$$

Sometimes we will also use the notation $\mathcal{I}_i, \mathfrak{i}_i, \delta_i$, where $i: \Delta_2 \to \Delta_1$ is the inclusion of the corresponding diagonals.

14.1.2. *Exactness*

Let $T \in S$ be a subset and $p_T : X^S \to X^T$ be the corresponding projection. Call an $H \in \mathcal{D}\text{-mod}_{X^S}$ *T-exact* if H is locally free as a $p_T^{-1}\mathcal{O}_{X^T}$-module. Let $i: \Delta_e \to X^S$ be a diagonal and let $T \subset S$ be such that the through map $T \to S \to S/e$ is a bijection.

Let $M \in \mathcal{D}\text{-mod}_{\Delta_e}$. Write

$$i_H(M) = i^{\wedge}(M) \otimes H.$$

Claim 14.1. (1) *Let the functor H be T-exact. Then the functor*

$$i_H(\cdot) : \mathcal{D}\text{-mod}_{\Delta_e} \to \mathcal{D}\text{-mod}_{X^S}$$

is exact.
(2) *Let*

$$0 \to \mathcal{H}_1 \to \mathcal{H}_2 \to \mathcal{H}_3 \to 0$$

be an exact sequence of T-exact modules. Then the sequence

$$0 \to i_{H_1}(M) \to i_{H_2}(M) \to iH_3(M) \to 0$$

is exact for all $M \in \mathcal{D}\text{-mod}_{\Delta_e}$.

Proof. Obvious. □

Note that $\mathcal{B}_S, \mathcal{C}_S, i_{\Delta*}\mathcal{O}_\Delta$ are $\{s\}$-exact for any one-element subset $s \subset S$ (here $i_\Delta : \Delta \subset X^S$ is the smallest diagonal). This immediately implies that the functors $\mathfrak{i}, \mathcal{I}, \delta$ are exact.

14.1.3. *Filtration*

Let $\Delta_e \subset \Delta_f \subset \Delta_g$. Let $p\colon S/g \to S/f$ and $q\colon S/f \to S/e$. For $x \in S/e$ let g_x be the equivalence relation on $(qp)^{-1}x$ induced by g. We have a projection

$$p_x\colon (qp)^{-1}x \to (qp)^{-1}x/g_x \cong q^{-1}(x)$$

induced by p. We have a map

$$J_{gfe}\colon \delta_{\Delta_g\Delta_f}\mathcal{I}_{\Delta_f\Delta_e} \to \mathcal{I}_{\Delta_g\Delta_e}$$

defined as follows:

$$\begin{aligned}\delta_{\Delta_g\Delta_f}\mathcal{I}_{\Delta_f\Delta_e}(M) &\cong i_{\Delta_f\Delta_g *}(i^{\wedge}_{\Delta_e\Delta_f}(M) \otimes \boxtimes_{i\in S/e} C_{q^{-1}(x)})\\ &\cong i^{\wedge}_{\Delta_e\Delta_g}(M) \otimes (i_{\Delta_f\Delta_g *}(\boxtimes_{i\in S/e} C_{q^{-1}(x)}))\\ &\cong i^{\wedge}_{\Delta_e\Delta_g}(M) \otimes \boxtimes_{x\in S_e} i_{\Delta_{g_x} X^{(qp)^{-1}x} *} C_{\Delta_{g_x}}\\ &\to i^{\wedge}_{\Delta_e\Delta_g}(M) \otimes \boxtimes_{x\in S_e} C_{(qp)^{-1}x}.\end{aligned}$$

This map is injective for all M. Indeed, this needs to be checked only for the last arrow, which follows from:

(1) injectivity of the arrow

$$i_{\Delta_{g_x} X^{(qp)^{-1}x} *} C_{\Delta_{g_x}} \to \boxtimes_{x\in S_e} C_{(qp)^{-1}x};$$

(2) both terms of this arrow are $\mathcal{O}_T$ exact where $T \subset S/e$ is such that $T \to S/e \to S/g$ is a bijection, as well as the cokernel of this arrow.

The above results imply that these inclusions, for all f such that $g \geq f \geq e$, define a perfect filtration on $\mathcal{I}_{\Delta_g\Delta_e}$. We denote by $(\mathcal{I}_{\Delta_g\Delta_e})_f$ the correponding term of this filtration.

14.1.4. *This filtration is perfect*

Let $p\colon S/g \to S/e$ be the projection. For $i \in S/e$, let $(S/g)_i = p^{-1}i$. The ordered set of equivelence relations f such that $g \geq f \geq e$ is isomorphic to the product $\prod_{x\in S/e} \mathfrak{Diag}((S/g)_x)$. Denote this ordered set by $[e,g]$. The filtration on $\mathcal{I}_{\Delta_g\Delta_e}$ is induced by the perfect $\prod_{x\in S/e} \mathfrak{Diag}((S/g)_x) = [e,g]$-filtrations on $\boxtimes_{x\in S/e}\mathcal{C}_{(S/g)_x}$. Denote by

$$F\colon [e,g] \to \mathcal{D}\text{-mod}_{X^{S/g}}$$

the functor determined by these filtrations:

$$F(\{f_x\}_{x\in S/e}) = \boxtimes_{x\in S/e}\mathcal{C}_{(S/g)_x,f_x}.$$

The statement we are proving follows immediately from the following one:

Let $T \subset S/g$ be a subset such that the through map $T \to S/g \to S/e$ is a bijection and

$$A\colon([e,g])^{\mathrm{op}} \to \mathbf{Vect}.$$

Then $F \otimes_{[e,g]} A$ is T-exact.

Let us prove this statement. Indeed, we have seen that F has a filtration F_n such that each F_n/F_{n+1} is perfect. Furthermore, as it follows from Corollary 11.14,

$$F_n/F_{n+1} \otimes_{[e,g]} A = \bigoplus_{t \in (\prod_{x \in S/e} \mathrm{Eq}((S/g)_x)))_n} \mathcal{G}_t \otimes A(t),$$

where each $\mathcal{G}_t$ is T-free.

Therefore, since each F_n/F_{n+1} is perfect, the filtration on F induces a filtration on $F \otimes_{[e,g]} A$, its associated graded quotient being isomorphic to $F_n/F_{n+1} \otimes A$, which are, as we have seen T-free. Therefore, $F \otimes A$ is also T-free.

14.1.5. Thus, $\mathcal{I}_{\Delta_g \Delta_e}$ is a perfect functor from the category Δ_{Δ_e}-modules to the category of $[e,g]$-perfectly filtered Δ_{Δ_g}-modules. We have a canonical isomorphism

$$\mathrm{Gr}_f(\mathcal{I}_{\Delta_g \Delta_e}(M)) \cong \delta_{\Delta_g \Delta_f} \mathfrak{i}_{\Delta_f \Delta_e}(M).$$

14.1.6. *Asymptotic decompositions*

The asymptotic maps $\mathfrak{As}_{S,e}$ from (12) define maps:

$$\mathfrak{As}_{fge}\colon \mathcal{I}_{\Delta_g \Delta_e} \to \mathcal{I}_{\Delta_g \Delta_f} \mathfrak{i}_{\Delta_f \Delta_e}$$

in the obvious way.

The compatibility of $\mathfrak{As}_{S,e}$ with the filtration implies that the map $\mathfrak{As}_{fge}$ is compatible with the filtrations in the following sence:

$$\mathfrak{As}_{fge}(\mathcal{I}_{\Delta_g \Delta_e})_{f'} = 0$$

if $f' \notin [f,g]$. Otherwise

$$\mathfrak{As}_{fge}(\mathcal{I}_{\Delta_g \Delta_e})_{f'} \subset (\mathcal{I}_{\Delta_g \Delta_f})_{f'} \mathfrak{i}_{\Delta_f \Delta_e}. \tag{28}$$

Furthermore, we have

$$(\mathcal{I}_{\Delta_g \Delta_e})_{f'} \cong \delta_{\Delta_g \Delta_{f'}} \mathcal{I}_{\Delta_{f'} \Delta_e}$$

and the above inclusion (28) is given by the map

$$\begin{aligned} \delta_{\Delta_g \Delta_{f'}} \mathcal{I}_{\Delta_{f'} \Delta_e} &\to \delta_{\Delta_g \Delta_{f'}} \mathcal{I}_{\Delta_{f'} \Delta_f} \mathfrak{i}_{\Delta_f \Delta_e} \\ &\cong (\mathcal{I}_{\Delta_g \Delta_f})_{f'} \mathfrak{i}_{\Delta_f \Delta_e}. \end{aligned}$$

Compute the associated graded map

$$\mathrm{Gr}_{f'} \mathcal{I}_{\Delta_g \Delta_e} \to (\mathrm{Gr}_{f'} \mathcal{I}_{\Delta_g \Delta_f}) \mathfrak{i}_{\Delta_f \Delta_e}. \tag{29}$$

We have

$$\mathrm{Gr}_{f'}\mathcal{I}_{\Delta_g\Delta_e} \cong \delta_{\Delta_g\Delta_{f'}}\mathfrak{i}_{\Delta_{f'}\Delta_e};$$

therefore the map (29) is given by the map

$$\delta_{\Delta_g\Delta_{f'}}\mathfrak{i}_{\Delta_{f'}\Delta_e} \to \delta_{\Delta_g\Delta_{f'}}\mathfrak{i}_{\Delta_{f'}\Delta_f}\mathfrak{i}_{\Delta_f\Delta_e}$$

induced by the asymptotic map

$$\mathfrak{i}_{f'e} \to \mathfrak{i}_{f'f}\mathfrak{i}_{fe}.$$

14.1.7. Let $S_\alpha, \alpha \in A$ be a finite family of finite sets. Let $e_\alpha, f_\alpha \in \mathfrak{Diag}(S_\alpha)$; $e_\alpha \leq f_\alpha$; let M_α be some $\mathcal{D}_{\Delta_{e_\alpha}}$-modules. Let $S = \sqcup_{\alpha\in A} S_\alpha$; $e := \sqcup_{\alpha\in A} e_\alpha$; $f = \sqcup_{\alpha\in A} f_\alpha$.

We then have a natural map

$$\boxtimes_{\alpha\in A}\mathcal{I}_{\Delta_{f_\alpha}\Delta_{e_\alpha}}(M_\alpha) \to \mathcal{I}_{\Delta_f\Delta_e}(\boxtimes_{\alpha\in A} M_\alpha).$$

15. Resolution

We will focus our study on the functors $\mathfrak{i}$ and $\mathcal{I}$. We will need the following properties. Let $\Delta_1 \subset \Delta_2 \subset \Delta_3 \subset \Delta_4$ be a flag of diagonals.

(1) We have natural transformations:

$$\mathcal{I}_{\Delta_i\Delta_j} \to \mathfrak{i}_{\Delta_i\Delta_j}, \quad i > j;$$

$$\mathfrak{i}_{\Delta_3\Delta_1} \to \mathfrak{i}_{\Delta_3\Delta_2}\mathfrak{i}_{\Delta_2\Delta_1};$$

$$\mathcal{I}_{\Delta_3\Delta_1} \to \mathcal{I}_{\Delta_3\Delta_2}\mathfrak{i}_{\Delta_2\Delta_1};$$

$$\boxtimes_{\alpha\in A}\mathcal{I}_{\Delta_{f_\alpha}\Delta_{e_\alpha}}(M_\alpha) \to \mathcal{I}_{\Delta_f\Delta_e}(\boxtimes_{\alpha\in A} M_\alpha).$$

$$\boxtimes_{\alpha\in A}\mathfrak{i}_{\Delta_{f_\alpha}\Delta_{e_\alpha}}(M_\alpha) \to \mathfrak{i}_{\Delta_f\Delta_e}(\boxtimes_{\alpha\in A} M_\alpha).$$

(2) The properties are:
   (a) The functors $\mathfrak{i}$ satisfy the axioms of system (see (2.3.3)).
   (b) The following diagrams commute:

$$\begin{array}{ccccc} \mathcal{I}_{\Delta_4\Delta_1} & \longrightarrow & \mathcal{I}_{\Delta_4\Delta_2}\mathfrak{i}_{\Delta_2\Delta_1} & \longrightarrow & \mathcal{I}_{\Delta_4\Delta_3}\mathfrak{i}_{\Delta_3\Delta_2}\mathfrak{i}_{\Delta_2\Delta_1}; \\ \downarrow & & & \nearrow & \\ \mathcal{I}_{\Delta_4\Delta_3}\mathfrak{i}_{\Delta_3\Delta_1} & & & & \end{array} \tag{30}$$

$$\begin{array}{ccccccc} \boxtimes_{\alpha\in A}\mathcal{I}_{\Delta_{3\alpha}\Delta_{1\alpha}}(M_\alpha) & \longrightarrow & \boxtimes_{\alpha\in A}\mathcal{I}_{\Delta_{3\alpha}\Delta_{2\alpha}}\mathfrak{i}_{\Delta_{2\alpha}\Delta_{1\alpha}}(M_\alpha) & \longrightarrow & \mathcal{I}_{\Delta_3\Delta_2}\boxtimes_{\alpha\in A}\mathfrak{i}_{\Delta_{2\alpha}\Delta_{1\alpha}}(M_\alpha) & \longrightarrow & \mathcal{I}_{\Delta_3\Delta_2}\mathfrak{i}_{\Delta_2\Delta_1}M \\ \downarrow & & & & & \nearrow & \\ \mathcal{I}_{\Delta_3\Delta_1}(M) & & & & & & \end{array} \tag{31}$$

15.0.1. Let $f > e$ be equivalence relations on S. Let $\mathbf{Zebra}(f, e)$ be the ordered set defined as follows. Elements of $\mathbf{Zebra}(S)$ are sequences $s := (e_1 i_{12} e_2 i_{23} e_3 i_{34} \cdots i_{n-1n} e_n)$, where $f = e_1 > e_2 > \cdots > e_n = e$ is a flag of equivalence relations and each i_{pp+1} is one of the symbols $\mathfrak{i}$ or $\mathcal{I}$. Let $s' = (e'_1 i'_{12} \cdots e'_{n'})$ be another element of $\mathbf{Zebra}(f, e)$. We write $e \geq e'$ if:

(1) for all $k = 1, 2, \ldots, n'$, there exists n_k such that $e'_k = e_{n_k}$ (in particular, $n_1 = 1; n_{n'} = n$;
(2) if $i'_{kk+1} = \mathfrak{i}$, then $i_{pp+1} = \mathfrak{i}$ for all $p = n_k, n_k + 1, \ldots, n_{k+1} - 1$;
(3) if $i'_{kk+1} = \mathcal{I}$, then $i_{pp+1} = \mathfrak{i}$ for all $p = n_k + 1, \ldots, n_{k+1} - 1$ (it is possible that $i_{n_k n_k+1} = \mathcal{I}$).

Let $\mathfrak{j}^i_{\Delta_1\Delta_2} = \mathfrak{i}_{\Delta_1\Delta_2}$ if $i = \mathfrak{i}$ and $\mathfrak{j}^i_{\Delta_1\Delta_2} = \mathcal{I}_{\Delta_1\Delta_2}$ if $i = \mathcal{I}$. For $s \in \mathbf{Zebra}(f, e)$ write

$$\mathfrak{j}(s) := \mathfrak{j}^{i_{12}}_{\Delta_1\Delta_2} \mathfrak{j}^{i_{23}}_{\Delta_2\Delta_3} \cdots \mathfrak{j}^{i_{n-1n}}_{\Delta_{n-1}\Delta_n}.$$

The above properties imply that $\mathfrak{j}$ is a functor from the category determined by the ordered set $\mathbf{Zebra}(f, e)$ to the category of functors $\mathcal{D}_{\Delta_e}$-mod $\to \mathcal{D}_{\Delta_f}$-mod; our agreement is that whenever $x' \leq x$, $x', x \in \mathbf{Zebra}(f, e)$, we have an arrow from $\mathfrak{j}(x') \to \mathfrak{j}(x)$.

15.1. *Filtration on the functor* $\mathfrak{j}$ *: Zebra* $\to$ *Funct*($\mathcal{D}_{\Delta_e}$-mod, $\mathcal{D}_{\Delta_f}$-mod)

To define such a filtration we need some combinatorics.

15.1.1. Define the ordered set $\mathbf{Segments}(f, e)$. To this end, we need a notion of *segment* in an arbitrary ordered set X, which is just an arbitrary pair of elements $x, y \in X$ such that $x > y$. We denote such a segment by $[x, y]$. Given two segments $[a, b]$ and $[c, d]$, we say that $[a, b] > [c, d]$ iff $b \geq c$ (in which case $a > b \geq c > d$). Define the set $\mathbf{Segments}(X)$ whose elements are arbitrary flags of segments

$$[a_0, b_0] > [a_1, b_1] > \cdots > [a_n, b_n].$$

Of course, this simply means that

$$a_0 > b_0 \geq a_1 > b_1 \geq a_2 > b_2 \geq \cdots \geq a_n > b_n.$$

Introduce an order on the set $\mathbf{Segments}(X)$ according to the following rule.

Let

$$u = ([a_0, b_0] > [a_1, b_1] > \cdots > [a_n, b_n])$$

and

$$v = ([a'_0, b'_0] > [a'_1, b'_1] > \cdots > [a'_m, b'_m])$$

be elements in **Segments**(X). We say that $u \leq v$ iff for every segment $[a'_i, b'_i]$ there exists a segment $[a_j, b_j]$ such that $a_j = a'_i > b'_i \geq b_j$.

Let $f \geq e$ be equivalence relations on S. Let $\mathfrak{Diag}(f, e)$ be the set of all equivalence relations g such that $f \geq g \geq e$.

Set

$$\mathbf{Segments}(e, f) := \mathbf{Segments}(\mathfrak{Diag}(f, e)).$$

For $s \in$ **Zebra**(f, e), where $s = e_1 i_{12} \cdots e_n$, we will define an element $\nu(s) \in$ **Segments**(f, e) by setting

$$\nu(s) = ([e_{k_1}, e_{k_1+1}] > [e_{k_2}, e_{k_2+1}] > \cdots > [e_{k_r}, e_{k_r+1}]),$$

where $k_1 < k_2 < \cdots < k_r$ is a sequence of all numbers such that $i_{k,k+1} = \mathcal{I}$.

15.1.2. Let $s \in$ **Zebra**(f, e),

$$s = e_1 i_{12} \cdots e_n,$$

and let $t \in$ **Segments**(f, e) be an element such that $t \geq \nu(s)$. Let

$$t = ([a_1, b_1] > [a_2, b_2] > \cdots > [a_k, b_k]).$$

Assume that $i_{p,p+1} = \mathcal{I}$. Then there are two possibilities:

(1) either there exists p' such that $e_p = a_{p'}$, $e_p = a_{p'} < b_{p'} \leq e_{p+1}$. In this case write

$$j'_p = \delta_{\Delta_{a_{p'} b_{p'}}} \mathcal{I}_{b_{p'} e_{p+1}};$$

(2) there are no segments $[a_{p'}, b_{p'}]$ as in (1).

We then set

$$j'_p = \mathfrak{j}^{i_{pp+1}}_{\Delta_{e_p} \Delta_{e_{p+1}}}.$$

Define:

$$F^t \mathfrak{j}(s) = j'_1 j'_2 \cdots j'_p.$$

If it is not true that $t \geq \nu(s)$, we then set $F^t \mathfrak{j}(s) = 0$.

Claim 15.1. *For every s, F is a perfect filtration on $\mathfrak{j}(s)$.*

Proof. Let $\mathbf{Segments}(f,e)_s = \{t \in \mathbf{Segments}[f,e] | t \geq s\}$. Let $u \in \mathbf{Segments}(f,e)$. We see that $\mathfrak{j}(s)_u = 0$ whenever $u \notin N_s$. Therefore,

$$\mathfrak{j}(s) \otimes_{\mathbf{Segments}(f,e)} A \cong \mathfrak{j}(s) \otimes_{\mathbf{Segments}(f,e)_s} A$$

for every $A\colon \mathbf{Segments}(f,e) \to \mathbf{Vect}$. Thus, it suffices to show that the $\mathbf{Segments}(f,e)_s$-filtration on $\mathfrak{j}(s)$ is perfect. Let

$$s = ([a_1, b_1] > [a_2, b_2] > \cdots > [a_n b_n])$$

be an element in $\mathbf{Segments}(f,e)$. Then we have an isomorphism

$$\mathbf{Segments}(f,e)_s \cong \prod_i [a_i, b_i]_{\mathfrak{Diag}(S)}.$$

Indeed, let $a_i \geq u_i \geq b_i$. Let $i_1 > i_2 > \cdots > i_r$ be the subsequence of all numbers such that $a_i > u_{i_k}$. Then the corresponding flag of segments is given by the formula

$$[a_{i_1}, u_{i_1}] > [a_{i_2}, u_{i_2}] > \cdots > [a_{i_r}, u_{i_r}].$$

Consider two cases.

Case 1 $a_1 = f$. Define an element $s' \in \mathbf{Segments}(b_1, e)$ by the formula

$$s' = ([a_2, b_2] > \cdots > [a_n b_n]).$$

We then have

$$\mathfrak{j}(s) = \mathcal{I}_{\Delta_{a_1}\Delta_{b_1}} \mathfrak{j}(s_1).$$

We have

$$\mathbf{Segments}(f,e)_s \cong [a_1, b_1]_{\mathfrak{Diag}(S)} \times \mathbf{Segments}(b_1, e)_{s'},$$

where a pair (u, r), where $a_1 \geq u \geq b_1$ and $r \in \mathbf{Segments}(b_1, e)_{s'}$,

$$r = [a'_2, b'_2] > [a'_3, b'_3] > \cdots > [a'_l, b'_l]$$

determine the flag of segments f, where

$$f = ([a_1, u] > [a'_2, b'_2] > \cdots > [a'_n, b'_n])$$

if $a_1 > u$ and

$$f = ([a'_2, b'_2] > \cdots > [a'_n, b'_n])$$

if $u = b_1$.

The filtration on $\mathfrak{j}(s)$ is induced by the corresponding filtrations on $\mathcal{I}_{\Delta_f \Delta_{b_1}}$ and $\mathfrak{j}(s')$.

We are going to use induction, so we can assume that we have already proven that the filtration on $\mathfrak{j}(s')$ is perfect. We denote by the same letter the functors determined by the corresponding filtrations on $\mathfrak{j}(s), \mathfrak{j}(s')$ and $\mathcal{I}_{\Delta_f \Delta_{b_1}}$.

Denote $j_n := \mathcal{I}_{\Delta_f \Delta_{b_1} n}\mathfrak{j}(s')$. Since the quotient

$$\mathcal{I}_{\Delta_f \Delta_{b_1} n}/\mathcal{I}_{\Delta_f \Delta_{b_1}(n+1)}$$

is T-free for every finite set $T \subset S/f$ such that $T \to S/f \to S/b_1$ is bijection, we have:

$$j_n/j_{n+1} \cong (\mathcal{I}_{\Delta_f \Delta_{b_1} n}/\mathcal{I}_{\Delta_f \Delta_{b_1}(n+1)})\mathfrak{j}(s').$$

Using (11.14), we obtain

$$j_n/j_{n+1} \otimes_{\mathbf{Segments}(f,e)_s} A \cong \oplus_{t\in[b_1 f]_n}(G_t j'(s')) \otimes_{\mathbf{Segments}(b_1,f)_{s'}} A(t,s')$$

therefore, j_n/j_{n+1} is perfect, hence $j(s)$ is also perfect.

Case 2 $f > a_1$ is similar. □

15.2. *Description of* $\mathbf{Gr}^t\mathfrak{j}$

15.2.1. Let $t \in \mathbf{Segments}(f,e)$; let

$$\mathbf{Zebra}(f,e)_t = \{g \in \mathbf{Zebra}(f,e) | \nu(g) = t\}.$$

We consider $\mathbf{Zebra}(f,e)_t$ as an ordered subset of $\mathbf{Zebra}(f,e)$. Let $i : \mathbf{Zebra}_t \to \mathbf{Zebra}(f,e)$ be the inclusion.

Let $\mathbf{Funct}(\mathbf{Zebra}(f,e)_t, \mathcal{C})$ be the category of functors from $\mathbf{Zebra}_t$ to an arbitrary abelian category $\mathcal{C}$. We have the restriction functor

$$i^{-1} : \mathbf{Funct}(\mathbf{Zebra}(f,e),\mathcal{C}) \to \mathbf{Funct}(\mathbf{Zebra}(f,e)_t,\mathcal{C}).$$

Let i_* be the right adjoint functor. It can be constructed as follows.

Let $F : \mathbf{Zebra}(f,e)_t \to \mathcal{C}$ and $s \in \mathbf{Zebra}(f,e)$. There are two cases:

(1) It is false that $\nu(s) \le t$, then $i_*F(s) = 0$;
(2) $\nu(s) \le t$. Then there exists the least element $s_t \in \mathbf{Zebra}(f,e)_t$ among the elements in $\mathbf{Zebra}(f,e)_t$ which are $\ge s$ (we will show it in the next paragraph). Set $i_*F(s) = F(s_t)$. It is clear that if $\nu(s_1), \nu(s_2) \le t$ and $s_1 \le s_2$, then $(s_1)_t \le (s_2)_t$. This determines the functor structure on i_*F.

We will now construct the element s_t. Let

$$s = (e = e_1 i_{12} e_2 i_{23} \cdots i_{n-1n} e_n = f).$$

Let

$$t = ([a_1,b_1] > [a_2,b_2] > \cdots > [a_m,b_m]).$$

The condition $\nu(s) \le t$ means that for every $\mu = 1,2,\ldots,m$ there exists a number k_i such that $e_{k_\mu} = a_\mu > b_\mu \ge e_{k_{\mu+1}}$ and $i_{k(k_\mu+1)} = \mathcal{I}$.

Let $e = u_1 > u_2 > \cdots > u_N = f$ be a flag of equivalence relations determined by the condition

$$\{u_1, u_2, \ldots, u_N\} = \{e_1, e_2, \ldots, e_n\} \cup \{a_1, b_1, a_2, b_2, \ldots, a_m, b_m\}.$$

Define the symbols $i'_{k,k+1}$, where $k = 1, 2, \ldots, N-1$, according to the rule:

if $u_k = e_m$ and $u_{k+1} = e_{m+1}$, then $i'_{k,k+1} = \mathfrak{i}$;

if $u_k = e_m = a_r$ and $u_{k+1} = b_r$, then $i'_{k,k+1} = \mathcal{I}$;

if $u_k = b_r$ and $u_{k+1} = e_l$, then $i'_{k,k+1} = \mathfrak{i}$. As we exhausted all the possibilities, we can now define

$$s_t = (u_1 i'_{12} u_2 i'_{23} \cdots u_N).$$

15.2.2. We have $\mathrm{Gr}^t \mathfrak{j} \cong i_* i^{-1} \mathrm{Gr}^t(\mathfrak{j})$ and it remains to describe $G := i^{-1}\mathrm{Gr}^t(\mathfrak{j})$. Let s be such that $\nu(s) = t$; $s = (e_1 i_{12} \cdots e_n)$.

Set $c^u = i$ if $u = \mathfrak{i}$; $c^u = \delta$ if $u = \mathcal{I}$. Then

$$G(s) = c^{i_{12}}_{\Delta_{e_1}\Delta_{e_2}} \cdots c^{i_{n-1n}}_{\Delta_{e_{n-1}}\Delta_{e_n}}.$$

15.2.3. *The functors* $\mathcal{P}_{\Delta_f \Delta_e}$

We will study the functor

$$\mathcal{P}_{\Delta_f \Delta_e} := \mathrm{liminv}_{s \in \mathbf{Zebra}(f,e)} \mathfrak{j}(s).$$

Set

$$\mathcal{P}_{\Delta_f, \Delta_e, t} := \mathrm{liminv}_{s \in \mathbf{Zebra}(f,e)} \mathfrak{j}(s)_t,$$

where $t \in \mathbf{Segments}(f, e)$. Our goal is to show that the functor $t \mapsto \mathcal{P}_{\Delta_f, \Delta_e, t}$ is

(1) a filtration on $\mathcal{P}_{\Delta_f, \Delta_t}$;
(2) a perfect functor on the category $\mathbf{Segments}(f, e)$.

Since these properties are the case for the functor $t \mapsto \mathfrak{j}_t$; it suffices to show that the derived functors $R^i\, \mathrm{liminv}_{\mathbf{Zebra}(f,e)}$, $i \geq 1$, vanish on $\mathrm{Gr}_t \mathfrak{j}$. This is what we are going to do.

15.2.4. Let I be a small category and $H : I \to \mathcal{C}$ be a functor, where $\mathcal{C}$ is an arbitrary k-linear category. Let I^- be the abelian category of functors $I \to \mathbf{Vect}$ Let $h_H(X) := \hom_I(X, H) \in \mathcal{C}$, where $X \in I^-$.

H is called *flabby* if the functor h_H is exact. It is clear that flabby functors are adjusted to the functor liminv_I and that there are enough flabby objects in the abelian category of functors $I \to \mathcal{C}$. The functor i_* is exact and maps flabby functors to flabby (this follows from the existence of an exact left adjoint functor i^{-1}, therefore

$$h_{i_*H}(X) = \hom_{\mathbf{Zebra}(f,e)}(X, i_*H) \cong \hom_{\mathbf{Zebra}(f,e)_t}(i^{-1}X, H),$$

which implies that i_*H is flabby).

Therefore,

$$R\,\mathrm{liminv}_{\mathbf{Zebra}(f,e)}\mathrm{Gr}^t(j) \cong R\,\mathrm{liminv}_{\mathbf{Zebra}(f,e)_t}G.$$

The category $\mathbf{Zebra}(f,e)_t$ has an initial object t_i, which is

$$t_i = (f\mathfrak{i}a_1\mathcal{I}b_1\mathfrak{i}a_2\mathcal{I}\mathfrak{b}_2\mathfrak{i}\cdots b_n\mathfrak{i}f),$$

where we assume that in the case $e = a_1$, or $b_i = a_{i+1}$, or $b_n = f$, the fragment $f\mathfrak{i}a_1$ (respectively $b_i\mathfrak{i}a_{i+1}$, respectively $b_n\mathfrak{i}f$) is replaced with f (respectively b_i, respectively f).

Therefore,

$$R\,\mathrm{liminv}_{\mathbf{Zebra}(f,e)}\mathrm{Gr}^t(j) = G(t_i).$$

15.2.5. *Conclusion*

As was mentioned above, these facts imply that we have a filtration on $\mathcal{P}_{\Delta_f\Delta_e}$ by subfunctors $\mathcal{P}_{\Delta_f\Delta_e,t}$ and that this filtration is perfect. We will also denote $F^t\mathcal{P}_{\Delta_f\Delta_e} := \mathcal{P}_{\Delta_f\Delta_e}$.

15.2.6. *Lemma*

We will prove a lemma which will only be used in the next section. We have an element $e\mathcal{I}f \in \mathbf{Zebra}(f,e)$. Let

$$\mathbf{Zebra}^0(f,e) := \mathbf{Zebra}(f,e)\backslash\{e\mathcal{I}f\}.$$

Let $\mathcal{P}^0_{fe} := \mathrm{liminv}_{\mathbf{Zebra}^0(f,e)}\mathfrak{j}$. We have natural maps

$$0 \to \delta_{fe} \to \mathcal{P}_{fe} \to \mathcal{P}^0_{fe} \to 0. \tag{32}$$

Lemma 15.2. *The sequence* (32) *is exact.*

Proof. It is easy to check that the composition of the arrows is zero. Let us now prove the exactness. Let $t \in \mathbf{Segments}(f,e)$. Set

$$\mathcal{P}^0(f,e)_t := \mathrm{liminv}_{s\in\mathbf{Zebra}^0(f,e)}\mathfrak{j}(s)_t.$$

The same argument as above shows that:

(1) $t \mapsto \mathcal{P}^0(f,e)_t$ is a filtration on $\mathcal{P}^0(f,e)_t$;
(2) the lowest element of the filtration is zero: $\mathcal{P}^0(f,e)_{[f,e]} = 0$.
(3) the induced map

$$\mathrm{Gr}^t\mathcal{P}(f,e) \to \mathrm{Gr}^t\mathcal{P}^0(f,e)$$

is an isomorphism for all $t \neq [f,e]$. If $t = 0$, then the induced map is a surjection (onto zero).

The lemma then follows easily. □

15.3. *Formalism $\delta, \mathcal{P}$*

We are going to describe a structure possessed by the functors $\delta, \mathcal{P}$. Let us first introduce the elements of this structure and then describe their properties.

15.3.1. *Decompositions*

Define a map

$$\alpha : \mathcal{P}_{\Delta_1\Delta_3} \to \mathcal{P}_{\Delta_1\Delta_2}\mathcal{P}_{\Delta_2\Delta_3}$$

as follows. Let $s_1 \in \mathbf{Zebra}(\Delta_1, \Delta_2)$ and $s_2 \in \mathbf{Zebra}(\Delta_2, \Delta_3)$. Let $(s_1 s_2) \in \mathbf{Zebra}(\Delta_1, \Delta_2)$ be the obvious concatenation. Set

$$(p_{s_1} \times p_{s_2})\alpha = p_{(s_1 s-2)}.$$

It is immediate that this definition is correct.

15.3.2. *Concatenations*

Let $s \in \mathbf{Zebra}(f,e)$; $s = (e_1 i_{12} \cdots e_n)$. Define a map

$$c : \mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathcal{P}_{\Delta_3\Delta_4} \to \mathcal{P}_{\Delta_1\Delta_4}$$

by setting

$$p_s c = 0$$

if the following is wrong:

There exists an m such that $i_{mm+1} = I$ and

$$\Delta_{e_m} = \Delta_2 \supseteq \Delta_3 \supseteq \Delta_{e_{m+1}}.$$

Let $\mathbf{Zebra}(f,e)_{\Delta_2\Delta_3} \subset \mathbf{Zebra}(e,f)$ be the set of s for which this condition is true. For $s \in \mathbf{Zebra}(f,e)_{\Delta_2\Delta_3}$ we define the elements $s_1 \in \mathbf{Zebra}(\Delta_1\Delta_2)$ and $s_2 \in \mathbf{Zebra}(\Delta_3, \Delta_4)$ (we do not distinguish between a diagonal in X^S and an equivalence relation on S by which it is determined) according to the rule:

$$s_1 = e_1 i_{12} \cdots e_m$$

and

$$s_2 = \Delta_3 I e_{m+1} \cdots e_n.$$

We then have a composition:

$$c_s : \mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathcal{P}_{\Delta_3\Delta_4} \to j_{s_1}\delta_{\Delta_2\Delta_3}\mathfrak{j}_{s_2} \to \mathfrak{j}_s.$$

Define c by the condition $p_s c = c_s$. Show that this definition is correct.

Let $t > s$. Let $a_{ts} : \mathfrak{j}_s \to \mathfrak{j}_t$ be the induced map. We need to check that $c_t = a_{ts}c_s$. There are several cases.

Case 1. $s \notin \mathbf{Zebra}(f,e)_{\Delta_2\Delta_3}$. Since $t > s$, $t \notin \mathbf{Zebra}(f,e)_{\Delta_2\Delta_3}$; the correctness is obvious;

Case 2. $t \notin \mathbf{Zebra}(f,e)_{\Delta_2\Delta_3}$; $s \in \mathbf{Zebra}(f,e)_{\Delta_2\Delta_3}$. This means that t contains an element ρ such that

$$\Delta_2 = e_m > \rho > e_{m+1},$$

but it is not true that

$$\Delta_3 \geq \rho.$$

In this case, the composition

$$\delta_{\Delta_2\Delta_3}\mathcal{I}_{\Delta_3 e_{m+1}} \to \mathcal{I}_{\Delta_2 e_{m+1}} \to \mathcal{I}_{\Delta_2\rho}\mathfrak{i}_{\rho e_{m+1}}$$

is zero, therefore $a_{ts}c_s = 0$, and the correctness condition is satisfied.

