Institut Camille Jordan

Laboratoire de recherche en mathématiques Lyon/Saint-Étienne

Inversion of the divergence and Hodge systems

PhD thesis manuscript
2020

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N°d'ordre NNT :
2020LYSE1157

THESE de DOCTORAT DE L'UNIVERSITE DE LYON
opérée au sein de
l'Université Claude Bernard Lyon 1

Ecole Doctorale InfoMaths (ED 512)

Spécialité : Mathèmatiques

Soutenue publiquement le 16/10/2020, par :

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Inversion de la divergence et systèmes de Hodge

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Acknowledgements

I want to thank to my scientific advisors. I am grateful to Professor Petru Mironescu for his patience and his scientific support during these years. He shared with me many scientific ideas during illuminating discussions and I am grateful for his serious involvement. His rigorous guidance and constant flow of energy helped to improve considerably the form and the content of my work. I am also deeply grateful for the personal support during my stay in Lyon, especially the thoughtful advices during the last year.

I am grateful to Professor Emmanuel Russ for his tolerance and scientific guidance. During these years he was restless in correcting my manuscripts and helping me to improve them. He was always understanding even if, usually, I completed the work after the limit term. I am also thankful for the insightful discussions that provided me with new perspectives on various subjects. His careful reading of the thesis and valuable suggestions raised the quality of this work.

I want to acknowledge here the valuable suggestions of Professor Haïm Brezis. The problem he pointed to us became the subject of Chapter 2 and Chapter 3. I thank him for this and for his inspiring work.

I am thankful to the members of the supervising committee for their useful advice. Aside the above mentioned scientific advisors, the other members of the committee were Professors Pierre Bousquet and Lorenzo Brandolese. In particular, I want to thank Professor Pierre Bousquet for his interest concerning the present work and for his insightful observations on Chapter 7.

I express my gratitude towards Professors Sylvie Benzoni-Gavage, Galia Dafni, Sandrine Grellier, Gabriela Marinoschi and Peter Sternberg who honored me by accepting to be part of the jury. Also, I am deeply grateful to Professor Po-Lam Yung and Professor El Maati Ouhabaz who accepted to act as referees and carefully read this thesis. Their corrections were very useful.

I am grateful to Professor Constantin Zălinescu, Professor Cătălin Lefter and Professor Florin Iacob from University “Al. I. Cuza” of Iaşi. Their support facilitated my arrival in France.

Résumé

L'objectif principal de cette thèse est d'étudier l'existence de solutions de systèmes de Hodge indéterminés dans des espaces fonctionnels "critiques". L'exemple le plus simple est l'équation de la divergence :

$$\operatorname{div} u = f, \text{ sur } \mathbb{R}^d, \tag{$*$}$$

où f est une fonction donnée et u un champ vectoriel. Si $1 < p < \infty$ et f est une fonction L^p à support compact d'intégrale nulle, alors la théorie elliptique standard implique l'existence d'une solution de $(*)$ dont le gradient appartient à L^p. En revanche, lorsque $p = 1$ ou $p = \infty$, il existe des fonctions f dans L^p, à support compact et d'intégrale nulle, telles que $(*)$ n'a pas de solutions u à gradient dans L^p. Ces résultats de non-existence ont été prouvés par Wojciechowski (1999), Bourgain-Brezis (2003), pour le cas $p = 1$, et par Preiss (1997), McMullen (1998), pour le cas $p = \infty$.

Nous obtenons des résultats similaires de non-existence dans le cas plus général des systèmes de Hodge indéterminés de la forme

$$du = f, \text{ sur } \mathbb{R}^d, \tag{$**$}$$

où f est une l-forme fermée prescrite et u est une $(l-1)$-forme.

En utilisant un nouveau résultat d'approximation pour les fonctions dans les espaces Sobolev critiques, Bourgain et Brezis (2007) ont montré que si f a les coefficients L^d, alors il existe une solution u de $(**)$, dont les coefficients sont bornés et dont le gradient appartient à L^d. En utilisant leur idée, Wang, Yung (2014) ont étendu ce résultat au cas plus général des groupes homogènes stratifiés, Ultérieurement, Bousquet, Russ, Wang, Yung (2017) ont obtenu une version euclidienne du résultat de Bourgain et Brezis, dans les espaces de Sobolev critiques avec une plus grande régularité. Nous unifions les deux résultats mentionnés ci-dessus, en obtenant une version pour les espaces de Sobolev critiques avec une plus grande régularité, dans le contexte des groupes stratifiés homogènes.

D'autres sujets connexes sont étudiés. Nous étudions l'équation de divergence avec, comme terme source, une mesure positive, nous fournissons une version améliorée du résultat de non-existence de Preiss et McMullen, et nous analysons les multiplicateurs de Fourier dans les espaces de Sobolev homogènes $\dot{W}^{k,p}(\mathbb{R}^d)$, lorsque $p = 1$ ou $p = \infty$ et $k \geq 1$ est un entier. Par ailleurs, nous étudions un probleme concernant les relèvements BV-minimaux des fonctions complexes unimodulaires.

La thèse comprend trois parties.

Partie I. Dans cette partie, nous étudions des systèmes de Hodge dont les termes sources sont dans L^1 ou L^∞, ou sont des mesures non négatives. La plus part de résultats que nous obtenons sont des résultats négatifs, concluant à la non-existence de solutions avec la régularité maximale attendue. Nous présentons également plusieurs résultats d'existence pour des solutions légèrement moins régulières, qui illustrent l'optimalité des résultats de non-existence.

La Partie I est formée des quatre chapitres.

Dans le Chapitre 1, notre objectif est de généraliser le théorème suivant de non-existence pour l'équation de divergence avec des termes sources non négatifs :

Théorème. *Soit μ une mesure de Radon non négative sur $\mathbb{R}^d$, et un paramètre $1 \leq p \leq d/(d-1)$. Si l'équation* $\operatorname{div} F = \mu$ *a une solution $F \in L^p(\mathbb{R}^d, \mathbb{R}^d)$, alors $\mu \equiv 0$.*

Nous montrons que ce résultat de non-existence se généralise à des espaces invariants par réarrangement (r. i. pour faire court). Sans donner ici une définition de ces espaces, nous citons quelques exemples d'espaces fonctionnels largement utilisés qui sont r. i. : les espaces de Lebesgue L^p, les espaces de Lorentz $L^{p,q}$ $(1 \leq p < \infty, 1 \leq q \leq \infty)$ et les espaces d'Orlicz $\Phi(L)$. Dans cette direction, nous obtenons:

Théorème. *Soit μ une mesure de Radon non négative sur $\mathbb{R}^d$, et X un espace r. i. de fonctions sur $\mathbb{R}^d$ tel que $|x|^{1-d}\mathbb{1}_{B^c}$ n'appartienne pas à X. Si l'équation* $\operatorname{div} F = \mu$ *a une solution $F \in X(\mathbb{R}^d, \mathbb{R}^d)$, alors $\mu \equiv 0$.*

De plus, nous montrons que la condition "$|x|^{1-d}\mathbb{1}_{B^c}$ n'appartient pas à X" dans le théorème ci-dessus est optimale. En effet, soit ϕ une fonction non triviale non négative de $L^\infty_c(\mathbb{R}^d)$ et définissons $\mu := \phi m$ (où m est la mesure de Lebesgue), qui est une mesure positive non triviale. Si $|x|^{1-d}\mathbb{1}_{B^c} \in X$, alors on montrons que l'équation $\operatorname{div} F = \mu$ a une solution F dans $X(\mathbb{R}^d, \mathbb{R}^d)$.

Nous étudions également le lien entre l'existence de solutions pour l'équation de divergence dans r. i. et les indices de Boyd associés à ces espaces.

Les Chapitres 2 et 3 sont consacrés à un même résultat de non existence pour les systèmes de Hodge, obtenu par deux méthodes différentes. Soit $N \geq 2$. Si $g \in L^1_c(\mathbb{R}^N)$ est d'integrale nulle, alors en général il n'est pas possible de résoudre l'équation $\operatorname{div} X = g$ avec $X \in W^{1,1}_{loc}(\mathbb{R}^N; \mathbb{R}^N)$ (Wojciechowski 1999), ou même $X \in L^{N/(N-1)}_{loc}(\mathbb{R}^N; \mathbb{R}^N)$ (Bourgain et Brezis 2003). En utilisant ces résultats, nous prouvons que, pour $N \geq 3$ et $2 \leq \ell \leq N-1$, il existe une ℓ-forme $f \in L^1_c(\mathbb{R}^N; \Lambda^\ell)$ avec les coefficients d'integrale nulle, satisfaisant la condition $df = 0$ et telle que l'équation $d\lambda = f$ n'ait pas de solution $\lambda \in W^{1,1}_{loc}(\mathbb{R}^N; \Lambda^{\ell-1})$. Ceci donne une réponse négative à une question posée par Baldi, Franchi et Pansu (2019).

Dans les deux chapitres, le problème est réduit au problème de l'équation de divergence. Dans le Chapitre 2, cette réduction est faite en utilisant l'hypoellipticité de l'operateur de Laplace, tandis que dans le Chapitre 3 la réduction est faite en utilisant la continuité des opérateurs de Calderón-Zygumnd sur des espaces de Besov homogènes.

Dans le Chapitre 4, notre point de départ est le résultat suivant de non-existence : il existe $g \in L^\infty(\mathbb{T}^2)$, d'integrale nulle et telle que l'équation $(*)$ n'ait pas de solution $f = (f_1, f_2) \in W^{1,\infty}(\mathbb{T}^2)$. Ce résultat a été obtenu indépendamment par Preiss (1997), en utilisant un argument géométrique délicat, et par McMullen (1998), via la non-inégalité d'Ornstein. Nous améliorons substantiellement ce résultat, en montrant qu'en général $(*)$ n'a pas de solution satisfaisant $\partial_2 f_2 \in L^\infty$, avec f "un peu mieux" que L^1. Notre démonstration est basée sur les produits Riesz, dans l'esprit de l'approche de Wojciechowski (1999) pour l'étude de $(*)$ avec source $g \in L^1$. La démonstration est élémentaire et évite completement l'utilisation de la non-inégalité d'Ornstein.

Voici par exemple une conséquence, simple à énoncer, du résultat principal de ce chapitre :

Théorème. *Soit $\varepsilon > 0$ fixé. Il existe une fonction $g \in L^\infty(\mathbb{T}^2)$ telle que l'équation*

$$g = f_0 + \partial_1 f_1 + \partial_2 f_2$$

n'ait pas de solution satisfaisant f_0, f_1, $f_2 \in H^\varepsilon(\mathbb{T}^2)$ et $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$.

Ce résultat se généralise aux dimensions $d \geq 3$.

Partie II. Dans cette partie, nous étudions la possibilité d'obtenir des solutions pour des systèmes de Hodge indéterminés, un peu plus réguliers (bornés et avec la régularité différentielle attendue) que ceux fournis par la théorie classique. Cette partie contient les Chapitres 5 et 6.

Le Chapitre 5 traite la généralisation commune de deux résultats d'approximation pour des fonctions dans des espaces critiques de Sobolev. D'une part, il s'agit d'un résultat pour les espaces de régularité différentiable 1 obtenu dans le cas général des groupes stratifiés homogènes par Wang et Yung (2014). D'autre part, d'un résultat d'approximation similaire obtenu par Bousquet, Russ, Wang, Yung (2017), pour des espaces de Sobolev de régularité plus élevée, mais uniquement dans le cas euclidien. Nous obtenons un résultat d'approximation dans le cas d'espaces de Sobolev de grande régularité sur des groupes stratifiés homogènes.

Pour simplifier la présentation, nous énonçons ci-dessous le résultat principal adapté au groupe de Heisenberg $\mathbb{H}^n$.