Case 3. $s, t \in \mathbf{Zebra}(f,e)_{\Delta_2\Delta_3}$ — straightforward.

15.3.3. *Factorization maps*

Let S_a,$a \in A$ be a finite family of finite sets. Let $f_a \geq e_a$ be equivalence relations on S_a. Let $S = \sqcup_a S_a$; $f = \sqcup_a f_a$; $e = \sqcup_a e_a$; $f \geq e$. Let $M_a \in \mathcal{D}_{\Delta_{e_a}}$. Define a natural transformation

$$\mu \colon \boxtimes_a \mathcal{P}_{f_a e_a}(M_a) \to \mathcal{P}_{fe}(\boxtimes_a M_a).$$

Let g be such that $f \geq G \geq e$. Any such an equivalence relation can be represented as $g = \sqcup_a G_a$, where $f_a \geq g_a \geq e_a$.

Let

$$\Phi = (g_1 i_{12} g_2 \cdots i_{n-1n} g_n);$$

$\Phi \in \mathbf{Zebra}(f,e)$. Let $g_r = \sqcup_a g_{ra}$. Let

$$\Phi'_a = (g_{1a} i_{12} g_{2a} \cdots i_{n-1n} g_{na}).$$

After deletion of repeating terms we get an element $\Phi_a \in \mathbf{Zebra}(f_a, e_a)$. Define

$$\begin{aligned} p_\Phi \mu \colon \boxtimes_a \mathcal{P}_{f_a e_a}(M_a)) &\to \boxtimes_a p_{\Phi_a} \mathcal{P}_{f_a e_a}(M_a) \\ &\cong \boxtimes_a (\mathfrak{i}^{i_{12}}_{g_{1a} g_{2a}} \mathfrak{i}^{i_{23}}_{g_{2a} g_{3a}} \cdots \mathfrak{i}^{i_{n-1n}}_{g_{n-1a} g_{na}}(M_a)) \\ &\cong (\boxtimes_a \mathfrak{i}^{i_{12}}_{g_{1a} g_{2a}})(\boxtimes_a \mathfrak{i}^{i_{23}}_{g_{2a} g_{3a}}) \cdots (\boxtimes_a \mathfrak{i}^{i_{n-1n}}_{g_{n-1a} g_{na}})(\boxtimes_a M_a) \\ &\to \mathfrak{i}^{i_{12}}_{g_1 g_2} \mathfrak{i}^{i_{23}}_{g_2 g_3} \cdots \mathfrak{i}^{i_{n-1n}}_{g_{n-1} g_n})(\boxtimes_a M_a). \end{aligned}$$

This defines the map μ. This completes the description of the elements. Now let us pass to the properties.

15.3.4. *Concatenation + factorization*

It follows that the map

$$\boxtimes_a(\mathcal{P}_{f_a e_a}(M_a)) \to \mathcal{P}_{fe}(\boxtimes_a M_a) \to \mathcal{P}_{eg}\mathcal{P}_{gf}(\boxtimes_a M_a)$$

is equal to the map

$$\begin{aligned} \boxtimes_a(\mathcal{P}_{f_a e_a}(M_a)) &\to \boxtimes_a(\mathcal{P}_{f_a g_a}\mathcal{P}_{g_a e_a}(M_a)) \\ &\to \mathcal{P}_{fg} \boxtimes_a \mathcal{P}_{g_a e_a}(M_a) \to \mathcal{P}_{fg}\mathcal{P}_{ge}(\boxtimes_a M_a). \end{aligned}$$

15.3.5. The map

$$\begin{aligned} \mathcal{F} : \mathcal{P}_{fg'}\delta_{g'g''}\mathcal{P}_{g''e}(M) \boxtimes (\mathcal{P}_{f_1 e_1}(M_1)) &\\ \to \mathcal{P}_{fe}(M) \boxtimes (\mathcal{P}_{f_1 e_1}(M_1)) &\\ \to \mathcal{P}_{f\sqcup f_1, e\sqcup e_1}(M \boxtimes M_1) & \end{aligned} \tag{33}$$

is equal to the sum of the maps $\mathcal{F}_{g_1}$, where $f_1 \geq g_1 \geq e_1$:

$$\begin{aligned} f_g : \mathcal{P}_{fg'}\delta_{g'g''}\mathcal{P}_{g''e}(M) \boxtimes (\mathcal{P}_{f_1 e_1}(M_1)) &\\ \to \mathcal{P}_{fg'}\delta_{g'g''}\mathcal{P}_{g''e}(M) \boxtimes (\mathcal{P}_{f_1 g_1}\mathcal{P}_{g_1 e_1}(M_1)) &\\ \to \mathcal{P}_{f\sqcup f_1, g'\sqcup g_1}(\delta_{g'g''}\mathcal{P}_{g''e}(M) \boxtimes \mathcal{P}_{g_1 e_1}(M_1)) &\\ \to \mathcal{P}_{f\sqcup f_1, g'\sqcup g_1}\delta_{g'\sqcup g_1, g''\sqcup g_1}\mathcal{P}_{g''\sqcup g_1, e\sqcup e_1}(M \boxtimes M_1) &\\ \to \mathcal{P}_{f\sqcup f_1, e\sqcup e_1}(M \boxtimes M_1). & \end{aligned}$$

Let us prove this statement. We need to show that for every $s \in \mathbf{Zebra}(f \sqcup f_1, e \sqcup e_1)$,

$$p_s\mathcal{F} = p_s \sum_{g_1} \mathcal{F}_{g_1}.$$

Let

$$s = ((k_0, h^0) > (k_1, h_1) > \cdots (k_n, h_n)),$$

where

$$f = k_0 \geq k_1 \geq \cdots \geq k_n = e$$

and

$$f_1 = h_0 \geq h_1 \geq \cdots \geq h_n = e_1.$$

Let

$$s' = (f = k'_0 \geq k'_1 \geq \cdots \geq k'_{n'} = e)$$

and

$$s'' = (f_1 = h'_0 \geq h'_1 \geq \cdots \geq h'_{m'} = e_1)$$

be obtained from

$$f = k_0 \geq k_1 \geq \cdots \geq k_n = e$$

and

$$f_1 = h_0 \geq h_1 \geq \cdots \geq h_n = e_1$$

by deleting repeating terms.

Let us compute $p_s\mathcal{F}$. To this end we first compute the composition

$$\mathcal{P}_{fg'}\delta_{g'g''}\mathcal{P}_{g''e} \to \mathcal{P}_{fe} \to \mathcal{I}_{k'_0k'_1}\mathcal{I}_{k'_1k'_2}\cdots\mathcal{I}_{k'_{n'-1}k'_n}, \tag{34}$$

which does not vanish only if there exists an index α' such that $k'_{\alpha'} = g' \geq g'' \geq k'_{\alpha'+1}$. This is equivalent to the existence of an index α such that

$$k_\alpha = g' \geq g'' \geq k_{\alpha+1}.$$

The composition 34 is then equal to

$$\begin{aligned}\mathcal{P}_{fg'}\delta_{g'g''}\mathcal{P}_{g''e} \to \mathcal{P}_{fe} &\to \mathcal{I}_{k_0k_1}\cdots\mathcal{I}_{k_{\alpha-1}k_\alpha}\delta_{g'g''}\mathcal{I}_{g''k'_{\alpha+1}}\mathcal{I}_{k_{\alpha+1}k_{\alpha+2}}\cdots\mathcal{I}_{k_{n-1}k_{n'}}\\ &\to \mathcal{I}_{k_0k_1}\mathcal{I}_{k_1k_2}\cdots\mathcal{I}_{k_{n-1}k_n}\\ &\cong \mathcal{I}_{k'_0k'_1}\mathcal{I}_{k'_1k'_2}\cdots\mathcal{I}_{k'_{n-1}k'_n}.\end{aligned}$$

The projection $p_s\mathcal{F}$ is then equal to:

$$\begin{aligned}
&\mathcal{P}_{fg'}\delta_{g'g''}\mathcal{P}_{g''e}(M)\boxtimes\mathcal{P}_{f_1e_1}(M_1)\\
&\quad\to(\mathcal{I}_{k_0k_1}\cdots\mathcal{I}_{k_{\alpha-1}k_a}(\delta_{g'g''}\mathcal{I}_{g''k_{a+1}}\cdots\mathcal{I}_{k_{n-1}k_n}(M))\\
&\qquad\boxtimes(\mathcal{I}_{h_0h_1}\mathcal{I}_{h_1h_2}\cdots\mathcal{I}_{h_{n-1}h_n}(M_1))\\
&\quad\to\mathcal{I}_{(k_0,h_0)(k_1,h_1)}\cdots\mathcal{I}_{(k_{\alpha-1},h_{\alpha-1}),(k_\alpha,h_\alpha)}\{((\delta_{g'g''}\mathcal{I}_{g''k_{a+1}})\cdots\mathcal{I}_{k_{n-1}k_n}(M))\\
&\qquad\boxtimes(\mathcal{I}_{h_\alpha h_{\alpha+1}}\cdots\mathcal{I}_{h_{n-1}h_n}(M_1))\}\\
&\quad\to\mathcal{I}_{(k_1,h_1)}\cdots\mathcal{I}_{(k_n,h_n)}(M\boxtimes M_1).
\end{aligned}$$

Where the last map is induced by the map

$$\begin{aligned}
&((\delta_{g'g''}\mathcal{I}_{g''k_{\alpha+1}})\cdots\mathcal{I}_{k_{n-1}k_n}(M))\boxtimes(\mathcal{I}_{h_\alpha h_{\alpha+1}}\cdots\mathcal{I}_{h_{n-1}h_n}(M_1))\\
&\quad\to(\mathcal{I}_{g'k_{\alpha+1}}\cdots\mathcal{I}_{k_{n-1}k_n}(M))\boxtimes(\mathcal{I}_{h_\alpha h_{\alpha+1}}\cdots\mathcal{I}_{h_{n-1}h_n}(M_1))\\
&\quad\to\mathcal{I}_{(k_\alpha,h_\alpha)(k_{\alpha+1},h_{\alpha+1})}\cdots\mathcal{I}_{(k_{n-1},h_{n-1})(k_n,h_n)}(M\boxtimes M_1).
\end{aligned}$$

This map is equal to the map:

$$\begin{aligned}
&((\delta_{g'g''}\mathcal{I}_{g''k_{\alpha+1}})\cdots\mathcal{I}_{k_{n-1}k_n}(M))\boxtimes(\mathcal{I}_{h_\alpha h_{\alpha+1}}\cdots\mathcal{I}_{h_{n-1}h_n}(M_1))\\
&\quad\to(\delta_{g'g''}\mathcal{I}_{g''k_{\alpha+1}}\cdots\mathcal{I}_{k_{n-1}k_n}(M))\boxtimes(\mathcal{I}_{\lambda_a\lambda_a}\mathcal{I}_{h_\alpha h_{\alpha+1}}\cdots\mathcal{I}_{h_{n-1}h_n}(M_1))\\
&\quad\to\delta_{(g',h_\alpha)(g'',h_a}\mathcal{I}_{(g'',h_\alpha)(k_{\alpha+1},h_{\alpha+1})}\cdots\mathcal{I}_{(k_{n-1},h_{n-1})(k_n,h_n)}(M\boxtimes M_1)\\
&\quad\to\mathcal{I}_{(g',h_\alpha)(k_{\alpha+1},h_{\alpha+1})}\cdots\mathcal{I}_{(k_{n-1},h_{n-1})(k_n,h_n)}(M\boxtimes M_1).
\end{aligned}$$

The map $p_s\mathcal{F}$ can then be rewritten as follows:

$$\begin{aligned}
&\mathcal{P}_{fg'}\delta_{g'g''}\mathcal{P}_{g''e}(M)\boxtimes\mathcal{P}_{f_1e_1}(M_1)\\
&\quad\to(\mathcal{I}_{k_0k_1}\cdots\mathcal{I}_{k_{\alpha-1}k_\alpha}\delta_{g'g''}\mathcal{I}_{g''k_{\alpha+1}}\cdots\mathcal{I}_{k_{n-1}k_n}(M))\\
&\qquad\boxtimes(\mathcal{I}_{h_0h_1}\cdots\mathcal{I}_{h_{\alpha-1}h_\alpha}\mathcal{I}_{h_\alpha h_\alpha}\mathcal{I}_{h_\alpha h_{\alpha+1}}\cdots\mathcal{I}_{h_{n-1}h_n}(M_1))\\
&\quad\to\mathcal{I}_{(k_0,h_0)(k_1h_1}\cdots\mathcal{I}_{(k_{\alpha-1}h_{\alpha-1})(k_\alpha h_\alpha)}\delta_{(g',h_\alpha)(g'',h_\alpha)}\mathcal{I}_{(g'',h_\alpha)(k_{\alpha+1},h_{\alpha+1})}\\
&\qquad\cdots\mathcal{I}_{(k_{n-1}h_{n-1})(k_nh_n)}(M\boxtimes M_1).
\end{aligned}\tag{35}$$

Let us now compute $p_s\mathcal{F}_{g_1}$. It follows that such a composition is not zero only if there exists an α such that

$$(k_\alpha,h_\alpha)=(g',g_1)\geq(g'',g_1)\geq(k_{\alpha+1},h_{\alpha+1}).$$

Since $g'>g''$, $k_\alpha>k_{\alpha+1}$. There exists at most one α such that $k_\alpha>k_{\alpha+1}$ and $k_a=g'$. Then g_1 is uniquely determined and equals g_1.

In other words, there exists at most one g_1 such that $p_s\mathcal{F}_{g_1} \neq 0$. If such a g_1 does not exist, then there is no α such that $k_\alpha = g' \geq g'' \geq k\alpha + 1$, therefore, $p_s\mathcal{F} = 0$. Thus, in this case $p_s\mathcal{F} = p_s \sum F_{g_1}$.

If there exists an α such that $k_\alpha = g' \geq g'' \geq k_{\alpha+1}$, then $p_s \sum \mathcal{F}_{g_1} = p_s\mathcal{F}_{h_a}$. It is not hard to see that $p_s\mathcal{F}_{h_a}$ coincides with the map (35) which is the same as $p_s\mathcal{F}$, whence the statement.

15.3.6. *Concatenation + concatenation*

Let $\Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_6$. Consider the following maps

$$a_1 : \mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathcal{P}_{\Delta_3\Delta_4}\delta_{\Delta_4\Delta_5}\mathcal{P}_{\Delta_5\Delta_6} \to \mathcal{P}_{\Delta_1\Delta_4}\delta_{\Delta_4\Delta_5}\mathcal{P}_{\Delta_5\Delta_6} \to \mathcal{P}_{\Delta_1\Delta_6}$$

and

$$a_2 : \mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathcal{P}_{\Delta_3\Delta_4}\delta_{\Delta_4\Delta_5}\mathcal{P}_{\Delta_5\Delta_6} \to \mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathcal{P}_{\Delta_3\Delta_6} \to \mathcal{P}_{\Delta_1\Delta_6}.$$

In the case when $\Delta_3 = \Delta_4$, we also have a map

$$a_3 : \mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathcal{P}_{\Delta_3\Delta_4}\delta_{\Delta_4\Delta_5}\mathcal{P}_{\Delta_5\Delta_6} \to \mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_5}\mathcal{P}_{\Delta_5\Delta_6} \to \mathcal{P}_{\Delta_1\Delta_6}.$$

In the case $\Delta_3 \neq \Delta_4$ set $a_3 = 0$.

Proposition 15.3. *We have*

$$a_2 = a_1 + a_3.$$

Proof. Straightforward. □

15.3.7. *Concatenation + decomposition*

Let $\Delta_1 \supseteq \cdots \supseteq \Delta_4$ be diagonals and let E be another diagonal such that $\Delta_1 \supseteq E \supseteq \Delta_4$. Compute the composition:

$$\mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathcal{P}_{\Delta_3\Delta_4} \to \mathcal{P}_{\Delta_1\Delta_4} \to \mathcal{P}_{\Delta_1 E}\mathcal{P}_{E\Delta_4}.$$

Proposition 15.4. *If* $\Delta_1 \supseteq E \supseteq \Delta_2$, *then this composition is equal to the composition*:

$$\mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathcal{P}_{\Delta_3\Delta_4} \to \mathcal{P}_{\Delta_1 E}\mathcal{P}_{E\Delta_2}\delta_{\Delta_2\Delta_3}\mathcal{P}_{\Delta_3\Delta_4} \to \mathcal{P}_{\Delta_1 E}\mathcal{P}_{E\Delta_4};$$

if $\Delta_3 \supseteq E \supseteq \Delta_4$, *then this composition is equal to the composition*:

$$\mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathcal{P}_{\Delta_3\Delta_4} \to \mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathcal{P}_{\Delta_3 E}\mathcal{P}_{E\Delta_4} \to \mathcal{P}_{\Delta_1 E}\mathcal{P}_{E\Delta_4};$$

otherwise this composition is zero.

Proof. Straightforward. □

15.4. *Filtrations*

We will study the relationship of the above introduced structure with the filtration on the functors $\mathcal{P}_{E\Delta}$ (see Sec. 15.2.5), whenever $E \supset \Delta$. We will see how it interacts with the maps introduced in the previous section.

Let $t_1 \in \mathbf{Zebra}(f,e)_{\Delta_1\Delta_2}$ and $t_2 \in \mathbf{Zebra}(f,e)_{\Delta_3\Delta_4}$. Let

$$t_1 = (e_1 a_{12} e_2 a_{23} \cdots e_n)$$

and

$$t_2 = (e'_1 a'_{12} e'_2 a'_{23} \cdots e'_{n'}).$$

Define the concatenation

$$t_1 \delta t_2 := (e_1 a_{12} \cdots e_n \delta_{\Delta_2\Delta_3} e'_1 a'_{12} \cdots e'_{n'}).$$

We say that t_2 starts with δ if $a'_{12} = \delta$. In this case we define one more concatenation

$$t_1 \delta \circ t_2 := (e_1 a_{12} \cdots e_n \delta_{\Delta_{e_n}\Delta_{e'_2}} e'_2 a'_{23} e'_3 \cdots e'_{n'}).$$

Proposition 15.5. *If t_1 does not terminate in δ, then*

$$c(F^{t_1}\mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}F^{t_2}\mathcal{P}_{\Delta_3\Delta_4}) \subset F^{t_1\delta t_2}\mathcal{P}_{\Delta_1\Delta_4};$$

if t_1 terminates in δ, then

$$c(F^{t_1}\mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}F^{t_2}\mathcal{P}_{\Delta_3\Delta_4}) \subset F^{t_1\delta t_2}\mathcal{P}_{\Delta_1\Delta_4} + F^{t_1\delta\circ t_2}\mathcal{P}_{\Delta_1\Delta_4}.$$

Proof. Straightforward. □

We have the induced maps

$$\mathrm{Gr}^{t_1}\mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathrm{Gr}^{t_2}\mathcal{P}_{\Delta_3\Delta_4} \to \mathrm{Gr}^{t_1\delta t_2}\mathcal{P}_{\Delta_1\Delta_4},$$

if t_2 does not start with δ; and

$$c(\mathrm{Gr}^{t_1}\mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathrm{Gr}^{t_2}\mathcal{P}_{\Delta_3\Delta_4}) \to \mathrm{Gr}^{t_1\delta t_2}\mathcal{P}_{\Delta_1\Delta_4} \oplus \mathrm{Gr}^{t_1\delta\circ t_2}\mathcal{P}_{\Delta_1\Delta_4},$$

if t_2 starts with δ.

We see that

$$\mathrm{Gr}^{t_1}\mathcal{P}_{\Delta_1\Delta_2}\delta_{\Delta_2\Delta_3}\mathrm{Gr}^{t_2}\mathcal{P}_{\Delta_3\Delta_4} \cong \mathrm{Gr}^{t_1\delta t_2}\mathcal{P}_{\Delta_1\Delta_4} \cong \mathrm{Gr}^{t_1\delta\circ t_2}\mathcal{P}_{\Delta_1\Delta_4},$$

whenever t_2 starts with δ; otherwise we have only the first isomorphism in this chain.

The above map (on the graded components) is induced by this isomorphism in the case when t_2 does not start with δ; otherwise the above map is induced by the direct sum of our isomorphisms.

15.5. *Resolution*

Fix two equivalence relations $f \geq e$ on S. We are going to construct a resolution of $\mathfrak{i}_{\Delta_f \Delta_e}$.

Denote by **Flags**(f, e) the set of all "non-strict" flags of the form

$$f = a_0 \geq b_0 \geq a_1 \geq b_1 \cdots \geq b_n = e,$$

where $n \geq 0$ and $b_i \neq a_{i+1}$ for all i. For $i = 0, 1, \ldots, n-1$ set

$$A_{(k)} = a_0 b_0 \cdots a_{k-1} b_{k-1} a_k b_{k+1} a_{k+2} b_{k+2} \cdots a_n b_n,$$

(we delete b_k and a_{k+1}). In the case $a_{k+1} = b_{k+1}$ set

$$A_{[k]} := a_0 b_0 \cdots a_k b_k a_{k+2} b_{k+2} \cdots a_n b_n,$$

where we delete a_{k+1} and b_{k+1}.

Denote $|A| := n$ and set

$$\mathcal{R}(A) := \mathcal{P}_{a_1 b_1} \delta_{b_1 a_2} \mathcal{P}_{a_2 b_2} \cdots \mathcal{P}_{a_n b_n}.$$

Let $\mathcal{R}_n = \bigoplus_{|A|=n} \mathcal{R}(A)$.

Denote

$$x_k : \mathcal{P}_{a_k b_k} \delta_{b_k a_{k+1}} \mathcal{P}_{a_{k+1} b_{k+1}} \to \mathcal{P}_{a_k b_{k+1}}.$$

Let $X_k : \mathcal{R}(A) \to \mathcal{R}(A_{(k)})$ be the map induced by x_k.

We also need maps Y_k defined as follows. In the case when $a_{k+1} = b_{k+1}$ we have an isomorphism $Y_k : \mathcal{R}(A) \to \mathcal{R}(A_{[k]})$.

In the case $a_{k+1} \neq b_{k+1}$ set $Y_k = 0$.

For example: Let $|A| = 2$, then the above theorem implies that $X_1(X_1 + Y_1 - X_2) = 0$ as a map $\mathcal{R}(A) \to \mathcal{P}_{fe}$.

Define the map $d_n : \mathcal{R}_n \to \mathcal{R}_{n-1}$ by the formula

$$d_n = X_1 + Y_1 - X_2 - Y_2 + X_3 + Y_3 + \cdots (-1)^n Y_{n-1} + (-1)^{n+1} X_n.$$

The above identity implies that $d_{n-1} d_n = 0$; thus, $(\mathcal{R}_\bullet, d)$ is complex.

We have a natural map $\mathfrak{v} : \mathcal{R}_0 \to \mathfrak{i}_{fe}$; we have $\mathfrak{v} d_0 = 0$.

Theorem 15.6. (1) *The homology $H_i(\mathcal{R}_\bullet, d) = 0$ for all $i > 0$.*
(2) *The map $\mathfrak{v}$ identifies $H_0(\mathcal{R}_\bullet)$ with $\mathfrak{i}_{fe}$.*

Proof. We are going to consider the associated graded complex with respect to a certain filtration which we are going to define.

Define the set **Segments**$(fe)^0$ whose elements are flags of segments

$$[a_1, b_1] > [a_2, b_2] > \cdots > [a_n, b_n]$$

such that

$$f \geq a_1 > b_1 > a_2 > b_2 > a_3 > \cdots > b_n \geq e.$$

For each $t \in \mathbf{Segments}(f, e)$,

$$t = ([a'_1, b'_1] > [a'_2, b'_2] > \cdots > [a'_n, b'_n])$$

define an element from $\mathbf{Segments}(f, e)^0$,

$$\nu(t) = ([a_1, b_1] > [a_2, b_2] > \cdots > [a_k, b_k])$$

according to the rule: the sequence $a_1 b_1 a_2 b_2 \cdots b_k$ is obtained from the sequence $a'_1 b'_1 \cdots a'_n b'_n$ by deleting all its repeating terms.

For an $s \in \mathbf{Segments}(FE)^0$, we set

$$\Phi^s \mathcal{R}_\bullet = \bigoplus_{\nu(t)=s} F^t(\mathcal{R}_\bullet).$$

We see that F is a filtration on $\mathcal{R}_\bullet$ and that the associated graded complex can be computed by the formula

$$\mathrm{Gr}^s_\Phi \mathcal{R}_\bullet = \bigoplus_{\nu(t)=s} \mathrm{Gr}^t(\mathcal{R}_\bullet).$$

Let $f' > e'$ be a pair of equivalence relations on S. Let $\mathbf{o} \in \mathbf{Segments}(f'e')^0$ be the least element, which is simply $[f'e']$. Denote $\mathfrak{R}^{\mathbf{o}}{}_{f'e'} := \mathrm{Gr}^{\mathbf{o}}_F \mathcal{R}_\bullet$. Let $s \in \mathbf{Segments}(fe)^0$ be an arbitrary element;

$$s = [a_1, b_1] > [a_2, b_2] > \cdots > [a_n b_n].$$

We then have

$$\mathcal{R}^s_\bullet \cong \mathfrak{i}_{fa_1} \mathfrak{R}^{\mathbf{o}}{}_{a_1 b_1} \mathfrak{i}_{b_1 a_2} \mathfrak{R}^{\mathbf{o}}{}_{a_2 b_2} \cdot \cdots \cdot \mathfrak{i}_{\mathfrak{b}_n e}.$$

This implies that our task is reduced to proving that $\mathfrak{R}^{\mathbf{o}}{}_{fe}$ is acyclic, which will be done in the next subsection.

15.5.1. We see that the complex $\mathfrak{R}^{\mathbf{o}}{}_{fe}$ is isomorphic to the complex $R_{fe\bullet} \otimes \delta_{fe}$, where $R_{fe\bullet}$ is a complex of vector spaces; the vector space R_{fen} has a basis labelled by the elements

$$H = (f = e_1 u_{12} e_2 u_{23} e_3 \cdots e_N = e),$$

where $e_1 > \cdots > e_N$ each u_{kk+1} is either p or δ and the total number of deltas is n. Denote $|H| := N$. The differential is given by the sum of several terms which we are now going to describe. Let $A_k H$ be zero if $u_{kk+1} = p$ and let it change u_{kk+1} from δ to p otherwise.

Let $B_k H$ be nonzero only if $u_{kk+1} = \delta, u_{k+1k+2} = \delta$, in which case it replaces the fragment $u_{kk+1}e_{k+1}u_{k+1k+2}$ with δ_{kk+2}.

Let $C_k H$ be nonzero only if $u_{kk+1} = \delta$ and $u_{k+1k+2} = p$, in which case the fragment

$$u_{kk+1}e_{k+1}u_{k+1k+2}$$

is going to be replaced with p.

Denote by d_k the number of symbols δ before a_{kk+1}. It follows that the differential on R is given by

$$d = \sum(-1)^{d_k}(A_k + B_k + C_k).$$

Set

$$F_N R := \oplus_{|H| \leq N} \mathbb{C}H.$$

It is clear that F is a filtration on R the associated graded complex has the basis labelled by the same elements, the differential is given by

$$d' = \sum(-1)^{d_k} A_k.$$

Let

$$\Phi = (f = e_1 > e_2 > \cdots > e_n = e)$$

be a flag and let $R_\Phi \subset \mathrm{Gr}_F R$ be the subcomplex spanned by the elements

$$H = (e_1 u_{12} \cdots e_n),$$

with arbitrary u_{ii+1} (it is clear that it is a subcomplex).

We have $\mathrm{Gr}_F R = \oplus_F R_F$. Furthermore, let $V = \mathbb{C}\langle\delta, p\rangle$ be a complex in which $|\delta| = 1$; $|p| = 0$ and $d\delta = 0$. Then $R_F \cong T^N V$ and is therefore acyclic. □

15.6. *The structure of system on the collection of functors $\mathcal{R}_{fe}$*

15.6.1. Let $f \geq g \geq e$ be a sequence of equivalence relations. Define the decomposition map

$$\mathfrak{As}_{fge} : \mathcal{R}_{fe} \to \mathcal{R}_{fg}\mathcal{R}_{ge}.$$

Let

$$A = (f = a_1 b_1 a_2 \cdots a_n b_n = e)$$

be an elements of **Flags**(fe). If there exists k such that $a_k \supseteq g \supseteq b_k$, then set

$$a : \mathcal{P}(A) \to \mathcal{P}(a_1 b_1 \cdots b_{k-1} a_k g)\mathcal{P}(g b_k \cdots a_n b_n)$$

is induced by the decomposition map

$$\mathcal{P}_{a_k b_k} \to \mathcal{P}_{a_k g}\mathcal{P}_{g b_k}.$$

Otherwise we set

$$a|_{P(A)} = 0.$$

15.6.2. *Factorization maps*

We will first study

15.7. *Factorization maps for $\mathcal{R}$*

15.7.1. We keep the notations of the previous subsection. Let $F_\alpha \in \mathbf{Flags}(f_\alpha, e_\alpha)$, $\alpha \in A$ and $F \in \mathbf{Flags}(f, e)$. We are going to define the map

$$\mu(\{F_\alpha\}_{\alpha \in A}; F) \colon \boxtimes_\alpha \mathcal{R}(F_\alpha) \to \mathcal{R}(F).$$

This map is zero for all F_a, F except those determined by the following conditions.

Let

$$F = (f = \mathfrak{a}_1 \geq \mathfrak{b}_1 > a_2 \geq \mathfrak{b}_2 \cdots \mathfrak{a}_n \geq \mathfrak{b}_n = e).$$

We then require that

(1) For every i: $\mathfrak{b}_{i\alpha} = \mathfrak{a}_{i+1\alpha}$ for all α except exactly one (denote it by α_i);
(2) Fix α and consider the sequence

$$\mathfrak{a}_{1\alpha} \geq \mathfrak{b}_{1\alpha} \geq \mathfrak{a}_{2\alpha} \geq \cdots \geq \mathfrak{a}_{n\alpha} \geq \mathfrak{b}_{n\alpha}.$$

Construct a subsequence

$$F(\alpha) = (\mathfrak{a}_{M_1\alpha} \mathfrak{b}_{N_1\alpha} \mathfrak{a}_{M_2\alpha} \mathfrak{b}_{N_2\alpha} \cdots \mathfrak{a}_{M_r\alpha} \mathfrak{b}_{N_r\alpha})$$

according to the following rule: we delete every pair $\mathfrak{b}_{i\alpha} \geq \mathfrak{a}_{i+1\alpha}$ in which $\mathfrak{b}_{i\alpha} = \mathfrak{a}_{i+1\alpha}$. We have: $M_1 = 1$; $N_r = n$; $\mathfrak{b}_{N_i\alpha} \neq \mathfrak{a}_{M_{i+1}\alpha}$; $M_{r+1} = N_r + 1$. Therefore, $F(\alpha) \in \mathbf{Flags}(f_\alpha, e_\alpha)$. Our second condition is then $F_\alpha = F(\alpha)$ for all α.

15.7.2. We have a natural map

$$r_\alpha \colon \mathcal{R}(F(\alpha)) \to \mathcal{P}_{\mathfrak{a}_{1\alpha}\mathfrak{b}_{1\alpha}} \delta_{\mathfrak{b}_{1\alpha}\mathfrak{a}_{2\alpha}} \cdots \mathcal{P}_{\mathfrak{a}_{n\alpha}\mathfrak{b}_{n\alpha}},$$

induced by the maps

$$\mathcal{P}_{\mathfrak{a}_{M_{i\alpha}}\mathfrak{b}_{N_{i\alpha}}} \to \mathcal{P}_{\mathfrak{a}_{M_{i\alpha}}\mathfrak{b}_{M_{i\alpha}}} \mathcal{P}_{\mathfrak{a}_{M_{i\alpha}+1}\mathfrak{b}_{M_{i_\alpha}+1}} \cdots \mathcal{P}_{\mathfrak{a}_{N_{i\alpha}}\mathfrak{b}_{N_{i\alpha}}}$$

which induce maps

$$\begin{aligned}
&\mathcal{P}_{\mathfrak{a}_{M_{1\alpha}}\mathfrak{b}_{N_{1\alpha}}} \delta_{\mathfrak{b}_{N_{1\alpha}}\mathfrak{a}_{M_{2\alpha}}} \cdots \mathcal{P}_{\mathfrak{a}_{M_{r\alpha}}\mathfrak{b}_{N_{r\alpha}}} \\
&\quad \to \mathcal{P}_{\mathfrak{a}_{M_{1\alpha}}\mathfrak{b}_{M_{1\alpha}}} \mathcal{P}_{\mathfrak{a}_{M_{1\alpha}+1}\mathfrak{b}_{M_{1\alpha}+1}} \cdots \mathcal{P}_{\mathfrak{a}_{N_{1\alpha}}\mathfrak{b}_{N_{1\alpha}}} \delta_{\mathfrak{b}_{N_{1\alpha}}\mathfrak{a}_{M_{2\alpha}}}
\end{aligned}$$

$$\begin{aligned}&\times \mathcal{P}_{\mathfrak{a}_{M_{2\alpha}}\mathfrak{b}_{M_{2\alpha}}}\mathcal{P}_{\mathfrak{a}_{M_{2\alpha}+1}\mathfrak{b}_{M_{2\alpha}+1}}\cdots\mathcal{P}_{\mathfrak{a}_{N_{2\alpha}}\mathfrak{b}_{N_{2\alpha}}}\delta_{\mathfrak{b}_{N_{2\alpha}}a_{M_{3\alpha}}}\cdots\\ &\times \mathcal{P}_{\mathfrak{a}_{M_{r\alpha}}\mathfrak{b}_{M_{r\alpha}}}\mathcal{P}_{\mathfrak{a}_{M_{r\alpha}+1}\mathfrak{b}_{M_{r\alpha}+1}}\cdots\mathcal{P}_{\mathfrak{a}_{N_{r\alpha}}\mathfrak{b}_{N_{r\alpha}}}\\ \cong{}& \mathcal{P}_{\mathfrak{a}_1\mathfrak{b}_1}\delta_{\mathfrak{b}_1\mathfrak{a}_2}\mathcal{P}_{\mathfrak{a}_2\mathfrak{b}_2}\cdots\mathcal{P}_{\mathfrak{a}_n\mathfrak{b}_n}.\end{aligned}$$

We then define

$$\begin{aligned}\boxtimes_\alpha\mathcal{R}(F_\alpha)&\xrightarrow{\prod_\alpha r_\alpha}\boxtimes_\alpha\mathcal{P}_{\mathfrak{a}_{1\alpha}\mathfrak{b}_{1a}}\delta_{\mathfrak{b}_{1\alpha}\mathfrak{a}_{2\alpha}}\cdots\mathcal{P}_{\mathfrak{a}_{n\alpha}\mathfrak{b}_{n\alpha}}\\ &\to\mathcal{P}_{\mathfrak{a}_1\mathfrak{b}_1}\delta_{\mathfrak{b}_1\mathfrak{a}_2}\cdots\mathcal{P}_{\mathfrak{a}_n\mathfrak{b}_n}\\ &\cong\mathcal{R}(F).\end{aligned}$$

15.7.3. *Signs*

The function $i\mapsto s_i$ defines a partition of the set $\{1,2,\ldots,n\}$. Fix an orientation of S and denote by $s(F)$ the sign of this partition.

15.7.4. *Definition of the map*

Define

$$\mu=\sum_F s(F)\mu(\{F(\alpha)\}_{\alpha\in A},F).$$

15.7.5. We are going to check that μ commutes with the differential.

This follows from the several statements we are going to formulate.

We assume that F satisfies the conditions from the previous section.

(1) Let

$$F\langle i\rangle=(\mathfrak{a}_1\mathfrak{b}_1\cdots\mathfrak{a}_i\mathfrak{b}_{i+1}\mathfrak{a}_{i+2}\cdots\mathfrak{b}_n);$$

Let $X_i^F:\mathcal{R}(F)\to\mathcal{R}(F\langle i\rangle)$ be induced by the map

$$\mathcal{P}_{\mathfrak{a}_i\mathfrak{b}_i}\delta_{\mathfrak{b}_i\mathfrak{a}_{i+1}}\mathcal{P}_{\mathfrak{a}_{i+1}\mathfrak{b}_{i+1}}\to\mathcal{P}_{\mathfrak{a}_i\mathfrak{b}_{i+1}}.$$

Let $U(F,i)$ be the set of all $F'\in\mathbf{Flags}(f,e)$ which are obtained from F by changing $\mathfrak{b}_i,\mathfrak{a}_{i+1}$ only in such a way that α_i, $\mathfrak{b}_{i\alpha_i}$ and $\mathfrak{a}_{i+1\alpha_i}$ do not change.

This means that every F' is of the form

$$\mathfrak{a}_1\mathfrak{b}_1\mathfrak{a}_2\mathfrak{b}_2\cdots\mathfrak{a}_i\mathfrak{b}'_i\mathfrak{a}'_{i+1}\mathfrak{b}_{i+1}\mathfrak{a}_{i+2}\cdots\mathfrak{a}_n\mathfrak{b}_n,$$

where $\mathfrak{b}'_{i,\alpha_i}=\mathfrak{b}_{i,\alpha_i}$; $\mathfrak{a}'_{i+1,\alpha_i}=\mathfrak{a}_{i+1,\alpha_i}$, and for all $\alpha\neq\alpha_i$, $\mathfrak{b}'_{i\alpha}=\mathfrak{a}'_{i+1\alpha}$.

Let j be such that $N_{j\alpha_i}=i$ (such a j always exists and is unique because $\mathfrak{b}_{i\alpha_i}\neq\mathfrak{a}_{i+1}$). We then have:

$$\sum F'\in U(F,i)X_i^{F'}\mu(F'(\alpha)_{\alpha\in A},F')=\mu(\{F(\alpha)_{\alpha\neq\alpha_i};F(\alpha_i)\langle j\rangle\};F\langle i\rangle)X_j^{F_{\alpha_i}}.$$

To check this identity it suffices to consider the case $n = 2$, in which case the statement follows immediately from (33).

(2) Let

$$F = (\mathfrak{a}_1\mathfrak{b}_1\mathfrak{a}_2\mathfrak{b}_2\cdots\mathfrak{a}_n\mathfrak{b}_n).$$

Assume that $\mathfrak{a}_i = \mathfrak{b}_i$ and set

$$F[i] = (\mathfrak{a}_1\mathfrak{b}_1\cdots\mathfrak{b}_{i-1}\mathfrak{a}_{i+1}\mathfrak{b}_{i+1}\cdots\mathfrak{a}_n\mathfrak{b}_n).$$

We then have a natural map

$$Y_i : \mathcal{R}(F) \to \mathcal{R}(F[i]).$$

There are two cases:

Case 1. $\alpha_{i-1} = \alpha_i$. Let j be such that $N_{j\alpha_i} = i$. In this case we have:

$$Y_i\mu(F(\alpha)_{\alpha\in A}; F) = \mu(\{F(\alpha)_{\alpha\neq\alpha_i}, F(\alpha)[j]\}; F[i])Y_j^{F(\alpha_i)}.$$

Case 2. $\alpha_{i-1} \neq \alpha_i$. In this case define

$$F' = (\mathfrak{a}_1\cdots\mathfrak{b}_{i-1}\mathfrak{a}'_i\mathfrak{b}'_i\mathfrak{a}_{i+1}\mathfrak{b}_{i+1}\cdots\mathfrak{a}_n\mathfrak{b}_n)$$

in such a way that $\alpha'_i := \alpha_i^{F'} = \alpha_{i+1}$, $\alpha'_{i+1} = \alpha_i$ and $\mathfrak{a}'_{i\alpha'_i} = \mathfrak{a}_{i\alpha_{i+1}}$.

We then have

$$Y_i^F\mu(\{F(\alpha)_{\alpha\in A}\}, F) = Y_i^{F'}\mu(\{F'(\alpha)_{\alpha\in A}\}, F').$$

These facts imply that the factorization map commutes with the differential.

15.8. The factorization commutes with the asymptotic decomposition. We omit the proof as it is straightforward.

15.9. *The system $\mathfrak{m}$ and a map $\langle\mathcal{R}\rangle \to \langle\mathfrak{m}\rangle$*

Was discussed in detail the above.

16. Bogoliubov–Parasyuk Theorem

Let $\langle\mathcal{R}\rangle$ be the resolution of the system $\langle\mathfrak{i}\rangle$ constructed in the previous section and let M be a cofibrant dg-Δ_X-sheaf endowed with an OPE-product over $\langle\mathfrak{i}\rangle$.

Theorem 16.1. *There exists an OPE structure on M over $\langle\mathcal{R}\rangle$ which lifts that over $\langle\mathfrak{i}\rangle$.*

The proof will occupy the rest of the section.

16.1. *Unfolding the definition of an OPE-algebra over* $\langle\mathcal{R}\rangle$

Let $p\colon S \to T$ be a surjection of finite sets and N a Δ_{X^T}-module. We have $\mathcal{R}^0_p(N) \cong \mathcal{P}_p(N) \to \mathcal{I}_p(N)$. This produces a natural transformation (whose differential is not zero):

$$\pi_p\colon \mathcal{R}_p \to \mathcal{I}_p.$$

Thus, we have an induced map $M^{\boxtimes S} \to \mathcal{I}_p(M^{\boxtimes T})$.

We also have a map of systems

$$\mathcal{R} \to \mathfrak{m}$$

which induces a strong homotopy *-Lie algebra structure on M. It turns out that the maps π_p and the *-SHLA structure on M completely determine the OPE-structure on M. The precise formulation will be given below.

16.1.1. Suppose that for every surjective map $p_S\colon S \to \mathrm{pt}$, we are given a map

$$\mathfrak{a}_S\colon M^{\boxtimes S} \to \mathcal{I}_{p_S}(M),$$

such that:

— for $\#S = 1$ we have: $\mathfrak{a}_S = \mathrm{Id}$;
— $\mathfrak{a}_S$ is equivariant with respect to bijections of finite sets.

Assume, in addition, that we are given some maps

$$C_S\colon M^{\boxtimes S} \to \delta_{p_S}(M)$$

of degree 1, where $\#S > 1$, C_S are equivariant with respect to bijections of finite sets.

We shall impose certain conditions on these maps which will allow us to construct an OPE-structure on M using these maps.

16.1.2. *Condition* 1

Let $q\colon S \to T$ be a surjection of finite sets. As usual, the product of maps $\mathfrak{a}_S$ gives rise to maps

$$\mathfrak{a}_q\colon M^{\boxtimes S} \to \mathcal{I}_q(M^{\boxtimes T}).$$

Our first condition is as follows.

Condition 16.2. C1 *Let*

$$S \xrightarrow{r} R \xrightarrow{s} T$$

be a sequence of surjections and $q = sr$. *Then the following diagram should commute*:

$$\begin{array}{ccccc} M^{\boxtimes S} & \longrightarrow & \mathcal{I}_r(M^{\boxtimes R}) & \longrightarrow & \mathcal{I}_r\mathcal{I}_s(M^{\boxtimes T}) \\ \downarrow & & & & \downarrow \\ \mathcal{I}_q(M^{\boxtimes T}) & & \longrightarrow & & \mathcal{I}_r\mathfrak{i}_s(M^{\boxtimes T}) \end{array}$$

One sees that it suffices to check this condition for all $p : S \to \mathrm{pt}$.

This condition implies the following fact. Let

$$\mathfrak{a}'_p : M^{\boxtimes S} \to \mathcal{I}_p M^{\boxtimes T} \to \mathfrak{i}_p M^{\boxtimes T}.$$

Let $\mathbf{p} := (p_i : S_i \to S_{i+1})$, $i = 0, \ldots, n-1$, be a sequence of surjections, where $S_0 = S$, $S_n = T$, and

$$p_{n-1}p_{n-2}\cdots p_0 = p.$$

Let $\mathfrak{a}^{\mathcal{I}} := \mathfrak{a}$ and let $\mathfrak{a}^{\mathfrak{i}} := \mathfrak{a}'$. Let $\mathfrak{j} := (\mathfrak{j}_1, \mathfrak{j}_2, \ldots, \mathfrak{j}_{n-1})$ be an arbitrary sequence of elements from the set $\{\mathcal{I}, \mathfrak{i}\}$.

Define the map

$$\mathfrak{j}_{\mathbf{p}} : M^{\boxtimes S} \to (\mathfrak{j}_1)_{p_1}(\mathfrak{j}_2)_{p_2}\cdots(\mathfrak{j}_{n-1})_{p_{n-1}}(M^{\boxtimes T})$$

by the formula:

$$M^{\boxtimes S} \xrightarrow{\mathfrak{a}^{\mathfrak{j}_1}_{p_1}} (\mathfrak{j}_1)_{p_1}(M^{S_1}) \xrightarrow{\mathfrak{a}^{\mathfrak{j}_2}_{p_2}} (\mathfrak{j}_1)_{p_1}(\mathfrak{j}_2)_{p_2}(M^{\boxtimes S_2}) \longrightarrow \cdots \longrightarrow (\mathfrak{j}_1)_{p_1}(\mathfrak{j}_2)_{p_2} \cdots (\mathfrak{j}_{n-1})_{p_{n-1}}(M^{\boxtimes T}).$$

Condition C1 implies that the collection of maps $\mathfrak{j}_{\mathbf{p}}$ for all $\mathfrak{j}$ and $\mathbf{p}$ determines a map $\mathfrak{ope}_p : M^{\boxtimes S} \to \mathcal{P}_p(M^{\boxtimes T})$.

16.1.3. *Condition* 2

Let us now formulate the condition on the collection of maps C_S which is equivalent to the fact that this collection endows $M[-1]$ with a structure of *-SHLA.

We will formulate this condition in a slightly unusual way. Let $p : S \to T$ be a surjection. Define the map

$$C_p : M^{\boxtimes S} \to \delta_p(M^{\boxtimes T})$$

according to the following rule.

(1) The map C_p is not equal to zero only if there exists a unique $t_p \in T$ such that $\#(p^{-1}t_p) > 1$ (in which case $\#(p^{-1}(t)) = 1$ for all $t \neq t_p$).

(2) If the above condition holds, then C_p is defined as follows. Let $S_p := p^{-1}t_p$ and $S' := S\backslash S_p$. Then C_p is defined as the composition:

$$M^{\boxtimes S} \cong M^{\boxtimes S_p} \boxtimes M^{\boxtimes S'}$$
$$\xrightarrow{C_{S_p}\boxtimes \mathrm{Id}} \delta_{S_p}(M) \boxtimes M^{\boxtimes S'} \cong \delta_p(M^{\boxtimes T}),$$

where the last arrow is constructed via the natural identification $T \cong S' \sqcup \mathrm{pt}$.

Now let $p\colon S \to T$ be a surjection and let Σ_p be the set of all isomorphism classes of splittings

$$S \xrightarrow{q} R \xrightarrow{r} T\ ,$$

where r, q are surjections and $p = rq$. Then the *-SHLA axiom can be formulated as follows:

Condition 16.3. C2 *For any surjection* $p\colon S \to T$, *we have*:

$$dC_p + \sum_{(r,q)\in\Sigma_p} C_r C_q = 0,$$

where we pick one representative for each element in Σ_p.

It is clear that if this condition is satisfied for all $p\colon S \to \mathrm{pt}$, then it is satisfied for all p.

16.1.4. *Condition* 3

This condition describes the differential of the maps $\mathfrak{a}_p$. Let p, q, r be the same as in the previous subsection. We have the natural transformation

$$f_{qr}\colon \delta_q\mathcal{I}_r \to \mathcal{I}_{qr}.$$

Using this transformation, define a map

$$\phi_{qr} : M^{\boxtimes S} \xrightarrow{C_q} \delta_q(M^{\boxtimes R}) \xrightarrow{\mathfrak{a}_r} \delta_q\mathcal{I}_r(M^{\boxtimes T}) \xrightarrow{f_{qr}} \mathcal{I}_p(M^{\boxtimes T})$$

Condition 16.4. C3 *For every surjection* $p\colon S \to T$, *we have*:

$$d\mathfrak{a}_p + \sum_{(q,r)\in\Sigma_p} \phi_{q,r} = 0.$$

As in the previous subsection, if this condition holds for all $p\colon S \to \mathrm{pt}$, then it holds for all p.

16.1.5. We will show how, having the maps $\mathfrak{a}_S, C_S$ satisfying conditions C1–C3, one can construct an OPE structure on M over $\langle\mathcal{R}\rangle$.

The definition of $\langle\mathcal{R}\rangle$ implies that to define an OPE-structure over $\langle\mathcal{R}\rangle$, we have to prescribe maps

$$M^{\boxtimes S} \to \mathcal{P}_{a_1}\delta_{b_1}\mathcal{P}_{a_2}\delta_{b_2}\mathcal{P}_{a_3}\cdots\mathcal{P}_{a_n}\delta_{b_n}\mathcal{P}_{a_{n+1}}(M^T), \tag{36}$$

where $a_i : S_{2(i-1)} \to S_{2i-1}$; $b_i : S_{2i-1} \to S_{2i}$ are surjections; $S_0 = S$, $S_{2n+1} = T$, and b_i are not bijections. We define the map (36) as the composition

$$M^{\boxtimes S} \xrightarrow{\mathfrak{ope}_{a_1}} \mathcal{P}_{a_1}M^{\boxtimes S_1} \xrightarrow{C_{b_1}} \mathcal{P}_{a_1}\delta_{b_1}M^{\boxtimes S_2} \longrightarrow$$
$$\cdots \longrightarrow \mathcal{P}_{a_1}\delta_{b_1}\cdots\mathcal{P}_{a_n}\delta_{b_n}\mathcal{P}_{a_{n+1}}(M^T).$$

One checks straightforwardly that all the conditions are satisfied.

16.2. *Proof of the Bogoliubov–Parasyuk theorem*

We are going to use induction. To this end introduce a notion of N-OPE-*structure on* M (over $\langle\mathcal{R}\rangle$), where $N \geq 2$ is an integer. This means that the maps $\mathfrak{a}_S, C_S$ are only defined when $\#S \leq N$ and the conditions C1–C3 are satisfied for all surjections p such that $\forall\, i \#(p^{-1}(i)) \leq N$.

The theorem follows from two statements:

(1) (base of induction). There exists a 2-OPE structure on M such that the composition

$$M \boxtimes M \xrightarrow{\mathfrak{a}_{\{1,2\}}} \mathcal{I}_{\{1,2\}}(M) \twoheadrightarrow \mathfrak{i}_{\{1,2\}}(M)$$

equals to $\mathfrak{ope}_{\{1,2\}}$.

(2) (transition). Assume there exists an N-OPE structure on M such that for every finite set S with $\#S \leq N$ the composition

$$M^{\boxtimes S} \xrightarrow{\mathfrak{a}_S} \mathcal{I}_S(M) \longrightarrow \mathfrak{i}_S(M) \tag{37}$$

coincides with $\mathfrak{ope}_S$. Then there exists an $(N+1)$-OPE-structure on M such that for all S with $\#S \leq N$ the maps $\mathfrak{a}_S, C_S$ coincide with the existing ones and the composition (37) coincides with $\mathfrak{ope}_S$ for all S with $\#S \leq N+1$.

Statement 1 follows from surjectivity of the map $\mathcal{I}_{\{1,2\}}(M) \to \mathfrak{i}_{\{1,2\}}(M)$ (because M is cofibrant). Therefore, the induced map

$$r : \hom(M \boxtimes M, \mathcal{I}_{\{1,2\}}(M))^{S_{\{1,2\}}} \to \hom(M \boxtimes M, \mathfrak{i}_{\{1,2\}}(M))^{S_{\{1,2\}}}$$

is also surjective. Let $\mathfrak{a}_{\{1,2\}}$ be any lifting of $\mathfrak{ope}_{\{1,2\}}$. Then $r(d\mathfrak{a}_{\{1,2\}}) = 0$, therefore, the image of $d\mathfrak{a}_{\{1,2\}}$ is $\delta_{\{1,2\}}(M)$. Set $C_{\{1,2\}} = -da_{\{1,2\}}$. It is clear that $(\mathfrak{a}_{\{1,2\}}, C_{\{1,2\}})$ determine a 2-OPE-structure.

Statement 2. Let $\mathfrak{j}$ be the functor from the category $\mathbf{Zebra}(p_S)$ to the category of functors $\text{D-mod}_X \to \text{D-mod}_{X^S}$ as in (15.2.6) and let

$$\mathcal{P}^0 := \mathcal{P}^0_S = \operatorname{liminv}_{s \in \mathbf{Zebra}^0(p_S)} \mathfrak{j}(s).$$

Let

$$\mathcal{P} := \mathcal{P}_S = \operatorname{liminv}_{s \in \mathbf{Zebra}(p_S)} \mathfrak{j}(s).$$

The existing N-OPE product defines an equivariant map

$$\mathfrak{a}^0_S : M^{\boxtimes S} \to \mathcal{P}^0(M).$$

According to the lemma from (15.2.6), the map

$$\mathcal{P}(M) \to \mathcal{P}^0(M)$$

is surjective. Therefore, there exists an equivariant lifting

$$\mathfrak{a}^1 : M^{\boxtimes S} \to \mathcal{P}(M)$$

of $\mathfrak{a}^0$. Define $\mathfrak{a}_S$ as the composition

$$M^{\boxtimes S} \to \mathcal{P}(M) \to \mathfrak{j}(M).$$

The condition C1 is then automatically satisfied. The map C_S can be uniquely found from condition $C3$. Indeed, let $p_S : S \to \text{pt}$. Let

$$\Sigma^0_S := \Sigma_{p_S} \backslash \{(p_S, \text{Id}_S)\}.$$

Then C3 reads as:

$$\phi_{p_S, \text{Id}_S} = -d\mathfrak{a}_S - \sum_{(q,r) \in \Sigma^0_S} \phi_{q,r}.$$

The right-hand side is uniquely determined by the existing N-OPE structure and by the chosen map $\mathfrak{a}_S$. It is only the left-hand side that depends on C_S. One can find a unique C_S satisfying C3 iff the right-hand side is a map whose image is contained in $\delta_S(M) \subset \mathcal{I}_S(M)$. Let us show that this is indeed the case. Denote the map specified by the right-hand side by $u : M^{\boxtimes S} \to \mathcal{I}_S(M)$. The image of u lies in δ_S iff for every $(q,r) \in \Sigma^0_S$, the through map

$$M^{\boxtimes S} \xrightarrow{u} \mathcal{I}_S(M) \longrightarrow \mathcal{I}_q \mathfrak{i}_r(M)$$

is zero. This can be checked directly.

With such a choice of C_S the condition C3 is satisfied.

Condition C2 is satisfied as well, as follows from the direct computation. Bogoliubov–Parasyuk theorem is proven.

17. The Maps $\mathbb{R}_{f\sqcup g} \to \mathbb{R}_{f\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$

17.1. *Notations*

17.1.1. Let $\phi: S \to T$, $g: A \to B$ be surjections. Define a functor $\mathfrak{i}\mathfrak{I}_{\phi\sqcup g}$ from the category of $\mathcal{D}_{X^{T\sqcup B}}$-modules to the category of $\mathcal{D}_{X^{S\sqcup A}}$-modules by:

$$\mathfrak{i}\mathfrak{I}_{\phi\sqcup g}(M) = i^{\wedge}_{\phi\sqcup g}(M) \otimes_{\mathcal{O}_{X^{S\sqcup A}}} (\mathcal{B}_\phi \boxtimes \mathcal{C}_g).$$

One can also define $\mathfrak{i}\mathfrak{I}_{\phi\sqcup g}$ as a quotient of $\mathcal{I}_{\phi\sqcup g}$ with the sum of images of all maps

$$\delta_{\phi_1\sqcup \mathrm{Id}}\mathcal{I}_{\phi_2\sqcup g} \to \mathcal{I}_{\phi\sqcup g},$$

where $\phi = \phi_2\phi_1, \phi_1, \phi_2$ are surjections, and ϕ_1 is not bijective.

17.1.2. We then have natural maps

$$\mathcal{I}_{\psi\phi\sqcup g} \to \mathcal{I}_\phi \mathfrak{i}\mathfrak{I}_{\psi\sqcup g}, \tag{38}$$

which shall be denoted by $a_{\psi\phi\times g}$.

17.2. *Map* $\xi(\phi, g): \mathcal{P}_{\phi\sqcup g} \to \mathcal{I}_{\phi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$

Let

$$\phi: S \to T; \quad g: A \to B$$

be surjections. We shall define a map

$$\xi(\phi, g): \mathcal{P}_{\phi\sqcup g} \to \mathcal{I}_{\phi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$$

recursively. The parameter of the recursion will be $|g| = \#A - \#B$. Since g is surjective, $|g| \geq 0$. To describe the recursive procedure we need to introduce some notation.

Suppose we are given an (arbitrary) collection of maps

$$\xi(\phi, g)$$

for all ϕ and all g with $|g| < N$. Fix a g with $|g| = N$. We then construct a map

$$X(\phi, g): \mathcal{P}_{\phi\sqcup g} \to \mathcal{I}_{\phi\sqcup \mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}$$

by means of the formulas:

$$X(\phi, g) = U(\phi, g) - \sum_{g = g_1\circ g_2, g_1 \neq g} F(\phi, g_1, g_2),$$

where

$$U(\phi, g) : \mathcal{P}_{\phi \sqcup g} \to \mathcal{I}_{\phi \sqcup g} \to \mathcal{I}_{\phi \sqcup \mathrm{Id}} \mathcal{I}_{\mathrm{Id} \sqcup g};$$
$$F(\phi, g_1, g_2) : \mathcal{P}_{\phi \sqcup g} \to \mathcal{P}_{\phi \sqcup g_1} \mathcal{P}_{\mathrm{Id} \sqcup g_2} \to \mathcal{I}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g_1} \mathcal{I}_{\mathrm{Id} \sqcup g_2} \to \mathcal{I}_{\phi \sqcup \mathrm{Id}} \mathcal{I}_{\mathrm{Id} \sqcup g}. \tag{39}$$

The recursive procedure will be now described by means of:

Definition-Proposition 17.1. *There exists a unique collection of maps $\xi(\phi, g)$ for all surjections ϕ, g such that*

(1) *If $|g| = 0$, i.e. g is a bijection, then $\xi(\phi, g)$ is the natural isomorphism induced by g.*
(2) *The composition*

$$\mathcal{P}_{\phi \sqcup g} \to \mathcal{I}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g} \to \mathcal{I}_{\phi \sqcup \mathrm{Id}} \mathcal{I}_{\mathrm{Id} \sqcup g}$$

equals $X(\phi, g)$.

Proof. We shall prove by induction in $|g|$ that given a natural N, the required maps $\xi(\phi, g)$ can be constructed for all g with $|g| \leq N$.

The base of induction, $N = 0$, is evident. Let us now pass to the transition. Pick a g with $|g| = N$ and assume that our statement is the case for all g' with $|g'| < N$.

We will then show that for every decomposition $g = lk$, where k, l are proper surjections (i.e. surjections but not bijections) the through map

$$X(\phi, g) : \mathcal{P}_{\phi \times g} \to \mathcal{I}_{\phi} \mathcal{I}_g \to \mathcal{I}_{\phi} \mathcal{I}_k \mathfrak{i}_l \tag{40}$$

is zero. Indeed, we have the following commutative diagrams:

(I)

$$\begin{array}{ccccc} P_{\phi \sqcup g} & \xrightarrow{U(\phi \sqcup g)} & \mathcal{I}_{\phi \sqcup \mathrm{Id}} \mathcal{I}_{\mathrm{Id} \sqcup g} & \longrightarrow & \mathcal{I}_{\phi \sqcup \mathrm{Id}} \mathcal{I}_{\mathrm{Id} \sqcup k} \mathfrak{i}_{\mathrm{Id} \sqcup l} \\ \downarrow & & & \nearrow_{U(\phi, k)} & \\ \mathcal{P}_{\phi \sqcup k} \mathcal{I}_{\mathrm{Id} \sqcup l} & & & & \end{array}$$

(II) The composition

$$\mathcal{P}_{\phi \sqcup g} \xrightarrow{F(\phi, g_1, g_2)} \mathcal{I}_{\phi \sqcup \mathrm{Id}} \mathcal{I}_{\mathrm{Id} \sqcup g} \longrightarrow \mathcal{I}_{\phi \sqcup \mathrm{Id}} \mathcal{I}_{\mathrm{Id} \sqcup k} \mathfrak{i}_{\mathrm{Id} \sqcup l} \tag{41}$$

does not vanish only if one can decompose $g = lug_1$ in such a way that $g_2 = lu$ and $k = ug_1$. In this case the map (41) is equal to the composition:

$$\mathcal{P}_{\phi \sqcup g} \to \mathcal{P}_{\phi \sqcup g_1} \mathcal{I}_{\mathrm{Id} \sqcup lu} \to \mathcal{P}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g_1} \mathcal{I}_{\mathrm{Id} \sqcup u} \mathfrak{i}_{\mathrm{Id} \sqcup l} \to \mathcal{I}_{\phi \sqcup \mathrm{Id}} \mathcal{I}_{\mathrm{Id} \sqcup g_1 u} \mathfrak{i}_{\mathrm{Id} \sqcup l}.$$

Therefore, the composition (40) is equal to

$$\mathcal{P}_{\phi \sqcup g} \longrightarrow \mathcal{P}_{\phi \sqcup k} \mathcal{P}_{\mathrm{Id} \sqcup l} \xrightarrow{V} \mathcal{I}_{\phi \sqcup \mathrm{Id}} \mathcal{I}_{\mathrm{Id} \sqcup k} \mathfrak{i}_{\mathrm{Id} \sqcup l}$$

where the arrow V is induced by the map

$$W : \mathcal{P}_{\phi\sqcup k} \to \mathcal{I}_{\phi\sqcup \mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup l}$$

given by the formula

$$W = U_{\phi,k} - \sum_{k=g_2 g_1, g_1\neq g} F(\phi, g_1, g_2) - \xi_{\phi,k}.$$

The induction assumption implies $W = 0$, therefore the map (40) vanishes as well.

Thus, the map $X(\phi, g)$ actually passes through $\mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$ thus defining a map

$$\xi(\phi, g) : \mathcal{P}_{\phi\sqcup g} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}.$$

This accomplishes the definition of $\xi(\phi, g)$.

17.2.1. *Claim*

Define maps

$$F(\phi, \psi, g_1, g_2) : \mathcal{P}_{\phi\psi\times g} \to \mathcal{P}_{\phi\times g_1}\mathcal{P}_{\psi\times g_2} \to \mathcal{I}_{\phi}\delta_{g_1}\mathfrak{i}\mathfrak{I}_{\psi\times g_2} \to \mathcal{I}_{\phi}\mathfrak{i}\mathfrak{I}_{\psi\times g}.$$

Let

$$a(\phi, \psi, g) : \mathcal{P}_{\phi\psi\times g} \to \mathcal{I}_{\phi}\mathfrak{i}\mathfrak{I}_{\psi\times g}$$

be as in Sec. 17.1. □

Claim 17.2.

$$a(\phi, \psi, g) = \sum_{g=g_2 g_1} F(\phi, \psi, g_1, g_2).$$

Proof. (1) If ψ is bijective, then the statement follows directly from Proposition 17.1.

(2) For an arbitrary ψ, let

$$D(\phi, \psi, g) := a(\phi, \psi, g) - \sum_{g=g_1 g_2} F(\phi, \psi, g_1, g_2)$$

be the difference. It then suffices to show that the composition

$$\mathcal{P}_{\phi\psi\times g} \xrightarrow{D} \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathfrak{i}\mathfrak{I}_{\psi\sqcup g} \longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}\mathfrak{i}_{\psi\sqcup\mathrm{Id}}$$

vanishes, by virtue of injectivity of the map

$$\mathfrak{i}\mathfrak{I}_{\psi\sqcup g} \to \mathcal{I}_{\mathrm{Id}\sqcup g}\mathfrak{i}_{\psi\sqcup\mathrm{Id}}.$$

We have the following facts.

(I) The diagram

$$\begin{array}{ccccc}
\mathcal{P}_{\psi\phi\sqcup g} & \xrightarrow{a(\phi,\psi,g)} & \mathcal{I}_{\phi\sqcup \mathrm{Id}}\mathfrak{i}\mathfrak{I}_{\psi\sqcup g} & \longrightarrow & \mathcal{I}_{\phi\sqcup \mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}\mathfrak{i}_{\mathrm{Id}\sqcup\psi} \\
\downarrow & & & \nearrow & \\
\mathcal{P}_{\phi\sqcup g}\mathcal{P}_{\psi\sqcup \mathrm{Id}} & \longrightarrow & \mathcal{I}_{\phi\sqcup g}\mathfrak{i}_{\psi\sqcup \mathrm{Id}} & &
\end{array}$$

commutes.

(II) The following diagram is commutative:

$$\begin{array}{ccccc}
\mathcal{P}_{\psi\phi\sqcup g} & \xrightarrow{F(\phi,\psi,g_1,g_2)} & \mathcal{I}_{\phi\sqcup \mathrm{Id}}\mathfrak{i}\mathfrak{I}_{\psi\sqcup g} & \longrightarrow & \mathcal{I}_{\phi\sqcup \mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}\mathfrak{i}_{\psi\sqcup \mathrm{Id}} \\
\downarrow & & & \nearrow & \\
\mathcal{P}_{\phi\sqcup g_1}\mathcal{P}_{g_2}\mathcal{P}_{\psi\sqcup \mathrm{Id}} & \longrightarrow & \mathcal{I}_{\phi}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{I}_{\mathrm{Id}\sqcup g_2}\mathfrak{i}_{\psi\sqcup \mathrm{Id}} & &
\end{array}$$

Using I, II we see that the statement follows from the case when $\psi = \mathrm{Id}$. □

17.3. *Claim*

Introduce a terminology. Let $g : A \to B$ be a surjection. Let e be an equivalence relation on A determined by g. A *decomposition* $g = g_k \cdots g_2 g_1$ is by definition a diagram

$$S \xrightarrow{g_1} S/e_1 \xrightarrow{g_2} S/e_2 \cdots \xrightarrow{g_{k-1}} S/e_{k-1} \xrightarrow{g_k} T,$$

where

$$e_1 \geq e_2 \geq \cdots \geq e_{k-1} \geq e$$

are equivalence relations on S and q_i are natural maps.