Soient $X_1,..,X_n$ et $Y_1,..,Y_n$ les champs vectoriels standard sur le groupe $\mathbb{H}^n$, définis par :

$$X_j := \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t},\ \ Y_j := \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t},\ \text{ pour } \ j = 1,2,...,n.$$

Soit

$$\nabla_b := (X_1,..,X_n,Y_1,..,Y_n).$$

Théorème. *Soit $Q := 2n+2$. Nous considérons les paramètres $1 < p,q < \infty$ et $\alpha := Q/p$. Si $J_1, J_2 \subset \{1,...,n\}$ sont deux ensembles tels que $|J_1| + |J_2| < \min(p,2n)$, alors, pour chaque fonction f, Schwartz sur $\mathbb{H}^n$, et chaque $\delta > 0$ il existe une fonction F telle que :*

$$\sum_{j\in J_1} \left\|X_j(f-F)\right\|_{\dot F_q^{\alpha-1,p}(\mathbb{H}^n)} + \sum_{j\in J_2} \left\|Y_j(f-F)\right\|_{\dot F_q^{\alpha-1,p}(\mathbb{H}^n)} \le \delta \left\|\nabla_b f\right\|_{\dot F_q^{\alpha,p}(\mathbb{H}^n)}$$

et

$$\|F\|_{L^\infty(\mathbb{H}^n)} + \|F\|_{\dot F_q^{\alpha,p}(\mathbb{H}^n)} \le C_\delta \left\|\nabla_b f\right\|_{\dot F_q^{\alpha,p}(\mathbb{H}^n)},$$

où C_δ est une constante qui ne dépend que de δ.

Ici, les espaces $\dot F_q^{\alpha,p}(\mathbb{H}^n)$ sont les analogues naturels des espaces du type Triebel-Lizorkin. Si α est un entier et $q = 2$, ces espaces coïncident avec les espaces de Sobolev standard sur $\mathbb{H}^n$.

Le Chapitre 6 aborde un problème géométrique. Nous étudions les solutions des systèmes de Hodge dans les espaces critiques de Sobolev, avec des conditions aux limites du type Dirichlet.

Partie III. Dans cette partie, nous étudions deux problèmes différents. Le premier problème, étudié au Chapitre 7, fait référence à la "généricité" des fonctions unimodulaires complexes qui ont un unique relèvement BV-minimal. Le deuxième problème, étudié au Chapitre 8, est étroitement lié à l'existence de solutions d'équations différentielles et concerne les multiplicateurs de Fourier sur les espaces de Sobolev pathologiques.

Le but du Chapitre 7 est de répondre à une question posée par Brezis et Mironescu sur les relèvements BV-minimaux pur les fonctions unimodulaires complexes. Étant donné $u \in W^{1,1}(\Omega,\mathbb{S}^1)$ (ici, Ω est un domaine lisse, borné et simplement connexe dans $\mathbb{R}^2$), nous appelons relèvement BV une fonction $\phi \in BV(\Omega,\mathbb{R})$ de sorte que $u = e^{i\phi}$ (l'existence d'un telle function ϕ est connue). On dit que un relèvement BV ϕ de u est BV-minimal si la seminorme BV de ϕ est minimale. La question que nous nous posons est la suivante : l'ensemble des fonctions $u \in W^{1,1}(\Omega,\mathbb{S}^1)$ qui admettent un relèvement BV-minimal unique ($mod\, 2\pi$), est-il résiduel dans $W^{1,1}(\Omega,\mathbb{S}^1)$? Nous montrons que la réponse à cette question est oui. En fait, nous obtenons cette réponse comme une conséquence du résultats suivant :

Théorème. *Soit Ω un domaine lisse, borné et simplement connexe dans $\mathbb{R}^2$. Soit k un entier positiv. L'ensemble des vecteurs $a = (a_1,...,a_k) \in \Omega^k$ pour lesquels chaque $u \in W^{1,1}(\Omega,\mathbb{S}^1) \cap C(\Omega\setminus\{a_1,...,a_k\})$ admet un relèvement BV-minimal unique ($mod\, 2\pi$) est de pleine mesure dans Ω^k.*

Nous démontrons ce résultat en réduisant le problème à l'étude des propriétés algébriques des distances entre les points du domaine Ω.

Au Chapitre 8, nous généralisons le résultat suivant obtenu par Kazaniecki et Wojciechowski (2013) concernant les multiplicateurs de Fourier sur l'espace homogène de Sobolev $\dot W^{1,1}$:

Théorème. *Soit $d \ge 2$. Si m est un multiplicateur de Fourier sur $\dot W^{1,1}(\mathbb{R}^d)$, alors $m \in C_b(\mathbb{R}^d)$.*

Nous démontrons la généralisation suivante du résultat ci-dessus :

Théorème. *Soient $d \geq 2$ et $l \geq 1$ deux entiers. Si m est un multiplicateur de Fourier sur $\dot{W}^{l,1}\left(\mathbb{R}^d\right)$, alors $m \in C_b\left(\mathbb{R}^d\right)$.*

On obtient également un résultat similaire dans le cas de l'espace $\dot{W}^{l,\infty}\left(\mathbb{R}^d\right)$. Nous démontrons ces résultats en utilisant une version de la méthode utilisée par Kazaniecki et Wojciechowski.

Abstract

The main purpose of the present thesis is to study the existence of solutions of underdetermined Hodge systems in "critical" function spaces. The simplest Hodge system is the (single) divergence equation:

$$\operatorname{div} u = f, \text{ on } \mathbb{R}^d, \tag{$*$}$$

where f is a given function and u a vector field. As long as $1 < p < \infty$, if f is an L^p compactly supported function with zero integral, the standard elliptic theory provides a solution u to $(*)$ whose gradient belongs to L^p. On the other hand, when $p = 1$ or $p = \infty$, there exist functions f in L^p which are compactly supported of integral zero, and such that $(*)$ does not have solutions u with gradient in L^p. These nonexistence results were proved by Wojciechowski (1999), Bourgain-Brezis (2003) in the case where $p = 1$, and by Preiss (1997), McMullen (1998) in the case where $p = \infty$.

We obtain similar nonexistence results in the case of more general undeterminated Hodge systems of the form

$$du = f, \text{ on } \mathbb{R}^d, \tag{$**$}$$

where f is a prescribed closed l-form and u is an $(l-1)$-form.

Using a new type of approximation result for functions in critical Sobolev spaces, Bourgain and Brezis (2007), showed that if f has L^d coefficients then there exists an $(l-1)$-form u, solution of $(**)$, whose coefficients are bounded and have the gradient in L^d. Following their idea, Wang, Yung (2014) extended the result to the more general case of stratified homogeneous groups and later Bousquet, Russ, Wang, Yung (2017) obtained an Euclidean version for higher regularity Sobolev spaces. We unify under a common roof the two aforementioned results, obtaining a version for higher regularity Sobolev spaces in the context of stratified homogeneous groups.

We also investigate several other related topics. We study the divergence equation when the source term is a nonnegative measure, we obtain improved versions of the nonexistence result of Preiss and McMullen and we analyze the multipliers of the homogeneous Sobolev spaces $\dot{W}^{k,p}\left(\mathbb{R}^d\right)$, when $p = 1$ or $p = \infty$ and $k \geq 1$ is an integer. Aside from these topics, we study a problem concerning minimal BV-liftings of complex unimodular maps.

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Introduction

Overview

The main theme of this manuscript is the inversion of the divergence equation, or, more generally of underdetermined Hodge systems in "critical" function spaces. The central question is whether or not these differential systems admit solutions which are sufficiently "smooth". Classical regularity theory provides satisfying positive answers in most of the cases. However, there are several limit situations in which the classical theories (e.g., the Calderón-Zygmund theory) cannot be applied. In some of these cases, we expect the nonexistence of solutions "as smooth as the source term allows". On the other side, there are cases where we expect to find solutions "smoother" than the ones provided by the classical theory.

We address several such questions. For equations or systems falling into the first category, we either significantly enlarge the function space in which nonexistence occurs, or we extend known nonexistence results from the divergence equation to Hodge systems. For the latter category, we generalize the existing positive results obtained on Euclidean spaces or for low regularity source terms to stratified homogeneous groups and to source terms with "critical regularity".

In different directions, we investigate smoothness properties of multipliers in homogeneous spaces, and the generic uniqueness of minimal liftings of unimodular maps.

1. The divergence equation

The simplest underdetermined system is the ubiquitous divergence equation

$$\operatorname{div} Y = f, \tag{0.1}$$

where f is a given function (or, more generally, distribution) defined on a domain of $\mathbb{R}^d$. The function f will be called source term. The problems we are interested in here are related to local regularity. In order to discard the possible influence of the boundary, we work with functions and vector fields on the d-dimensional torus $\mathbb{T}^d$ or on the d-dimensional Euclidean space $\mathbb{R}^d$. Here, $\mathbb{T}^d$ is the group $\mathbb{R}^d/(2\pi\mathbb{Z})^d$. In most cases, it will be identified with the set $[-\pi,\pi)^d$ endowed with the usual Lebesgue measure.

In general, it is easy to transfer existence results from $\mathbb{T}^d$ to $\mathbb{R}^d$ (or conversely), and from $\mathbb{T}^d$ to smooth bounded domains.

If $d \ge 2$, the equation (0.1) is underdetermined. For example, if Y is a solution of (0.1) and Y' is another vector field satisfying $\operatorname{div} Y' = 0$ (i.e., Y' is "divergence-free"), then

$$Y'' := Y + Y' \tag{0.2}$$

is also a solution of (0.1). There are many divergence-free vector fields. For example, if $d = 2$, any vector field Y' of the form $Y' = \left(-\partial_2\phi, \partial_1\phi\right)$, where ϕ is an arbitrary distribution, is divergence-free.

The case $d = 1$ is easy; in this case, (0.1) becomes

$$Y' = f \text{ on } \mathbb{T}. \tag{0.3}$$

Let us note some regularity results in this trivial case; they will guide us in the higher dimensional case. Consider some $1 \le p \le \infty$. If $f \in L^p(\mathbb{T})$ has zero integral, then there exists a unique function $Y \in W^{1,p}(\mathbb{T},\mathbb{R})$ with zero integral satisfying (0.1). This Y has one extra degree of regularity compared with f. This is a natural property that we expect to occur also in the case $d \ge 2$ whenever we have a solution, at least for one solution ((0.2) shows that this cannot hold for *all* the solutions and that, even in the Sobolev class, no uniqueness of solutions can be expected in any reasonable sense).

From now on we assume that $d \geq 2$.

The case $1 < p < \infty$**. Classical theory.** Let us fix $1 < p < \infty$. We first observe that, if $f \in L^p(\mathbb{T}^d)$ and there exists Y in $W^{1,p}(\mathbb{T}^d, \mathbb{R}^d)$ such that (0.1) holds, then

$$\int_{\mathbb{T}^d} f = \int_{\mathbb{T}^d} \operatorname{div} Y = 0. \tag{0.4}$$

Hence, we have to impose the necessary condition (0.4), i.e., the source term must have zero integral. We thus let f belong to $L^p_\sharp(\mathbb{T}^d)$, the space of all $L^p(\mathbb{T}^d)$-functions with zero integral. If $f \in L^p_\sharp(\mathbb{T}^d)$, then it is well-known that there always exists a solution Y in $W^{1,p}(\mathbb{T}^d, \mathbb{R}^d)$ of (0.1). This can be easily seen by applying the standard Calderón-Zygmund theory. In fact, we have the following explicit solution:

$$Y := \nabla \triangle^{-1} f, \tag{0.5}$$

where $\triangle : \mathscr{D}'(\mathbb{T}^d) \to \mathscr{D}'(\mathbb{T}^d)/\mathbb{R}$ is the Laplacian on $\mathbb{T}^d$, and $\triangle^{-1} : \mathscr{D}'(\mathbb{T}^d)/\mathbb{R} \to \mathscr{D}'(\mathbb{T}^d)$ is its inverse.

Indeed, one may see that this Y satisfies (0.1) on $\mathbb{T}^d$ in the sense of distributions:

$$\operatorname{div} Y = \operatorname{div} \nabla \triangle^{-1} f = \triangle \triangle^{-1} f = f.$$

Also, if we write $Y = (Y_1, ..., Y_d)$, we have

$$\partial_i Y_j = \partial_i \partial_j \triangle^{-1} f = R_i R_j f, \text{ for all } i, j = 1, 2, ..., d, \tag{0.6}$$

where $R_1, ..., R_d$ are the Riesz transforms on $\mathbb{T}^d$. Here, the operators R_j are defined by the relations

$$\widehat{R_j \psi}(n) := \frac{n_j}{|n|} \widehat{\psi}(n), \text{ for any } n \in \mathbb{Z}^d \setminus \{0\}, \ \widehat{R_j \psi}(0) = 0, \tag{0.7}$$

where ψ is any trigonometric polynomial on $\mathbb{T}^d$ with

$$\widehat{\psi}(0) = 0. \tag{0.8}$$

Notice that this gives

$$\widehat{R_i R_j \psi}(n) = \frac{n_i n_j}{|n|^2} \widehat{\psi}(n) = \widehat{\partial_i \partial_j \triangle^{-1} \psi}(n), \text{ for any } n \in \mathbb{Z}^d \setminus \{0\},$$

which formally justifies the formula (0.6).

One may still define R_i for distributions ψ on $\mathbb{T}^d$ satisfying (0.8) (which has to be understood as $\langle \psi, 1 \rangle = 0$), via formula (0.7). It is not difficult to see that, in this case, $R_i \psi$ is again a distribution

The operators R_j are bounded on $L^p(\mathbb{T}^d)$ (see below). Hence, we get $\partial_i Y_j \in L^p_\sharp(\mathbb{T}^d)$, for all $i, j = 1, 2, ..., d$, i.e., for each $j = 1, 2, ..., d$, we have $Y_j \in W^{1,p}(\mathbb{T}^d, \mathbb{R}^d)$. We also obtain the estimate

$$\|Y\|_{W^{1,p}(\mathbb{T}^d)} \leq C_p \|f\|_{L^p(\mathbb{T}^d)},$$

were C_p is a constant depending only on p and d.

We now briefly recall why R_i acts on $L^p_\sharp(\mathbb{T}^d)$, when $1 < p < \infty$. A *Calderón-Zygmund kernel* on $\mathbb{R}^d$ is a measurable function $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ for which there exists a constant $B > 0$, such that (see [**22**, p. 166]):

(i) $|K(x)| \leq B|x|^{-d}$, for all $x \in \mathbb{R}^d \setminus \{0\}$;

(ii) $\displaystyle\int_{|x|>2|y|} |K(x) - K(x-y)|\, dx \leq B$, for all $y \in \mathbb{R}^d \setminus \{0\}$;

(iii) $\displaystyle\int_{s<|x|<t} K(x) dx = 0$, for all $0 < s < t < \infty$.

With such a kernel we can associate an operator T, formally defined by $T\psi = K * \psi$ for Schwartz functions ψ; T is called a *Calderón-Zygmund operator*. We have the following fundamental theorem of Calderón and Zygmund (see [**22**, Theorem 7.5]):

THEOREM 0.1. *Suppose T is a Calderón-Zygmund operator as above. Then, for every $1 < p < \infty$ one can extend T to a bounded operator on $L^p(\mathbb{R}^d)$ with the bound $\|T\|_{L^p\to L^p} \le CB$, where $C = C(p,d)$ is a constant only depending on p and d.*

The usual Riesz transforms R_j on $\mathbb{R}^d$ are Calderón-Zygmund operators whose kernels are respectively defined by

$$k_j(x) = c_d \frac{x_j}{|x|^{d+1}}, \text{ for } x \in \mathbb{R}^d \setminus \{0\},$$

where c_d is a constant such that

$$\widehat{k_j}(\xi) = \frac{\xi_j}{|\xi|}, \text{ for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

Hence, each $R_j : L^p\left(\mathbb{R}^d\right) \to L^p\left(\mathbb{R}^d\right)$ is well-defined and bounded (see [**22**, Section 7.2]). Without much difficulty, this result implies also the boundedness of the Riesz transforms on $L^p\left(\mathbb{T}^d\right)$ via a transference method (see for example [**14**, Theorem 3.6.7]).

Concerning the divergence equation, the case of $\mathbb{R}^d$ is very similar. This time $\triangle$ and R_i will be the Laplacian and the Riesz transforms on $\mathbb{R}^d$. The operator $\triangle^{-1}$ will be defined by the formula:

$$\triangle^{-1}\psi = E * \psi,$$

for any Schwartz function ψ, where E is "the" fundamental solution of $\triangle$.

However, since $\mathbb{R}^d$ is not compact, some care is needed when defining the right spaces to work with. For example if $f \in L^p\left(\mathbb{R}^d\right)$ and the tempered distribution Y is given by (0.5), then we have again that each component of ∇Y is L^p, i.e., $Y \in \dot{W}^{1,p}\left(\mathbb{R}^d, \mathbb{R}^d\right)$ (see Section 5 for notation). Yet, we do not have in general that $Y \in L^p\left(\mathbb{R}^d, \mathbb{R}^d\right)$. However, such a Y satisfies $Y \in L^p_{loc}(\mathbb{R}^d)$ [**15**, Theorem 4.5.8].

It is worth mentioning that Bogovskiĭ ([**2**], 1980) found an explicit formula (see (0.9) below) for an inverse of the divergence operator on quite general domains.

Let Ω be a bounded domain in $\mathbb{R}^d$. The question that we ask in this case is the following: given a function $f \in L^p(\Omega)$ with zero integral on Ω, does there exist a vector field $Y \in W^{1,p}_0(\Omega)$ such that (0.1) holds in the sense of distributions on Ω? We recall that $W^{1,p}_0(\Omega)$ is the closure of $C^\infty_c(\Omega)$ under the $W^{1,p}$-norm. Note that a solution Y satisfying the weaker condition $Y \in W^{1,p}(\Omega)$ can be obtained as above. Indeed, we can extend f by letting $f = 0$ outside Ω and then let Y as in (0.5). The stronger condition $Y \in W^{1,p}_0(\Omega)$ amounts to requiring in addition that, in some generalized sense, we have $Y = 0$ on $\partial\Omega$.

In order to explicitly construct such a Y, we work on domains Ω that are star-like with respect to a ball. More specifically, we assume that there exists a ball $B(x_0,r)$ such that $\overline{B}(x_0,r) \subset \Omega$ and Ω is star-like with respect to every point of $B(x_0,r)$, i.e., for any $x' \in B(x_0,r)$ and any $y' \in \Omega$ the whole segment $\left[x', y'\right]$ is contained in Ω.

Consider now a function $\eta \in C^\infty_c(B(x_0,r))$ such that

$$\int_{B(x_0,r)} \eta = 1.$$

For a function $f \in L^p(\Omega)$, satisfying

$$\int_\Omega f = 0,$$

we define the vector field

$$Y(x) := \int_\Omega f(y)\left[(x-y)\int_1^\infty \eta(y + t(x-y))t^{d-1}dt\right]dy. \tag{0.9}$$

This vector field has the remarkable property that $Y \in W^{1,p}_0(\Omega)$ with

$$\|Y\|_{W^{1,p}(\Omega)} \lesssim \|f\|_{L^p(\Omega)}, \tag{0.10}$$

and it satisfies (0.1) (see [**13**, Lemma III.3.1, p. 162]). Here and in what follows, $A(x) \lesssim B(x)$ stands for $A(x) \le CB(x)$, for some constant $C < \infty$ independent of x. In particular, in (0.10) we have , $\|Y\|_{W^{1,p}(\Omega)} \le C\|f\|_{L^p(\Omega)}$ for some $C < \infty$ independent of f.

To check the estimate (0.10) is a matter of Calderón-Zygmund theory. Since the argument needs some careful computations we skip it. Also, the verification of (0.1) is somewhat lengthy and will be omitted. For both results, we refer the reader to [**13**] (see the proof of Lemma III.3.1, p. 162.)

Finally, we note that it is intuitively clear that Y "vanishes" on the boundary. Indeed, suppose for simplicity that f is smooth. In this case, after a more careful look at formula (0.9), one can see that the vector field Y is also smooth (in the whole $\mathbb{R}^d$). Consider now a point $x \in \partial\Omega$, or, more generally, a point x which does not belong to Ω. In order to see that $Y(x) = 0$, it suffices to show that

$$\eta(y + t(x - y)) = 0,$$

for any $y \in \Omega$ and for any $t \ge 1$. Suppose this is not the case. If y and t are fixed, then we must have $y + t(x - y) = b$ for some $b \in B(x_0, r)$. However, we get from this that x is a convex combination of y and b:

$$x = \left(1 - \frac{1}{t}\right) y + \frac{1}{t} b.$$

Since Ω is star-like with respect to $b \in B(x_0, r)$, we get $x \in [y, b] \subset \Omega$ which contradicts our hypothesis that $x \notin \Omega$.

This method can be extended to John domains (see [**1**]), or even to general bounded domains, considering weighted L^p spaces (see [**12**]).

The cases $p = 1$, $p = \infty$. "Pathological" sources. We saw that, as long as $1 < p < \infty$, we always have solutions of expected regularity. A natural question is to ask what happens in the remaining cases. It is well-known that the Riesz transforms are not bounded on L^1 or L^∞ (see, for example, [**22**, Exercise 7.5]).

Therefore, in these cases formula (0.5) need not provide solutions with the expected regularity of the divergence equation. This suggests that, when $p = 1$ or $p = \infty$, $W^{1,p}$ solutions of the divergence equation may not exist for a general $f \in L^p_\sharp$.

It turns out that this is indeed the case. More precisely, we have the following negative results (which for technical reasons are formulated for the d-dimensional torus):

THEOREM 0.2. *Assume $d \ge 2$. There exists $f \in L^1_\sharp(\mathbb{T}^d)$ such that there is no vector field $Y \in W^{1,1}(\mathbb{T}^d)$ with* $\operatorname{div} Y = f$.

THEOREM 0.3. *Assume $d \ge 2$. There exists $f \in L^\infty_\sharp(\mathbb{T}^d)$ such that there is no vector field $Y \in W^{1,\infty}(\mathbb{T}^d)$ with* $\operatorname{div} Y = f$.

Theorem 0.2 was first proved by Wojciechowski in 1998 (see [**31**]). His proof is by contradiction, and relies on Riesz products. A simpler proof was given by Bourdaud-Wojciechowski ([**7**]) in 2000 and Bourgain-Brezis in 2003 ([**3**]) by showing that a stronger conclusion holds: there exists $f \in L^1_\sharp(\mathbb{T}^d)$ such that there is no vector field $Y \in L^{d'}(\mathbb{T}^d)$ with $\operatorname{div} Y = f$. Here $d' := d/(d-1)$ is the conjugate exponent of d.

Theorem 0.3 was initially proved by Preiss in 1997 ([**26**]) using a geometrical argument and by McMullen in 1998 ([**21**]), via Ornstein's L^1-non-inequality ([**23**]). The proof of Theorem 0.3 that we present below is essentially McMullen's one.

Both theorems were rediscovered by Dacorogna, Fusco and Tartar in [**10**]. They also provided several different proofs of Theorem 0.2.

Since the proofs of Theorem 0.2 and Theorem 0.3 are instructive and simple, we recall them below, following the presentation in [**3**].

PROOF OF THEOREM 0.2. Suppose, by contradiction, that the statement of the theorem is false. That is, for any $f \in L^1_\sharp(\mathbb{T}^d)$ there exists a vector field $Y \in W^{1,1}(\mathbb{T}^d)$ such that $\operatorname{div} Y = f$.

We find that the operator $\operatorname{div} : W^{1,1}_\sharp(\mathbb{T}^d) \to L^1_\sharp(\mathbb{T}^d)$ is continuous and surjective. By the open mapping theorem, for every $f \in L^1_\sharp(\mathbb{T}^d)$, there exists some $Y \in W^{1,1}(\mathbb{T}^d)$ satisfying

$$\operatorname{div} Y = f \text{ and } \|Y\|_{W^{1,1}} \lesssim \|f\|_{L^1}. \tag{0.11}$$

Combining (0.11) with Gagliardo's embedding $W^{1,1} \hookrightarrow L^{d'}$, we find that, for every $f \in L^1_\sharp(\mathbb{T}^d)$, there exists some Y satisfying

$$\operatorname{div} Y = f \text{ and } \|Y\|_{L^{d'}} \lesssim \|f\|_{L^1}. \tag{0.12}$$

Let us fix $\varphi \in C^\infty(\mathbb{T}^d)$ with zero integral. There exists $f \in L^1_\sharp(\mathbb{T}^d)$ with $\|f\|_{L^1} = 1$ such that $\|\varphi\|_{L^\infty} \lesssim \langle \varphi, f\rangle$. With Y as in (0.12), we have

$$\|\varphi\|_{L^\infty} \lesssim \langle \varphi, f\rangle = \langle \varphi, \operatorname{div} Y\rangle = -\langle \nabla\varphi, Y\rangle \le \|\nabla\varphi\|_{L^d} \|Y\|_{L^{d'}} \lesssim \|\nabla\varphi\|_{L^d}. \tag{0.13}$$

From (0.13), we easily obtain the embedding $W^{1,d}(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$, which is well-known to be false when $d \ge 2$. This contradiction completes the proof of Theorem 0.2. □

PROOF OF THEOREM 0.3. For simplicity we prove the theorem in the case $d = 2$. The general case is very similar.