Let $g = g_2 g_1$ be a decomposition. Define a map

$$Y(\phi,\psi,g_1,g_2) : \mathcal{P}_{\psi\phi\sqcup g} \to \mathcal{P}_{\phi\sqcup g_1}\mathcal{P}_{\psi\sqcup g_2} \to \mathcal{I}_{\phi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{I}_{\psi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2}$$
$$\to \mathcal{I}_{\phi\sqcup \mathrm{Id}}\mathcal{I}_{\psi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}.$$

Set

$$Z(\phi,\psi,g) = \sum_{g=g_2g_1} Y(\phi,\psi,g_1,g_2).$$

Claim 17.3. *The map $Z(\phi,\psi,g)$ coincides with the composition*

$$\mathcal{P}_{\psi\phi\sqcup g} \longrightarrow \mathcal{I}_{\psi\phi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \longrightarrow \mathcal{I}_{\phi\sqcup \mathrm{Id}}\mathfrak{i}_{\psi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}.$$

Proof. Denote this composition by $W(\phi,\psi,g)$. We shall use induction in $|g|$. Let $g = g_2 g_1$. Define a map

$$Z(\phi,\psi,g_1,g_2) : \mathcal{P}_{\phi\psi\times g} \to \mathcal{I}_{\phi}\mathfrak{i}_{\psi}\mathcal{I}_{g}$$

as follows:

$$Z(\phi,\psi,g_1,g_2)\colon \mathcal{P}_{\psi\phi\sqcup g} \longrightarrow \mathcal{P}_{\psi\phi\sqcup g_1}\mathcal{P}_{\mathrm{Id}\sqcup g_2} \xrightarrow{Z(\phi,\psi,g_1)} \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathfrak{i}_{\psi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{I}_{\mathrm{Id}\sqcup g_2}$$
$$\longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathfrak{i}_{\psi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}.$$

Define maps $W(\phi,\psi,g_1,g_2)$ in the similar way (using $W(\phi,\psi,g)$ instead of $Z(\phi,\psi,g)$). By the induction assumption,

$$Z(\phi,\psi,g_1,g_2) = W(\phi,\psi,g_1,g_2)$$

whenever $g_2g_1 = g$ and $g_1 \neq g$. Therefore, it suffices to show that

$$\sum_{g_2g_1=g} Z(\phi,\psi,g_1,g_2) = \sum_{g_2g_1=g} W(\phi,\psi,g_1,g_2).$$

Let L be the sum on the L.H.S. and R be the sum on the R.H.S. It follows that L equals the sum, over all decompositions $g = g_3g_2g_1$, of the following maps

$$\mathcal{P}_{\psi\phi\sqcup g} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{P}_{\psi\sqcup g_3g_2}$$
$$\xrightarrow{\alpha} \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{P}_{\psi\sqcup g_2}\mathcal{I}_{\mathrm{Id}\sqcup g_3} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{I}_{\psi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2}\mathcal{I}_{\mathrm{Id}\sqcup g_3}$$
$$\xrightarrow{\omega} \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{I}_{\psi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g_3g_2} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\psi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{I}_{\mathrm{Id}\sqcup g_3g_2}$$
$$\to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathfrak{i}_{\psi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}.$$

Fix g_1 and set $g^2 = g_3g_2$. The previous claim implies that the sum of the compositions of arrows from α to ω, over all decompositions $g^2 = g_3g_2$, equals the following composition:

$$\mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{P}_{\psi\sqcup g^2} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{I}_{\psi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g^2}.$$

Therefore, L equals the sum over all decompositions $g = g_2g_1$ of the following maps:

$$\mathcal{P}_{\psi\phi\sqcup g} \to \mathcal{P}_{\phi\sqcup g_1}\mathcal{P}_{\psi\sqcup g_2} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{I}_{\psi\sqcup g_2} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\psi\sqcup g}$$
$$\to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\psi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathfrak{i}_{\psi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}.$$

This can be rewritten as follows:

$$\mathcal{P}_{\psi\phi\sqcup g} \to \mathcal{P}_{\phi\sqcup g_1}\mathcal{P}_{\psi\sqcup g_2} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathcal{I}_{\psi\sqcup g_2} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\psi\sqcup g}$$
$$\to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathfrak{i}\mathfrak{I}_{\psi\sqcup g} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathfrak{i}_{\psi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}.$$

According to the previous statement, the sum of these maps equals the following composition:

$$\mathcal{P}_{\psi\phi\sqcup g} \to \mathcal{I}_{\psi\phi\sqcup g} \to \mathcal{I}_{\psi\sqcup\mathrm{Id}}\mathfrak{i}\mathfrak{I}_{\phi\sqcup g} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathfrak{i}_{\psi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}.$$

This composition, in turn, is equal to:

$$\mathcal{P}_{\psi\phi\sqcup g} \to \mathcal{I}_{\psi\phi\sqcup g} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\psi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathfrak{i}_{\psi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}.$$

It easily follows that this sum equals R. This completes the proof. □

17.3.1. *Compatibility with* $\boxtimes$

Claim 17.4. *The following diagram is commutative*:

$$\begin{array}{ccccc} \mathcal{P}_{f_1\sqcup g_1}(M_1)\boxtimes\mathcal{P}_{f_2\sqcup g_2}(M_2) & \longrightarrow & \mathcal{I}_{f_1\sqcup\mathrm{Id}}\delta_{Id\sqcup g_1}(M_1)\boxtimes\mathcal{I}_{f_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2}(M_2) & \longrightarrow & \mathcal{I}_{f_1\sqcup f_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2) \\ \downarrow & & & \nearrow & \\ \mathcal{P}_{f_1\sqcup f_2\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2) & & & & \end{array}$$

Proof. We shall use induction. The composition

$$\begin{aligned} &\mathcal{P}_{f_1\sqcup g_1}(M_1)\boxtimes\mathcal{P}_{f_2\sqcup g_2}(M_2) \to \mathcal{P}_{f_1\sqcup f_2\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2) \\ &\quad \to \mathcal{I}_{f_1\sqcup f_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2) \to \mathcal{I}_{f_1\sqcup f_2\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2) \end{aligned}$$

equals the negative of the sum over all decompositions $g_1 = h_2h_1$, $g_2 = h_4h_3$, $(h_2\sqcup h_4 \neq \mathrm{Id})$ of the following maps:

$$\begin{aligned} &\mathcal{P}_{f_1\sqcup g_1}(M_1)\boxtimes\mathcal{P}_{f_2\sqcup g_2}(M_2) \to \mathcal{P}_{f_1\sqcup f_2\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2) \\ &\quad \to \mathcal{P}_{f_1\sqcup f_2\sqcup h_1\sqcup h_2}\mathcal{I}_{\mathrm{Id}\sqcup h_3\sqcup h_4}(M_1\boxtimes M_2) \\ &\quad \to \mathcal{I}_{f_1\sqcup f_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup h_1\sqcup h_2}\mathcal{I}_{\mathrm{Id}\sqcup h_3\sqcup h_4}(M_1\boxtimes M_2) \\ &\quad \to \mathcal{I}_{f_1\sqcup f_2\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2), \end{aligned}$$

which is (due to the induction assumption) the same as:

$$\begin{aligned} &\mathcal{P}_{f_1\sqcup g_1}(M_1)\boxtimes\mathcal{P}_{f_2\sqcup g_2}(M_2) \to \mathcal{P}_{f_1\sqcup h_1}\mathcal{I}_{h_3}(M_1)\boxtimes\mathcal{P}_{f_2\sqcup h_2}\mathcal{I}_{h_4}(M_2) \\ &\quad \to \mathcal{P}_{f_1\sqcup f_2\sqcup h_1\sqcup h_2}\mathcal{I}_{\mathrm{Id}\sqcup h_3\sqcup h_4}(M_1\boxtimes M_2) \\ &\quad \to \mathcal{I}_{f_1\sqcup f_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup h_1\sqcup h_2}\mathcal{I}_{\mathrm{Id}\sqcup h_3\sqcup h_4}(M_1\boxtimes M_2) \\ &\quad \to \mathcal{I}_{f_1\sqcup f_2\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2), \end{aligned}$$

which, in turn, equals:

$$\begin{aligned} &\mathcal{P}_{f_1\sqcup g_1}(M_1)\boxtimes\mathcal{P}_{f_2\sqcup g_2}(M_2) \to \mathcal{P}_{f_1\sqcup h_1}\mathcal{I}_{\mathrm{Id}\sqcup h_3}(M_1)\boxtimes\mathcal{P}_{f_2\sqcup h_2}\mathcal{I}_{\mathrm{Id}\sqcup h_4}(M_2) \\ &\quad \to \mathcal{I}_{f_1\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup h_1}\mathcal{I}_{\mathrm{Id}\sqcup h_3}(M_1)\boxtimes\mathcal{I}_{f_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup h_2}\mathcal{I}_{\mathrm{Id}\sqcup h_4}(M_2) \\ &\quad \to \mathcal{I}_{f_1\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g_1}(M_1)\boxtimes\mathcal{I}_{f_2\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g_2}(M_2) \\ &\quad \to \mathcal{P}_{f_1\sqcup f_2\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2). \end{aligned}$$

The sum of all such maps over all decompositions $g_1 = h_3 h_1, g_2 = h_4 h_2$, is zero. Therefore, the negative of the sum over all decompositions with $h_3 \sqcup h_4 \neq \mathrm{Id}$ is equal to the map in which $h_1 = g_1$, $h_2 = g_2$, $h_3 = \mathrm{Id}$, $h_4 = \mathrm{Id}$, which immediately implies the commutativity of the diagram in question. ◻

17.4. *Maps* $c(\phi, g) : \mathcal{P}_{\phi \sqcup g} \to \mathcal{P}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g}$

First, we define maps

$$\xi(\phi_1, \phi_2, \ldots, \phi_n, g) : \mathcal{P}_{\phi \sqcup g} \to \mathcal{I}_{\phi_1 \sqcup \mathrm{Id}} \mathcal{I}_{\phi_2 \sqcup \mathrm{Id}} \cdots \mathcal{I}_{\phi_n \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g},$$

where $\phi = \phi_n \phi_{n-1} \cdots \phi_1$, as the sum over all decompositions $g = g_n g_{n-1} \cdots g_1$ of the maps:

$$\begin{aligned} \mathcal{P}_{\phi \sqcup g} &\to \mathcal{P}_{\phi_1 \sqcup g_1} \mathcal{P}_{\phi_2 \sqcup g_2} \cdots \mathcal{P}_{\phi_n \sqcup g_n} \\ &\to \mathcal{I}_{\phi_1 \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g_1} \mathcal{I}_{\phi_2 \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g_2} \cdots \mathcal{I}_{\phi_n \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g_n} \\ &\to \mathcal{I}_{\phi_1 \sqcup \mathrm{Id}} \mathcal{I}_{\phi_2 \sqcup \mathrm{Id}} \cdots \mathcal{I}_{\phi_n \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g}. \end{aligned}$$

The previous claim implies that the collection of maps $\xi(\phi_1, \phi_2, \ldots, \phi_n, g)$ for all decompositions $\phi = \phi_n \phi_{n-1} \cdots \phi_1$ gives rise to a map

$$c(\phi, g) : \mathcal{P}_{\phi \sqcup g} \to \mathcal{P}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g}.$$

17.4.1. *Claim*

Claim 17.5. *The composition*

$$\mathcal{P}_{\psi\phi \sqcup g} \to \mathcal{P}_{\psi\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g} \to \mathcal{P}_{\phi \sqcup \mathrm{Id}} \mathcal{P}_{\psi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g}$$

is equal to the sum, over all decompositions $g = g_2 g_1$, *of the maps*

$$\mathcal{P}_{\phi\psi \sqcup g} \to \mathcal{P}_{\phi \sqcup g_1} \mathcal{P}_{\psi \sqcup g_2} \to \mathcal{P}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g_1} \mathcal{P}_{\psi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g_2} \to \mathcal{P}_{\phi \sqcup \mathrm{Id}} \mathcal{P}_{\psi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g}.$$

Proof. Clear. ◻

17.5. *Composition*

Claim 17.6. *The following diagram is commutative*:

$$\begin{array}{ccc} \mathcal{P}_{\phi \sqcup g \sqcup h} & \longrightarrow & \mathcal{P}_{\phi \sqcup \mathrm{Id} \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g \sqcup h} \\ \downarrow & \nearrow & \\ \mathcal{P}_{\phi \sqcup g \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup \mathrm{Id} \sqcup h} & & \end{array}$$

Proof. First, let us prove that the diagram

$$\begin{array}{ccc} \mathcal{P}_{\phi\sqcup g\sqcup h} & \longrightarrow & \mathcal{I}_{\phi\sqcup \mathrm{Id}\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g\sqcup h} \\ \downarrow & \nearrow & \\ \mathcal{P}_{\phi\sqcup g\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h} & & \end{array} \tag{42}$$

is commutative. Denote the composition

$$\mathcal{P}_{\phi\sqcup g\sqcup h} \to \mathcal{P}_{\phi\sqcup g\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h} \to \mathcal{I}_{\phi\sqcup \mathrm{Id}\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g\sqcup h} \to \mathcal{I}_{\phi\sqcup \mathrm{Id}\sqcup \mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h}$$

by $U(\phi, g, h)$.

Let $g = g_2 g_1$ be a decomposition. Define a map

$$U(\phi, g_1, g_2, h) : \mathcal{P}_{\phi\sqcup g\sqcup h} \to \mathcal{I}_{\phi\sqcup \mathrm{Id}\sqcup \mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h}$$

as the following composition:

$$\begin{aligned} U(\phi, g_1, g_2, h) : \mathcal{P}_{\phi\sqcup g\sqcup h} &\to \mathcal{P}_{\phi\sqcup g\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h} \\ &\to \mathcal{P}_{\phi\sqcup g_1\sqcup \mathrm{Id}}\mathcal{P}_{\mathrm{Id}\sqcup g_2\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h} \to \mathcal{I}_{\phi\sqcup \mathrm{Id}\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1\sqcup \mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g_2\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h} \\ &\to \mathcal{I}_{\phi\sqcup \mathrm{Id}\sqcup \mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h}. \end{aligned}$$

Let also

$$A(\phi, g, h) : \mathcal{P}_{\phi\sqcup g\sqcup h} \to \mathcal{I}_{\phi\sqcup g\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h} \to \mathcal{I}_{\phi\sqcup \mathrm{Id}\sqcup \mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h}.$$

Then, by definition,

$$U(\phi, g, h) = A(\phi, g, h) - \sum_{g = g_2 g_1, g_1 \neq g} U(\phi, g_1, g_2, h). \tag{43}$$

The map $U(\phi, g_1, g_2, h)$ equals, in turn, the sum over all decompositions $h = h_2 h_1$ of the maps:

$$\begin{aligned} U(\phi, g_1, g_2, h_1, h_2) : \mathcal{P}_{\phi\sqcup g\sqcup h} & \\ \to \mathcal{P}_{\phi\sqcup g_1\sqcup h_1}\mathcal{P}_{\mathrm{Id}\sqcup g_2\sqcup h_2} &\xrightarrow{U(\phi, g_1, h_1)} \mathcal{I}_{\phi\sqcup \mathrm{Id}\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h_1}\mathcal{I}_{\mathrm{Id}\sqcup g_2\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h_2} \\ \to \mathcal{I}_{\phi\sqcup \mathrm{Id}\sqcup \mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h}. & \end{aligned}$$

Then

$$U(\phi, g, h) = A(\phi, g, h) - \sum_{g = g_2 g_1, h = h_2 h_1, g_1 \neq g} U(\phi, g_1, g_2, h_1, h_2). \tag{44}$$

Set

$$U(\phi, g, h_1, h_2) := U(\phi, g, \mathrm{Id}, h_1, h_2) : \mathcal{P}_{\phi\sqcup g\sqcup h} \to \mathcal{I}_{\phi\sqcup \mathrm{Id}\sqcup \mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup \mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup \mathrm{Id}\sqcup h}$$

to be

$$\mathcal{P}_{\phi\sqcup g\sqcup h} \to \mathcal{P}_{\phi\sqcup g\sqcup h_1}\mathcal{P}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h_2} \xrightarrow{U(\phi,g,h_1)} \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h_1}\mathcal{I}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h_2} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h}.$$

Similarly, let

$$A(\phi,g,h_1,h_2) : \mathcal{P}_{\phi\sqcup g\sqcup h} \to \mathcal{P}_{\phi\sqcup g\sqcup h_1}\mathcal{P}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h_2} \xrightarrow{A(\phi,g,h_1)} \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h_1}\mathcal{I}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h_2} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h}$$

and

$$\begin{aligned} U(\phi,g_1,g_2,h_1,h_2,h_3) &: \mathcal{P}_{\phi\sqcup g\sqcup h} \to \mathcal{P}_{\phi\sqcup g\sqcup h_2h_1}\mathcal{P}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h_3} \\ &\xrightarrow{U(\phi,g_1,g_2,h_1,h_2)} \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h_2h_1}\mathcal{I}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h_3} \\ &\to \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h}. \end{aligned}$$

Equation (43) implies that

$$\sum_{h=h_2h_1} U(\phi,g,h_1,h_2) = \sum_{h=h_2h_1} A(\phi,g,h_1,h_2) - \sum_{g=g_2g_1,g_1\neq g;h=h_3h_2h_1} U(\phi,g_1,g_2,h_3,h_2,h_1). \quad (45)$$

The map

$$\sum_{h=h_2h_1} A(\phi,g,h_1,h_2)$$

is equal to the following one:

$$X(\phi,g,h) : \mathcal{P}_{\phi\sqcup g\sqcup h} \to \mathcal{I}_{\phi\sqcup g\sqcup h} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h}.$$

The map

$$Y(\phi,g_1,g_2,h_1,h_2) := \sum_{h^2=h_3h_2} U(\phi,g_1,g_2,h_1,h_2,h_3)$$

equals

$$\mathcal{P}_{\phi\sqcup g\sqcup h} \to \mathcal{P}_{\phi\sqcup g_1\sqcup h_1}\mathcal{P}_{\mathrm{Id}\sqcup\mathfrak{g}_2\sqcup h^2} \xrightarrow{U(\phi,g_1,h_1)} \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h_1}\mathcal{I}_{\mathrm{Id}\sqcup g_2\times h_2} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h}.$$

Therefore,

$$Y(\phi,g_1,g_2,h_1,h_2) = U(\phi,g_1,g_2,h_1,h_2).$$

Equation (45) can now be rewritten as:

$$\sum_{h=h_1h_2} U(\phi, g, h_1, h_2) = X(\phi, g, h) - \sum_{g=g_1g_2, h=h_1h_2, g_1\neq g} U(\phi, g_1, g_2, h_1, h_2). \quad (46)$$

Note that

$$U(\phi, g, h_1, h_2) = U(\phi, g, \mathrm{Id}, h_1, h_2).$$

Therefore (46) implies that

$$U(\phi, g, h, \mathrm{Id}) = X(\phi, g, h) - \sum_{g=g_2g_1, h=h_2h_1, g_1\sqcup h_1\neq g_2\sqcup h_2} U(\phi, g_1, g_2, h_1, h_2).$$

The induction assumption implies that $U(\phi, g_1, h_1) = c(\phi, g_1 \sqcup h_1)$ if $g_1 \sqcup h_1 \neq g \sqcup h$. This implies that the right-hand side equals the composition:

$$\mathcal{P}_{\phi\sqcup g\sqcup h} \xrightarrow{c(\phi, g\sqcup h)} \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g\sqcup h} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h}.$$

By definition, the left-hand side equals the composition:

$$\begin{aligned}&\mathcal{P}_{\phi\sqcup g\sqcup h} \to \mathcal{P}_{\phi\sqcup g\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h}\\ &\qquad \to \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h}.\end{aligned}$$

Therefore, the diagram (42) is commutative. The original statement can now be proven straightforwardly using 17.5. □

17.6. *Compositions $\mathcal{P}\delta\mathcal{P} \to \mathcal{P} \to \mathcal{I}\delta$*

Claim 17.7. (1) *The composition*

$$\mathcal{P}_{\phi_1\sqcup g_1}\delta_{\phi_2\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \to \mathcal{P}_{\phi\sqcup g} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \quad (47)$$

vanishes if $\phi_1 \sqcup g_1 \neq \mathrm{Id}$ *and* $\phi_1 \neq \phi$ *or* $g_3 \neq \mathrm{Id}$. *In the cases when it does not vanish we have the following rules*:

(2) *In the case* $\phi_1 \sqcup g_1 = \mathrm{Id}$, *this composition equals*:

$$\delta_{\phi_2\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \to \delta_{\phi_2\sqcup g_2}\mathcal{I}_{\phi_3\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_3} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}.$$

(3) *In the case* $\phi_1 = \phi, g_3 = \mathrm{Id}$, *this composition equals* $-A$, *where*

$$A : \mathcal{P}_{\phi\sqcup\mathfrak{g}_1}\delta_{\mathrm{Id}\sqcup g_2} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\delta_{\mathrm{Id}\sqcup g_2} \to \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}.$$

Proof. We shall use induction in g. Compute the composition

$$\mathcal{P}_{\phi_1\sqcup g_1}\delta_{\phi_2\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \to \mathcal{P}_{\phi\sqcup g} \xrightarrow{F(\phi, g^1, g^2)} \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}, \quad (48)$$

where $F(\phi, g^1, g^2)$ is as in (39). This composition is equal to:

$$\mathcal{P}_{\phi_1\sqcup g_1}\delta_{\phi_2\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \longrightarrow \mathcal{P}_{\phi\sqcup g} \longrightarrow \mathcal{P}_{\phi\sqcup g^1}\mathcal{P}_{\mathrm{Id}\sqcup g^2} \longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g^1}\mathcal{I}_{\mathrm{Id}\sqcup g^2}$$

$$\longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}.$$

This composition does not vanish only if

A: $g_3 = g^2 u$, $g^1 = u g_2 g_1$;
B: $\phi_1 = \phi$; $g^1 = g_1$, $g^2 = g_3 g_2$.

Consider several cases.

Case 1. $\phi_1 \neq \phi$ and $\phi_1 \times g_1 \neq \mathrm{Id}$. The induction assumption implies that the composition (48) vanishes whenever $g_2 \neq \mathrm{Id}$. Therefore, the composition

$$\mathcal{P}_{\phi_1\sqcup g_1}\delta_{\phi_2\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \longrightarrow \mathcal{P}_{\phi\sqcup g} \longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}$$

equals

$$\mathcal{P}_{\phi_1\sqcup g_1}\delta_{\phi_2\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \longrightarrow \mathcal{I}_{\phi\sqcup g} \longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}.$$

This composition vanishes because $\phi_1 \sqcup g_1 \neq \mathrm{Id}$.

Case 2. $\phi_1 \sqcup g_1 = \mathrm{Id}$, $g \neq \mathrm{Id}$. In this case B is again excluded. By the inductive assumption, the composition (48) equals

$$A(u, g^2) : \delta_{\phi_2\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \xrightarrow{F(\phi_3, u, g^2)} \delta_{\phi_2\sqcup g_2}\mathcal{I}_{\phi_3} I_{\mathrm{Id}\sqcup g^2 u} \longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}.$$

The composition

$$\delta_{\phi_2\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \longrightarrow \mathcal{I}_{\phi\sqcup g} \longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}$$

equals

$$B : \ \delta_{\phi_2\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \longrightarrow \delta_{\phi_2\sqcup g_3}\mathcal{I}_{\phi_3\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g_3} \longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}.$$

Therefore, the composition

$$\delta_{\phi_2\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \longrightarrow \mathcal{P}_{\phi\sqcup g} \longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{\mathrm{Id}\sqcup g}$$

equals

$$B - \sum_{g_3 = g^2 u, g^2 \neq \mathrm{Id}} A(u, g^2).$$

It follows that this composition equals:

$$\delta_{\phi_2\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \longrightarrow \delta_{\phi_2\sqcup g_2}\mathcal{I}_{\phi_3\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_3} \longrightarrow \mathcal{I}_{\phi\sqcup\mathrm{Id}}\mathcal{I}_{g\sqcup\mathrm{Id}}$$

which is what is predicted by 2.

We have the last remaining case $\phi_1 = \phi$, $g \neq \mathrm{Id}$, where we have to add contributions from A and B.

Then the contribution from A is equal to zero if $g_3 = \mathrm{Id}$. Otherwise, according to the inductive assumption, it equals to $-C$, where

$$C : \mathcal{P}_{\phi \sqcup g_1} \delta_{\mathrm{Id} \sqcup g_2} \mathcal{P}_{\mathrm{Id} \sqcup g_3} \to \mathcal{I}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g_1} \delta_{\mathrm{Id} \sqcup g_2} \mathcal{I}_{\mathrm{Id} \sqcup g_3} \to \mathcal{I}_{\phi \sqcup \mathrm{Id}} \mathcal{I}_{\mathrm{Id} \sqcup g}.$$

We see that the contribution from B equals C. Therefore, the composition (48) is zero if $g_3 \neq \mathrm{Id}$ and C otherwise. Note that the composition

$$C : \mathcal{P}_{\phi \sqcup g_1} \delta_{\mathrm{Id} \sqcup g_2} \mathcal{P}_{\mathrm{Id} \sqcup g_3} \twoheadrightarrow \mathcal{I}_{\phi \sqcup g}$$

is always zero. Therefore, the map (47) is zero if $g_3 \neq \mathrm{Id}$ and $-C$ otherwise. This completes the proof. □

17.7. *Compositions $\mathcal{P}\delta\mathcal{P} \to \mathcal{P} \to \mathcal{P}\delta$*

Claim 17.8. *Consider the composition*

$$\mathcal{P}_{\phi_1 \sqcup g_1} \delta_{\phi_2 \sqcup g_2} \mathcal{P}_{\phi_3 \sqcup g_3} \to \mathcal{P}_{\phi \sqcup g} \to \mathcal{P}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g}. \tag{49}$$

If $\phi_2 \neq \mathrm{Id}$, this composition is equal to the following composition:

$$\mathcal{P}_{\phi_1 \sqcup g_1} \delta_{\phi_2 \sqcup g_2} \mathcal{P}_{\phi_3 \sqcup g_3} \to \mathcal{P}_{\phi_1 \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g_1} \delta_{\phi_2 \sqcup g_2} \mathcal{P}_{\phi_3 \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g_3}$$
$$\to \mathcal{P}_{\phi_1 \sqcup \mathrm{Id}} \delta_{\phi_2 \sqcup \mathrm{Id}} \mathcal{P}_{\phi_3 \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g} \to \mathcal{P}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g}.$$

Otherwise, this composition is equal to zero except the following cases:

(a) $\phi_1 \times g_1 = \mathrm{Id}$, *in which case our composition equals*

$$\delta_{\mathrm{Id} \sqcup g_2} \mathcal{P}_{\phi \sqcup g_3} \to \delta_{\mathrm{Id} \sqcup g_2} \mathcal{P}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g_3} \to \mathcal{P}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g}; \tag{50}$$

(b) $\phi_3 \sqcup g_3 = \mathrm{Id}$, *in which case the composition is equal to*

$$-C, \tag{51}$$

where

$$C : \mathcal{P}_{\phi \sqcup g_1} \delta_{\mathrm{Id} \sqcup g_2} \to \mathcal{P}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g_1} \delta_{\mathrm{Id} \sqcup g_2} \to \mathcal{P}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g}.$$

Proof. We will prove the statement by induction in ϕ.

It suffices to check that

The composition (49) coincides with the maps (50), (51) after composing each of them

(1) with the map

$$P_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g} \to \mathcal{I}_{\phi \sqcup \mathrm{Id}} \delta_{\mathrm{Id} \sqcup g}$$

(2) with the maps

$$P_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \to \mathcal{P}_{\phi_1\sqcup\mathrm{Id}}\mathcal{P}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g},$$

where $\phi_1, \phi_2 \neq \mathrm{Id}$.

(1) Can be checked straightforwardly:

(1a) $\phi_2 \neq \mathrm{Id}$.
If $\phi_1 \neq \mathrm{Id}$, then both compositions are immediately zero.
If $\phi_1 = \mathrm{Id}$, $g_1 \neq \mathrm{Id}$, then again both compositions are zero (the composition (50) is zero because the corresponding map $\mathcal{P}_{\phi_1\sqcup g_1} \to \mathcal{P}_{\phi_1\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}$ is zero).
If $\phi_1 \sqcup g_1 = \mathrm{Id}$, then the two compositions coincide.

(1b) $\phi_2 = \mathrm{Id}$. If none of ϕ_1, ϕ_3 is identity, then both compositions are clearly equal to zero.
If $\phi_1 = \mathrm{Id}$ and $g_1 \neq \mathrm{Id}$, then both compositions are zero.
If $\phi_1 \sqcup g_1 = \mathrm{Id}$, then both compositions do clearly coincide.
If $\phi_3 = \mathrm{Id}$ and $g_3 \neq \mathrm{Id}$, then both compositions are zero.
If $\phi_3 = \mathrm{Id}$ and $g_3 = \mathrm{Id}$, then both compositions coincide.

(2a) $\phi_2 \neq \mathrm{Id}$. Compute the composition

$$\mathcal{P}_{\phi_1\sqcup g_1}\delta_{\phi_2\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \to \mathcal{P}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \to \mathcal{P}_{\phi^1\sqcup\mathrm{Id}}\mathcal{P}_{\phi^2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}.$$

This composition vanishes except the following two cases:

(i) $\phi_1 = u\phi^1$.
(ii) $\phi_3 = \phi^2 u$.

In both cases the coincidence is obvious.

(2b) $\phi_2 = \mathrm{Id}$,
$\phi_1, \phi_3 \neq \mathrm{Id}$. If $\phi^1 \neq \phi_1$, both compositions are obviously zero.
Assume $\phi_1 = \phi^1$, $\phi_3 = \phi^2$. Then, according to (17.5) the composition

$$\mathcal{P}_{\phi_1\sqcup g_1}\delta_{\mathrm{Id}\sqcup g_2}\mathcal{P}_{\phi_3\sqcup g_3} \to \mathcal{P}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \to \mathcal{P}_{\phi_1\sqcup\mathrm{Id}}\mathcal{P}_{\phi_3\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$$

is equal to the sum of two maps which annihilate each other. The second composition is also zero.
If $\phi_1 = \mathrm{Id}$ or $\phi_3 = \mathrm{Id}$, then the two compositions do clearly coincide.
This completes the proof. □

17.7.1. *Compatibility with* $\boxtimes$

Claim 17.9. *The following diagram is commutative*:

$$\begin{array}{ccc}
\mathcal{P}_{f_1\sqcup g_1}(M_1)\boxtimes\mathcal{P}_{f_2\sqcup g_2}(M_2) & \longrightarrow & \mathcal{P}_{f_1\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}(M_1)\boxtimes\mathcal{P}_{f_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2}(M_2) \\
\downarrow & & \downarrow \\
\mathcal{P}_{f_1\sqcup f_2\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2) & \longrightarrow & \mathcal{P}_{f_1\sqcup f_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2)
\end{array}$$

Direct application of formulas yields the commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{P}_{f_1\sqcup g_1}(M_1)\boxtimes\mathcal{P}_{f_2\sqcup g_2}(M_2) & \longrightarrow & \mathcal{P}_{f_1\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}(M_1)\boxtimes\mathcal{P}_{f_2\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2}(M_2) \\
\downarrow & & \downarrow \\
\mathcal{P}_{f_1\sqcup f_2\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2) & & \mathcal{P}_{f_1\sqcup f_2\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2) \\
\downarrow & & \downarrow \\
\mathcal{I}_{u_1\sqcup v_1\sqcup \mathrm{Id}}\mathcal{I}_{u_2\sqcup v_2\sqcup \mathrm{Id}}\cdots\mathcal{I}_{u_n\sqcup v_n\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1\sqcup g_2} & \longrightarrow & \mathcal{I}_{u_1\sqcup v_1\sqcup \mathrm{Id}}\mathcal{I}_{u_2\sqcup v_2\sqcup \mathrm{Id}}\cdots\mathcal{I}_{u_n\sqcup v_n\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1\sqcup g_2}
\end{array}$$

which proves the statement.

17.8. *The maps* $s(\phi, g) : \mathbb{R}_{\phi\sqcup g} \to \mathbb{R}_{\phi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$

17.8.1. *Definition*

Define a map

$$S(\phi_1\sqcup g_1, \phi^1\sqcup g^1, \ldots, \phi_n\sqcup g_n) : \mathcal{P}_{\phi_1\sqcup g_1}\delta_{\phi^1\sqcup g^1}\cdots\mathcal{P}_{\phi_n\sqcup g_n}$$

$$\to \mathcal{P}_{\phi_1\sqcup \mathrm{Id}}\delta_{\phi^1\sqcup \mathrm{Id}}\cdots\delta_{\phi^{n-1}\sqcup \mathrm{Id}}\mathcal{P}_{\phi_n\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g},$$

where $g = g_n g_{n-1}\cdots g_1$, as follows:

$$\mathcal{P}_{\phi_1\sqcup g_1}\delta_{\phi^1\sqcup g^1}\cdots\mathcal{P}_{\phi_n\sqcup g_n} \to \mathcal{P}_{\phi_1\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\delta_{\phi^1\sqcup \mathrm{Id}}\cdots\delta_{\phi^{n-1}\sqcup \mathrm{Id}}\mathcal{P}_{\phi_n\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \phi^n}$$

$$\to \mathcal{P}_{\phi_1\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g^1}\cdots\delta_{\mathrm{Id}\sqcup \phi^{n-1}}\mathcal{P}_{\phi_n\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}.$$

Let $S'(\phi_1\sqcup g_1, \phi^1\sqcup g^1, \ldots, \phi_n\sqcup g_n) = 0$ if at least one of ϕ^i is identity. Otherwise set

$$S'(\phi_1\sqcup g_1, \phi^1\sqcup g^1, \ldots, \phi_n\sqcup g_n) = S(\phi_1\sqcup g_1, \phi^1\sqcup g^1, \ldots, \phi_n\sqcup g_n).$$

The sum of all possible $S'(\phi_1\sqcup g_1, \phi^1\sqcup g^1, \ldots, \phi_n\sqcup g_n)$ produces a map

$$s(\phi, g) : \mathbb{R}_{\phi\sqcup g} \to \mathbb{R}_{\phi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}.$$

Let us study its properties.

17.8.2. Denote

$$s(g_1, \phi, g_2) : \mathbb{R}_{\phi\sqcup g} \to \mathbb{R}_{\mathrm{Id}\sqcup g_1}\mathbb{R}_{\phi\sqcup g_2} \to \delta_{\mathrm{Id}\sqcup g_1}\mathbb{R}_{\phi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2} \to \mathbb{R}_{\phi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g};$$

$$s(\phi, g_1, g_2) : \mathbb{R}_{\phi\sqcup g} \to \mathbb{R}_{\phi\sqcup g_1}\mathbb{R}_{\mathrm{Id}\sqcup g_2} \to \mathbb{R}_{\phi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup \mathfrak{g}_1}\delta_{\mathrm{Id}\sqcup g_2} \to \mathbb{R}_{\phi\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}.$$

Claim 17.10.

$$ds(\phi, g) = \sum_{g = g_2 g_1} (s(g_1, \phi, g_2) - s(\phi, g_1, g_2))$$

Proof. Follows directly from Sec. 17.7. □

17.8.3. **Claim 17.11.** *The following diagram is commutative*:

$$\begin{array}{ccc}\mathbb{R}_{\phi\sqcup g\sqcup h} & \longrightarrow & \mathbb{R}_{\phi\sqcup g\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup\mathrm{Id}\sqcup h} \\ & \searrow & \downarrow \\ & & \mathbb{R}_{\phi\sqcup \mathrm{Id}\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g\sqcup h}\end{array}$$

Proof. Follows directly from Sec. 17.5. □

17.8.4. **Claim 17.12.** *Assume that ϕ is not bijective. Then the composition*

$$\mathcal{P}_{\phi\sqcup g} \to \mathcal{P}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \to \delta_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$$

equals

$$\mathcal{P}_{\phi\sqcup g} \to \delta_{\phi\sqcup g}.$$

17.8.5. Introduce a map

$$K(\phi_1,\phi_2,g_1,g_2)\colon \mathbb{R}_{\phi_2\phi_1\sqcup g_2g_1} \to \mathbb{R}_{\phi_1\sqcup g_1}\mathbb{R}_{\phi_2\sqcup g_2} \to \mathbb{R}_{\phi_1\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathbb{R}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2}$$
$$\to \mathbb{R}_{\phi_1\sqcup\mathrm{Id}}\mathbb{R}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2g_1}.$$

Claim 17.13. *The map*

$$\mathbb{R}_{\phi_2\phi_1\sqcup g} \to \mathbb{R}_{\phi_2\phi_1\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \to \mathbb{R}_{\phi_1\sqcup\mathrm{Id}}\mathbb{R}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$$

is equal to

$$\sum_{g_2g_1=g} K(\phi_1,\phi_2,g_1,g_2).$$

Proof. It suffices to check that the two maps coincide when compose with the maps

(1)

$$\mathbb{R}_{\phi_1\sqcup\mathrm{Id}}\mathbb{R}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \to \delta_{\phi_1\sqcup\mathrm{Id}}\delta_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$$

(2)

$$\mathbb{R}_{\phi_1\sqcup\mathrm{Id}}\mathbb{R}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \to \mathbb{R}_{\psi_1\sqcup\mathrm{Id}}\mathbb{R}_{\psi_2\sqcup\mathrm{Id}}\mathbb{R}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g},$$

where $\psi_2\psi_1 = \phi_1$ and $\psi_1,\psi_2 \neq \phi_1$;

(3)

$$\mathbb{R}_{\phi_1\sqcup\mathrm{Id}}\mathbb{R}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \to \mathbb{R}_{\phi_1\sqcup\mathrm{Id}}\mathbb{R}_{\chi_1\sqcup\mathrm{Id}}\mathbb{R}_{\chi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g},$$

where $\chi_2\chi_1 = \phi_2$ and $\chi_1, \chi_2 \neq \phi_2$.

Let us check (1). The composition

$$\mathcal{P}_{\psi_1\sqcup g_1}\delta_{\psi^1\sqcup g^1}\cdots\mathcal{P}_{\psi_n\sqcup g_n} \to \mathbb{R}_{\psi\sqcup g} \to \mathbb{R}_{\phi\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g} \to \mathbb{R}_{\phi_1\sqcup\mathrm{Id}}\mathbb{R}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$$
$$\to \delta_{\phi_1\sqcup\mathrm{Id}}\delta_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g}$$

does not vanish iff the leftmost term is

$$\mathcal{P}_{\mathrm{Id}}\delta_{\phi_1\sqcup g_1}\mathcal{P}_{\mathrm{Id}}\delta_{\phi_2\sqcup g_2}\mathcal{P}_{\mathrm{Id}},$$

in which case it is

(a) zero if $\phi_1 = \mathrm{Id}$ or $\phi_2 = \mathrm{Id}$;
(b) identity otherwise.

Let us now examine the composition:

$$\mathbb{R}_{\phi\sqcup g} \to \mathbb{R}_{\phi_1\sqcup g_1}\mathbb{R}_{\phi_2\sqcup g_2} \to \mathbb{R}_{\phi_1\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\mathbb{R}_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2} \to \delta_{\phi_1\sqcup\mathrm{Id}}\delta_{\phi_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}\delta_{\mathrm{Id}\sqcup g_2}.$$

According to the previous statement, this composition vanishes if $\phi_1 = \mathrm{Id}$ or $\phi_2 = \mathrm{Id}$. Otherwise, this composition equals:

$$\mathbb{R}_{\phi\sqcup g} \to \mathbb{R}_{\phi_1\sqcup g_1}\mathbb{R}_{\phi_2\sqcup g_2} \to \delta_{\phi_1\sqcup g_1}\delta_{\phi_2\sqcup g_2}.$$

We see that the two maps coincide.