Suppose by contradiction that the statement of the theorem is false. That is, for any $f \in L^\infty_\sharp(\mathbb{T}^2)$ there exists a vector field $Y \in W^{1,\infty}(\mathbb{T}^2)$ such that $\operatorname{div} Y = f$. Using the open mapping theorem, Y can be chosen such that

$$\|Y\|_{W^{1,\infty}(\mathbb{T}^2)} \lesssim \|f\|_{L^\infty(\mathbb{T}^2)}.$$

Let us fix $\varphi \in C^\infty(\mathbb{T}^2)$. There exists $f \in L^\infty_\sharp(\mathbb{T}^2)$ with $\|f\|_{L^\infty} = 1$ such that $\|\partial_1\partial_2\varphi\|_{L^1} \lesssim \langle \partial_1\partial_2\varphi, f\rangle$. Let Y be as above. Then we have

$$\begin{aligned}\|\partial_1\partial_2\varphi\|_{L^1} &\lesssim \langle \partial_1\partial_2\varphi, f\rangle = \langle \partial_1\partial_2\varphi, \operatorname{div} Y\rangle = \langle \partial_1\partial_2\varphi, \partial_1 Y_1\rangle + \langle \partial_1\partial_2\varphi, \partial_2 Y_2\rangle \\ &= \langle \partial_1^2\varphi, \partial_2 Y_1\rangle + \langle \partial_2^2\varphi, \partial_1 Y_2\rangle \le \|\partial_1^2\varphi\|_{L^1} \|\partial_2 Y_1\|_{L^\infty} + \|\partial_2^2\varphi\|_{L^1} \|\partial_1 Y_2\|_{L^\infty} \\ &\lesssim \|\partial_1^2\varphi\|_{L^1} + \|\partial_2^2\varphi\|_{L^1}.\end{aligned}$$

However, as Ornstein proved (see [**23**]), this inequality is false in general.

Actually, this argument shows that one cannot take $\partial_1 Y_2$ and $\partial_2 Y_1$ in L^∞, which is weaker than requiring Y to be in $W^{1,\infty}$. □

Let us make some observations concerning the above proofs. In the proof of Theorem 0.3, we have used the following relatively difficult non-inequality of Ornstein ([**23**], 1962):

$$\|\partial_1\partial_2\varphi\|_{L^1} \not\lesssim \|\partial_1^2\varphi\|_{L^1} + \|\partial_2^2\varphi\|_{L^1},\ \varphi \in C^\infty(\mathbb{T}^2). \tag{0.14}$$

Following the same idea as in the proof of Theorem 0.3, it is possible to prove Theorem 0.2 using the following non-inequality (see for example [**10**]):

$$\|\partial_1\partial_2\varphi\|_{L^\infty} \not\lesssim \|\partial_1^2\varphi\|_{L^\infty} + \|\partial_2^2\varphi\|_{L^\infty},\ \varphi \in C^\infty(\mathbb{T}^2). \tag{0.15}$$

This non-inequality is easier than (0.14). In a more general form, it was first proved by de Leeuw and Mirkil (see [**20**]) before Ornstein proved (0.14). The proof in [**20**] relies on relatively simple duality methods. Also, some explicit constructions can be given. For example, Mityagin gave in 1958 (see [**18**]) the following example illustrating the failure of (0.15). Consider the function

$$g(x_1, x_2) := 3x_1x_2 - x_1 \ln(x_1^2 + x_2^2) - x_2^2 \arctan\left(\frac{x_1}{x_2}\right) - x_1^2 \arctan\left(\frac{x_2}{x_1}\right), \text{ on } (\mathbb{R}\setminus\{0\})^2,$$

extended by continuity to $\mathbb{R}^2$.

One can check that

$$\partial_1^2 g(x_1,x_2) = -2\arctan\left(\frac{x_2}{x_1}\right),\ \partial_2^2 g(x_1,x_2) = -2\arctan\left(\frac{x_1}{x_2}\right),\ \partial_1\partial_2 g(x_1,x_2) = -\ln\left(x_1^2+x_2^2\right),$$

in the classical sense in $(\mathbb{R}\setminus\{0\})^2$, and in the sense of distributions in $\mathbb{R}^2$.

Now choose a function $\eta \in C_c^\infty(B(0,1))$ such that $\eta \equiv 1$ on $B(0,1/2)$. By defining $\varphi := g\eta$, we have

$$\partial_1^2\varphi, \partial_2^2\varphi \in L^\infty \text{ and } \partial_1\partial_2\varphi \notin L^\infty.$$

Identifying $\mathbb{T}^2$ with $[-\pi,\pi)^2$ we get that smooth approximations of φ will provide examples for (0.15).

Compared to (0.15), it is much more difficult to illustrate the failure of (0.14); actually, the validity of (0.14) was an open problem of L. Schwartz, negatively solved by Ornstein via a delicate explicit construction, in the first part of its seminal contribution [**23**]. By the duality arguments presented above, this suggests that Theorem 0.2) is easier than Theorem 0.3.

The case $p = d$. Critical spaces. Let us now turn to another "limiting" case for the exponent p, namely $p = d$. As we already saw, if $f \in L^d_\sharp\left(\mathbb{T}^d\right)$, then there exists $Y \in W^{1,d}\left(\mathbb{T}^d\right)$ which satisfies (0.1). It is important to recall that $W^{1,d}\left(\mathbb{T}^d\right)$ is not embedded in $L^\infty\left(\mathbb{T}^d\right)$. (Recall that $d \geq 2$.) An explicit example of function in $W^{1,d}\left(\mathbb{T}^d\right)\setminus L^\infty\left(\mathbb{T}^d\right)$ is provided by

$$g_\alpha(x) := |\ln|x||^\alpha \eta(x),$$

where $\eta \in C_c^\infty(B(0,1))$ is a function such that $\eta(0) = 1$ and $0 < \alpha < d'$. One may observe that $g_\alpha \in W^{1,d}\left(\mathbb{T}^d\right)$, while, clearly, g_α is not bounded.

Let us consider the function

$$h_\alpha(x) := x_1 g_\alpha(x), \text{ on } \mathbb{T}^d,$$

and define the vector field $Y_\alpha := \nabla h_\alpha$. Clearly, we have

$$\operatorname{div} Y_\alpha = \triangle h_\alpha,$$

and it is easy to see that Y_α is the solution to the divergence equation with source term $\triangle h_\alpha$ provided by the formula (0.5). Clearly, $Y_\alpha := (g_\alpha + x_1\partial_1 g_\alpha, x_1\partial_2 g_\alpha, \ldots, x_1\partial_d g_\alpha)$ and, since $g_\alpha + x_1\partial_1 g_\alpha \notin L^\infty\left(\mathbb{T}^d\right)$, we have that $Y_\alpha \notin L^\infty\left(\mathbb{T}^d\right)$. By a direct computation, we see that

$$\left|\nabla^2 h_\alpha(x)\right| \lesssim \frac{1}{|x|}|\ln|x||^{\alpha-1}, \text{ on } B(0,1),$$

and since

$$\frac{1}{|x|}|\ln|x||^{\alpha-1} \in L^d_{loc}$$

and h_α is supported in $B(0,1)$, we get that $\nabla^2 h_\alpha \in L^d\left(\mathbb{T}^d\right)$. In particular, this gives us that $Y_\alpha \in W^{1,d}\left(\mathbb{T}^d\right)$ and that $\triangle h_\alpha \in L^d\left(\mathbb{T}^d\right)$.

The above example is due to L. Nirenberg and appears in [**3**]. It shows that, in general, if $f \in L^d_\sharp\left(\mathbb{T}^d\right)$, the solution to the divergence equation provided by formula (0.5) is not necessarily bounded.

However, it is possible to conclude by other means that, for this type of source term, the divergence equation admits a bounded solution. (This does not contradict the above example, since the divergence equation is underdetermined.) More precisely, we have:

THEOREM 0.4. *For any $f \in L^d_\sharp\left(\mathbb{T}^d\right)$ there exists a vector field $Y \in L^\infty\left(\mathbb{T}^d\right)$ satisfying* $\operatorname{div} Y = f$ *and*

$$\|Y\|_{L^\infty(\mathbb{T}^d)} \lesssim \|f\|_{L^d(\mathbb{T}^d)}. \tag{0.16}$$

This was first proved by Bourgain and Brezis in [**3**] (2003). Since their proof is simple and short we recall it below.

PROOF. Recall that we have Gagliardo's embedding

$$\|u\|_{L^{d'}(\mathbb{T}^d)} \lesssim \|\nabla u\|_{L^1(\mathbb{T}^d)}, \tag{0.17}$$

for any smooth function u on $\mathbb{T}^d$ with zero integral. Consider now the normed subspace

$$V := \left\{\nabla u \mid u \text{ smooth on } \mathbb{T}^d \text{ with zero integral}\right\} \subset L^1\left(\mathbb{T}^d, \mathbb{R}^d\right),$$

and let $f \in L^d_\sharp\left(\mathbb{T}^d\right)$. Define the functional $L_f : V \to \mathbb{R}$ by

$$L_f(\nabla u) := -\langle f, u\rangle,$$

whenever u is smooth on $\mathbb{T}^d$ with zero integral. Note that L_f is well-defined, since $u \mapsto \nabla u$ is one-to-one for such u's. Moreover, L_f is clearly linear.

The inequality (0.17) gives us that L_f is bounded on V, and that

$$\|L_f\| \lesssim \|f\|_{L^d}. \tag{0.18}$$

By using the Hahn-Banach theorem, we can find a bounded extension $\widetilde{L}_f \in \left(L^1\left(\mathbb{T}^d, \mathbb{R}^d\right)\right)^* = L^\infty\left(\mathbb{T}^d, \mathbb{R}^d\right)$ of L_f such that $\|\widetilde{L}_f\| = \|L_f\|$. Let $Y \in L^\infty\left(\mathbb{T}^d, \mathbb{R}^d\right)$ be a vector field representing $\widetilde{L}_f$. We have

$$-\langle f, u\rangle = L_f(\nabla u) = \widetilde{L}_f(\nabla u) = \langle Y, \nabla u\rangle = -\langle \operatorname{div} Y, u\rangle,$$

for any u smooth on $\mathbb{T}^d$ with zero integral. Hence, Y is a bounded solution (0.1) in the sense of distributions on $\mathbb{T}^d$. Estimate (0.16) follows from (0.18). □

Observe that in the above theorem the solution is obtained by a nonconstructive argument. In their paper [**3**], Bourgain and Brezis also proved that the bounded solution Y whose existence is given by Theorem 0.4 cannot depend linearly on f. Equivalently, there is no *bounded linear* map $T : L^d_\sharp(\mathbb{T}^d) \to L^\infty(\mathbb{T}^d; \mathbb{R}^d)$ satisfying $\operatorname{div} Tf = f$, $\forall f \in L^d_\sharp(\mathbb{T}^d)$. This is in contrast with the explicit and linear formula (0.5).