(2) and (3) are immediate by induction. □

17.8.6. *Compatibility with* $\boxtimes$

Claim 17.14. *The following diagram is commutative*:

$$\begin{array}{ccccc}
\mathbb{R}_{f_1\sqcup g_1}(M_1)\boxtimes\mathbb{R}_{f_2\sqcup g_2}(M_2) & \longrightarrow & \mathbb{R}_{f_1\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1}(M_1)\boxtimes\mathbb{R}_{f_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_2}(M_2) & \longrightarrow & \mathbb{R}_{f_1\sqcup f_2\sqcup\mathrm{Id}}\delta_{\mathrm{Id}\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2) \\
\downarrow & & & \nearrow & \\
\mathbb{R}_{f_1\sqcup f_2\sqcup g_1\sqcup g_2}(M_1\boxtimes M_2) & & & &
\end{array}$$

Proof. Similar to the previous one. □

17.9. *Direct images with respect to projections*

The reformulation of the properties that were proven in the previous subsections in terms of direct image functors with respect to projections is given in Sec. 5.3. We are now passing to giving an appropriate formalism for description of structures that we have encountered.

18. Formalism for Description of Different Structures on a Collection of Functors

18.1. *Definition of skeleton*

18.1.1. Let $\mathcal{C}$ be a category (for example, the category of finite sets). We consider it as a 2-category with trivial 2-morphisms.

A skeleton over $\mathcal{C}$ is a 2-category $\mathcal{S}$ with the following features:

objects of $\mathcal{S}$ are the same as in $\mathcal{C}$;
all categories $\mathcal{S}(S, T)$ are groupoids;
we have a 2-functor $P : \mathcal{S} \to \mathcal{C}^{\mathrm{op}}$.

Let us decode this definition. Note that P induces maps of groupoids

$$P(T, S) : \mathcal{S}(T, S) \to \mathcal{C}^{\mathrm{op}}(T, S).$$

For $F : S \to T$ being an arrow in $\mathcal{C}$, let $\mathcal{S}(F) := P(T, S)^{-1}(F)$. Since $\mathbf{setf}^{\mathrm{op}}(S, T)$ is a trivial groupoid (with only identity morphisms), we have an isomorphism of groupoids:

$$\mathcal{S}(T, S) = \sqcup_{F : S \to T} \mathcal{S}(F).$$

The rest of the structure can be reformulated as follows:

For every pair of $\mathcal{C}$-arrows $F : S \to R$ and $G : R \to T$, there should be given composition functors

$$\circ(F, G) : \mathcal{S}(F) \times \mathcal{S}(G) \to \mathcal{S}(GF);$$

for every triple of $\mathcal{C}$-arrows $F : S \to R$, $G : R \to P$, $H : P \to T$, there should be given isomorphism $i(F, G, H)$ of functors

$$\mathcal{S}(F) \times \mathcal{S}(G) \times \mathcal{S}(H) \to \mathcal{S}(GF) \times \mathcal{S}(H) \to \mathcal{S}(HGF)$$

and

$$\mathcal{S}(F) \times \mathcal{S}(G) \times \mathcal{S}(H) \to \mathcal{S}(F) \times \mathcal{S}(HG) \to \mathcal{S}(HGF).$$

These isomorphisms should satisfy the pentagon axiom.

Namely, let

$$S \xrightarrow{F} P \xrightarrow{G} Q \xrightarrow{H} R \xrightarrow{K} S$$

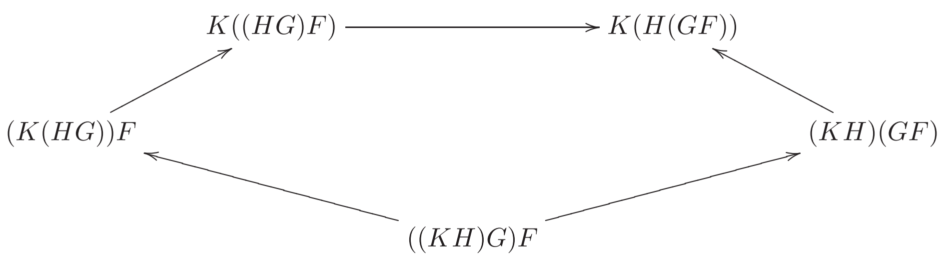
be a sequence of maps of finite sets. Every bracketing of the product $KHGF$ specifies a functor

$$\mathcal{B}(F) \otimes \mathcal{B}(G) \otimes \mathcal{B}(H) \otimes \mathcal{B}(K) \to \mathcal{B}(KHGF):$$

for example, the bracketing $(KH)(GF)$ corresponds to the functor

$$[(KH)(GF)] : \mathcal{B}(F) \otimes \mathcal{B}(G) \otimes \mathcal{B}(H) \otimes \mathcal{B}(K) \xrightarrow{\circ(G,F)\otimes\circ(K,H)} \mathcal{B}(GF)$$
$$\otimes\, \mathcal{B}(KH) \xrightarrow{\circ(KH,GF)} \mathcal{B}(KHGF).$$

The other bracketings produce the corresponding functors in a similar way. Total there are five such bracketings. The associativity maps induce isomorphisms between these functors as shown on the following diagram:



The pentagon axioms requires that this pentagon be commutative.

18.2. *Body*

A body $\mathcal{B}$ built on a skeleton $\mathcal{S}$ is an arbitrary dg-2-category with the following features:

Objects of $\mathcal{B}$ are the same as in $\mathcal{C}$;
$\mathrm{Ob}\mathcal{B}(T,S) = \mathrm{Ob}\mathcal{S}(T,S)$;

There exists a 2-functor

$$s : \mathcal{S} \to \mathcal{B}$$

identical on objects and on $\mathrm{Ob}\mathcal{S}(T,S)$ for all T, S;

There exists a 2-functor $P_B : \mathcal{B} \to \mathbf{setf}^{\mathrm{op}}$ such that $P_B\mathfrak{s} = P$.

This definition is equivalent to the following one.

A body $\mathcal{B}$ is a collection of dg-categories $\mathcal{B}(F)$ for all $\mathcal{C}$-arrows $F : S \to T$ with the following features:

(1) $\mathrm{Ob}\mathcal{B}(F) = \mathrm{Ob}\mathcal{S}(F)$;
(2) There are given functors $\mathfrak{s} := \mathfrak{s}(F) : k[\mathcal{S}(F)] \to \mathcal{B}(F)$ identical on objects;
(3) There are given functors $\circ_{\mathcal{B}}(G,F) : \mathcal{B}(F) \times \mathcal{B}(G) \to \mathcal{B}(GF)$ which coincide on the level of objects with $\circ(G,F)$ and such that

$$\circ_{\mathcal{B}}(G,F)(\mathfrak{s}(a) \times s(b)) = \mathfrak{s}(\circ(F,G)(a \times b));$$

where a is an arrow in $\mathcal{S}(G)$ and b is an arrow in $\mathcal{S}(F)$.
(4) There are given associativity constraints $c_{\mathcal{B}}(H,G,F)$ which satisfy the pentagon axiom and are compatible with $c(F,G,H)$ in the obvious way, that is: given

arrows a, b, c in respectively $\mathcal{S}(H), \mathcal{S}(G), \mathcal{S}(F)$, one has:

$$\mathfrak{s}(c_{\mathcal{S}}(a, b, c)) = c_{\mathcal{B}}(\mathfrak{s}(a), \mathfrak{s}(b), \mathfrak{s}(c)).$$

18.2.1. To define a body one has to prescribe complexes $\mathcal{B}(X, Y)$ for all $X, Y \in \mathcal{S}(F)$ and certain poly-linear maps between these complexes. Assume that the isomorphism classes of $\mathcal{C}$ and the isomorphism classes of $\mathcal{S}(F)$ form a (countable) set for any $\mathcal{C}$-arrow F. Then it is clear that the structure of a body with skeleton $\mathcal{S}$ is equivalent to a structure of an algebra over a certain colored operad with a (countable) set of colors. Denote this colored operad by $\mathbf{body}(\mathcal{S})$. The countability hypothesis will always be the case in our constructions.

Thus, given a fixed skeleton, we have notions of a free body, a body generated by generators and relations etc.

18.2.2. *Example*

In this example the objects of $\mathcal{S}(F)$ will not form a set.

For $F : S \to T$ we set $B(F)$ to be the category of all functors from the category of $\mathcal{D}_{X^T}$-modules to the category of $\mathcal{D}_{X^S}$-modules. Let $\mathcal{S}(F)$ be the groupoid of isomorphisms of $B(F)$. The rest of the structure is defined in an obvious way. Denote such a body by **FULL**.

18.2.3. A map of bodies is naturally defined; a map $B \to \mathbf{FULL}$ is referred to as *a representation.*

18.3. *Construction of a skeleton*

We will mainly use a skeleton **Ske**, which will now be described. We set $\mathcal{C} := \mathbf{setf}$ to be the category of finite sets. Let $F : S \to T$. Objects of $\mathcal{S}(F)$ are sequences

$$S \xhookrightarrow{i} U_0 \xrightarrow{p_1} U_1 \xrightarrow{p_2} U_2 \xrightarrow{p_3} \cdots \xrightarrow{p_n} U_n = T,$$

where $p_n \cdots p_1 i = F$ and each p_k is a proper surjection (i.e. is not a bijection). Such objects will also be denoted by

$$\mathfrak{p}_i \mathbb{R}_{p_1} \mathbb{R}_{p_2} \cdots \mathbb{R}_{p_n}.$$

We shall also use a notation

$$\mathbb{R}_{p_1} \mathbb{R}_{p_2} \cdots \mathbb{R}_{p_n}$$

instead of

$$\mathfrak{p}_{\mathrm{Id}_{U_0}} \mathbb{R}_{p_1} \mathbb{R}_{p_2} \cdots \mathbb{R}_{p_n}.$$

We do not exclude the case $n = 0$, in which case the corresponding object will be written simply as $\mathfrak{p}_i$.

18.3.1. Define isomorphisms in this groupoid. Let

$$Y = \mathfrak{p}_j \mathbb{R}_{q_1} \mathbb{R}_{q_2} \cdots \mathbb{R}_{q_m}$$

be another object in $\mathbf{Ske}(F)$, where $q_j : V_{j-1} \to V_j$ and $V_m = T$.

The set $\mathbf{Ske}(F)(X, Y)$ is non-empty only if $n = m$.

An isomorphism $p : X \to Y$ is a collection of bijections $b_k : U_k \to V_k$ for all k satisfying the following natural compatibility properties:

(1) $b_n = \mathrm{Id}_T$;

(2) For $k < k'$ set

$$p_{k'k} := p_{k'} p_{k'-1} \cdots p_{k+1};$$

$$q_{k'k} := q_{k'} q_{k'-1} \cdots q_{k+1}.$$

Then the diagram

$$\begin{array}{ccc} U'_k & \xrightarrow{b_{k'}} & V_{k'} \\ \big\downarrow{\scriptstyle p_{k'k}} & & \big\downarrow{\scriptstyle q_{k'k}} \\ U_k & \xrightarrow{b_k} & V_k \end{array}$$

commutes.

(3) The diagrams

$$\begin{array}{ccc} S & \xrightarrow{i} & U_k \\ & \searrow^{j} & \big\downarrow{\scriptstyle b_k} \\ & & V_k \end{array}$$

commute.

The composition law is obvious.

18.3.2. Let

$$S \xrightarrow{F} T \xrightarrow{G} R.$$

The composition morphisms

$$\circ_{\mathbf{ske}}(G, F)$$

are defined as follows.

Let

$$X = \mathfrak{p}_i \mathbb{R}_{p_0} \mathbb{R}_{p_1} \cdots \mathbb{R}_{p_n},$$

where $i : S \to U_0$, $p_k : U_k \to U_{k+1}$, $U_{n+1} = T$, $F = p_n p_{n-1} \cdots p_0 i$. Let

$$Y = \mathfrak{p}_j \mathbb{R}_{q_0} \mathbb{R}_{q_1} \cdots \mathbb{R}_{q_m},$$

where $j : T \to V_0$, $q_k : V_k \to V_{k+1}$, $V_{m+1} = R$, $G = q_m q_{m-1} \cdots q_0 j$.

Let $Z := V_0 \backslash j(T)$. Set $U'_k := U_k \sqcup Z$, $p'_k := p_k \sqcup \mathrm{Id}_Z$, $j' : U'_{n+1} \to V_0$, $j' := j \sqcup i_Z$, where $i_Z : Z \to V_0$ is the natural embedding, j' is then bijective. Set $i' : S \to U_0 \to U'_0$ to be the natural map.

Set $\circ_{\mathbf{ske}}(G, F)(X, Y)$ to be

$$\mathfrak{p}_{i'} \mathbb{R}_{p'_0} \mathbb{R}_{p'_1} \cdots \mathbb{R}_{jp'_n} \mathbb{R}_{q_0} \mathbb{R}_{q_1} \cdots \mathbb{R}_{q_m}.$$

We shall write XY instead of $\circ_{\mathbf{ske}}(G, F)(X, Y)$.

18.4. *Bodies* $\mathcal{B}_{as}, \mathcal{B}_{\mathbf{presymm}}, \mathcal{B}_{\mathbf{symm}}$

We are going to define the bodies which axiomatize the situations we are working with: those of a system ($\mathcal{B}_{as}$); of a pre-symmetric system ($\mathcal{B}_{\mathbf{presymm}}$) and of a symmetric system ($\mathcal{B}_{\mathbf{symm}}$). All these bodies are constructed on the skeleton **Ske**.

18.4.1. *Body* $\mathcal{B}_{as}$

It is generated by the maps $\mathfrak{as}(q, p) : \mathbb{R}_{pq} \to \mathbb{R}_q \mathbb{R}_p$ of degree zero with zero differential, where $q : S \to R$ and $p : R \to T$ and the relation:

The compositions

$$\mathbb{R}_{pqr} \xrightarrow{\mathfrak{as}(r,qp)} \mathbb{R}_r \mathbb{R}_{pq} \xrightarrow{\mathfrak{as}(q,p)} \mathbb{R}_r \mathbb{R}_q \mathbb{R}_p$$

and

$$\mathbb{R}_{pqr} \to \mathbb{R}_{qr} \mathbb{R}_p \to \mathbb{R}_r \mathbb{R}_q \mathbb{R}_p$$

coincide.

18.5. *Explicit description of the complexes* $\mathbf{hom}_{\mathcal{B}_{as}(F)}(X, Y)$

Let

$$X = \mathfrak{p}_i \mathbb{R}_{p_0} \mathbb{R}_{p_1} \cdots \mathbb{R}_{p_n},$$

where $i : S \to U_0, p_k : U_k \to U_{k+1}, U_{n+1} = T$, $F = p_n p_{n-1} \cdots p_0 i$. Let

$$Y = \mathfrak{p}_j \mathbb{R}_{q_0} \mathbb{R}_{q_1} \cdots \mathbb{R}_{q_m},$$

where $j : S \to V_0, q_k : V_k \to V_{k+1}, V_{m+1} = T, F = q_m q_{m-1} \cdots q_0 j$.

The space $\hom_{\mathcal{B}_{as}(F)}(X, Y)$ is non-empty only if for every U_k there exists a V_l such that $\#U_k = \#V_l$. Define the set $S(X, Y)$ whose each element f is a collection of bijections $f_{kl} : U_k \to V_l$, whenever $\#U_k = \#V_l$ satisfying all the properties from Sec. 18.3.1. Set

$$\hom_{\mathcal{B}_{as}(F)}(X, Y) := k[S(X, Y)].$$

The composition law in $\mathcal{B}_{as}(F)$ and the inclusion functor $\mathbf{Ske}(F) \to \mathcal{B}_{as}(F)$ are immediate.

18.5.1. *The body* $\mathcal{B}_{\mathbf{presymm}}$

It is generated over $\mathcal{B}_{as}$ by the elements of two types:

Type 1. Consider a commutative triangle

$$\begin{array}{ccc} S & \overset{p}{\twoheadrightarrow} & T \\ {\scriptstyle i}\uparrow & \overset{j}{\nearrow} & \\ R & & \end{array}$$

in which i, j are injections and p is a proper surjection. We then have a degree $+1$ map

$$L(i,p) : \mathfrak{p}_i \mathbb{R}_p \to \mathfrak{p}_j.$$

Type 2. Consider a commutative square

$$\begin{array}{ccc} R & \overset{p}{\twoheadrightarrow} & T \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ S & \underset{q}{\twoheadrightarrow} & P \end{array} \tag{52}$$

in which i, j are injections and p, q are proper surjections. Call such a square *suitable* if the following is satisfied: Let $T_1 = T \backslash T_2$ be the subset of all $t \in T$ such that $p^{-1}t \cap i(S)$ consists of ≥ 2 elements. Then $p^{-1}(T_1) \subset i(S)$, i.e.

$$\#(p^{-1}t \cap i(S)) \geq 2 \Rightarrow p^{-1}(t) \subset i(S).$$

We then have a degree zero map

$$A(i,p,j,q) : \mathfrak{p}_i \mathbb{R}_p \to \mathbb{R}_q \mathfrak{p}_j,$$

where $\mathbb{R}_q \mathfrak{p}_j := \circ_{\mathbf{Ske}}(\mathbb{R}_q, \mathfrak{p}_j)$.

18.5.2. *Relations*

(1) Let

$$\begin{array}{ccc} R & \overset{p}{\twoheadrightarrow} & T \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ S & \underset{q}{\twoheadrightarrow} & P \end{array}$$

be a suitable square and $q = q_2 q_1$, where q_1, q_2 are surjections.

Define the set $X(q_1, q_2)$ of isomorphism classes of commutative diagrams

$$\begin{array}{ccccc} R & \xrightarrow{p_1} & U & \xrightarrow{p_2} & T \\ {\scriptstyle j}\uparrow & & {\scriptstyle j'}\uparrow & & {\scriptstyle j}\uparrow \\ S & \xrightarrow{q_1} & V & \xrightarrow{q_2} & P \end{array}$$

We will refer to such a diagram as (p_1, p_2, j'). Both squares in every such a diagram are automatically suitable. Therefore, every element $x := (p_1, p_2, j') \in X(q_1, q_2)$ determines a map

$$m_x : \mathfrak{p}_i \mathbb{R}_p \to \mathfrak{p}_i \mathbb{R}_{p_1} \mathbb{R}_{p_2} \to \mathbb{R}_{q_1} \mathfrak{p}_{j'} \mathbb{R}_{p_2} \to \mathbb{R}_{q_1} \mathbb{R}_{q_2} \mathfrak{p}_j.$$

Then the relation says that the composition

$$\mathfrak{p}_i \mathbb{R}_p \to \mathbb{R}_q \mathfrak{p}_j \to \mathbb{R}_{q_1} \mathbb{R}_{q_2} \mathfrak{p}_j$$

equals

$$\sum_{x \in X(q_1, q_2)} m_x.$$

(2) Consider the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{p} & T \\ {\scriptstyle i_2}\uparrow & & \uparrow{\scriptstyle j_2} \\ S_1 & \xrightarrow{q} & P_1 \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle j_1} \\ S & \xrightarrow{r} & P \end{array}$$

in which both small squares are suitable. Then the large square is also suitable and the following maps coincide:

$$\mathfrak{p}_{i_2 i_1} \mathbb{R}_p \to \mathbb{R}_r \mathfrak{p}_{j_2 j_1}$$

and

$$\mathfrak{p}_{i_2 i_1} \mathbb{R}_p \to \mathfrak{p}_{i_1} \mathfrak{p}_{i_2} \mathbb{R}_p \to \mathfrak{p}_{i_1} \mathbb{R}_q \mathfrak{p}_{j_2} \to \mathbb{R}_r \mathfrak{p}_{j_1} \mathfrak{p}_{j_2} \to \mathbb{R}_r \mathfrak{p}_{j_2 j_1}.$$

(3) Consider the following commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{p} & T \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ S & \xrightarrow{q} & P \\ {\scriptstyle k}\uparrow & \nearrow{\scriptstyle l} & \\ Q & & \end{array}$$

where the upper square is suitable. Then the following maps coincide:

$$\mathfrak{p}_{ik}\mathbb{R}_p \to \mathfrak{p}_k\mathfrak{p}_i\mathbb{R}_p \to \mathfrak{p}_k\mathbb{R}_q\mathfrak{p}_j \to \mathfrak{p}_{qk}\mathfrak{p}_j = \mathfrak{p}_l$$

and

$$\mathfrak{p}_{ik}\mathbb{R}_p \to \mathfrak{p}_{pik} = \mathfrak{p}_l.$$

(4) Consider the following commutative diagram

$$\begin{array}{ccc} S \sqcup U & \longrightarrow & T \sqcup U \\ \uparrow & & \uparrow \\ S & \longrightarrow & T \end{array}$$

this diagram is suitable and we require that the corresponding map $A(i,p,j,q)$ be equal to the corresponding isomorphism in **Ske**.
(5) Let

$$\begin{array}{ccc} R & \xrightarrow{p} & T \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ S & \xrightarrow{q} & X \end{array}$$

and

$$\begin{array}{ccc} R_1 & \xrightarrow{p_1} & T_1 \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle j_1} \\ S_1 & \xrightarrow{q_1} & X_1 \end{array}$$

be suitable squares and let $s\colon S \to S_1$, $r\colon R \to R_1$, $t\colon T \to T_1$, $x\colon X \to X_1$ be bijections fitting the two squares into a commutative cube. Then the map $A(i,p,j,q)$ coincides with the map

$$\begin{aligned} \mathfrak{p}_i\mathbb{R}_p &\cong \mathfrak{p}_s\mathfrak{p}_{i_1}\mathfrak{p}_{r^{-1}}\mathfrak{p}_r\mathbb{R}_{p_1}\mathfrak{p}_{t_1^{-1}} \cong \mathfrak{p}_s\mathfrak{p}_{i_1}\mathbb{R}_{p_1}\mathfrak{p}_{t_1^{-1}} \to \mathfrak{p}_s\mathbb{R}_{q_1}\mathfrak{p}_{j_1}\mathfrak{p}_{t_1^{-1}} \\ &\cong \mathfrak{p}_s\mathbb{R}_{q_1}\mathfrak{p}_{x^{-1}}\mathfrak{p}_x\mathfrak{p}_{j_1}\mathfrak{p}_{t_1^{-1}}\mathbb{R}_q\mathfrak{p}_j. \end{aligned}$$

18.5.3. *Differentials*

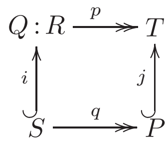
(1) The differential of the map $L(i,p)$ is computed as follows. Consider the set of all equivalence classes of decompositions $p = p_2p_1$, where p_1, p_2 are surjections and p_1i is injection. We then have a map

$$l(p_1,p_2)\colon \mathfrak{p}_i\mathbb{R}_p \to \mathfrak{p}_i\mathbb{R}_{p_1}\mathbb{R}_{p_2} \to \mathfrak{p}_{p_1i}R_{p_2} \to \mathfrak{p}_{p_2p_1i} = \mathfrak{p}_{pi}.$$

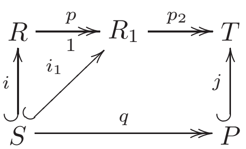
We then have

$$dL(i,p) + \sum_{(p_1,p_2)} l(p_1,p_2) = 0.$$

(2) Let

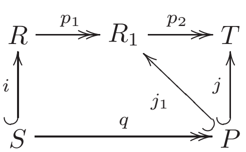


be a suitable square. Define two sets $L(Q)$ and $R(Q)$ as follows. The set $L(Q)$ is the set of all isomorphism classes of diagrams:



such that $p = p_1 p_2$. It is clear that the internal commutative square in this diagram is also suitable.

Define the set $R(Q)$ as the set of isomorphisms classes of diagrams



where $p = p_1 p_2$. The internal square in such a diagram is always suitable as well.

Every element $l := (p_1, p_2, i_1) \in L(Q)$ determines a map

$$f_l : \mathfrak{p}_i \mathbb{R}_p \to \mathfrak{p}_i \mathbb{R}_{p_1} \mathbb{R}_{p_2} \to \mathbb{R}_q \mathfrak{p}_{i_1} \mathbb{R}_{p_2} \to \mathbb{R}_q \mathfrak{p}_{p_2 i_1} = \mathbb{R}_q \mathfrak{p}_j.$$

Every element $r = (p_1, p_2, j_1) \in R(Q)$ determines a map

$$g_r : \mathfrak{p}_i \mathbb{R}_p \to \mathfrak{p}_i \mathbb{R}_{p_1} \mathbb{R}_{p_2} \to \mathfrak{p}_{p_1 i} \mathbb{R}_{p_2} = \mathfrak{p}_{j_1} \mathbb{R}_{p_2} \to \mathbb{R}_q \mathfrak{p}_j.$$

We then have

$$dA(i, p, j, q) = \sum_{l \in L(Q)} f_l - \sum_{r \in R(Q)} g_r.$$

This completes the definition. We need to check that $d^2 = 0$ and that d preserves the ideal generated by the relations, which is left to the reader.

18.5.4. The system $\langle \mathbb{R} \rangle$ with its properties provides for a representation of $\mathcal{B}_{\mathbf{presymm}}$.

18.5.5. *Explicit description of the categories $\mathcal{B}_{\mathbf{presymm}}(F)$*

Consider two objects X, Y in $A(F)$:

$$S \overset{i}{\hookrightarrow} U_0 \overset{p_0}{\twoheadrightarrow} U_1 \overset{p_1}{\twoheadrightarrow} \cdots \quad U_k \overset{p_k}{\twoheadrightarrow} U_{k+1}$$

and

$$S \overset{j}{\hookrightarrow} V_0 \overset{q_0}{\twoheadrightarrow} V_1 \overset{q_1}{\twoheadrightarrow} \cdots \quad V_{l-1} \overset{q_l}{\twoheadrightarrow} U_{l+1}.$$

Define the set $M(X,Y)$ whose each element is a collection of injections

$$j_r : V_{m_r} \hookrightarrow U_r,$$

where $r = 0,1,\ldots,k+1, m_0 = 0$, $0 \leq m_{r+1} - m_r \leq 1$, $m_{k+1} = l+1$. The following conditions should be satisfied:

(1) if $m_{r+1} = m_r$, then the diagram

$$\begin{array}{ccc} U_r & \xrightarrow{p_r} & U_{r+1} \\ {\scriptstyle j_r}\uparrow & \nearrow{\scriptstyle j_{r+1}} & \\ V_{m_r} & & \end{array}$$

must be commutative.

(2) If $m_{r+1} = m_r + 1$, then the diagram

$$\begin{array}{ccc} U_r & \xrightarrow{p_r} & U_{r+1} \\ {\scriptstyle j_r}\uparrow & & \uparrow{\scriptstyle j_{r+1}} \\ V_{m_r} & \xrightarrow{q_r} & V_{m_r+1} \end{array}$$

must be commutative and suitable.

Every element $m = (j_1, j_2, \ldots, j_{k+1})$ in $M(X,Y)$ defines a map

$$A(m) : \mathfrak{p}_i \mathbb{R}_{p_0} \mathbb{R}_{p_1} \cdots \mathbb{R} p_k \to \mathfrak{p}_j \mathbb{R}_{q_0} \mathbb{R}_{q_1} \cdots \mathbb{R}_{q_l},$$

where we set $\mathbb{R}_{\mathrm{Id}} = \mathrm{Id}$, as follows. Define

$$q'_{m_r} : V_{m_k} \to V_{m_{r+1}}$$

to be $\mathrm{Id}_{V_{m_r}}$ if $m_r = m_{r+1}$ and q_{m_r} if $m_{r+1} = m_r + 1$. We then have maps

$$F_k : \mathfrak{p}_{j_r} \mathbb{R}_{p_r} \to \mathbb{R}_{q'_{m_r}} \mathfrak{p}_{j_{r+1}},$$

where $F_k = L(j_r, p_r, j_{r+1})$ if $m_{r+1} = m_r$, and $F_k = C(j_r, p_r, j_{r+1}, q'_{m_r})$ if $m_{r+1} = m_r + 1$.

Set

$$A(m) : \mathfrak{p}_i \mathbb{R}_{p_0} \mathbb{R}_{p_1} \cdots \mathbb{R}_{p_l} \to \mathfrak{p}_j \mathfrak{p}_{j_0} \mathbb{R}_{p_0} \mathbb{R}_{p_1} \cdots \mathbb{R}_{p_l}$$
$$\xrightarrow{A'(j_0,p_0,j_1,q'_{m_0})} \mathfrak{p}_j \mathbb{R}_{q'_{m_0}} \mathfrak{p}_{j_1} \mathbb{R}_{p_1} \cdots \mathbb{R}_{p_l}$$

$$\xrightarrow{A'(j_1,p_1,j_2,q'_{m_1})} \mathfrak{p}_j \mathbb{R}_{q'_{m_0}} \mathbb{R}_{q'_{m_1}} \mathfrak{p}_{j_2} \mathbb{R}_{p_2} \mathbb{R}_{p_3} \cdots \mathbb{R}_{p_l}$$

$$\to \cdots \to \mathfrak{p}_j \mathbb{R}_{q'_{m_0}} \mathbb{R}_{q'_{m_1}} \cdots \mathbb{R}_{q'_{m_{k+1}}} \cong \mathfrak{p}_j \mathbb{R}_{q_0} \mathbb{R}_{q_1} \cdots \mathbb{R}_{q_l},$$

where

$$A'(j_u, p_u, j_{u+1}, q'_{m_u}) = A(j_u, p_u, j_{u+1}, q'_{m_u})$$

if $q'_{m_u} \neq \mathrm{Id}$. Otherwise

$$A'(j_u, p_u, j_{u+1}, q'_{m_u}) = l(j_u, p_u, j_{u+1}).$$

Let $N(X,Y) := k[M(X,Y)]$. Let

$$H(X,Y) = \oplus_Z N(Z,Y),$$

where the sum is taken over all refinements Z of X. We have an obvious map $H(X,Y) \to \hom(X,Y)$. Set $A_F(X,Y) := H(X,Y)$. The relations given in the previous section provide us with a composition law $H(X,Y) \otimes H(Y,Z) \to H(X,Z)$ and a differential.

18.5.6. *Body* $\mathcal{B}_{\mathbf{symm}}$

The definition of the body $\mathcal{B}_{\mathbf{symm}}$ is exactly the same as the one of the body $\mathcal{B}_{\mathbf{presymm}}$ except that the maps $A(i,p,j,q)$ are defined for all commutative squares

$$\begin{array}{ccc} R & \overset{p}{\twoheadrightarrow} & T \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ S & \underset{q}{\twoheadrightarrow} & P \end{array}$$

not necessarily suitable; the relations are the same except that we lift everywhere the restriction of suitability; the formulas for the differential remain the same.

It is clear that we have a map of bodies

$$i : \mathcal{B}_{\mathbf{presymm}} \to \mathcal{B}_{\mathbf{symm}}$$

We are going to study this map.

18.5.7. Explicit expression for $\hom_{\mathcal{B}_{\mathbf{symm}}(F)}(X,Y)$, where $X, Y \in \mathbf{Ske}(F)$ is exactly the same as for $\mathcal{B}_{\mathbf{presymm}}$.

The further study of $\mathcal{B}_{\mathbf{symm}}$ is facilitated by the statement we are going to consider

18.5.8. Let

$$S \overset{i}{\hookrightarrow} R \overset{p}{\twoheadrightarrow} T$$

be a diagram. Call it *super-surjective* if for every $t \in T$:
either $p^{-1}t \cap i(S)$ has at least two elements
or $p^{-1}t$ is a one-element subset of $i(S)$.

Claim 18.1. *Let*

$$\begin{array}{ccc} R & \overset{p}{\twoheadrightarrow} & T \\ {\scriptstyle i}\Big\uparrow & & \Big\uparrow{\scriptstyle j} \\ S & \underset{q}{\twoheadrightarrow} & P \end{array}$$

be a commutative diagram.

Then there exists a decomposition:

$$\begin{array}{ccc} R & \overset{p}{\twoheadrightarrow} & T \\ {\scriptstyle i_2}\Big\uparrow & & \Big\uparrow{\scriptstyle j} \\ U & \overset{r}{\twoheadrightarrow} & P \\ {\scriptstyle i_1}\Big\uparrow & \nearrow{\scriptstyle q} & \\ S & & \end{array}$$

where the diagram is commutative, $i = i_2 i_1$, *the square* (i_2, p, j, r) *is suitable and the pair* (i_1, q) *is super-surjective.*

Such a decomposition is unique upto an isomorphism.

Proof. *Existence*. Call an element $t \in T$ *good* if $p^{-1}t$ satisfies the condition of the definition. Let $G_T \subset T$ be the subset of all good elements. Let $U := i(S) \cup p^{-1}G_T$. Let i_1, i_2 be the natural inclusions. By definition, for every $t \in G_t$, the intersection $p^{-1}t \cap i(S)$ is non-empty. Hence, $G_T \subset pi(S) = jq(S)$ and, therefore, $p(U) = jq(S)$.

Thus, $p(U) = \operatorname{Im} j$, which implies that the map $p|_U : U \to T$ uniquely decomposes as jr, where $r : U \twoheadrightarrow P$. It is clear that all the conditions are satisfied.

Uniqueness is also clear. □

18.5.9. *Corollary*

Let $X \in \mathbf{Ske}(F)$ be an object of the form

$$\mathfrak{p}_j \mathfrak{p}_{i_1} \mathbb{R}_{p_1} \mathfrak{p}_{i_2} \mathbb{R}_{p_2} \cdots \mathfrak{p}_{i_n} \mathbb{R}_{p_n},$$

where every pair (i_k, p_k), $i_k : U_k \to A_k$, $p_k : A_k \to U_{k+1}$, is super-surjective. Let

$$Y = \mathfrak{p}_j \mathbb{R}_{p_1 i_1} \mathbb{R}_{p_2 i_2} \cdots \mathbb{R}_{p_n i_n}.$$

The maps $\mathfrak{p}_{i_k} \mathbb{R}_{p_k} \to \mathbb{R}_{p_k i_k}$ induce a map $f_X : X \to Y$. Call X *a super-surjective decomposition* of Y. Let $\mathbf{super} - \mathbf{sur}(Y)$ be the groupoid of all super-symmetric decompositions of Y and their isomorphisms (i.e. collections of isomorphisms $U_k \to U'_k$ fitting into the commutative diagrams...). It is clear that if $a : X_1 \to X_2$ is such an isomorphism, then $f_{X_1} = f_{X_2} a$.

Let $Z, Y \in \mathbf{Ske}(F)$. Define a functor $h_Z : \mathbf{super} - \mathbf{sur}(Y) \to$ complexes by the formula $h_Z(X) = \hom_{\mathcal{B}_{\mathbf{presymm}}(F)}(Z, X)$.