The striking fact that was proved in [**3**] is that we can *simultaneously* satisfy the conditions $Y \in L^\infty\left(\mathbb{T}^d\right)$ and $Y \in W^{1,d}\left(\mathbb{T}^d\right)$. More precisely, we have

THEOREM 0.5. *For any* $f \in L^d_\sharp\left(\mathbb{T}^d\right)$ *there exists a vector field* $Y \in L^\infty\left(\mathbb{T}^d\right) \cap W^{1,d}\left(\mathbb{T}^d\right)$ *satisfying* $\operatorname{div} Y = f$ *and*

$$\|Y\|_{L^\infty(\mathbb{T}^d)} + \|Y\|_{W^{1,d}(\mathbb{T}^d)} \lesssim \|f\|_{L^d(\mathbb{T}^d)}. \tag{0.19}$$

This result was proved by an involved approximation argument using the Littlewood-Paley square function. We will not describe the argument here. We mention instead that the complicated construction used in [**3**] can also be used in more general situations. Following the ideas in [**3**], Bousquet, Mironescu and Russ proved in [**5**] (2014) the following generalization of Theorem 0.5 in the scale of Triebel-Lizorkin spaces:

THEOREM 0.6. *Suppose that* $2 \le q \le p < \infty$ *and* $s > -1/2$ *are such that* $(s+1)p = d$. *For any* $f \in F^{s,p}_q\left(\mathbb{T}^d\right)$, *there exists a vector field* $Y \in L^\infty\left(\mathbb{T}^d\right) \cap F^{s+1,p}_q\left(\mathbb{T}^d\right)$ *satisfying* $\operatorname{div} Y = f$ *and*

$$\|Y\|_{L^\infty(\mathbb{T}^d)} + \|Y\|_{F^{s+1,p}_q(\mathbb{T}^d)} \lesssim \|f\|_{F^{s,p}_q(\mathbb{T}^d)}. \tag{0.20}$$

In order to keep the presentation simple, we omit here the precise definition of the Triebel-Lizorkin spaces, and refer the interested reader to Section 5. It is worth noting that the scale of these spaces includes the classical Sobolev spaces $W^{k,p}$, $k \in \mathbb{N}$, $1 < p < \infty$.

The existence of a vector field Y satisfying one of the estimates $\|Y\|_{L^\infty(\mathbb{T}^d)} \lesssim \|f\|_{L^d(\mathbb{T}^d)}$ *or* $\|Y\|_{W^{1,d}(\mathbb{T}^d)} \lesssim \|f\|_{L^d(\mathbb{T}^d)}$ (implied by (0.20)) follows from standard results in harmonic analysis. Indeed, it suffices to adapt the proof of Theorem 0.4 for the first estimate (and to use the adapted Sobolev type embedding), respectively to apply Calderón-Zygmund theory (whose validity for Triebel-Lizorkin spaces is well-established) for the latter one. As in the case of Theorem 0.5, the difficulty consists of finding Y satisfying *both* estimates.

In [**5**], the authors

also proved a version of Theorem 0.6 on smooth domains.

THEOREM 0.7. *Suppose that* $2 \le q \le p < \infty$ *and* $s > -1/2$ *are such that* $(s+1)p = d$. *Let* Ω *be a smooth bounded domain in* $\mathbb{R}^d$ *and let* $f \in F_q^{s,p}(\Omega)$ *be such that*

$$\int_\Omega f = 0.$$

Then, there exists a vector field $Y \in L^\infty(\Omega) \cap F_q^{s+1,p}(\Omega)$ *satisfying* $\operatorname{div} Y = f$ *and* $trY = 0$ *on* $\partial\Omega$. *Moreover, we can choose* Y *such that*

$$\|Y\|_{L^\infty(\Omega)} + \|Y\|_{F_q^{s+1,p}(\Omega)} \lesssim \|f\|_{F_q^{s,p}(\Omega)}.$$

Sources which are nonnegative measures. We also consider the case where sources are measures rather than L^p functions. This is related to the "pathological" case where the sources were L^1, however, we are interested here in the decay at infinity of the solutions, rather than their differential regularity.

We consider the equation

$$\operatorname{div} F = \mu \quad \text{on} \quad \mathbb{R}^d, \tag{0.21}$$

with μ a nonnegative Radon measure on $\mathbb{R}^d$.

Let us prove, by a simple argument[1], that, if the above equation has a solution in *certain* L^p spaces, then we necessarily have $\mu = 0$.

For this purpose, suppose μ is as above and let F be a solution of (0.21) that belongs to $L^p(\mathbb{R}^d)$ for some $1 \le p \le d'$. For simplicity, we suppose that F is smooth, but this is not relevant for the final conclusion. Even without the smoothness assumption on F, we can "smooth" the problem by taking convolutions with smooth compactly supported functions, and then argue as below.

By applying the Gauss-Ostrogradskii theorem, we get, for any $R > 0$,

$$\mu(B(0,R)) = \int_{B(0,R)} d\mu = \int_{B(0,R)} \operatorname{div} F \, dx = \int_{S(0,R)} F \cdot \nu \, d\sigma,$$

where $S(0,R)$ is the boundary of $B(0,R)$ and ν is the unit outward normal at $S(0,R)$. We immediately obtain that

$$\mu(B(0,R)) \le \int_{S(0,R)} |F| \, d\sigma,$$

and, by applying Hölder's inequality, we have

$$\mu(B(0,R)) \lesssim R^{(d-1)/p'} \left(\int_{S(0,R)} |F|^p \, d\sigma \right)^{1/p},$$

i.e.,

$$\frac{\mu^p(B(0,R))}{R^{(d-1)(p-1)}} \lesssim \int_{S(0,R)} |F|^p \, d\sigma.$$

Integrating in R this last inequality, we get

$$\int_{R_0}^\infty \frac{\mu^p(B(0,R))}{R^{(d-1)(p-1)}} dR \lesssim \int_{R_0}^\infty \int_{S(0,R)} |F|^p \, d\sigma dR = \|F\|^p_{L^p(\mathbb{R}^d \setminus B(0,R_0))} < \infty, \tag{0.22}$$

for any $R_0 > 0$.

We now observe that, since $p \le d' = d/(d-1)$, we have $(d-1)(p-1) \le 1$, and thus

$$\int_{R_0}^\infty \frac{1}{R^{(d-1)(p-1)}} dR = \infty. \tag{0.23}$$

Since μ is nonnegative, we have that $\mu(B(0,R)) \ge \mu(B(0,R_0)) \ge 0$ for all $R \ge R_0$ and

$$\int_{R_0}^\infty \frac{\mu^p(B(0,R))}{R^{(d-1)(p-1)}} dR \ge \mu^p(B(0,R_0)) \int_{R_0}^\infty \frac{1}{R^{(d-1)(p-1)}} dR,$$

[1] We thank to P. Mironescu for this argument.

which together with (0.22) and (0.23) gives us that $\mu(B(0,R_0)) = 0$. Since we can choose R_0 arbitrarily large, we get that $\mu \equiv 0$ on $\mathbb{R}^d$.

To summarize, by the above argument we have obtained the following result:

THEOREM 0.8. *Let $1 \le p \le d/(d-1)$ and let μ be a nonnegative Radon measure on $\mathbb{R}^d$. If there exists a vector field $F \in L^p(\mathbb{R}^d,\mathbb{R}^d)$ such that* $\operatorname{div} F = \mu$ *on $\mathbb{R}^d$, then $\mu \equiv 0$.*

The above result was proved by Phuc and Torres (see [**24**, Theorem 3.1]) by a different method than the one given above. They have obtained Theorem 0.8 by a direct application of the Calderón-Zygmund theory. They also proved that the exponent $d/(d-1)$ above is sharp. We point out that this result treats only the case of nonnegative measures. In the more general case of signed Radon measures the situation is more complicated and little is known in this direction.

2. Differential forms and Hodge systems in $\mathbb{R}^d$ and in $\mathbb{H}^n$

General facts. Hodge systems in $\mathbb{R}^d$. A natural generalisation of the divergence equation is a (underdetermined) Hodge system: given a l-differential form λ on $\mathbb{R}^d$ whose coefficients are elements of some function space on $\mathbb{R}^d$, we ask for the existence of an $(l-1)$-differential form u on $\mathbb{R}^d$, with coefficients in some "appropriate" function space such that:

$$du = \lambda. \tag{0.24}$$

Here, l is an integer with $1 \le l \le d$ and du stands for the exterior derivative of u, defined as follows. When $1 \le k \le d$, we write $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$ for any increasing sequence $i_1 < \dots < i_k$ in $\{1,\dots,d\}$ and $I := \{i_1,\dots,i_k\}$. With this notation, if

$$u := \sum_{\substack{I\subseteq\{1,\dots,d\}\\ |I|=l-1}} u_I dx_I,$$

then

$$du = \sum_{\substack{I\subseteq\{1,\dots,d\}\\ |I|=l-1}} \sum_{1\le i\le d} \partial_i u_I dx_i \wedge dx_I. \tag{0.25}$$

Since we will work in spaces of distributions, as in the case treated before of the divergence equation, all the derivatives in (0.25) will be considered in the sense of distributions.

Let us quickly explain why, when $l = d$, the system (0.24) is equivalent to the divergence equation. We have exactly one subset of $\{1,\dots,d\}$ whose cardinality is d; namely, the set $\{1,\dots,d\}$ itself. Hence, any d-form λ can be written as

$$\lambda = f\,dx_1 \wedge \dots \wedge dx_d, \tag{0.26}$$

for some function f. We have exactly d subsets of $\{1,\dots,d\}$ of cardinality $d-1$; namely $\{1,\dots,j-1,j+1,\dots,d\}$ for $1 \le j \le d$. Hence, any $(d-1)$-form u can be written as

$$u = \sum_{j=1}^{d} u_j dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dx_d,$$

for some functions u_j. Using the definition (0.25), we obtain

$$\begin{aligned} du &= \sum_{j=1}^{d}\sum_{i=1}^{d} \partial_i u_j dx_i \wedge dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dx_d \\ &= \sum_{j=1}^{d} \partial_j u_j dx_j \wedge dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dx_d \\ &= \left(\sum_{j=1}^{d} (-1)^{j-1} \partial_j u_j\right) dx_1 \wedge \dots \wedge dx_d, \end{aligned} \tag{0.27}$$

where we have used the fact that $dx_i \wedge dx_i = 0$. Combining (0.27) with (0.26), we obtain that in the case $l = d$ the system (0.24) becomes

$$\sum_{j=1}^{d} (-1)^{j-1} \partial_j u_j = f.$$

Clearly, this is equivalent to a divergence equation with the source f.

As in the smooth case, we have $d^2 u = 0$ (this time in the sense of distributions), and thus in order to be able to solve (0.24) we have to impose the compatibility condition

$$d\lambda = 0 \text{ in the sense of distributions.} \tag{0.28}$$

Note that, as long as we work on $\mathbb{R}^d$, the above compatibility condition is vacuous in the case $l = d$.

If

$$\varphi = \sum_{\substack{I \subseteq \{1,\ldots,d\} \\ |I|=l}} \varphi_I dx_I \quad \text{and} \quad \phi = \sum_{\substack{I \subseteq \{1,\ldots,d\} \\ |I|=l}} \phi_I dx_I$$

are some l-forms in $L^2(\mathbb{R}^d)$, formally, we write

$$\langle \varphi, \phi \rangle := \sum_{\substack{I \subseteq \{1,\ldots,d\} \\ |I|=l}} \int_{\mathbb{R}^d} \varphi_I \phi_I dx.$$

The formal adjoint of the exterior derivative d will be denoted by d^*. Hence, we have:

$$\langle d\psi_1, \psi_2 \rangle = -\langle \psi_1, d^* \psi_2 \rangle$$

for all $(l-1)$-forms ψ_1 and l-forms ψ_2 with smooth and compactly supported coefficients. It turns out that we can compute d^* explicitly, as explained below. For φ as above we have

$$d^* \varphi = \sum_{|I|=l} \sum_{1 \le i \le d} \partial_i \varphi_I \partial_i \rfloor dx_I, \tag{0.29}$$

where the expressions $\partial_i \rfloor dx_I$ are $(l-1)$-forms defined as follows.

Suppose $I = \{i_1, \ldots, i_l\}$, where $1 \le i_1 < \ldots < i_l \le d$. If $i \in I$, and $1 \le k \le l$ is such that $i = i_k$, then we set

$$\partial_i \rfloor dx_I = \partial_{i_k} \rfloor dx_I := (-1)^{k-1} dx_{i_1} \wedge \ldots \wedge dx_{i_{k-1}} \wedge dx_{i_{k+1}} \wedge \ldots \wedge dx_{i_l}.$$

Moreover, if $i \notin I$, then $\partial_i \rfloor dx_I := 0$ if $i \notin I$. (See [**11**, Example, (3.12)].)

By a direct computation we can verify that $\triangle = d\,d^* + d^*\,d$. In other words, for any differential form φ with smooth coefficients we have:

$$\triangle \varphi = dd^* \varphi + d^* d\varphi, \tag{0.30}$$

where $\triangle$ acts on differential forms "component-wise":

$$\triangle \varphi := \sum_{|I|=l} \triangle \varphi_I dx_I,$$

for any form

$$\varphi = \sum_{|I|=l} \varphi_I dx_I.$$

In a similar way one can define the action of $\triangle^{-1}$ on differential forms with smooth compactly supported coefficients:

$$\triangle^{-1} \varphi := \sum_{|I|=l} \triangle^{-1} \varphi_I dx_I.$$

It is easy to see that $\triangle$ and $\triangle^{-1}$ are commuting with the operators d and d^*. If φ is a form with smooth compactly supported coefficients, then from (0.30) we get

$$\varphi = \triangle^{-1} dd^* \varphi + \triangle^{-1} d^* d\varphi = d \triangle^{-1} d^* \varphi + d^* \triangle^{-1} d\varphi. \tag{0.31}$$

Note that $d\triangle^{-1}d^*\varphi$ (respectively $d^*\triangle^{-1}d\varphi$) is a closed (respectively co-closed) form, i.e., we have

$$d(d\triangle^{-1}d^*\varphi) = 0 \text{ and } d^*(d^*\triangle^{-1}d\varphi) = 0. \tag{0.32}$$

In view of (0.32), (0.31) asserts that any smooth compactly supported form can be decomposed as a sum of a closed form and a co-closed form.

It is also possible to give an L^p-version of (0.31).[2] In the case of $\mathbb{R}^d$, we have the following simple Hodge decomposition formula (see for example [**27**]):

$$v = d\triangle^{-1}d^*v + d^*\triangle^{-1}dv \tag{0.33}$$

for any l-form v with L^p coefficients where $1 < p < \infty$.

Indeed, using (0.25) and the explicit formula of d^*, (0.29), it is easy to see that the operators $d\triangle^{-1}d^*$ and $d^*\triangle^{-1}d$ are linear combinations of operators of the form R_iR_j, where R_j are the Riesz transforms on $\mathbb{R}^d$. Hence, each term in the right hand side of (0.33) is well-defined and we have

$$\left\|d\triangle^{-1}d^*v\right\|_{L^p(\mathbb{R}^d)} \lesssim \|v\|_{L^p(\mathbb{R}^d)} \quad \text{and} \quad \left\|d^*\triangle^{-1}dv\right\|_{L^p(\mathbb{R}^d)} \lesssim \|v\|_{L^p(\mathbb{R}^d)}.$$

In some cases we can use the decomposition formula in (0.33) to construct solutions to Hodge systems. We illustrate this by the following simple proposition (see for example [**27**]).

PROPOSITION 0.9. *Let $1 \le l \le d$ be an integer and $1 < p < \infty$. Suppose λ is an l-form with $d\lambda = 0$ and whose coefficients are L^p functions on $\mathbb{R}^d$. Then, there exists an $(l-1)$-form u with $\dot{W}^{1,p}$ coefficients on $\mathbb{R}^d$ and such that (0.24) is satisfied.*

Indeed, one can construct explicitly the solution

$$u := \triangle^{-1}d^*\lambda. \tag{0.34}$$

To see, at least formally, that u solves (0.24), we rely on (0.33) and find

$$du = d\triangle^{-1}d^*\lambda = d\triangle^{-1}d^*\lambda + d^*\triangle^{-1}d\lambda = \lambda.$$

The expression (0.34) is very similar to the one in (0.5) used to explicitly construct solutions for the divergence equation. As in the case of (0.5), using the Calderón-Zygmund theory, we infer that each coefficient of u is a distribution in $\dot{W}^{1,p}\left(\mathbb{R}^d\right)$.

Knowing the nonexistence results for the divergence equation for $p = 1$ or $p = \infty$, described in Theorem 0.2 and 0.3, it is natural to ask if similar results hold true for more general Hodge systems (0.24). In other words, is it true that there exists an l-form f on $\mathbb{R}^d$ with L^1 coefficients and satisfying $df = 0$, such that there is no $(l-1)$-form u with $\dot{W}^{1,1}$ coefficients and satisfying (0.24)? The same question makes sense if we replace the space L^1 with L^∞ and $\dot{W}^{1,1}$ with $\dot{W}^{1,\infty}$. We will address these questions in Chapters 2 and 3.

Hodge systems in $\mathbb{R}^d$. The case of critical function spaces. We have the following analogue of Theorem 0.4, in the case of Hodge systems:

THEOREM 0.10. *Let $2 \le l \le d$. If $\lambda \in L^d(\mathbb{R}^d)$ is an l-form with $d\lambda = 0$, then there exists an $(l-1)$-form $u \in L^\infty(\mathbb{R}^d)$ such that $du = \lambda$ on $\mathbb{R}^d$. Also, we can choose u such that*

$$\|u\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\lambda\|_{L^d(\mathbb{R}^d)}.$$

[2] Whenever X is a normed function space on $\mathbb{R}^d$, we write, for all l-forms φ,

$$\|\varphi\|_X := \sum_{\substack{I\subseteq\{1,\dots,d\}\\|I|=l}} \|\varphi_I\|_X.$$

Usually, for the sake of simplicity, when each coefficient φ_I belongs to some function space X, we say that φ belongs to X (and we write $\varphi \in X$).

Notice that the condition $l \geq 2$ in the above result is necessary. An analogue result for the case $l = 1$ does not hold. Indeed, the case $l = 1$ corresponds to the gradient equation. Actually, if we use the following standard identifications: a 0-form u is a function, and its exterior differential du is identified with ∇u, a 1-form λ is a vector field, and its exterior differential $d\lambda$ is identified with $\operatorname{curl}\lambda$, then solving $du = \lambda$ for a 1-form λ satisfying $d\lambda = 0$ amounts to the following: given a vector field λ with $\operatorname{curl}\lambda = 0$, and whose components are L^d functions, we ask if there exists an L^∞ function u such that

$$\nabla u = \lambda, \text{ on } \mathbb{R}^d.$$

Such a function u does not always exists. Clearly, if $\lambda := \nabla v$ where v is an unbounded function in $\dot{W}^{1,d}(\mathbb{R}^d)$, then $u - v$ is constant and hence u is not bounded.

We mention that Theorem 0.10 is a direct consequence of Theorem 0.13 below which was obtained by quite complicated means. However, even the weaker Theorem 0.10 is interesting on its own. The natural question here is whether there is a simple(r) proof for this weaker result. In the case $l = d$, where the Hodge system reduces to the divergence equation, this is a direct consequence of Gagliardo's embedding (see Theorem 0.4 and its proof). In the case $2 \leq l < d$, we can prove Theorem 0.10 by following the idea in the proof of Theorem 0.4 and the next estimate (see [**33**, Theorem 3]):

THEOREM 0.11. *Let $1 \leq l \leq d-1$. Suppose that $f \in C_c^\infty(\mathbb{R}^d)$ is an l- form such that $df = 0$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ is a $(d-l)$-form. Then,*

$$\left| \int_{\mathbb{R}^d} f \wedge \varphi \right| \leq C \|f\|_{L^1(\mathbb{R}^d)} \left\| d\varphi \right\|_{L^d(\mathbb{R}^d)}, \tag{0.35}$$

where C is a constant independent of f and φ.

It turns out that Theorem 0.11 is equivalent to Theorem 0.10 (see [**33**] for details). The advantage of the statement of Theorem 0.11, as was shown by Van Schaftingen in [**33**], is that (0.35) can be proved by much simpler means than Theorem 0.13. The technique is based on some embeddings for Morrey spaces. There is also a relative resemblance with the standard method for proving the classical Gagliardo embedding. In fact, Theorem 0.11 implies the following Gagliardo-type embedding for differential forms:

$$\|u\|_{L^{d'}(\mathbb{R}^d)} \lesssim \|du\|_{L^1(\mathbb{R}^d)} + \left\|d^* u\right\|_{L^1(\mathbb{R}^d)},$$

for any l-form u, provided that $2 \leq l \leq d-2$ and the result does not hold for $l = 1, d-1$ (see [**33**] for details). This result was also independently obtained by Lanzani and Stein in [**19**].

The technique in [**3**], used in the proof of Theorem 0.5, was developed to a higher level of generality by the same authors in [**4**]. The main tool in [**4**] is a new approximation result (see [**4**, Theorem 11]).

THEOREM 0.12. *Given $\delta > 0$ and $f \in \dot{W}^{1,d}(\mathbb{R}^d)$ there exists some $F \in L^\infty(\mathbb{R}^d) \cap \dot{W}^{1,d}(\mathbb{R}^d)$ such that, for all $j = 1, 2, \ldots, d-1$,*

$$\left\|\partial_j (f - F)\right\|_{L^d(\mathbb{R}^d)} \leq \delta \|f\|_{\dot{W}^{1,d}(\mathbb{R}^d)}$$

and

$$\|F\|_{\dot{W}^{1,d}(\mathbb{R}^d)} \leq C_\delta \|f\|_{\dot{W}^{1,d}(\mathbb{R}^d)},$$
$$\|F\|_{L^\infty(\mathbb{R}^d)} \leq C_\delta \|f\|_{\dot{W}^{1,d}(\mathbb{R}^d)},$$

for a constant C_δ that depends only on δ and d.

This approximation result is sufficiently robust so that, using it in conjunction with an iterative method, it is possible to prove existence results for more general Hodge systems than the divergence equation. It can be applied even to other differential systems (see [**33**]). We illustrate how the argument works in the case of Hodge systems. Suppose λ is an l-form (where $l \geq 2$) on $\mathbb{R}^d$

with L^d coefficients, satisfying $d\lambda = 0$. Then, thanks to Proposition 0.9 we can find an $(l-1)$-form φ on $\mathbb{R}^d$,

$$\varphi = \sum_{\substack{I\subseteq\{1,\dots,d\}\\ |I|=l-1}} \varphi_I dx_I,$$

solving $d\varphi = \lambda$, such that $\varphi_I \in \dot{W}^{1,d}$ for each I. This φ can be chosen such that

$$\left\|\varphi_I\right\|_{\dot{W}^{1,d}(\mathbb{R}^d)} \lesssim \|\lambda\|_{L^d(\mathbb{R}^d)}, \text{ for all } I. \tag{0.36}$$

Let $\delta > 0$. Using the approximation result given by Theorem 0.12 and (0.36) we can find, for each I, a function $F_I \in L^\infty(\mathbb{R}^d) \cap \dot{W}^{1,d}(\mathbb{R}^d)$ such that

$$\left\|\partial_j\left(\varphi_I - F_I\right)\right\|_{L^d(\mathbb{R}^d)} \le \delta \|\lambda\|_{L^d(\mathbb{R}^d)}, \text{ for all } j \notin I,$$

and

$$\|F_I\|_{\dot{W}^{1,d}(\mathbb{R}^d)} \le C_\delta \|\lambda\|_{L^d(\mathbb{R}^d)},$$
$$\|F_I\|_{L^\infty(\mathbb{R}^d)} \le C_\delta \|\lambda\|_{L^d(\mathbb{R}^d)}.$$

It is possible to apply here Theorem 0.12, thanks to the fact that $l \ge 2$. Indeed, since $l-1 \ge 1$, each set I with $|I| = l-1$ is nonempty. With no loss of generality, we may assume that $d \in I$, and then the existence of F_I follows from Theorem 0.12.

Note that, by definition,

$$d\varphi = \sum_{\substack{I\subseteq\{1,\dots,d\}\\ |I|=l-1}} \sum_{1\le i\le d} \partial_i\varphi_I dx_i \wedge dx_I.$$

Let observe that in the above formula, the expressions like $dx_i \wedge dx_I$ are zero if $i \in I$. Hence, we can write

$$d\varphi = \sum_{\substack{I\subseteq\{1,\dots,d\}\\ |I|=l-1}} \sum_{\substack{1\le i\le d\\ i\notin I}} \partial_i\varphi_I dx_i \wedge dx_I.$$

If we set

$$F := \sum_{\substack{I\subseteq\{1,\dots,d\}\\ |I|=l-1}} F_I dx_I,$$

and we use the triangle inequality, we get

$$\begin{aligned}\|\lambda - dF\|_{L^d(\mathbb{R}^d)} &= \left\|d\varphi - dF\right\|_{L^d(\mathbb{R}^d)}\\ &\le \sum_{\substack{I\subseteq\{1,\dots,d\}\\ |I|=l-1}} \sum_{\substack{1\le i\le d\\ i\notin I}} \left\|\partial_i\varphi_I - \partial_i F_I\right\|_{L^d(\mathbb{R}^d)}\\ &\lesssim \delta \|\lambda\|_{L^d(\mathbb{R}^d)}.\end{aligned}$$

Note that we also have

$$\|F\|_{\dot{W}^{1,d}(\mathbb{R}^d)} \le C_\delta \|\lambda\|_{L^d(\mathbb{R}^d)},$$
$$\|F\|_{L^\infty(\mathbb{R}^d)} \le C_\delta \|\lambda\|_{L^d(\mathbb{R}^d)}.$$

Hence, if δ is sufficiently small, we have

$$\|\lambda - dF\|_{L^d(\mathbb{R}^d)} \le \frac{1}{2} \|\lambda\|_{L^d(\mathbb{R}^d)},$$

and

$$\|F\|_{\dot{W}^{1,d}(\mathbb{R}^d)} \le C \|\lambda\|_{L^d(\mathbb{R}^d)},$$
$$\|F\|_{L^\infty(\mathbb{R}^d)} \le C \|\lambda\|_{L^d(\mathbb{R}^d)},$$

for some constant C depending only on d.