The collection of maps f_X induces a functor

$$\mathrm{limdir}_{\mathbf{super}-\mathbf{sur}(Y)} h_Z \to \hom_{\mathcal{B}_{\mathbf{symm}}(F)}(Z, Y).$$

Claim 18.2. *This map is an isomorphism.*

18.5.10. *One more lemma*

Let $Y \in \mathbf{Ske}(F)$. Let $F = F_2 F_1$ be a decomposition and assume that we have an isomorphism $t : Y \to Y_2 Y_1$, where $Y_i \mathbf{Ske}(F_i)$. We then have a natural functor:

$$\mathbf{super} - \mathbf{sur}(Y_2) \times \mathbf{super} - \mathbf{sur}(Y_1) \to \mathbf{super} - \mathbf{sur}(Y).$$

Lemma 18.3. *This functor is an equivalence of groupoids.*

Proof. Clear. □

18.6. *Pseudo-tensor bodies*

Let $\mathcal{B}$ be a body. A pseudo-tensor structure on $\mathcal{B}$ is a collection of several pieces of data, the first one being functors

$$\Psi(\{F_i\}_{i \in I}) : \otimes_{i \in I} \mathcal{B}(F_i) \otimes \mathcal{B}(F)^{\mathrm{op}} \to \text{complexes},$$

where $F = \sqcup_{i \in I F_i}$, for all $n > 0$ and all collections $F_i : S_i \to T_i$ of maps of finite sets indexed by an arbitrary finite non-empty set I. Let $X_i \in \mathcal{B}(F_i)$ and $X \in \mathcal{B}(F)$. We then denote

$$\hom(\{X_i\}_{i \in I}; X) := \Psi(\{F_i\}_{i \in I})((\otimes_{i \in I} X_i) \otimes X).$$

Let $\pi : I \to J$ be a surjection of finite sets. Let $F_i : S_i \to T_i$, $i \in I$ be maps of finite sets. For a $j \in J$ set

$$F_j := \sqcup_{i \in \pi^{-1} j} F_i.$$

Let $X_i \in \mathcal{B}(F_i)$, $Y_j \in \mathcal{B}(F_j)$, where $i \in I$, $j \in J$. Set

$$\hom_\pi(\{X_i\}_{i \in I}; \{Y_j\}_{j \in J}) := \otimes_{j \in J} \hom(\{X_i\}_{i \in I}, \{Y_j\}_{j \in J}).$$

An element f in this complex will also be written as

$$f : \{X_i\}_{i \in I} \to \{Y_j\}_{j \in J}.$$

Let $X, Y \in \mathcal{B}(G)$, where $G : S \to T$ be a map of finite sets. It is assumed that $\hom(X, Y) = \hom_{\mathcal{B}(G)}(X, Y)$.

18.6.1. *Composition of the first type*

Let $F = \sqcup_{i \in I} F_i$. The second feature of a pseudo-tensor structure is a collection of *composition maps of the first kind*:

$$C_1(\{X_i\}_{i\in I}; \{Y_j\}_{j\in J}; Z)\colon \hom_\pi(\{X_i\}_{i\in I}; \{Y_j\}_{j\in J}) \otimes \hom(\{Y_j\}_{j\in J}; Z) \\ \to \hom(\{X_i\}_{i\in I}; Z).$$

Let $\sigma\colon J \to K$ be another surjection and set $F_k := \sqcup_{j\in\pi^{-1}k}$. Pick objects $Z_k \in \mathcal{B}(F_k)$, $k \in K$. Define a map

$$C_1(\{X_i\}_{i\in I}; \{Y_j\}_{j\in J}; \{Z_k\}_{k\in K})\colon \hom_\pi(\{X_i\}_{i\in I}; \{Y_j\}_{j\in J}) \\ \otimes \hom_\sigma(\{Y_j\}_{j\in J}; \{Z_k\}_{k\in K}) \to \hom_{\sigma\pi}(\{X_i\}_{i\in I}; \{Z_k\}_{k\in K})$$

as the tensor product

$$\otimes_{k\in K} C_1(\{X_i\}_{i\in(\sigma\pi)^{-1}k}; \{Y_j\}_{j\in\sigma^{-1}k}; Z_k).$$

18.6.2. *Compositions of the second kind*

Let $F_i\colon S_i \to T_i$, $G_i\colon T_i \to R_i$, $i \in I$ be a family of maps of finite sets. Let $F = \sqcup_{i\in I} F_i$, $G = \sqcup_{j\in J} G_j$. Let $X_i \in \mathcal{B}(F_i)$, $Y_i \in \mathcal{B}(G_i)$, $Z \in \mathcal{B}(F)$, $W \in \mathcal{B}(G)$. We then have objects $Y_i X_i \in \mathcal{B}(G_i F_i)$, $WZ \in \mathcal{B}(GF)$. The last feature of a pseudo-tensor structure is a prescription of *composition maps of the second kind*:

$$C_2(\{X_i\}_{i\in I}, Z; \{Y_i\}_{i\in I}, W)\colon \hom(\{X_i\}_{i\in I}; Z) \otimes \hom(\{Y_i\}_{i\in I}; W) \\ \to \hom(\{Y_i X_i\}_{i\in I}; WZ).$$

Let $\pi\colon I \to J$ be a surjection. Let $F_j = \sqcup_{i\in\pi^{-1}j} F_i$; $G_j = \sqcup_{i\in\pi^{-1}j} G_i$. Let $X_i \in \mathcal{B}(F_i), Y_i n \mathcal{B}(G_i), i \in I; Z_j \in \mathcal{B}(F_j), W_j \in \mathcal{B}(G_j), j \in J$. Define a map

$$C_2(\{X_i\}_{i\in I}; \{Y_i\}_{i\in I}; \{Z_j\}_{j\in J}; \{W_j\}_{j\in J})\colon \hom_\pi(\{X_i\}_{i\in I}, \{Z_j\}_{j\in J}) \\ \otimes \hom_\pi(\{Y_i\}_{i\in I}, \{W_j\}_{j\in J}) \to \hom_\pi(\{Y_i X_i\}_{i\in I}; \{W_j Z_j\}_{j\in J})$$

as the tensor product

$$\otimes_{j\in J} C_2(\{X_i\}_{i\in\pi^{-1}j}, Z_j; \{Y_i\}_{i\in\pi^{-1}j}, W_j).$$

18.6.3. *Axiom*

The only axiom is as follows. Let I be a finite set and consider an I-family of chains of maps

$$S_i^0 \xrightarrow{F_i^1} S_i^1 \xrightarrow{F_i^2} \cdots \xrightarrow{F_i^N} S_i^N,$$

where N is a fixed number. For $0 \le p < q \le N$, set $F_i^{qp}: S_i^p \to S_i^q$ to be the composition

$$F_i^q F_p^{q-1} \cdots F_i^{p+1}.$$

We also set $F_i^{qq} := \mathrm{Id}_{S_i^q}$.

We shall also need a chain of surjections

$$I = I_0 \xrightarrow{\pi_1} I_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_M} I_M$$

where M is a fixed natural number. For $0 \le u < v \le M$, denote by $\pi_{vu}: I_u \to I_v$ the composition

$$p_{vu} = \pi_v \pi_{v-1} \cdots \pi_{u+1};$$

set $\pi_{uu} := \mathrm{Id}_{I_u}$. For a $j \in I_k$ define a subset

$$\overline{j} := \pi_{k0}^{-1}(j)$$

of I_0. Take the following disjoint unions

$$S_j^r = \sqcup_{i \in \overline{j}} S_i^r; \quad F_j^r = \sqcup_{i \in \overline{j}} F_i^r; \quad F_j^{qp} = \sqcup_{i \in \overline{j}} F_j^{qp}.$$

Pick elements $X_j^k \in \mathcal{B}(F_j^k)$, for all $j \in I_u$, $0 \le u \le M$ and all $k = 1, \ldots, N$. For $0 \le p < q \le N$, set

$$X_j^{qp} = X_j^q X_j^{q-1} \cdots X_j^{p+1}$$

so that $X_j^{qp} \in \mathcal{B}(F_j^{qp})$.

Iterating various compositions of the two kinds in various ways, one can construct, *a priori*, several maps

$$\bigotimes_{u=1}^{M} \bigotimes_{k=1}^{N} \hom_{\pi_u}(\{X_j^k\}_{j \in I_{u-1}}; \{X_l^k\}_{l \in I_u}) \to \hom_{\pi_{M,1}}(\{X_s^{N,0}\}_{s \in I_1}, \{X_v^{N,0}\}_{v \in I_M}).$$

The axiom says that all these maps should coincide. Denote thus obtained unique map by $\mathbf{comp}\{X_s^k\}$.

18.6.4. Given a fixed skeleton $\mathcal{S}$, a structure of a pseudo-tensor body on this skeleton is equivalent to the one of algebra over a certain colored operad $\mathbf{body}_{\otimes}(\mathcal{S})$. Therefore, pseudo-tensor bodies can be specified by means of generators and relations.

18.6.5. *Example*

Introduce a pseudo-tensor structure on **FULL** as follows. Let $X_a \in \mathbf{FULL}(F_a)$ and $Y \in \mathbf{FULL}(F)$, where $F_a: S_a \to T_a$, $S = \sqcup_a S_a$, $F = \sqcup_a F_a$, etc.

Let

$$\boxtimes_a : \prod_a \text{D-mod}_{X^{S_a}} \to \text{D-mod}_{X^S},$$
$$\boxtimes_a : \prod_a \text{D-mod}_{X^{T_a}} \to \text{D-mod}_{X^T}$$

be the functors of the exterior tensor product. Set

$$\hom_{\mathbf{FULL}}(\{X_a\}; Y) := \hom(\boxtimes_a X_a; Y \circ \boxtimes_a),$$

where hom is taken in the category of functors:

$$\prod_a \text{D-mod}_{X^{T^a}} \to \text{D-mod}_{X^S}.$$

18.7. *Maps of pseudo-tensor bodies*

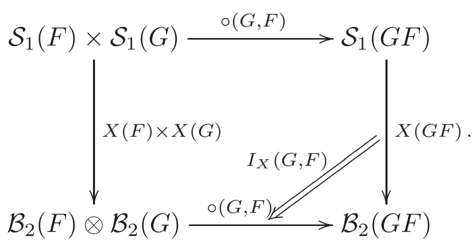
Let $\mathcal{B}_1, \mathcal{B}_2$ be pseudo-tensor bodies over skeletons respectively $\mathcal{S}_1$ and $\mathcal{S}_2$. Our goal is to define a notion of a map $R : \mathcal{B}_1 \to \mathcal{B}_2$. We shall give two equivalent definitions. The first definition is based on a notion of

18.7.1. *Induced skeleton*

Let $X : k[\mathcal{S}_1] \to \mathcal{B}_2$ be a 2-functor which maps $k[\mathcal{S}_1](F) \to \mathcal{B}_2(F)$ for all F. In the sequel we shall write $\mathcal{S}_1$ instead of $k[\mathcal{S}_1]$.

This structure is equivalent to the following one:

(1) we have functors $X(F) : \mathcal{S}_1(F) \to \mathcal{B}_2(F)$ for all F,
(2) for all composable pairs F, G, the natural transformation $I_X(G, F)$, shown on the diagram:



(3) The transformations I_X should be compatible with the associativity transformations of $\mathcal{S}_1$ and $\mathcal{B}_2$ in a natural way.

Using such an X we shall construct a body $X^{-1}\mathcal{B}_2$ on the skeleton $\mathcal{S}_2$. First of all, we set

$$\hom_{X^{-1}\mathcal{B}_2}(\{Y_a\}_{a\in A}; Z) := \hom_{\mathcal{B}_2}(\{X(Y_a)\}_{a\in A}, X(Z)).$$

The compositions of the first and second kinds on $X^{-1}\mathcal{B}_2$ are naturally induced by those on $\mathcal{B}_2$. Thus constructed pseudo-tensor body is called *induced*.

18.7.2. *Definition of a map* $f:\mathcal{B}_1 \to \mathcal{B}_2$

By definition, such a map is given by a 2-functor $X_f:\mathcal{S}_1 \to \mathcal{B}_2$ as above and by a map $f':\mathcal{B}_1 \to X_f^{-1}\mathcal{B}_2$, where the meaning of f' is as follows: since $\mathcal{B}_1$ and $X_f^{-1}\mathcal{B}_2$ are pseudo-tensor bodies over the same skeleton $\mathcal{S}_1$ they can be both interpreted as algebras over the operad $\mathbf{body}_{\otimes}(\mathcal{S}_1)$; f' is by definition a map of such algebras.

This definition will be now decoded.

18.8. *More straightforward approach*

To define a map $\mathcal{B}_1 \to \mathcal{B}_2$ one has to prescribe the following data:

(1) A collection of functors $R_F:\mathcal{B}_1(F) \to \mathcal{B}_2(F)$;

(2) For every sequence of maps of finite sets $F_i:S_{i-1} \to S_i$, $i = 1,2,\ldots,N$, such that $F_N F_{N-1}\cdots F_1 = F$, consider a diagram of functors:

$$\begin{array}{ccc} \otimes_i \mathcal{B}_1(F_i) & \xrightarrow{\otimes_i R_{F_i}} & \otimes_i \mathcal{B}_2 F_i \\ \Big\downarrow{\scriptstyle \circ(F_N,F_{N-1},\ldots,F_1)} & & \Big\downarrow{\scriptstyle \circ(F_N,F_{N-1},\ldots,F_1)} \\ \mathcal{B}_1(F) & \xrightarrow{R_F} & \mathcal{B}_2(F) \end{array}$$

There should be specified an isomorphism $I(F_1,F_2,\ldots,F_N)$ between the composition of the top arrow followed by the right arrow and the composition of the left arrow followed by the bottom arrow. As it is common in the theory of 2-categories, $I(F_1,F_2,\ldots,F_N)$ will be denoted by a double diagonal arrow:

$$\begin{array}{ccc} \otimes_i \mathcal{B}_1(F_i) & \xrightarrow{\otimes_i R_{F_i}} & \otimes_i \mathcal{B}_2 F_i \\ \Big\downarrow{\scriptstyle \circ(F_1,\ldots,F_N)} & \overset{I}{\swarrow\!\!\swarrow} & \Big\downarrow{\scriptstyle \circ(F_1,F_2,\ldots,F_N)} \\ \mathcal{B}_1(F) & \xrightarrow{R_F} & \mathcal{B}_2(F) \end{array}$$

(3) For every $\{X_a\}_{a\in A}$, $X_a \in \mathcal{B}_1(F_a)$, and every $Y \in \mathcal{B}_2(F)$, where $F = \sqcup_{a\in A} F_a$, there should be given a map of complexes:

$$\mathbb{R}_{\{X_a\}_{a\in A};Y} : \hom_{\mathcal{B}_1}(\{X_a\}_{a\in A};Y) \to \hom_{\mathcal{B}_2}(\{R_{F_a}(X_a)\}_{a\in A};R_F(Y)).$$

The axioms are as follows:

(1) Associativity axiom for $I(F_N,\ldots,F_1)$.

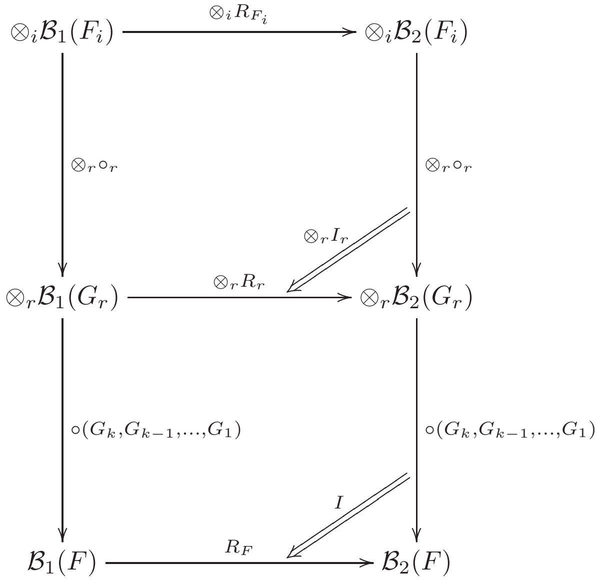
Pick a sequence $1 = i_1 \le i_2 \le \cdots i_k = N$. Set

$$G^r := F_{i_r} F_{i_r-1} \cdots F_{i_{r-1}+1}.$$

Let

$$\circ_r := \circ(F_{i_r}, F_{i_r-1}, \ldots, F_{i_{r-1}+1});$$

let $I_r := I(F_{i_r}, F_{i_r-1}, \ldots, F_{i_{r-1}+1})$. Let $I := I(G_k, G_{k-1}, \ldots, G_1)$. We then have the following diagram:



We then see that the two squares of this diagram are composable and the axiom requires that the composition be equal to $I(F_N, F_{N-1}, \ldots, F_1)$.

(2) Compatibility of $R(\{X_a\}_{a\in A}; Y)$ with compositions of the first type.

Let $p\colon A \to B$ be a surjection of finite sets. Let $F_a\colon S_a \to T_a$ be an A-family of maps of finite sets. Let

$$S_b = \sqcup_{a\in p^{-1}b} S_a, \quad T_b = \sqcup_{a\in p^{-1}b} T_a; \quad F_b = \sqcup_{a\in p^{-1}b} F_a,$$

so that $F_b\colon S_b \to T_b$. Let $X_a \in \mathcal{B}_1(F_a)$, $Y_b \in \mathcal{B}_1(F_b)$. Let

$$R_p(\{X_a\}_{a\in A}; \{Y_b\}_{b\in B})\colon \hom_{B_1,p}(\{X_a\}_{a\in A}; \{Y_b\}_{b\in B})$$
$$\to \hom_{B_2,p}(\{R_{F_a}(X_a)\}_{a\in A}; \{R_{F_b}(Y_b)\}_{b\in B})$$

be the tensor product

$$\otimes_{b\in B} R(\{X_a\}_{a\in p^{-1}b}; Y_b).$$

Let, finally, $q\colon B \to C$ be another surjection. Let $S_c = \sqcup_{a\in(qp)^{-1}c} S_a$; let T_c, F_c be similar disjoint unions. Let $Z_c \in \mathcal{B}_1(F_c)$. We then have the following diagram:

$$\begin{array}{ccc}
\hom_{\mathcal{B}_1,p}(\{X_a\};\{Y_b\}) \otimes \hom_{B_2,q}(\{Y_b\};\{Z_c\}) & \xrightarrow{C_1} & \hom_{\mathcal{B}_1,qp}(\{X_a\};\{Z_c\}) \\
\downarrow{\scriptstyle R(\{X_a\};\{Y_b\})} & & \downarrow{\scriptstyle R(\{X_a\};\{Z_c\})} \\
\hom_{\mathcal{B}_2,p}(\{R(X_a)\};\{R(Y_b)\}) \otimes \hom_{B_2,q}(\{R(Y_b)\};\{R(Z_c)\}) & \xrightarrow{C_1} & \hom_{\mathcal{B}_2,qp}(\{R(X_a)\};\{R(Z_c)\}).
\end{array}$$

The axiom says that this diagram should be commutative.

(3) Compatibility of $R(\{X_a\}_{a\in A};Y)$ with compositions of the second type.

Let $F_a : S_a \to T_a$, $G_a : T_a \to R_a$ be A-families of maps of finite sets. Let $F = \sqcup_a F_a$ and $G = \sqcup_a G_a$. Let $X_a \in \mathcal{B}_1(F_a)$, $Y_a \in \mathcal{B}_2(G_a)$. Let

$$I_a := I(G_a, F_a)(Y_a, X_a) : R(Y_a)R(X_a) \to R(Y_a X_a);$$
$$I : R(Y)R(X) \to R(YX).$$

We then have the following diagram:

$$\begin{array}{ccc}
\hom_{\mathcal{B}_1}(\{X_a\};X)\otimes \hom_{\mathcal{B}_1}(\{Y_a\};Y) & \longrightarrow & \hom_{\mathcal{B}_1}(\{Y_aX_a\};YX)\\
\downarrow & & \downarrow\\
\hom_{\mathcal{B}_2}(\{R(X_a)\};R(X))\otimes \hom_{\mathcal{B}_2}(\{R(Y_a)\};R(Y)) & & \hom_{\mathcal{B}_2}(\{R(Y_aX_a)\};R(YX))\\
\downarrow & & \downarrow\\
\hom_{\mathcal{B}_2}(\{R(Y_a)R(X_a)\};R(Y)R(X)) & \longrightarrow & \hom_{\mathcal{B}_2}(\{R(Y_a)R(X_a)\};R(YX))
\end{array}$$

The axiom requires the commutativity of this diagram.

18.9. *Pseudo-tensor structure on* $\mathcal{B}_{as}, \mathcal{B}_{\mathbf{presymm}}, \mathcal{B}_{\mathbf{symm}}$

18.9.1. $\mathcal{B}_{as}$

The pseudo-tensor body $\mathcal{B}_{as}$ is generated over the usual body $\mathcal{B}_{as}$ by the following generators and relations.

Generators: Let $f_k : R_{\mathfrak{k}} \to T_k$, $k \in K$ be a family of surjections and $i_k : S_k \to T_k$ be a family of injections. Let $f = \sqcup_{k\in K} f_k$ and $i = \sqcup_{k\in K} i_k$. We then have a generator

$$\mathbf{fact}(\{i_k, f_k\}_{k\in K}) : \{\mathfrak{p}_{i_k}\mathbb{R}_{f_k}\}_{k\in K} \to \mathfrak{p}_i\mathbb{R}_f.$$

Let $\pi : K \to L$ be a surjection. For $l \in L$ set

$$i_l = \sqcup_{k\in\pi^{-1}l} i_k;$$

$$f_l = \sqcup_{k\in\pi^{-1}l} f_k.$$

Set

$$\mathbf{fact}(\{i_k, f_k\}_{k\in K}, \{i_l, f_l\}_{l\in L}) : \{\mathfrak{p}_{i_k}\mathbb{R}_{f_k}\}_{k\in K} \to \{\mathfrak{p}_{i_l}R_{f_l}\}_{l\in L}$$

to be

$$\otimes_{l\in L}\mathbf{fact}(\{i_k, f_k\}_{k\in\pi^{-1}l}; i_l, f_l).$$

Relations:

(1) Let $\sigma : L \to M$ be a surjection. For $m \in M$ set

$$f_m = \sqcup_{k\in(\sigma\pi)^{-1}m} f_k;$$
$$i_m = \sqcup_{k\in(\sigma\pi)^{-1}m} i_k.$$

Then the composition of the first kind

$$\{\mathfrak{p}_{i_k}\mathbb{R}_{f_k}\}_{k\in K} \to \{\mathfrak{p}_{i_l}\mathbb{R}_{f_l}\}_{l\in L} \to \{\mathfrak{p}_{i_m}\mathbb{R}_{f_m}\}_{m\in M}$$

equals $C_1(\{\mathfrak{p}_{i_k}R_{f_k}\}_{k\in K}, \{\mathfrak{p}_{i_l}R_{f_l}\}_{l\in L})$.

(2) Let $f_k : S_k \to R_k$, $k \in K$ be surjections and $i_k : R_k \to T_k$ be injections. Let $Z_k := T_k \backslash i_k(R_k)$. Let $S'_k = S_k \sqcup Z_k$, $R'_k = R_k \sqcup Z_k$; let $i'_k : S_k \to S'_k$ be the natural inclusion. Let $f'_k : S'_k \to T_k$, $f'_k = f_k \sqcup i_{Z_k}$, where $i_{Z_k} : Z_k \to T_k$. We then have isomorphisms in $\mathbf{Ske}(i_k f_k)$:

$$\mathbb{R}_{f_k}\mathfrak{p}_{i_k} \to \mathfrak{p}_{i'_k}\mathbb{R}_{f'_k}.$$

Let $f = \sqcup_k f_k$, $i = \sqcup_k i_k$, $f' = \sqcup_k f'_k$, $i' = \sqcup_k i'_k$. We then have an isomorphism in $\mathbf{Ske}(if)$:

$$\mathbb{R}_f\mathfrak{p}_i \to \mathfrak{p}_{i'}\mathbb{R}_{f'}.$$

The relation says that the composition

$$\{\mathbb{R}_{f_k}\mathfrak{p}_{i_k}\}_{k\in K} \to \{\mathbb{R}_f\mathfrak{p}_i\} \to \{\mathfrak{p}_{i'}\mathbb{R}_{f'}\}$$

equals the following composition:

$$\{\mathbb{R}_{f_k}\mathfrak{p}_{i_k}\}_{k\in K} \to \{\mathfrak{p}_{i'_k}\mathbb{R}_{f'_k}\} \to \{\mathfrak{p}_i\mathbb{R}_f\}.$$

(3) Let $f_k : R_k \to T_k$, $k \in K$ be surjections. Let $f_k = g_k h_k$, where g_k, h_k are surjections. Let $f = \sqcup_{k\in K} f_k$, $g = \sqcup_{k\in K} g_k$, $h = \sqcup_{k\in K} h_k$.
Then the composition

$$\{\mathbb{R}_{f_k}\}_{k\in K} \to \mathbb{R}_f \to \mathbb{R}_h\mathbb{R}_g$$

equals the composition

$$\{\mathbb{R}_{f_k}\}_{k\in K} \to \{\mathbb{R}_{h_k}\mathbb{R}_{g_k}\}_{k\in K} \to \mathbb{R}_h\mathbb{R}_g.$$

18.9.2. $\mathcal{B}_{\mathbf{presymm}}$

The pseudo-tensor structure on $\mathcal{B}_{\mathbf{presymm}}$ is generated by the same generators as on $\mathcal{B}_{as}$, and the relations include those in $\mathcal{B}_{as}$ with an addition of the following relations:

(a) let i_k, p_k, j_k, q_k, $k \in K$ be a collection of suitable squares. Let $i = \sqcup_{k\in K} i_k$, $p = \sqcup_{k\in K} p_k$, $j = \sqcup_{k\in K} j_k$, $q = \sqcup_{k\in K} q_k$. Then the square i, p, j, q is also suitable and the following compositions coincide:

$$\{\mathfrak{p}_{i_k}\mathbb{R}_{p_k}\}_{k\in K} \to \{R_{q_k}\mathfrak{p}_{j_k}\}_{k\in K} \to \mathbb{R}_q\mathfrak{p}_j$$

and

$$\{\mathfrak{p}_{i_k}\mathbb{R}_{p_k}\} \to \mathfrak{p}_i\mathbb{R}_p \to \mathbb{R}_q\mathfrak{p}_j.$$

(b) Let $i_k : S_k \to R_k$, $k \in K$ be injections and $p_k : R_k \to T_k$, $k \in K$ be surjections such that $j_k := p_k i_k$ are injections. Assume that at least two of the maps p_k are proper surjections. Then the composition

$$\{\mathfrak{p}_{i_k}\mathbb{R}_{p_k}\}_{k\in K} \to \mathfrak{p}_i\mathbb{R}_p \to \mathbb{R}_j$$

vanishes.

If only one of the surjections p_k is proper, say p_κ, $\kappa \in K$, then the above composition equals

$$\{\mathfrak{p}_{i_k}\mathbb{R}_{p_k}\}_{k\in K} = \{\mathfrak{p}_{i_\kappa}\mathbb{R}_{p_\kappa}, \mathfrak{p}_{i_k}\mathbb{R}_{\{p_k\}_{k\in K}}\} \xrightarrow{L(i_\kappa,p_\kappa)} \{\mathfrak{p}_{j_\kappa}, \{\mathfrak{p}_{i_k}\mathbb{R}_{p_k}\}_{k\in K}\}$$
$$\to \{p_{j_k}\}_{k\in K} \to \mathfrak{p}_j.$$

18.9.3. $\mathcal{B}_{\mathbf{symm}}$

This pseudo-tensor body is generated by the same generators and relations as $\mathcal{B}_{\mathbf{presymm}}$ except that we lift the condition of suitability. We have a natural map

$$\mathcal{B}_{\mathbf{presymm}} \to \mathcal{B}_{\mathbf{symm}}. \tag{53}$$

18.9.4. It is clear that the system $\langle\mathbb{R}\rangle$ determines a map of pseudo-tensor bodies

$$\mathcal{B}_{\mathbf{presymm}} \to \mathbf{FULL},$$

any such a functor will be also called *representation.*

18.10. *Explicit form of pseudo-tensor maps*

18.10.1. *Category of special maps*

Consider a family of objects

$$X_a = \mathfrak{p}_{i^a}\mathbb{R}_{p_1^a}\mathbb{R}_{p_2^a}\cdots\mathbb{R}_{p_{n_a}^a}$$

indexed by a finite set A, where all p_k^a are proper surjections and

$$p_{n_a}^a \cdots p_2^a p_1^a i^a = F_a$$

so that $X_a \in \mathbf{Ske}(F_a)$. Let $N \geq j \geq i \geq 1$. Set $p_{ji}^a = p_j^a p_{j-1}^a \cdots p_{i+1}^a$ (if $i = j$, then we set $p_{ji}^a = \mathrm{Id}$). Let $\mathbf{u} := \{u_k^a\}$, where $a \in A$, $k = 0, 1, 2, \ldots, N$, be a sequence of numbers satisfying: $u_0^a = 0$,

$$0 \leq u_{k+1}^a - u_k^a \leq 1,$$

and $u^a_N = n_a$. Set

$$p_k(\mathbf{u}) := \sqcup_{a\in A} p^a_{u^a_k u^a_{k-1}}$$

Call $\mathbf{u}$ *proper* if such are all $p_k(\mathbf{u})$.

For proper $\mathbf{u}$ we set

$$X(\mathbf{u}) := p_{\sqcup_a i_a} \mathbb{R}_{p_1(\mathbf{u})} \mathbb{R}_{p_2(\mathbf{u})} \cdots \mathbb{R}_{p_N(\mathbf{u})}.$$

We have natural maps

$$\mathbf{fact}(\mathbf{u}) : \{X_a\}_{a\in A} \to X(\mathbf{u}).$$

For an $Y \in \mathbf{Ske}(\sqcup_a F_a)$ consider the groupoid G_Y whose objects are collections

$$(\{X_a \in \mathbf{Ske}(F_a)\}_{a\in A}, \mathbf{u}), \quad m : Y \to X(\mathbf{u}),$$

where the meaning of the ingredients is the same as above and m is an isomorphism; the isomorphisms in G_Y are isomorphisms of such collections. It is clear that G_Y is a trivial groupoid. Let $Z_a \in \mathbf{Ske}(f_a)$. We have a natural map

$$\operatorname{liminv}_{\{X_a\}_{a\in A} \in G_Y} \otimes_{a\in A} \hom_?(Z_a, X_a) \to \hom_?(\{Z_a\}_{a\in A}, Y).$$

We claim that this map is an isomorphism, where $? = \mathcal{B}_{as}, \mathcal{B}_{\mathbf{presymm}}, \mathcal{B}_{\mathbf{symm}}$.

18.10.2. We shall also need another form of decomposition of the pseudo-tensor maps in $\mathcal{B}_{\mathbf{symm}}$.

Let $\{X_a\}_{a\in A}$ be a family of objects $X_a \in \mathbf{Ske}(F_a)$. Let $F = \sqcup_{a\in A} F_a$. We then have the following natural functors

$$h^{\mathbf{presymm}}_{\{X_a\}_{a\in A}} : \mathcal{B}_{\mathbf{presymm}}(F) \to \text{complexes}$$

and

$$h^{\mathbf{symm}}_{\{X_a\}_{a\in A}} : \mathcal{B}_{\mathbf{symm}}(F) \to \text{complexes}$$

defined by the formulas:

$$h^{\mathbf{presymm}}_{\{X_a\}_{a\in A}}(Y) = \hom_{\mathcal{B}_{\mathbf{presymm}}}(\{X_a\}_{a\in A}; Y);$$

$$h^{\mathbf{symm}}_{\{X_a\}_{a\in A}}(Y) = \hom_{\mathcal{B}_{\mathbf{symm}}}(\{X_a\}_{a\in A}; Y).$$

We shall also need a functor

$$G^{\mathbf{symm}} : \mathcal{B}^{\mathrm{op}}_{\mathbf{presymm}}(F) \otimes \mathcal{B}_{\mathbf{symm}}(F) \to \text{complexes},$$

where $G^{\mathbf{symm}}(Z, U) = \hom_{\mathcal{B}_{\mathbf{symm}}}(Z, U)$. We then have a natural map:

$$h^{\mathbf{presymm}}_{\{X_a\}_{a\in A}} \otimes_{\mathcal{B}_{\mathbf{presymm}}(F)} G^{\mathbf{symm}} \to h^{\mathbf{symm}}_{\{X_a\}_{a\in A}}.$$

Lemma 18.4. *This map is an isomorphism of functors.*

Proof. Straightforward. □

18.11. *Linear span of a body*

Let $\mathcal{B}$ be a pseudo-tensor body. We shall construct a body $\mathcal{L}[B]$, over another skeleton, as follows. Set $\mathcal{L}[B](F)$ to be the category of functors $\mathcal{B}(F)^{\mathrm{op}} \to$ complexes. We shall start with the composition maps

$$\circ := \circ(G,F) : \mathcal{L}[B](F) \otimes \mathcal{L}[B](G) \to \mathcal{L}[B](GF).$$

Introduce an auxiliary functor

$$\mathcal{D} := \mathcal{D}^{G,F} : \mathcal{B}(F) \otimes \mathcal{B}(G) \otimes \mathcal{B}(GF)^{\mathrm{op}} \to \text{complexes},$$

where

$$\mathcal{D}(X,Y,Z) = \hom_{\mathcal{B}(GF)}(Z, YX).$$

Let now $U \in \mathcal{L}[B](F), V \in \mathcal{L}[B](G)$. Define

$$V \circ U := \mathcal{D} \otimes_{\mathcal{B}(G)\otimes\mathcal{B}(F)} V \boxtimes U.$$

Let us construct the associativity map. Define

$$\mathcal{D}_3 := \mathcal{D}^{H,G,F} : \mathcal{B}(H) \otimes \mathcal{B}(G) \otimes \mathcal{B}(F) \otimes \mathcal{B}(HGF)^{\mathrm{op}} \to \text{complexes}$$

by

$$\mathcal{D}_3(Z,Y,X,U) := \hom_{\mathcal{B}(HGF)}(U, ZYX).$$

Ioneda's lemma combined with the associativity maps implies isomorphisms

$$\mathcal{D}^{G,F} \otimes_{\mathcal{B}(GF)} \mathcal{D}^{H,GF} \xrightarrow{\sim} \mathcal{D}_3;$$
$$\mathcal{D}^{H,G} \otimes_{\mathcal{B}(HG)} \mathcal{D}^{HG,F} \xrightarrow{\sim} \mathcal{D}_3.$$

Let $U \in \mathcal{B}(F), V \in \mathcal{B}(G)$, and $W \in \mathcal{B}(H)$. Set

$$(WVU) := W \boxtimes V \boxtimes U \otimes_{\mathcal{B}(H)\otimes\mathcal{B}(G)\otimes\mathcal{B}(F)} D_3.$$

We then have isomorphisms

$$(WV)U \xrightarrow{\sim} (WVU) \xrightarrow{\sim} W(VU),$$

which furnish the associativity isomorphism.

For $X_i \in \mathcal{L}[B](F_i)$, $i \in I$ and $Y \in \mathcal{L}[B](\sqcup_{i\in I} F_i)$ set

$$\hom_{\mathcal{L}[B]}(\{X_i\}_{i\in I}; Y) := \hom(\boxtimes_{i\in I} X_i, \hom_{\mathcal{B}}(\{.\},.) \otimes_{\mathcal{B}_F} Y).$$

Define the compositions of the first kind. Let $\pi : I \to J$ be a surjection. Let $F_i : S_i \to T_i$ be a family of maps of finite sets. Let $F_j = \sqcup_{i\in\pi^{-1}j} F_i$.