Now we can use an iterative method. Let $F^0 := F$. Applying the above result for $\lambda - dF^0$ instead of λ, we obtain an $(l-1)$-form F^1 such that

$$\|\lambda - dF^0 - dF^1\|_{L^d(\mathbb{R}^d)} \le \frac{1}{2}\|\lambda - dF^0\|_{L^d(\mathbb{R}^d)} \le \frac{1}{4}\|\lambda\|_{L^d(\mathbb{R}^d)},$$

and F^1 satisfies

$$\|F^1\|_{\dot{W}^{1,d}(\mathbb{R}^d)} \le \frac{C}{2}\|\lambda\|_{L^d(\mathbb{R}^d)},$$
$$\|F^1\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{2}\|\lambda\|_{L^d(\mathbb{R}^d)}.$$

Now, as above we approximate $\lambda - dF^0 - dF^1$ and we find an $(l-1)$-form F^2 such that

$$\|\lambda - dF^0 - dF^1 - dF^2\|_{L^d(\mathbb{R}^d)} \le \frac{1}{2}\|\lambda - dF^0 - dF^1\|_{L^d(\mathbb{R}^d)} \le \frac{1}{8}\|\lambda\|_{L^d(\mathbb{R}^d)},$$

with

$$\|F^2\|_{\dot{W}^{1,d}(\mathbb{R}^d)} \le \frac{C}{4}\|\lambda\|_{L^d(\mathbb{R}^d)},$$
$$\|F^2\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{4}\|\lambda\|_{L^d(\mathbb{R}^d)}.$$

We continue this iteration scheme and we obtain a sequence of $(l-1)$-forms $F_0, F_1, \ldots, F_n, \ldots$, such that

$$\|\lambda - dF^0 - \ldots - dF^n\|_{L^d(\mathbb{R}^d)} \le \frac{1}{2^{n+1}}\|\lambda\|_{L^d(\mathbb{R}^d)}, \tag{0.37}$$

and

$$\|F^n\|_{\dot{W}^{1,d}(\mathbb{R}^d)} \le \frac{C}{2^n}\|\lambda\|_{L^d(\mathbb{R}^d)}, \tag{0.38}$$
$$\|F^n\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{2^n}\|\lambda\|_{L^d(\mathbb{R}^d)}. \tag{0.39}$$

We see that we can define the $(l-1)$-form

$$u := F^0 + F^1 + \ldots + F^n + \ldots.$$

Indeed, this series is absolutely convergent in L^∞ (which is a Banach space) thanks to (0.39). From (0.38), u also belongs to $\dot{W}^{1,d}(\mathbb{R}^d)$.

Quantitatively, we have

$$\|u\|_{\dot{W}^{1,d}(\mathbb{R}^d)} \le 2C\|\lambda\|_{L^d(\mathbb{R}^d)},$$
$$\|u\|_{L^\infty(\mathbb{R}^d)} \le 2C\|\lambda\|_{L^d(\mathbb{R}^d)}.$$

Also, (0.37) implies that $du = \lambda$.

To summarize, we have obtained the following (see [**4**]).

THEOREM 0.13. *Let $2 \le l \le d$. If $\lambda \in L^d(\mathbb{R}^d)$ is a l-form with $d\lambda = 0$, then there exists an $(l-1)$-form $u \in L^\infty(\mathbb{R}^d) \cap \dot{W}^{1,d}(\mathbb{R}^d)$ such that $du = \lambda$ on $\mathbb{R}^d$. Also, we can choose u such that*

$$\|u\|_{L^\infty(\mathbb{R}^d)} + \|u\|_{\dot{W}^{1,d}(\mathbb{R}^d)} \lesssim \|\lambda\|_{L^d(\mathbb{R}^d)}.$$

As we have already mentioned, this result implies in particular Theorem 0.10. Similar results holds for d^* instead of d.

Observe that Theorem 0.12 gives a result of approximation only for functions of differential regularity one. A similar approximation result, for higher order Sobolev spaces was obtained by Bousquet, Russ, Wang, Yung in [**6**] (2017). Following the ideas in [**4**], they were able to extend Theorem 0.12 to the more general case of the homogeneous spaces $\dot{F}^{d/p,p}_q(\mathbb{R}^d)$ (see Section 5 for their precise definition). More specifically, these authors have proved the following:

THEOREM 0.14. *Consider the parameters $1 < p, q < \infty$, $\alpha := d/p$ and let $\Bbbk$ be the largest positive integer with $\Bbbk < \min(p,d)$. Then, for every $\delta > 0$ there exists a constant $C_\delta > 0$ depending only on δ, such that for every function $f \in \dot{F}_q^{\alpha,p}(\mathbb{R}^d)$ there exists $F \in L^\infty(\mathbb{R}^d) \cap \dot{F}_q^{\alpha,p}(\mathbb{R}^d)$ satisfying the following estimates:*

$$\sum_{i=1}^{\Bbbk} \|\partial_i(f - F)\|_{\dot{F}_q^{\alpha-1,p}(\mathbb{R}^d)} \le \delta \|f\|_{\dot{F}_q^{\alpha,p}(\mathbb{R}^d)},$$
$$\|F\|_{L^\infty(\mathbb{R}^d)} + \|F\|_{\dot{F}_q^{\alpha,p}(\mathbb{R}^d)} \le C_\delta \|f\|_{\dot{F}_q^{\alpha,p}(\mathbb{R}^d)}.$$

Note that, for $\alpha = 1$ and $p = d$, Theorem 0.14 is exactly Theorem 0.12. As a consequence of Theorem 0.14, we obtain the following result, similar to Theorem 0.13, concerning Hodge systems [**6**, Theorem 1.2]:

THEOREM 0.15. *Consider the parameters $1 < p, q < \infty$, $\alpha := d/p$ and let $\Bbbk$ be the largest positive integer with $\Bbbk < \min(p,d)$. Let $d - \Bbbk + 1 \le l \le d$. If $\lambda \in \dot{F}_q^{\alpha-1,p}(\mathbb{R}^d)$ is an l-form with $d\lambda = 0$, then there exists an $(l-1)$-form $u \in L^\infty(\mathbb{R}^d) \cap \dot{F}_q^{\alpha,p}(\mathbb{R}^d)$ such that $du = \lambda$ on $\mathbb{R}^d$. Also, we can choose u such that*

$$\|u\|_{L^\infty(\mathbb{R}^d)} + \|u\|_{\dot{F}_q^{\alpha,p}(\mathbb{R}^d)} \lesssim \|\lambda\|_{\dot{F}_q^{\alpha-1,p}(\mathbb{R}^d)}.$$

Theorem 0.15 follows from Theorem 0.14, by an iterative argument, in the same way Theorem 0.13 follows from Theorem 0.12.

Hodge systems in $\mathbb{H}^n$. The case of critical function spaces. The results obtained in the Euclidean framework were generalized, to some extent, to the case of stratified homogeneous groups. This class of groups is large enough to contain, for example, the Euclidean space $\mathbb{R}^d$ and the Heisenberg group $\mathbb{H}^n$. One attempt of development in this context is due to Chanillo and Van Schaftingen in [**9**]. A more elaborated approach was proposed by Wang and Yung [**30**]. We will discuss their results in what follows.

In order to give a glimpse of the results in this framework of stratified homogeneous groups, for the sake of simplicity, we focus on the case of Heisenberg group $\mathbb{H}^n$, which arises quite often in analysis. Its non-abelian character makes the group $\mathbb{H}^n$ quite different from $\mathbb{R}^d$.

Before describing the results obtained in [**9**] and [**30**], we quickly recall some basic facts about $\mathbb{H}^n$. We follow [**28**, pp. 531–545].

Let $n \ge 1$ be an integer. Viewed as a set, we identify $\mathbb{H}^n$ with

$$\mathbb{C}^n \times \mathbb{R} = \{[\zeta, t] \mid \zeta \in \mathbb{C}^n,\ t \in \mathbb{R}\},$$

with the usual additive operation. We endow $\mathbb{H}^n$ with the multiplicative operation "$\circ$" given by

$$[\zeta, t] \circ [\eta, s] := [\zeta + \eta, t + s + 2\,\mathrm{Im}(\zeta \cdot \overline{\eta})],$$

where

$$\zeta \cdot \overline{\eta} := \zeta_1 \overline{\eta_1} + \ldots + \zeta_n \cdot \overline{\eta_n}.$$

We also define a dilation on $\mathbb{H}^n$, different from the one on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$: if $\lambda > 0$ and $[\zeta, t] \in \mathbb{H}^n$ then we set:

$$\lambda[\zeta, t] := [\lambda\zeta, \lambda^2 t]. \tag{0.40}$$

The dilation is consistent with the operation "$\circ$", in the sense that

$$\lambda([\zeta, t] \circ [\eta, s]) = (\lambda[\zeta, t]) \circ (\lambda[\eta, s]).$$

One can verify that $\mathbb{H}^n$ endowed with the operation "$\circ$" is a non-abelian group, with identity $[0,0]$ and the inverse in given by the rule $[\zeta, t]^{-1} = [-\zeta, -t]$. It also turns out that $\mathbb{H}^n$ is a Lie group. Its Lie algebra $\mathfrak{h}^n$ is generated by the following $2n+1$ left-invariant vector fields:

$$X_j := \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t},\ \ Y_j := \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},\ \text{ for } j = 1, 2, \ldots, n,$$

and

$$T := \frac{\partial}{\partial t}.$$

Note that we have the commutation relations

$$[X_j, X_k] = [Y_j, Y_k] = 0, \text{ for } j,k = 1,2,...,n,$$

and

$$[Y_j, X_k] = 4\delta_{jk} T,$$

where $\delta_{jk} = 1$ if $k = j$, and $\delta_{jk} = 0$ if $k \neq j$. Notice that, if we are allowed to take commutators of vector fields and linear combinations, then the $2n$ vector fields $X_1, ..., X_n, Y_1, ..., Y_n$ are sufficient in order to generate the full Lie algebra $\mathfrak{h}^n$.

After introducing the above vector fields, we can now define homogeneous Sobolev spaces on $\mathbb{H}^n$ similar to the usual ones defined on $\mathbb{R}^d$. First, by identifying $\mathbb{H}^n$ with the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, any function f defined on $\mathbb{H}^n$ can be seen as a function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. With this identification, we will say that a function f is Schwartz on $\mathbb{H}^n$ if f is Schwartz on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Similarly, by distributions on $\mathbb{H}^n$ we mean distributions on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Also, for each $1 \le p \le \infty$ we let $L^p(\mathbb{H}^n)$ be the usual space $L^p(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$. Next, we define the seminorm of the Sobolev space $\dot{N}L^{k,p}(\mathbb{H}^n)$, where k is an nonnegative integer. For this purpose, we consider the *subgradient* on $\mathbb{H}^n$ defined by

$$\nabla_b := (X_1, ..., X_n, Y_1, ..., Y_n).$$

Then, the $\dot{N}L^{1,p}(\mathbb{H}^n)$-seminorm is given by (the possibly infinite quantity)

$$\|f\|_{\dot{N}L^{1,p}(\mathbb{H}^n)} := \|\nabla_b f\|_{L^p(\mathbb{H}^n)},$$

for any distribution f on $\mathbb{H}^n$.