Let $K_i \in \mathcal{L}[B](F_i)$ and $L_j \in \mathcal{L}[B](F_j)$. Let

$$A(\pi)\colon \otimes_{i\in I} \mathcal{B}(F_i)^{\mathrm{op}} \otimes_{j\in J} \mathcal{B}(F_j) \to \text{complexes}$$

be given by:

$$A(\pi)(\{X_i\};\{Y_j\}) = \hom_{\mathcal{B}}(\{X_i\};\{Y_j\}).$$

Let $\mathcal{B}_I := \otimes_{i\in I}\mathcal{B}(F_i)$; $\mathcal{B}_J := \otimes_{j\in J}\mathcal{B}(F_j)$. Let $K : \mathcal{B}_I^{\mathrm{op}} \to$ complexes be $\boxtimes_{i\in I} K_i$. Let $L : \mathcal{B}_J^{\mathrm{op}} \to$ complexes be $\boxtimes_{j\in J} L_j$. We then have:

$$\hom_{\mathcal{L}[B],\pi}(\{K_i\}_{i\in I};\{L_j\}) \cong \hom_{\mathcal{B}_I^{\mathrm{op}}}(K, L\otimes_{\mathcal{B}_J} A(\pi)).$$

Let $\sigma : J \to K$ be the third surjection. For $k \in K$, let $F_k := \sqcup_{i\in(\sigma\pi)^{-1}k} F_i$. Let $M_k \in \mathcal{B}(F_k)$. Let $\mathcal{B}_K = \otimes_{k\in K}\mathcal{B}(F_k)$; let $M := \boxtimes_{k\in K} M_k$. Then

$$\hom_{\mathcal{L}[B],\sigma}(\{L_j\};\{M_k\}) \cong \hom_{\mathcal{B}_j^{\mathrm{op}}}(L, M\otimes_{\mathcal{B}_K} A(\sigma)).$$

To construct the composition of the first kind we shall also need an isomorphism

$$A(\sigma)\otimes_{\mathcal{B}_J} A(\pi) \to A(\sigma\pi),$$

where the isomorphism follows from the Ioneda's lemma.

In view of the above isomorphisms, the composition of the first kind reduces to:

$$\begin{aligned}
&\hom_{\mathcal{B}_I^{\mathrm{op}}}(K, L\otimes_{\mathcal{B}_J} A(\pi)) \otimes \hom_{\mathcal{B}_J^{\mathrm{op}}}(L, M\otimes_{\mathcal{B}_K} A(\sigma))\\
&\to \hom_{\mathcal{B}_I^{\mathrm{op}}}(K, M\otimes_{\mathcal{B}_K} A(\sigma)\otimes_{\mathcal{B}_J} A(\pi))\\
&\cong \hom_{\mathcal{B}_I^{\mathrm{op}}}(K, M\otimes_{\mathcal{B}_k} A(\sigma\pi)).
\end{aligned}$$

Lastly, let us define the compositions of the second kind. We shall keep the above notation. Let $G_i : T_i \to R_i$ be another family of maps of finite sets. Let $K_i' \in \mathcal{L}[B](G_i)$ and $L_j' \in \mathcal{L}[B](G_j)$. Let $a \in \hom(K, L)$ and $a' \in \hom(K', L')$. Let $A(\pi)$ be as above and let $A'(\pi)$ (respectively $A''(\pi)$) be the same as $A(\pi)$ but F_i are all replaced with G_i (respectively G_iF_i). Let $\mathcal{B}_I' = \otimes_{i\in I}\mathcal{B}(G_i)$; $\mathcal{B}_I'' = \otimes_{i\in I}\mathcal{B}(G_iF_i)$. Construct the composition $a'a \in \hom(K'K, L'L)$.

As was mentioned above, a determines a map

$$a : K \to L\otimes_{B_J} A(\pi)$$

and a' produces a map

$$a' : K' \to L'\otimes_{B_J'} A'(\pi).$$

To construct the compositions $K'K, L'L$, introduce functors

$$O_I : \mathcal{B}_I \otimes \mathcal{B}_I' \otimes (\mathcal{B}_I'')^{\mathrm{op}} \to \text{complexes};$$

$$O_J : \mathcal{B}_J \otimes \mathcal{B}_J' \otimes (\mathcal{B}_J'')^{\mathrm{op}} \to \text{complexes};$$

by setting

$$O_I(\{X_i\};\{X_i'\};\{X_i''\}) := \hom(\{X_i''\};\{X_i'X_i\});$$
$$O_J(\{X_j\};\{X_j'\};\{X_j''\}) := \hom(\{X_j''\};\{X_j'X_j\}).$$

Then

$$K'K = K' \boxtimes K \otimes_{B_I \otimes B_I'} O_I;$$
$$L'L = L' \boxtimes L \otimes_{B_J \otimes B_J'} O_J.$$

Application of a, a' yields a map:

$$K' \boxtimes K \otimes_{\mathcal{B}_I \otimes \mathcal{B}_I'} O_I \to L' \boxtimes L \otimes_{\mathcal{B}_J \otimes \mathcal{B}_J'} (A(\pi) \boxtimes A'(\pi)) \otimes_{\mathcal{B}_I \otimes \mathcal{B}_I'} O_I.$$

Next, by Ioneda's lemma, we have an isomorphism.

$$(A(\pi) \boxtimes A'(\pi)) \otimes_{\mathcal{B}_I \otimes \mathcal{B}_I'} O_I \to O_J.$$

If we apply this isomorphism to the previous map, we will get the desired second kind composition map:

$$K' \boxtimes K \otimes_{\mathcal{B}_I \otimes \mathcal{B}_I'} O_I \to L' \boxtimes L \otimes_{\mathcal{B}_J \otimes \mathcal{B}_J'} O_J.$$

This concludes the definition of the structure. Checking the axioms is straightforward.

18.11.1. A representation of a pseudo-tensor body B (i.e. a map $B \to \mathbf{FULL}$) naturally extends to a representation of $\mathcal{L}[B]$.

18.12. *Representation of a body in another body*

An arbitrary map of bodies $\mathcal{B}_1 \to k[\mathcal{B}_2]$ will be called *a representation of* $\mathcal{B}_1$ *in* $\mathcal{B}_2$. We shall construct

18.13. *Representation of $\mathcal{B}_{\mathbf{symm}}$ in $\mathcal{B}_{\mathbf{presymm}}$*

By constructing such a representation, we shall automatically obtain a map $\mathcal{B}_{\mathbf{symm}} \to \mathbf{FULL}$, i.e. a symmetric system.

First of all construct maps $R_F : \mathcal{B}_{\mathbf{symm}}(F) \to \mathcal{L}[\mathcal{B}_{\mathbf{presymm}}](F)$ by assigning

$$R_F(X)(Y) := \hom_{\mathcal{B}_{\mathbf{symm}}(F)}(Y, X).$$

On $R_F(X)$, we have a natural structure of functor from the category $\mathcal{B}_{\mathbf{symm}}(F)^{\mathrm{op}}$ to the category complexes given by the map $\mathcal{B}_{\mathbf{presymm}} \to \mathcal{B}_{\mathbf{symm}}$ as in (53). Let $X_i \in \mathcal{B}_{\mathbf{presymm}}(F_i)$. We then have a natural map

$$I(F_2, F_1) : R_{F_1}(X_1)R_{F_2}(X_2) \to R_{F_2F_1}(X_1X_2)$$

given by:

$$(R_{F_1}(X_1)R_{F_2}(X_2))(Z) = R_{F_1} \otimes R_{F_2} \otimes_{\mathcal{B}(F_1)\otimes\mathcal{B}(F_2)} \mathcal{D}^{F_2F_1}_{\mathcal{B}_{\mathbf{presymm}}}(Z)$$

$$\to R_{F_1} \otimes R_{F_2} \otimes_{\mathcal{B}(F_1)\otimes\mathcal{B}(F_2)} \mathcal{D}^{F_2F_1}_{\mathcal{B}_{\mathbf{symm}}}(Z) \cong R_{F_2F_1}(X_1X_2)(Z).$$

Furthermore, as follows from the decomposition (18.4), $I(F_2, F_1)$ is an isomorphism.

To defined the maps

$$R_{\{X_a\}_{a\in A};Y} : \hom_{\mathcal{B}_{\mathbf{symm}}}(\{X_a\}_{a\in A}; Y) \to \hom_{\mathcal{B}_{\mathbf{presymm}}}(\{R_{F_a}(X_a)\}_{a\in A}; R_F(Y))$$

we shall use Lemma 18.4. We have

$$R_F(X)(Y) = G^{\mathbf{symm}}(Y, X),$$

where $G^{\mathbf{symm}}$ is as in the statement of Lemma 18.4. Let

$$h^{\mathbf{presymm}} : \otimes_{a\in A} \mathcal{B}_{\mathbf{presymm}}(F_a)^{\mathrm{op}} \otimes \mathcal{B}_{\mathbf{presymm}}(F) \to \text{complexes}$$

be defined by the formula

$$h^{\mathbf{presymm}}(\{X_a\}_{a\in A}; Y) := \hom_{\mathcal{B}_{\mathbf{presymm}}}(\{X_a\}_{a\in A}; Y).$$

Then, by definition,

$$\hom_{\mathcal{B}_{\mathbf{presymm}}}(\{R_{F_a}(X_a)\}_{a\in A}; R_F(Y))$$
$$= \hom_{\otimes_{a\in A}\mathcal{B}_{\mathbf{presymm}}(F_a)^{\mathrm{op}}}(\boxtimes_{a\in A} R_F(X_a); h^{\mathbf{presymm}} \otimes_{\mathcal{B}_{\mathbf{presymm}}(F)} G^{\mathbf{symm}}).$$

The latter term is, by Lemma 18.4, isomorphic to

$$\hom_{\otimes_{a\in A}\mathcal{B}_{\mathbf{presymm}}(F_a)^{\mathrm{op}}}(\boxtimes_{a\in A} R_F(X_a); h^{\mathbf{symm}}),$$

where

$$h^{\mathbf{symm}}(\{X_a\}_{a\in A}; Y) := \hom_{\mathcal{B}_{\mathbf{symm}}}(\{X_a\}; Y).$$

Lastly, we have a natural map

$$\hom_{\otimes_{a\in A}\mathcal{B}_{\mathbf{symm}}(F_a)^{\mathrm{op}}}(\boxtimes_{a\in A} R_F(X_a); h^{\mathbf{symm}})$$
$$\to \hom_{\otimes_{a\in A}\mathcal{B}_{\mathbf{presymm}}(F_a)^{\mathrm{op}}}(\boxtimes_{a\in A} R_F(X_a); h^{\mathbf{symm}}),$$

and the first space is, by Ioneda's lemma, isomorphic to

$$\hom_{\mathcal{B}_{\mathbf{symm}}}(\{X_a\}; Y).$$

This completes the desired construction. Checking the axioms is straightforward.

18.13.1. As was mentioned above, the above construction provides us with a symmetric system. Denote it $\langle \mathbb{R}^{\mathbf{symm}} \rangle$. An explicit construction of $\langle \mathbb{R}^{\mathbf{symm}} \rangle$ is given in Sec. 7. Checking that this construction produces the same system as in the previous section is straightforward, and we omit it.

19. Realization of the System $\langle \mathbb{R}^{\mathbf{symm}} \rangle$ in the Spaces of Real-Analytic Functions

19.1. *Conventions and notation*

We do not consider sheaves in this sections, but only their global sections. By $\mathcal{D}_{X^S}$ we denote the algebra of polynomial differential operators on X^S. By a $\mathcal{D}_{X^S}$-module we mean a module over the algebra $\mathcal{D}_{X^S}$.

We denote by $\mathfrak{D}_{Y^S}$ the space of compactly supported infinitely-differentiable top-forms on Y^S, and by $\mathfrak{D}'_{Y^S}$ the space of distributions (= generalized functions) on $\mathfrak{D}_{Y^S}$. $\mathfrak{D}'_{Y^S}$ is a left $\mathfrak{D}_{X^S}$-module.

For simplicity, we fix a translation invariant top form ω on Y^S, and define ω_S to be a top form on Y^S which is the exterior product of copies of ω. Because $\dim Y$ is even, the order in this product does not matter.

The space $\mathfrak{D}_{Y^S}$ is then identified with the space of compactly supported infinitely differentiable functions on Y^S.

19.2. *Asymptotic decompositions of functions from $\mathcal{C}_S$*

19.2.1. *The main theorem*

Let S be a finite set. Let $T \subset S$ be a subset. Let $R := S\backslash T$. Pick an element $\tau \in T$. We shall refer to a point of Y^S as $(\{y_s\}_{s\in S})$, where $y \in Y$.

For a positive real λ set

$$U_\lambda((\{y_s\}_{s\in S}) = \left(\left\{y_\tau + \frac{y_t - y_\tau}{\lambda}\right\}_{t\in T}; \{y_r\}_{r\in R}\right).$$

This determines an action of the Lie group $\mathbb{R}^\times_{>0}$ on Y^S.

Let $F \in \mathcal{C}_S$.

Claim 19.1. *For every $g \in \mathfrak{D}_{Y^S}$ there exist: constants $A(F), B(F)$; distributions $C^F_{n,k} \in \mathfrak{D}'_{Y^S}$, for every $n \geq A(F)$ and every k such that $0 \leq k \leq B(F)$; such that for every N and every $g \in \mathfrak{D}_{Y^S}$, the following asymptotics takes place*:

$$\langle F, U_\lambda g\rangle = \sum_{A(F)\leq n\leq N, 0\leq k\leq B(F)} C^F_{n,k}(g)\lambda^n(\ln\lambda)^k + o(\lambda^N).$$

Proof. We shall use induction in $\#R$ to prove even stronger statement:

There exist: constants $A(F), B(F), K(F)$; distributions $C^F_{n,k}$ on the space of compactly supported $K(F)$-times differentiable functions, for every $n \geq A(F)$ and every k such that $0 \leq k \leq B(F)$; such that for every N there exists a constant $L := L(N, F)$ such that whenever

$$\phi \in C^L(Y^S), \quad g \in C^L_c(Y^S),$$

the following asymptotics takes place:

$$\langle F, \phi U_\lambda g\rangle = \sum_{A(F)\le n\le N, 0\le k\le B(F)} C^F_{n,k}((U_{\lambda^{-1}}\phi)g)\lambda^n(\ln\lambda)^k + o(\lambda^N). \tag{54}$$

Remark. We have

$$U_{\lambda^{-1}}\phi(\{x_s\}_{s\in S}) = \phi(\{x_\tau + \lambda(x_t - x_\tau)\}_{t\in T}, \{x_r\}_{r\in R}).$$

Therefore, for L large enough, we can replace $U_{\lambda^{-1}}\phi$ in (54) with a finite sum

$$\sum \lambda^k \phi_k.$$

Base: $\#R = 0$. We then have $\langle F, U_\lambda g\rangle = \langle U^*_\lambda F, g\rangle$. Let us study the action U^*_λ on $\mathcal{C}_S$. It is clear that this action preserves the filtration on $\mathcal{C}_S$ and that the associated graded action is diagonalizable. It then follows that for every $F \in C_S$,

$$U^*_\lambda F = \sum_{n,k} \lambda^n(\ln\lambda)^k F_{nk},$$

where the sum is finite and $F_{nk} \in C_S$.

The statement now follows immediately.

Now let R be arbitrary, and assume that the statement is the case whenever R has a smaller number of elements.

Let $R_1 \subset R$ be an arbitrary non-empty subset. Let $R_2 = S\backslash R_1$. Assume that

$$F = F_1F_2, \tag{55}$$

where $F_i \in C_{R_i}$.

We then claim that the required asymptotics is the case. Indeed, we have

$$\langle F_1F_2, \phi U_\lambda g\rangle = \langle F_1, \langle F_2, \phi U_\lambda g\rangle\rangle$$

and the statement follows from the corresponding statement for F_2 (which holds by virtue of the induction assumption).

Let us generalize this result. Let $I^{N,K}_{R',R''} \subset C^K(Y^S)$ be the subspace consisting of functions which vanish on each diagonal $x_{r_1} = x_{r_2}$, $r_i \in R_i$ upto the order N.

Let

$$Q_{R_1R_2} := \prod_{r_i\in R_i} q(x_{r_1} - x_{r_2}).$$

It is not hard to see that for every M, L there exist N, K such that for every $\chi \in I^{N,K}_{R',R''}$, we have

$$\chi = Q^M\psi,$$

where $\psi \in C^M(Y^S)$. Thus, for N, K sufficiently large, and $\chi \in I^{N,K}$ we have

$$\langle F, \chi\phi U_\lambda g\rangle = \langle FQ^M, \psi\phi U_\lambda g\rangle.$$

But for M large enough, FQ^M splits into a sum of elements of the form (55), whence the statement for all $\chi \in I^{N,K}_{R',R''}$.

Let us now define the space $J^{N,K} \subset C^K Y^S$, consisting of all functions which vanish on the diagonal

$$\forall\, t \in T : x_t = x_\tau$$

upto the order N. It is not hard to see that for N, K large enough,

$$\langle F, \chi\phi U_\lambda g\rangle = o(\lambda^n).$$

Therefore, there are large enough N, K such that whenever

$$\phi \in \sum_{R'} I^{N,K}_{R',R''} + J^{N,K},$$

the required asymptotics holds.

Let $A^{N,K} \subset C^K(Y^S)$ be the subspace of functions which vanish on the main diagonal in Y^S upto the order N.

By the Nullstellensatz, for some N', K',

$$A^{N',K'} \subset \sum_{R'} I^{N,K}_{R',R''} + J^{N,K}.$$

Therefore, the required asymptotics holds whenever

$$\chi \in A^{N',K'}.$$

Let us now pass to the original statement.

Let

$$\Xi_l = \sum_{t\in T} (x_t - x_\tau)\frac{\partial}{\partial x_t},$$

$$\Xi_r = \sum_{r\in R} (x_r - x_\tau)\frac{\partial}{\partial x_r}.$$

The action of the vector field $\Xi_l + \Xi_r$ on the space $\mathcal{C}_S$ preserves the filtration, and the induced action on the associated graded quotients is diagonalizeable, therefore we may assume that $(\Xi_l + \Xi_r - n)^N F = 0$ for some n, N.

Consider expressions

$$\left\langle F, P\left(\Xi_l, \Xi_r, \lambda\frac{d}{d\lambda}\right)\phi U_\lambda g)\right\rangle, \tag{56}$$

where P is a polynomial.

Let $U_M(z) = z(z-1)(z-2)\cdots(z-M)$.

Consider the following ideals in the ring of polynomials of three variables:

$$A_M = \left(U_M\left(\Xi_l - \lambda\frac{d}{d\lambda}\right)\right);$$

$$B_M = (U_M(\Xi_r));$$

$$C = ((\Xi_l + \Xi_r - n)^N).$$

It is not hard to see that for M large enough, whenever P is large enough, the expression (56) has the required asymptotics.

Indeed, consider for example the ideal A_M. We have:

$$U_M\left(\Xi_l - \lambda\frac{d}{d\lambda}\right)\phi U_\lambda(g) = (U_M(\Xi_l)\phi)U_\lambda(g)$$

and it is easy to see that $\chi := U_M(\Xi_l)\phi$ has at least the Mth order of vanishing along the main diagonal, whence the statement. The ideals B_M, C_M can be checked in a similar way.

Next, we see, by the Nullstellensatz, that for some M, L

$$U_M\left(\lambda\frac{d}{d\lambda} - n\right)^L \in A_M + B_M + C.$$

Therefore, we see that there exists the required asymptotics for

$$U_M\left(\lambda\frac{d}{d\lambda} - n\right)^L \langle F, \phi U_\lambda g\rangle.$$

The theory of ordinary differential equations now implies the statement. □

19.2.2. *A claim about the distributions* $C_{n,k}$

Let G be a function on Y^T which is invariant under translations by a vector from Y, with support compact modulo the action of Y.

Let H be a function on $Y^{R\sqcup\{\tau\}}$ with compact support.

We then have $U_\lambda(GH) = HU_\lambda(G)$.

Claim 19.2. *We have,*

$$C_{n,k}(HG) = D_{n,k}(G)(H),$$

where $D_{n,k}(G) \in \mathcal{C}_{R\sqcup\{\tau\}}$.

Furthermore, for every N, *the distributions* $D_{n,k}(G)$, *where* $n \le N$ *and* k *is arbitrary, span a finitely dimensional vector subspace.*

Proof. Use induction. If R is empty, there is nothing to prove.

Otherwise, let us split $R = R_1 \sqcup R_2$, in a nontrivial way.

We then see that for M large enough $D_{n,k}(GQ^M_{R_1\sqcup T, R_2})$ satisfy the statement by virtue of the induction statement.

Also, for M' large enough and any G vanishing on the diagonal $\forall\, t \in T : y_t = y_\tau$ up to the order M', $D_{n,k}(G) = 0$. The Nullstellensatz then implies that for L large enough, one can write

$$Q^L_{R_1,R_2} = P_1 Q^M_{T\sqcup R_1, R_2} + P_2,$$

where P_1, P_2 are polynomials and P_2 vanishes on the diagonal $\forall\, t \in T : y_t = y_\tau$ upto the order M'.

This implies that $D_{n,k}(Q^L_{R_1,R_2}G) = D_{n,k}(G)Q^L_{R_1,R_2}$ satisfy the statement.

Also, $D_{n,k}(G)$ are all translation invariants, therefore $D_{n,k}(G) \in \mathcal{C}_{R\sqcup\{\tau\}}$.

Let $\mathcal{C}'$ be the quotient of $\mathcal{C}_{R\sqcup\{\tau\}}$ by the distributions supported on the main diagonal.

It then follows that $D_{n,k}$ span a finitely dimensional space in $\mathcal{C}'$.

Let $Q_R = \prod_{r,s\in R, r\neq s} q(x_r - x_s)$. Then $D_{n,k}(G)Q_R$ also span a finitely dimensional space. Hence, $D_{n,k}(G)$ span a finitely-dimensional space in $\mathcal{C}_{R\sqcup\{\tau\}}$. □

19.2.3. Consider a decomposition $S = S_1 \sqcup S_2$ such that $T \subset S_1$. Consider an element $F \in \mathcal{C}_S$ which decomposes as a product $F = F_1F_2$, where $F_i \in \mathcal{C}_{S_i}$. We are going to express $D^F_{n,k}$ in terms of $D^{F_1}_{n,k}$.

Let G be as above (i.e. an infinitely differentiable function on Y^T invariant under shifts by Y and with compact support modulo these shifts).

Claim 19.3. *We then have*

$$D^F_{n,k}(G)(H) = \langle F_2, D^{F_1}_{n,k}(G)(H)\rangle.$$

Proof. Clear. □

19.2.4. Let S be a finite set with a marked point $\sigma \in S$. Let Ξ_r be the dilation vector field on X^S given by:

$$\sum_{s\in S}(x_s - x_\sigma)\frac{\partial}{\partial x_s}.$$

Denote $\mathfrak{D}'_{Y^S,n}$ the generalized eigenspace of Ξ_r with eigenvalue n. Let $\mathcal{C}_{S,n} := \mathcal{C}_S \cap \mathfrak{D}'_{Y^S,n}$.

We know that

$$\mathcal{C}_S = \oplus_{n\in\mathbb{Z}}\mathcal{C}_{S,n}.$$

Let us now go back to our situation in which we have a finite set S, its subset T and a marked point $\tau \in T$.

Consider a subspace $\mathfrak{D}'_{T,n} \subset \mathfrak{D}'_{Y^T,n}$ consisting of all elements which are nilpotent under translations by Y.

It is then not hard to see that

Lemma 19.4.

$$D_{n,k} \in \mathfrak{D}'_{T,n} \otimes_{\mathcal{O}_X} \mathcal{C}_{R\sqcup\{\tau\},N-n},$$

where the $\mathcal{O}_X$-action is on the τth components of both tensor factors.

19.2.5. Let

$$\mathfrak{D}'_{T,\geq n} := \oplus_{N\geq n}\mathfrak{D}'_{T,N};$$

let

$$\mathfrak{D}' = \oplus_N \mathfrak{D}'_T.$$

Let

$$E(S,T) := \text{liminv}_{n\to\infty} \mathfrak{D}'/\mathfrak{D}'_{T,\geq n} \otimes C_{R\sqcup\{\tau\},N-n}.$$

Given a function $G \in \mathfrak{D}_{Y^S}$ and an element

$$s_n \in \mathfrak{D}'_{T,\geq n} \otimes \mathcal{C}_{R\sqcup\{\tau\},N-n}$$

one has: $\langle s_n, U_\lambda G\rangle = o(\lambda^{n-1})$. Therefore, for every $s \in E(S,T)$ and $G \in \mathfrak{D}_{Y^S}$ we have an asymptotic series

$$\langle s, U_\lambda G\rangle.$$

Claim 19.5. *There exists a map*

$$\epsilon : C_S \to E(S,T)$$

uniquely determined by the condition that $\langle \epsilon(F), U_\lambda G\rangle$ is an asymptotic series for $\langle F, U_\lambda G\rangle$.

19.2.6. Let $\pi : S \to S/T$ be the natural surjection. Define a functor $\mathcal{A}_\pi$ from the category of $\mathfrak{D}_{X^{S/T}}$-modules to the category of $\mathfrak{D}_{X^S}$-modules by the formula

$$\mathcal{A}_\pi(M) = \text{liminv}_{n\to\infty} i^\wedge_\pi(M) \otimes_{\mathcal{O}_{X^T}} \mathfrak{D}'/\mathfrak{D}'_{T,\geq n}.$$

Then the above result can be rewritten as a map

$$\mathcal{C}_S \to \mathcal{A}_\pi(\mathcal{C}_{S/T}).$$

19.2.7. Let $q : S/T \to P$ be an arbitrary surjection. Let $\overline{\tau} \in S/T$ be the image of T. Let $\chi = \pi(\overline{\tau})$. For $p \in P$ set $S_p := (q\pi)^{-1}p$. Let $\sigma : S \to S/S_chi$ be the natural projection. We then have induced maps $q_1 : S/S_\chi \to P$.

Lemma 19.6. *The composition*

$$\mathcal{C}_S \to \mathcal{A}_\pi \mathcal{C}_{S/T} \to \mathcal{A}_\pi I_q(\mathcal{B}_{X^P})$$

equals the following composition:

$$\mathcal{C}_S \to i^\wedge_{q\pi}(\mathcal{B}_{X^P}) \otimes_{p\in P} (\mathcal{C}_{S_p}) \to i^\wedge_{q\pi}(\mathcal{B}_{X^P}) \otimes_{p\neq\chi} (\mathcal{C}_{S_p}) \otimes (\mathcal{A}_\sigma \mathcal{C}_{S_\chi/T})$$
$$\to \mathcal{A}_\pi(i^\wedge_q(\mathcal{B}_{X^P}) \otimes \otimes_{p\in P}(\mathcal{C}_{q^{-1}p})) = \mathcal{A}_\pi \mathcal{I}_q(\mathcal{B}_{X^P}).$$

Proof. Pick an F in $\mathcal{C}_S$ and show that its images under the two maps coincide.

First of all we note the following thing. Let $s_1, s_2 \in S$ be such that $q\pi(s_1) \neq q\pi(s_2)$. Then $q(x_{s_1} - x_{s_2})$ is invertible in $\mathcal{A}_\pi \mathcal{I}_q(\mathcal{B}_{X^P})$. Let us multiply F by a product of sufficiently large number of such factors. We shall then obtain an element in $\otimes_{p\in P} \mathcal{C}_{S_p}$, and it is sufficient to prove the statement for only such elements, (because $q(x_{s_1} - x_{s_2})$ are all invertible in the target space). In this case the statement follows directly from Lemma 19.3. □

19.2.8. Let $q\colon S \to P$ be an arbitrary surjection. Define a functor $\mathcal{A}_p$ from the category of $\mathfrak{D}_{X^P}$-modules to the category of $\mathfrak{D}_{X^S}$-modules by the formula

$$\operatorname{liminv}_{N\to\infty} i_p^{\wedge}(M) \otimes_{\mathcal{O}_{X^S}} \otimes_{p\in P} \mathfrak{D}'_{q^{-1}p}/\mathfrak{D}'_{X^{q^{-1}p},N}.$$

19.2.9. Let $p\colon S \to R$ be an arbitrary surjection. For $r \in R$ let $S_r := p^{-1}r$. Pick non-empty subsets $T_r \subset S_r$. Let $P := \sqcup_r S_r/T_r$. We then have a natural decomposition: Let $p = p_2 p_1$, where $p_1 : S \to P$, $p_2 : P \to R$. We also denote $p_r : S_r \to S_r/T_r$.

The above constructions allow us to define a map

$$\mathcal{I}_p \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}$$

as follows.

$$\begin{aligned}\mathcal{I}_p(M) = i_p^{\wedge}(M) \otimes (\boxtimes_{r\in R}\mathcal{C}_{S_r}) &\to i_p^{\wedge}(M) \otimes (\boxtimes_{r\in R}\mathcal{A}_{p_r}\mathcal{C}_{S_r/T_r})\\ &\to i_p^{\wedge}(M) \otimes \mathcal{A}_{p_1}(\boxtimes_{r\in R}\mathcal{C}_{p_2^{-1}r})\\ &\to \mathcal{A}_{p_1}(i_{p_2}^{\wedge}(M) \otimes \boxtimes_{r\in R}\mathcal{C}_{p_2^{-1}r}) = \mathcal{A}_{p_1}\mathcal{I}_{p_2}(M).\end{aligned}$$

19.2.10. It follows that the map

$$\mathcal{I}_p \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}$$

is defined for all decompositions $p = p_2 p_1$ such that p_1, p_2 are surjections and for every $t \in \operatorname{Im} p$, $p_2^{-1}t$ contains at most one element u such that $p_1^{-1}u$ consists of more than one element.

19.2.11. Let $p_2 = q_2 q_1$ be a decomposition, where q_2, q_1 are surjections.

Claim 19.7. *The following diagram is commutative*:

$$\begin{array}{ccccc} \mathcal{I}_p & \longrightarrow & \mathcal{A}_{p_1} I_{p_2} & \longrightarrow & \mathcal{A}_{p_1}\mathcal{I}_{q_1}\mathfrak{i}_{q_2} \\ \downarrow & & & \nearrow & \\ \mathcal{I}_{q_1 p_1}\mathfrak{i}_{q_2} & & & & \end{array}$$

Proof. Follows from Lemma 19.6. □

19.2.12. *Compositions* $\delta_{q_1}\mathcal{I}_{q_2} \to \mathcal{I}_p \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}$

Let $q_2 q_1 = p$ be a decomposition of p as a product of two surjections. We will investigate the composition

$$\delta_{q_1}\mathcal{I}_{q_2} \to \mathcal{I}_p \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}.$$

Let a be a universal surjection among those that p_1 and q_1 pass through a: $p_1 = p_1' a, q_1 = q_1' a$. The surjection a is uniquely determined by the condition $a(x) = a(y)$ iff $p_1(x) = p_1(y)$ and $q_1(x) = q_1(y)$. Let b be the universal surjection among those that $b = b_p p_1 = b_q q_1$.

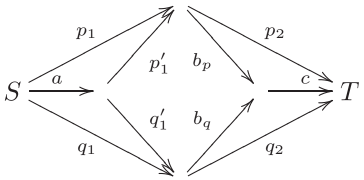
Let us describe b more concretely. For $t \in T$, let $R_t = p_2^{-1}t$ and $S_t = p^{-1}t$. Then there is at most one $r_t \in R_t$ such that $\#p_1^{-1}r_t > 1$. If there is no such an element pick r_t arbitrarily.

Let $P_t = p_1^{-1}r_t$. Then

$$S_t/P_t \xrightarrow{\sim} R_t.$$

Let e be the equivalence relation on S determined by q_1. The subsets S_t are not connected by this relation. Define the equivalence relation f which determines b. Let $u, v \in S_t$ we say $u \sim_f v$ if either $u \sim_e v$ or if there are $u', v' \in R_t$ such that $u \sim_e u'$ and $v \sim_e v'$.

We then have a commutative diagram:



We see that there is a natural map

$$\delta_{q_1'}\mathcal{A}_{b_q} \to \mathcal{A}_{p_1'}\delta_{b_p}.$$

Claim 19.8. *The composition*

$$\delta_{q_1}\mathcal{I}_{q_2} \to \mathcal{I}_p \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}$$

coincides with the composition:

$$\delta_{q_1}\mathcal{I}_{q_2} \to \delta_a\delta_{q_1'}\mathcal{I}_{cb_q} \to \delta_a\delta_{q_1'}\mathcal{A}_{b_q}\mathcal{I}_c \to \delta_a\mathcal{A}_{p_1'}\delta_{b_p}\mathcal{I}_c \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}.$$

Proof. Clear. □

19.3. *Maps* $\mathcal{P}_p \to \mathcal{A}_{p_1}\delta_{p_2}$

We always assume that p, p_1, p_2 are the same as above.

We are going to define maps $x(p_1, p_2) : \mathcal{P}_p \to \mathcal{A}_{p_1}\delta_{p_2}$ using induction in $|p_2| := \#R - \#P$.

The base is $|p_2| = 0$, i.e. a bijective p_2. Without loss of generality we can assume that $P = S/T$ and $p_2 = \mathrm{Id}$. The map $x(p_1, \mathrm{Id})$ is then defined as a composition

$$\mathcal{P}_p \to \mathcal{I}_p \to \mathcal{A}_p.$$

The transition is as follows. We begin with construction of a map $\xi(p_1, p_2) : \mathcal{P}_p \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}$. We then show that it passes through a unique map $x(p_1, p_2) : \mathcal{P}_p \to \mathcal{A}_{p_1}\delta_{p_2}$.

(1) Construction of $\xi(p_1, p_2)$. For every decomposition $p_2 = q_2 q_1$, all the maps being properly surjective, we define a map

$$\xi(p_1, q_1, q_2) : \mathcal{P}_p \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}$$

as the composition:

$$\mathcal{P}_p \to \mathcal{P}_{q_1 p_1}\mathcal{I}_{q_2} \to \mathcal{A}_{p_1}\delta_{q_1}\mathcal{I}_{q_2} \to \mathcal{A}_{p_1}\mathcal{I}_q.$$

We also set $K(p_1,p_2) : \mathcal{P}_p \to \mathcal{I}_{p_1}\mathcal{I}_{p_2} \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}$; $L(p_1,p_2) : \mathcal{P}_p \to \mathcal{I}_p \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}$ to be the natural maps.

We finally define a map

$$\xi(p_1,p_2) : \mathcal{P}_p \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}$$

as:

$$\xi(p_1,p_2) = L(p_1,p_2) - K(p_1,p_2) - \sum_{q=q_2q_1} \xi(p_1,q_1,q_2).$$

(2) We will now show that all compositions

$$\mathcal{P}_p \xrightarrow{\xi(p_1,p_2)} \mathcal{A}_{p_1}\mathcal{I}_{p_2} \xrightarrow{a} \mathcal{A}_{p_1}\mathcal{I}_{q_1}\mathfrak{i}_{q_2}$$

vanish, where $p_2 = q_2q_1$ is an arbitrary decomposition into a product of proper surjection. To show the vanishing, introduce a notation. For a map $L : \mathcal{P}_{q_1p_1} \to \mathcal{A}_{p_1}\mathcal{I}_{q_1}$ we set

$$L_! : \mathcal{P}_p \to \mathcal{P}_{q_1p_1}\mathfrak{i}_{p_2} \to \mathcal{A}_{p_1}\mathcal{I}_{q_1}\mathfrak{i}_{p_2}.$$

We then have (1) if $q_1 = q_3q^1$ and q_3, q^1 is proper,

$$a\xi(p_1,q^1,q_2q_3) = \xi(p_1,q^1,q_3)_!;$$

(2)

$$a\xi(p_1,q_1,q_2) = \xi(p_1,q_1)_!;$$

(3) $a\xi(q^1,q^2) = 0$ if q_1 does not pass through q^3;
(4)

$$aK(p_1,p_2) = K(p_1,q_1)_!;$$

(5)

$$aL(p_1,p_2) = L(p_1,q_1)_!.$$

Therefore,

$$a\xi(p_1,p_2) = \xi(p_1,q_1)_! - L(p_1,q_1)_! + K(p_1,q_1)_! + \sum_{q_1=q_3q^1} \xi(p_1,q^1,q_3)_! = 0,$$

by virtue of the induction assumption.

This implies that $\xi(p_1, p_2)$ passes through $\mathcal{A}_{p_1}\delta_{p_2}$. This completes the construction.

19.4. *Interaction with the maps $\mathcal{P}\delta\mathcal{P} \to \mathcal{P}$*

We are going to study the compositions

$$\mathcal{P}_{p^1}\delta_{p^2}\mathcal{P}_{p_3} \to \mathcal{P}_p \to \mathcal{A}_{p_1}\delta_{p_2}, \tag{57}$$

where p, p_1, p_2 are as above and $p = p^3p^2p^1$ is an arbitrary decomposition into a product of surjections and p^2 is proper.

19.4.1. We first of all note that the map $\mathcal{P}_p \to \mathcal{A}_{p_1}\delta_{p_2}$ passes through the direct sum of natural maps

$$\mathcal{P}_p \to \mathcal{I}_{p^1}\mathcal{I}_{p^2}\cdots\mathcal{I}_{p^k},$$

where $p^kp^{k-1}\cdots p^1 = p$ is a decomposition into a product of proper surjections, and $p^1 = ap_1$ for a surjection a.

This implies that the composition (57) vanishes except the following cases

(1) p_1 is bijective; (2) p_3 is bijective; (3) $p_1 = ap^1$.

Consider these cases.

(1) Investigate the composition

$$\delta_{q_1}\mathcal{P}_{q_2} \to \mathcal{P}_p \to \mathcal{A}_{p^1}\delta_{p^2}.$$

We shall use the notations from Sec. 19.2.12. We then claim that this composition equals:

$$\delta_{q_1}\mathcal{P}_{q_2} \to \delta_a\delta_{q_1'}\mathcal{P}_{cb_q} \to \delta_a\delta_{q_1'}\mathcal{A}_{b_q}\delta_c \to \delta_a\mathcal{A}_{p_1'}\delta_{b_p}\delta_c \to \mathcal{A}_{p_1}\delta_{p_2}.$$

(2) The composition $\mathcal{P}_{p_1}\delta_{p_2} \to \mathcal{P}_p \to \mathcal{A}_{p^1}\delta_{p^2}$ does not vanish only if $p_1 = ap^1$ for some a, in which case this map equals:

$$\mathcal{P}_{p_1}\delta_{p_2} \to \mathcal{A}_{p^1}\delta_a\delta_{p_2} \to \mathcal{A}_{p^1}\delta_{p^2}.$$

(3) In this case the composition vanishes. We shall use induction in $|p^2|$.

The base, i.e. the case when p^2 is bijective is clear.

Let us pass to the transition. We will show that the composition

$$\mathcal{P}_{ap^1}\delta_{p_2}\mathcal{P}_{p_3} \xrightarrow{u} \mathcal{P}_p \xrightarrow{\xi(p^1,p^2)} \mathcal{A}_{p^1}\mathcal{I}_{p^2}$$

vanishes.

We first consider the case when a is proper.

We see that $L(p^1, p^2)u = K(p^1, p^2)u = 0$. And that $\xi(p^1, q_1, q_2)u = 0$ unless (q_1, q_2) belong to the isomorphism class of $q_1 = a$ or $q_1 = ap_2$ in which cases these compositions mutually annihilate each other.

In the case $a = \mathrm{Id}, K(p^1, p^2) = 0, \xi(p^1, q_1, q_2) = 0$ every time except when the isomorphism class of (q_1, q_2) is given by $q_1 = p_1$. In this situation L and $\xi(p^1, p_2, p_3)$ annihilate each other.

19.4.2. *Composition* $\mathcal{P}_{fg\sqcup h} \to \mathcal{P}_{fg\sqcup \mathrm{Id}}\delta_{\mathrm{Id}\sqcup h} \to \mathcal{A}_{g\sqcup \mathrm{Id}}\delta_{f\sqcup h}$

We claim that this composition coincides with the map

$$\mathcal{P}_{fg\sqcup \mathrm{Id}} \to \mathcal{A}_{g\sqcup \mathrm{Id}} \to \delta_{f\sqcup \mathrm{Id}}.$$

19.5. *Interaction with the maps with $\mathcal{P} \to \mathcal{P}\mathcal{P}$*

The collection of functors $\mathcal{A}_p$ does not form a system because it may include very bad singularities which do not admit the required asymptotic decomposition.

One can, nevertheless, define a "correspondence". That is, for every decomposition $p = p_2p_1$. One can define a functor $\Gamma(p_1, p_2)$ such that

$$\Gamma(p_1, p_2)(M) \subset \mathcal{A}_p(M) \oplus \mathcal{A}_{p_1}\mathcal{A}_{p_2}(M).$$

This is what we are going to do.

19.5.1. *A subspace* $\Gamma_p \subset \mathfrak{D}'_S \oplus \mathcal{A}_p\mathfrak{D}'_T$

Let $p : S \to T$ be a surjection. We shall construct a subspace $\Gamma_p \subset \mathfrak{D}'_S \oplus \mathcal{A}_p\mathfrak{D}'_T$.

Pick a splitting $i : T \to S$ so that $pi = \mathrm{Id}_T$. For $\{x_s\}_{s\in S} \in Y^S$ and $\lambda > 0$ we set

$$V_\lambda(\{x_s\}_{s\in S}) = \left\{x_{ip(s)} + \frac{x_s - x_{ip(s)}}{\lambda}\right\}_{\lambda\in S}.$$

Pick an element $\tau \in T$; for a point $\{x_t\}_{t\in T} \in Y^T$ and $\mu > 0$ we set

$$U_\mu(\{x_t\}_{t\in T}) = \left\{x_\tau + \frac{x_t - x_\tau}{\mu}\right\}_{t\in T}.$$

Pick $f \in \mathfrak{D}_{Y^S}$, $g \in \mathfrak{D}_{Y^T}$ and $F \in \mathfrak{D}'_S$. We then have a function $A(\lambda, \mu) := \langle F, V_\lambda f U_\mu g\rangle$ in two variables λ, μ. This function is smooth for all $\lambda, \mu > 0$.

Let now $F' \in \mathcal{A}_p\mathfrak{D}'_T$. We can then construct an element

$$A' := \langle F', V_\lambda f U_\mu g\rangle \in \mathbb{C}[\ln \mu, \mu^{-1}, \mu][\ln \lambda, \lambda^{-1}, \lambda]$$

in the obvious way.

We say that A' is an asymptotic series for A if for every $P, Q > 0$ and every sufficiently large partial sum A'' of A'

$$A - A'' = \lambda^{P+1}x(\lambda, \mu) + \lambda^P\mu^Q y(\lambda, \mu),$$

where $x(\lambda, \mu)$ is continuous for all $\lambda \geq 0, \mu > 0$, and λ, μ is continuous for all $\lambda, \mu \geq 0$.

Define Γ_p as the set of all pairs F, F' such that A' is an asymptotic series for A for all f, g and all splittings i (one can actually show that if this is true for one splitting i, it is also true for every such a splitting).

The map $\Gamma_p \to \mathfrak{D}'_S$ is injective and closed under the action of dilations U^S_λ. We may, therefore, split $\Gamma_p = \oplus_n \Gamma_{p,n}$ into the direct sum of generalized eigenvalues of U^S_λ.

Let $p_1 : S \to R$, $p_2 : R \to T$ be surjections. For $t \in T$ let S_t, R_t be the preimages and let $p_{1t} : S_t \to R_t$ be the induced maps.

Set

$$\Gamma(p_1, p_2)(M) := \text{liminv}_N \left(\bigotimes_{t \in T} \Gamma_{p_{1t}} \Big/ \Gamma_{p_{1t},N} \right) \otimes i_p^\wedge(M).$$

The inclusions

$$\Gamma_{p_{1t}} \subset \mathfrak{D}'_p \oplus \mathcal{A}_{p_1} \mathfrak{D}'_{R_t}$$

induce the inclusions

$$\Gamma(p_1, p_2) \subset \mathcal{A}_p \oplus \mathcal{A}_{p_1} \mathcal{A}_{p_2}.$$

19.5.2. Let p, p_1, p_2 be as above. We then have maps

$$a : C_S \to \mathcal{A}_p(C_T)$$

and

$$b : C_S \to \mathcal{A}_{p_1} C_R \to \mathcal{A}_{p_1} \mathcal{A}_{p_2} C_T.$$

Claim 19.9. *The map $a \oplus b$ passes through $\Gamma(p_1, p_2) C_T$.*

19.5.3. *Asymptotic series modulo diagonals*

We will need a weaker version of the above definition. In the setting of the previous section, we say that F' is *an asymptotic series for F modulo diagonals in $X^{S/T}$* (*respectively in X^S*) if for every P, Q there exists an N such that whenever g vanishes on all generalized diagonals upto the order N (respectively f and g vanish on all generalized diagonals upto the order N), we have

$$A - A'' = \lambda^{P+1} x(\lambda, \mu) + \lambda^P \mu^Q y(\lambda, \mu),$$

where $x(\lambda, \mu)$ is continuous for all $\lambda \geq 0, \mu > 0, y(\lambda, \mu)$ is continuous for all $\lambda, \mu \geq 0$, and A'' is a partial sum of A' with sufficiently many terms.

Define $\Gamma^\circ(p_1, p_2)$ (respectively $\Gamma^{\circ\circ}(p_1, p_2)$) in the same way as $\Gamma(p_1, p_2)$ but using asymptotic series modulo diagonals in $X^{S/T}$ (respectively X^S).

Let $a, b \in S$ be such that $p_1(a) \neq p_1(b)$. $\Gamma^\circ(p_1, p_2)$ "does not feel sections supported on the diagonal $a = b$". Formal meaning is as follows. Let $\pi : S \to S/\{a, b\}$; let $p' : S/\{a, b\} \to T$. Let $H : i_{\pi *}\mathcal{A}_{p'} \to \mathcal{A}_p$ be the natural map. Then the functor

$$(H(i_{\pi *}\mathcal{A}_{p'}(M)), 0) \subset \Gamma^\circ(p_1, p_2).$$

Similarly, let $p_2 = p_4 p_3$ be a decomposition into a product of surjections, where p_3 is proper. Let $i_{p_3 *}\mathcal{A}_{p_4} \to \mathcal{A}_{p_2}$ be the natural map. Let

$$G : \mathcal{A}_{p_1} i_{p_3 *}\mathcal{A}_{p_4} \to \mathcal{A}_{p_1}\mathcal{A}_{p_2}$$

be the induced map. Then

$$(0, G(\mathcal{A}_{p_1} i_{p_3 *}\mathcal{A}_{p_4})(M)) \in \Gamma^\circ(p_1, p_2).$$

19.6. *Decomposition of the map $\mathcal{P}_p \to \mathcal{A}_{p_2}\delta_{p_1}$*

Let $p = p_2 p_1$ be as in Sec. 19.2.10. Choose a decomposition $p_2 = q_2 q_1$, where q_2, q_1 are surjections.

We are going to construct a map

$$\mathcal{P}_p \to \mathcal{A}_{q_1}\mathcal{A}_{q_2}\delta_{p_1}$$

such that its direct sum with the map

$$\mathcal{P}_p \to \mathcal{A}_{p_2}\delta_{p_1}$$

will pass through $\Gamma^\circ(q_1, q_2)\delta_{p_1}$.

19.7. For a surjection $u : A \to B$ let $B_m(p) \subset B$ be given by

$$B_m(u) = \{x \in B \mid \#p_1^{-1}(x) > 1\}.$$

Let $A_m(u) = u^{-1}B_m(u)$. Let $B = B_m(u) \sqcup B_s(u)$, $A = A_m(u) \sqcup A_s(u)$ be the decompositions. We then have $u = u_m \sqcup u_s$, where u_s is bijective and u_m is essentially surjective, i.e. $\#u_m^{-1}x > 1$ for all $x \in B_m(u)$.

19.7.1. Let $p_1 : S \to R, p_2 : R \to T$. Let $q_1 : S \to U$, $q_2 : U \to R$.

Let $S_m := S_m(p_1)$, $S_s = S_s(p_1)$. Then $q_1(S_s), p_1(S_s)$ are identified with S_s. Using this identification, we may assume that $U = U_m \sqcup S_s$ and that $q_1 = q_{1m} \sqcup \mathrm{Id}_{S_s}$; $R = R_m \sqcup S_s$, $q_2 = q_{2m} \sqcup \mathrm{Id}_{S_s}$ (see the diagrams below).

We will work with isomorphism classes of maps $v : U \to X$ which are

(1) injective on U_m,
(2) there exists $w : X \to T$ such that $wv = p_2 q_2$.

We may therefore assume that

$$X = U_m \sqcup Y$$

and that $v = \mathrm{Id} \sqcup v_s$.

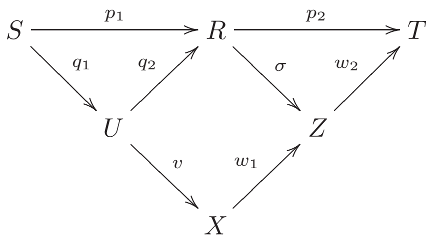
We see that equivalently, one can define a map v by a prescription of a map $v_s : S_s \to U_m \sqcup Y$ such that

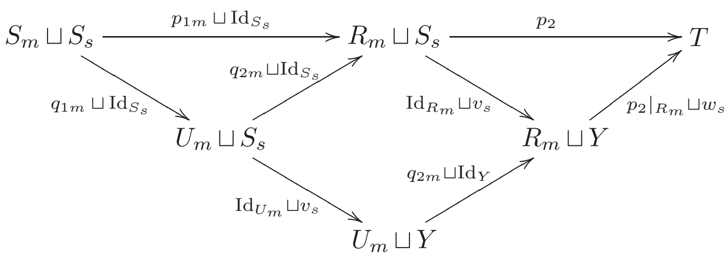
(1) $v_s(S_s) \supset Y$;
(2) there exists a map $w_s : Y \to T$ (it is then determined uniquely) such that the diagram below commutes.

We then have $w = p_2 q_{2m} \sqcup w_s$.

Let $Z = R_m \sqcup Y$. Let $w_1 : X \to Z$, $w_1 = q_{2m} \sqcup \mathrm{Id}_Y$; let $w_2 : Z \to T$, $w_2 = p_2|_{R_m} \sqcup w_s$.

Let $\sigma : R \to Z$ be given by $\mathrm{Id}_{R_m} \sqcup v_s$.





We also see that there is a natural transformation:

$$\delta_v \mathcal{A}_{w_1} \to \mathcal{A}_{q_2} \delta_\sigma.$$

Therefore, one constructs a map

$$\mu_v : \mathcal{P}_p \to \mathcal{P}_{vq_1} \mathcal{P}_{w_2 w_1} \to \mathcal{A}_{q_1} \delta_v \mathcal{A}_{w_1} \delta_{w_2} \to \mathcal{A}_{q_1} \mathcal{A}_{q_2} \delta_\sigma \delta_{w_2} \cong \mathcal{A}_{q_1} \mathcal{A}_{q_2} \delta_{p_2}.$$

Define a map

$$\mu(q_1, q_2) : \mathcal{P}_p \to \mathcal{A}_{q_1} \mathcal{A}_{q_2} \delta_{p_2}$$

as a sum of μ_v over the set of all isomorphism classes of maps v.

Let $\nu : \mathcal{P}_p \to \mathcal{A}_{p_1} \delta_{p_2}$.

Claim 19.10. *The map $\nu \oplus \mu$ passes through $\Gamma^\circ(q_1, q_2)\delta_{p_2}$.*

19.7.2. We need a lemma.

Let

$$L : \mathcal{P}_p \to \mathcal{I}_p \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}.$$

For v as in the previous section, set

$$\lambda_v : \mathcal{P}_p \to \mathcal{P}_{vq_1}\mathcal{P}_w \to \mathcal{A}_{q_1}\delta_v\mathcal{I}_w \to \mathcal{A}_{q_1}\mathcal{I}_{p_2q_2} \to \mathcal{A}_{q_1}\mathcal{A}_{q_2}\mathcal{I}_{p_2}.$$

Let $\lambda = \sum_v \lambda_v$.

Lemma 19.11. *$L - \lambda$ passes through $\Gamma^\circ(q_1, q_2)\mathcal{I}_{p_2}$*

Proof. Let Λ be given by:

$$\mathcal{P}_p \to \mathcal{I}_p \to \mathcal{A}_{q_1}\mathcal{I}_{p_2q_2} \to \mathcal{A}_{q_1}\mathcal{A}_{q_2}\mathcal{I}_{p_2}.$$

As we have seen above, $L - \Lambda$ passes through

$$\Gamma^\circ(q_1, q_2).$$

We can now focus on the difference $\Lambda - \lambda$. It suffices to show that it passes through $\mathcal{A}_{q_1}\mathcal{A}^\circ{}_{q_2}\mathcal{I}_{p_2}$.

Let

$$H : \mathcal{P}_p \to \mathcal{I}_p \to \mathcal{A}_{q_1}\mathcal{I}_{p_2q_2}.$$

Let

$$G_v : \mathcal{P}_p \to \mathcal{P}_{vq_1}\mathcal{P}_w \to \mathcal{A}_{q_1}\delta_v\mathcal{I}_w \to \mathcal{A}_{q_1}\mathcal{I}_{p_2q_2}.$$

We see that $\Lambda - \lambda$ equals the composition of $H - \sum_v G_v$ with the map

$$\mathcal{A}_{q_1}\mathcal{I}_{p_2q_2} \to \mathcal{A}_{q_1}\mathcal{A}_{q_2}\mathcal{I}_{p_2}.$$

Let $p_2q_2 = wv$ be a decomposition such that v is as above. Then it is not hard to see that the compositions of $H - \sum_v G_v$ with the map

$$\mathcal{I}_{p_2q_2} \to \mathcal{I}_v\mathfrak{i}_w$$

vanish. This implies that the composition of $H - \sum_v G_v$ with the map

$$\mathcal{A}_{q_1}\mathcal{I}_{p_2q_2} \to \mathcal{A}_{q_1}\mathcal{A}_{q_2}\mathcal{I}_{p_2}$$

passes through

$$\mathcal{A}_{q_1}\mathcal{A}^\circ{}_{q_2}\mathcal{I}_{p_2}.$$

This implies the statement. □

19.7.3. *Proof of the claim*

We shall use induction with respect to $|p|$. The base is clear. Let us pass to the transition.

By definition, the composition

$$\xi(p_1, p_2) : \mathcal{P}_p \to \mathcal{A}_{p_1}\delta_{p_2} \to \mathcal{A}_{p_1}\mathcal{I}_{p_2}$$

equals $-L + \sum \xi(p_1; r_1, r_2)$, where we changed q for r to avoid a confusion, and the sum is taken over all isomorphism classes of decompositions $p_2 = r_2 r_1$ such that r_2 is proper (so that K is included as the term corresponding to $r_1 = \mathrm{Id}$).

Define the map $\eta(p_1; r_1, r_2)$ as the composition:

$$\mathcal{P}_p \to \mathcal{P}_{r_1 p_1}\mathcal{P}_{r_2} \to \mathcal{A}_{q_1}\mathcal{A}_{q_2}\delta_{r_1}\mathcal{I}_{r_2} \to \mathcal{A}_{q_1}\mathcal{A}_{q_2}\mathcal{I}_{p_2}.$$

According to the induction assumption, the direct sum

$$\xi(p_1, r_1, r_2) + \eta(p_1; r_1, r_2)$$

passes through $\Gamma^\circ(q_1, q_2)\mathcal{I}_{p_1}$.

Let λ be as in the lemma. We then know that $L + \lambda$ passes through $\Gamma^\circ(q_1, q_2)$.

It now suffices to prove that $-\sum_v \lambda_v + \sum \eta(p_1, r_1, r_2) = 0$.

We, first of all see that

$$\sum \eta(p_1, r_1, r_2)$$

equals the sum of the maps of the form

$$\mathcal{P}_p \to \mathcal{P}_{vq_1}\mathcal{P}_{r_2 w_2' w_1} \to \mathcal{P}_{vq_1}\mathcal{P}_{w_2' w_1}\mathcal{P}_{r_2} \xrightarrow{\xi(w_1, w_2', r_2)} \mathcal{A}_{vq_1}\mathcal{A}_{w_1}\mathcal{I}_{r_2 w_2'}$$
$$\to \mathcal{A}_{q_1}\delta_v\mathcal{A}_{w_1}\mathcal{I}_{r_2 w_2'} \to \mathcal{A}_{q_1}\mathcal{A}_{q_2}\mathcal{I}_{p_2},$$

where $v, w_1, w_2 := r_2 w_2'$ are as in the previous section, and the decompositions $w_2 = r_2 w_2'$ are arbitrary, not necessarily proper.

The map λ equals

$$\mathcal{P}_p \to \mathcal{P}_{vq_1}\mathcal{P}_w \to \mathcal{P}_{vq_1}\mathcal{I}_w \xrightarrow{L} \mathcal{P}_{vq_1}\mathcal{A}_{w_1}\mathcal{I}_{w_2} \to \mathcal{A}_{q_1}\delta_v\mathcal{A}_{w_1}\mathcal{I}_{w_2} \to \mathcal{A}_{q_1}\mathcal{A}_{q_2}\mathcal{I}_{p_2}.$$

The statement now follows immediately.

19.8. *Maps $\mathfrak{p}_i\mathcal{P}_p \to \mathcal{A}_q\mathfrak{p}_j$*

19.8.1. *Definition*

Suppose we have a commutative square

$$\begin{array}{ccc} R & \xrightarrow{p} & T \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ S & \xrightarrow{q} & Q \end{array}$$

Let us define a map $\mathfrak{p}_i\mathcal{P}_p \to \mathcal{A}_q\mathfrak{p}_j$ in the following way.

Let $L := R\backslash i(S)$. We then have an identification $R = S \sqcup L$. Let $p_1 : R \to Q \sqcup L$ be just

$$R \xrightarrow{cong} S \sqcup L \xrightarrow{q\sqcup \mathrm{Id}_L} Q \sqcup L.$$

Let $p_2 : Q \sqcup L \to T$ be given by $j \sqcup p_{1L}$. We then have $p = p_2p_1$, where p_2, p_1 are surjections. We see that they satisfy the conditions which are necessary to define the map

$$\mathcal{P}_p \to \mathcal{A}_{p_1}\delta_{p_2}.$$

Finally, let $i_Q : Q \to Q \sqcup L$ be the inclusion. We then have a natural map

$$\mathfrak{p}_i\mathcal{A}_{p_1} \to \mathcal{A}_q\mathfrak{p}_{i_Q}.$$

The map $\mathfrak{p}_i\mathcal{P}_p \to \mathcal{P}_q\mathfrak{p}_j$ is now defined as the composition:

$$\mathfrak{p}_i\mathcal{P}_p \to \mathfrak{p}_i\mathcal{A}_{p_1}\delta_{p_2} \to \mathcal{A}_q\mathfrak{p}_{i_Q}\delta_{p_2} \to \mathcal{A}_q\mathfrak{p}_{p_2 i_Q} = \mathcal{A}_p\mathfrak{p}_j.$$

19.8.2. *Properties*

We shall translate the properties of the maps $\mathcal{P}_{p_2p_1} \to \mathcal{A}_{p_1}\delta_{p_2}$ into the language of the maps

$$\mathfrak{p}_i\mathcal{P}_p \to \mathcal{A}_q\mathfrak{p}_j.$$

(1) If the square (i, p, j, q) is suitable, then the diagram

$$\begin{array}{ccc} \mathfrak{p}_i\mathcal{P}_p & \longrightarrow & \mathcal{A}_q\mathfrak{p}_j \\ & \searrow & \uparrow \\ & & \mathcal{I}_q\mathfrak{p}_j \end{array}$$

is commutative.

(2) Let $p = p_3p_2p_1$ be a decomposition into a product of surjections, where p_2 is proper.

(3.1) The composition

$$\mathfrak{p}_i\mathcal{P}_{p_1}\delta_{p_2}\mathcal{P}_{p_3} \to \mathfrak{p}_i\mathcal{P}_p \to \mathcal{P}_q\delta_j$$

vanishes unless p_1 or p_3 are bijective.

(3.2) Investigate the composition

$$\mathfrak{p}_i\delta_{p_2}\mathcal{P}_{p_3} \to \mathfrak{p}_i\mathcal{P}_p \to \mathcal{P}_q\mathfrak{p}_j.$$

We can uniquely decompose $p_2 i = j_2q_2$, where j_2 is injective and q_2 is bijective. Furthermore, we can decompose $p_3j_2 = jq'$ for a surjection q'. The above composition is then:

$$\mathfrak{p}_i\delta_{p_2} \to \delta_{q_2}\mathfrak{p}_{j_2}\mathcal{P}_{p_3} \to \delta_{q_2}\mathcal{A}_{q'}\mathfrak{p}_j \to \mathcal{A}_q\mathfrak{p}_j.$$

(3.3) The composition

$$\mathfrak{p}_i \mathcal{P}_{p_1} \delta_{p_2} \to \mathfrak{p}_i \mathcal{P}_p \to \mathcal{P}_q \mathfrak{p}_j$$

does not vanish only if one can decompose $p_1 i = j_1 q$, where j_1 is injective, in which case it equals

$$\mathfrak{p}_i \mathcal{P}_{p_1} \delta_{p_2} \to \mathcal{P}_q \mathfrak{p}_{j_1} \delta_{p_2} \to \mathcal{P}_q \mathfrak{p}_j.$$

(4) Let $q = q^2 q^1$ be a decomposition into a product of surjections.

Consider the set of all isomorphism classes of the diagrams

$$\begin{array}{ccccc} R & \xrightarrow{p^1} & R_1 & \xrightarrow{p^2} & T \\ {\scriptstyle i}\uparrow & & {\scriptstyle j_1}\uparrow & & {\scriptstyle j}\uparrow \\ S & \xrightarrow{q^1} & P_1 & \xrightarrow{q^2} & P \end{array}$$

where $p^2 p^1 = p$. For every such a diagram D we have a map

$$u_D : \mathfrak{p}_i \mathcal{P}_p \to \mathfrak{p}_i \mathcal{P}_{p_1} \mathcal{P}_{p_2} \to \mathcal{A}_{q_1} \mathfrak{p}_{j_1} \mathcal{P}_{p_2} \to \mathcal{A}_{q_1} \mathcal{A}_{q_2} \mathfrak{p}_j.$$

Let u be the sum of u_D taken over the set of all diagrams D.

Let $v : \mathfrak{p}_i \mathcal{P}_p \to \mathcal{P}_q \mathfrak{p}_j$. Then the direct sum $u \oplus v$ passes through $\Gamma^\circ(q_1, q_2)\mathfrak{p}_j$.

19.9. *Maps* $\int_q : \mathbb{R}_q^{\mathbf{symm}} \to \mathcal{A}^\circ{}_q$

We define $\int_q = 0$ on all terms of cohomological degree <0. The terms of degree zero are all of the form $p_{i*}\mathcal{P}_p$, where $p_i = q$, i is injective and P is surjective. We define $\int_q |_{p_{i*}\mathcal{P}_p}$ as the composition:

$$p_{i*}\mathcal{P}_p \to \mathcal{A}_q \to \mathcal{A}^\circ{}_q.$$

Claim 19.12. $d\int_q = 0$.

Proof. We need to check that the composition

$$(\mathbb{R}_p^{\mathbf{symm}})^{-1} \xrightarrow{d} (\mathbb{R}^{\mathbf{symm}})_p^0 \xrightarrow{\int_p} \mathcal{A}_p$$

vanishes.

The functor $(\mathbb{R}_p^{\mathbf{symm}}) - 1$ is a direct sum of the terms $\mathfrak{p}_i \mathcal{P}_{p_1} \delta_{p_2} \mathcal{P}_{p_3}$, where $p = p_1 p_2 p_3 i$, where i is injective and p_1, p_2, p_3 are surjective and p_2 is proper.

Consider several cases.

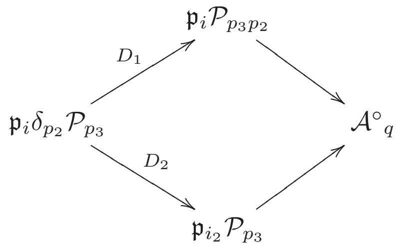
(1) p_1 is bijective. We may think that $p_1 = \mathrm{Id}$. The restriction of the differential onto this term equals the sum $-D_1 + D_2$, where

$$D_1 : \mathfrak{p}_i \delta_{p_2} \mathcal{P}_{p_3} \to \mathfrak{p}_i \mathcal{P}_{p_3 p_2}.$$

The map D_2 does not vanish only if $i_1 = p_2 i$ is injective, in which case

$$D_2 : \mathfrak{p}_i \delta_{p_2} \mathcal{P}_{p_3} \to \mathfrak{p}_{i_2} \mathcal{P}_{p_3}.$$

The check now reduces to showing that the diagram

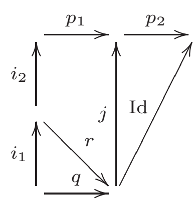


is commutative which follows from the Property 3.2.

(2) p_3 is bijective. We may assume $p_3 = \mathrm{Id}$. In this case, the restriction of the differential onto $\mathfrak{p}_i \mathcal{P}_{p_1} \delta_{p_2}$ equals $-D_1 + D_2$, where

$$D_1 : \mathfrak{p}_i \mathcal{P}_{p_1} \delta_{p_2} \to \mathfrak{p}_i \mathcal{P}_{p_2 p_1}.$$

The second term D_2 does not vanish only if $p_1 i = jq$, where j is injective. In this case we can construct a commutative diagram (uniquely upto an isomorphism):

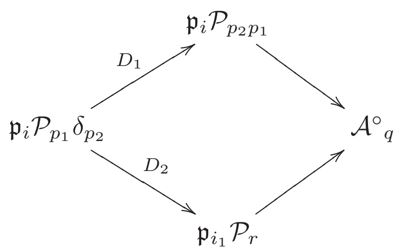


in which the square i_2, p, r, j is suitable.

The map D_2 is then:

$$\mathfrak{p}_i \mathcal{P}_{p_1} \delta_{p_2} \xrightarrow{\sim} \mathfrak{p}_{i_1} \mathfrak{p}_{i_2} \mathcal{P}_{p_1} \delta_{p_2} \to \mathfrak{p}_{i_1} \mathcal{P}_r \mathfrak{p}_j \delta_{p_2} \xrightarrow{\sim} \mathfrak{p}_{i_1} \mathcal{P}_r.$$

The Property 3.3, and 19.4.2 imply that the diagram



is commutative, whence the statement.

(3) p_1, p_3 are proper. In this case the restriction of the differential onto $\mathcal{P}_{p_1}\delta_{p_2}\mathcal{P}_{p_3}$ simply equals:

$$\mathfrak{p}_i\mathcal{P}_{p_1}\delta_{p_2}\mathcal{P}_{p_3} \to \mathfrak{p}_i\mathcal{P}_p.$$

The composition

$$\mathfrak{p}_i\mathcal{P}_{p_1}\delta_{p_2}\mathcal{P}_{p_3} \to \mathfrak{p}_i\mathcal{P}_p \to \mathcal{A}^\circ{}_q$$

vanishes according to Sec. 3.1. □

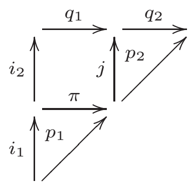
19.9.1. *Interaction with the maps* $\mathbb{R}^{\mathbf{symm}} \to \mathbb{R}^{\mathbf{symm}}\mathbb{R}^{\mathbf{symm}}$

Let $p = p_2 p_1$ be surjections. Let $\int_{p_1 p_2} : \mathbb{R}_p^{\mathbf{symm}} \to \mathbb{R}_{p_1}^{\mathbf{symm}}\mathbb{R}_{p_2}^{\mathbf{symm}} \to \mathcal{A}^\circ{}_{p_1}\mathcal{A}^\circ{}_{p_2}$

Claim 19.13. *The map* $\int_p \oplus \int_{p_1,p_2}$ *passes through* $\Gamma^{\circ\circ}(p_1, p_2)$.

Proof. Compute the restriction of the map $\mathbb{R}_p^{\mathbf{symm}} \to \mathbb{R}_{p_1}^{\mathbf{symm}}\mathbb{R}_{p_2}^{\mathbf{symm}}$ onto $\mathfrak{p}_i\mathcal{P}_q$.

By definition, such a restriction equals the sum of maps $m(q_1, q_2)$, where $q = q_2 q_1$ and $q_1 i = j p_2$, where j is injective. In this case one can construct a unique, upto an isomorphism, commutative diagram



where the square i_2, q_1, j, π is suitable.

The map $m(q_1, q_2)$ is then given by:

$$\mathfrak{p}_i\mathcal{P}_q \to \mathfrak{p}_{i_1}\mathfrak{p}_{i_2}\mathcal{P}_{q_1}\mathcal{P}_{q_2} \to \mathfrak{p}_{i_1}\mathcal{P}_\pi\mathfrak{p}_j\mathcal{P}_{q_2}.$$

The composition

$$\mathfrak{p}_i\mathcal{P}_q \to \mathfrak{p}_{i_1}\mathfrak{p}_{i_2}\mathcal{P}_{q_1}\mathcal{P}_{q_2} \to \mathfrak{p}_{i_1}\mathcal{P}_\pi\mathfrak{p}_j\mathcal{P}_{q_2} \to \mathcal{A}_{p_1}\mathcal{A}_{p_2}$$

equals, by virtue of Sec. 19.4.2,

$$u(q_1, q_2) : \mathfrak{p}_i\mathcal{P}_q \to \mathfrak{p}_i\mathcal{P}_{q_1}\mathcal{P}_{q_2} \to \mathcal{A}_{p_1}\mathfrak{p}_j\mathcal{A}_{q_2} \to \mathcal{A}_{p_1}\mathcal{A}_{p_2}.$$

The sum of all $u(q_1, q_2)$ is the map u from 4. Therefore, the direct sum of the composition

$$\mathfrak{p}_i\mathcal{P}_q \to \mathfrak{p}_{i_1}\mathfrak{p}_{i_2}\mathcal{P}_{q_1}\mathcal{P}_{q_2} \to \mathfrak{p}_{i_1}\mathcal{P}_\pi\mathfrak{p}_j\mathcal{P}_{q_2} \to \mathcal{A}_{p_1}\mathcal{A}_{p_2}$$

with the map

$$\mathfrak{p}_i\mathcal{P}_q \to \mathcal{A}_p$$

passes through $\mathcal{A}^\circ(p_1, p_2)$, whence the statement. □

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