For $k \ge 2$, the $\dot{N}L^{k,p}(\mathbb{H}^n)$-seminorm is given by the recurrence formula

$$\|f\|_{\dot{N}L^{k,p}(\mathbb{H}^n)} := \|\nabla_b f\|_{\dot{N}L^{k-1,p}(\mathbb{H}^n)},$$

for any distribution f on $\mathbb{H}^n$.

The function space $\dot{N}L^{k,p}(\mathbb{H}^n)$ consists of distributions on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ for which the $\dot{N}L^{k,p}(\mathbb{H}^n)$-seminorm is finite. E.g., we have

$$\dot{N}L^{1,p}(\mathbb{H}^n) := \{f \in \mathscr{D}'(\mathbb{H}^n) \mid \nabla_b f \in L^p\}.$$

In order to parallel the theory on $\mathbb{R}^d$, we next recall few facts related to differential forms on $\mathbb{H}^n$. We follow [**28**, pp. 594–595]. Let $d\overline{z_1}, ..., d\overline{z_n}$ be the basic $(0,1)$-forms on $\mathbb{H}^n$, where $z_j := x_j + i y_j$. If $I = \{j_1, ..., j_q\}$, with $1 \le j_1 < ... < j_q \le n$, we write

$$d\overline{z_I} := d\overline{z_{j_1}} \wedge ... \wedge d\overline{z_{j_q}}.$$

Suppose $1 \le q \le n$ is given. An expression of the form

$$\sum_{|I|=q} f_I d\overline{z_I},$$

where f_I are some complex-valued functions on $\mathbb{H}^n$, will be called $(0,q)$-form. We formally define the operator $\overline{\partial}_b$ by the relation

$$\overline{\partial}_b \left(\sum_{|I|=q} f_I d\overline{z_I} \right) := \sum_{j=1}^{n} \sum_{|I|=q} \overline{Z}_j (f_I) d\overline{z_j} \wedge d\overline{z_I},$$

where $\overline{Z}_j$ are the *left-invariant Cauchy-Riemann operators* defined by

$$\overline{Z}_j := \frac{\partial}{\partial \overline{z_j}} - i z_j \frac{\partial}{\partial t}.$$

Let $\overline{\partial}_b^*$ be the formal adjoint of $\overline{\partial}_b$. Thus $\overline{\partial}_b^*$ is characterized by the equality

$$\left\langle \overline{\partial}_b^* f, g \right\rangle := \left\langle f, \overline{\partial}_b g \right\rangle$$

for any smooth $(0,q)$-form f and any smooth $(0,(q-1))$-form g in $L^2(\mathbb{H}^n)$. Here, for any two $(0,q)$-forms φ and ψ in $L^2(\mathbb{H}^n)$, their scalar product is defined by

$$\langle\varphi,\psi\rangle := \sum_{|I|=q}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}}\varphi_I\psi_I dxdydt$$

Let $Q := 2n+2$. This number is the *homogeneous dimension* of $\mathbb{H}^n$, and differs from the dimension of $\mathbb{H}^n$, which is $2n+1$. As we will see below, this homogeneous dimension plays, to some extent, the role of the space dimension in the Euclidean setting.

To illustrate this, let us investigate the behaviour of the homogeneous space $\dot{N}L^{1,Q}(\mathbb{H}^n)$ under the action of the group of dilations. Suppose e.g. that f is a Schwartz function on $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}=\mathbb{R}^{2n}\times\mathbb{R}$ and $\lambda>0$ is given. We define f_λ by

$$f_\lambda(\zeta,t) := f(\lambda[\zeta,t]) = f\left([\lambda\zeta,\lambda^2 t]\right).$$

We have

$$\begin{aligned}\|f_\lambda\|^Q_{\dot{N}L^{1,Q}(\mathbb{H}^n)} &= \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}}\left|\nabla_b\left(f\left([\lambda\zeta,\lambda^2 t]\right)\right)\right|^Q d\zeta dt\\ &=\lambda^Q\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}}\left|\nabla_b f\left([\lambda\zeta,\lambda^2 t]\right)\right|^Q d\zeta dt\\ &=\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}}\left|\nabla_b f\left([\lambda\zeta,\lambda^2 t]\right)\right|^Q d(\lambda\zeta)d\left(\lambda^2 t\right)\\ &=\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}}|\nabla_b f([\zeta,t])|^Q d\zeta dt,\end{aligned}$$

i.e., $\|f_\lambda\|_{\dot{N}L^{1,Q}(\mathbb{H}^n)} = \|f\|_{\dot{N}L^{1,Q}(\mathbb{H}^n)}$. The same type of invariance holds in the case of $L^\infty(\mathbb{H}^n)$. From this point of view, the pair of spaces $\dot{N}L^{1,Q}(\mathbb{H}^n)$ and $L^\infty(\mathbb{H}^n)$ behave like their Euclidean counterparts: $\dot{W}^{1,d}(\mathbb{R}^n)$ and $L^\infty\left(\mathbb{R}^d\right)$. We will see that this is not a coincidence.

We have the following analogue of Theorem 0.10, which can be deduced from the work [**9**] of Chanillo and Van Schaftingen (see [**32**]).

THEOREM 0.16. *Suppose $n\geq 3$ is an integer and consider $Q := 2n+2$. Let r be an integer with $1\leq r<n-1$. For any $(0,r)$-form φ in $\dot{N}L^{1,Q}(\mathbb{H}^n)$, there exists a $(0,r)$-form Y in $L^\infty(\mathbb{H}^n)$ such that*

$$\bar{\partial}_b^* Y = \bar{\partial}_b^*\varphi$$

and

$$\|Y\|_{L^\infty(\mathbb{H}^n)} \lesssim \left\|\bar{\partial}_b^*\varphi\right\|_{L^Q(\mathbb{H}^n)}.$$

Note that, in the case of $\mathbb{H}^n$, the critical homogeneous Sobolev space $\dot{N}L^{1,Q}(\mathbb{H}^n)$ plays the same role as $\dot{W}^{1,d}\left(\mathbb{R}^d\right)$ plays in the case of $\mathbb{R}^d$.

Following the ideas in [**4**], Wang and Yung proved in [**30**] an analogue of Theorem 0.12 for the case of stratified homogeneous groups.[3] Adapted to $\mathbb{H}^n$, their result reads as follows.

THEOREM 0.17. *Suppose $J_1,J_2\subset\{1,\dots,n\}$ are two nonempty sets such that $|J_1|+|J_2|\leq 2n-1$. Then, for any Schwartz function f on $\mathbb{H}^n$ and any $\delta>0$ there exists a function F such that:*

$$\sum_{j\in J_1}\left\|X_j(f-F)\right\|_{L^Q(\mathbb{H}^n)} + \sum_{j\in J_2}\left\|Y_j(f-F)\right\|_{L^Q(\mathbb{H}^n)} \leq \delta\|\nabla_b f\|_{L^Q(\mathbb{H}^n)}$$

and

$$\|F\|_{L^\infty(\mathbb{H}^n)} + \|\nabla_b F\|_{L^Q(\mathbb{H}^n)} \leq C_\delta\|\nabla_b f\|_{L^Q(\mathbb{H}^n)},$$

where C_δ is a constant depending only on δ.

[3] The definition and some important properties of these groups, which include the Heisenberg group $\mathbb{H}^n$, will be recalled in Chapter 5.

The iterative method used in conjunction with Theorem 0.17 leads to the following improvement of Theorem 0.16.

THEOREM 0.18. *Suppose $n \geq 3$ is an integer. Let r be an integer with $1 \leq r < n-1$. For any $(0,r)$-form φ in $\dot{N}L^{1,Q}(\mathbb{H}^n)$, there exists a $(0,r)$-form Y in $L^\infty(\mathbb{H}^n) \cap \dot{N}L^{1,Q}(\mathbb{H}^n)$ such that*

$$\overline{\partial}_b^* Y = \overline{\partial}_b^* \varphi$$

and

$$\|Y\|_{L^\infty(\mathbb{H}^n)} + \|Y\|_{\dot{N}L^{1,Q}(\mathbb{H}^n)} \lesssim \left\|\overline{\partial}_b^* \varphi\right\|_{L^Q(\mathbb{H}^n)}.$$

Note that Theorem 0.17 and Theorem 0.12 only concern functions of differential regularity one. We will study in Chaper **5** higher order analogues of these results, in the more general context of stratified homogeneous groups.

3. Short description of the main contributions of the thesis

This manuscript is based on the following articles:

1. *On the existence of vector fields with nonnegative divergence in rearrangement-invariant spaces*, Indiana Univ. Math. J. 69, 87-104, 2020.
   This will form the content of Chapter 1.
2. *On the representation as exterior differentials of closed forms with L^1-coefficients*, C. R. Math. Acad. Sci. Paris, 357(4) :355-359, 2019.
   This will form the content of Chapter 2.
3. *The divergence equation with L^∞ source*, accepted at Annales de la Faculté des Sciences de Toulouse.
   This will form the content of Chapter 4.
4. *Approximation of critical regularity functions on stratified homogeneous groups*, accepted at Communications in Contemporary Mathematics.
   This will form the content of Chapter 5.
5. *Minimal BV-liftings of $W^{1,1}(\Omega,\mathbb{S}^1)$ maps in 2D are "often" unique*, in press at Nonlinear Analysis.
   This will form the content of Chapter 7.
6. Chapters 3 and 6 are original contributions that will not be published elsewhere.
7. Chapter 8 is the basis of a manuscript in preparation.

Part I. Hodge systems with "pathological" source terms. In this part, we study underdetermined Hodge systems whose source terms are in L^1 or L^∞, or are nonnegative measures. Many of the results that we obtain are negative results, concluding to the nonexistence of solutions with the maximal expected regularity. We also present several positive existence results, of slightly rougher solutions, that illustrate the sharpness of the nonexistence results.

Chapter 1. In this chapter, our goal was to generalise Theorem 0.8, by replacing the L^p spaces with more general *rearrangement-invariant spaces* (r. i. for short). Without providing here a definition of these spaces, we mention few examples of widely used function spaces that are r. i.: the Lebesgue spaces L^p, the Lorentz spaces $L^{p,q}$ ($1 \leq p < \infty$, $1 \leq q \leq \infty$) and the Orlicz spaces $\Phi(L)$.

Our first result is the following.

THEOREM 0.19. *Let μ be a nonnegative Radon measure on $\mathbb{R}^d$, and X an r. i. space of functions on $\mathbb{R}^d$ such that $|x|^{1-d}\mathbb{1}_{B^c}$ does not belong to X. If the equation* $\operatorname{div} F = \mu$ *has a solution $F \in X(\mathbb{R}^d,\mathbb{R}^d)$, then $\mu \equiv 0$.*

(Here, B^c is the complement of the unit ball and $\mathbb{1}_{B^c}$ is its characteristic function.)

Let us observe that in the case where $X = L^p$, the condition "$|x|^{1-d}\mathbb{1}_{B^c}$ does not belong to X" is equivalent to the fact that $1 \le p \le d'$, and thus Theorem 0.8 follows from Theorem 0.19. Indeed, we have

$$\int_{\mathbb{R}^d} \left| |x|^{1-d} \mathbb{1}_{B^c}(x) \right|^p dx = \int_{B^c} \frac{1}{|x|^{(d-1)p}} dx,$$

and this integral is divergent if and only if $1 \le p \le d'$.

The proof of Theorem 0.19 is elementary and uses only basic properties of the r. i. spaces. The main argument relies on a decomposition of $\mathbb{R}^d$ in dyadic shells and is similar to the argument we presented above, leading to the proof of Theorem 0.8.

Furthermore, we show that the condition "$|x|^{1-d}\mathbb{1}_{B^c}$ does not belong to X" in the above theorem is sharp. Indeed, let ϕ be a non trivial nonnegative function in $L^\infty_c(\mathbb{R}^d)$ and set $\mu := \phi m$ (where m is the Lebesgue measure), so that μ is a non trivial positive measure. If $|x|^{1-d}\mathbb{1}_{B^c} \in X$, then we prove that the equation $\operatorname{div} F = \mu$ has a solution F in $X(\mathbb{R}^d, \mathbb{R}^d)$.

Next, we are interested in obtaining some explicit (i.e., we construct F) and quantitative (i.e., we estimate F) versions of Theorem 0.19. In this direction, we obtain the following result.

THEOREM 0.20. *Let X be a r. i. space of functions on $\mathbb{R}^d$ such that $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$. If (0.21) has a solution $F \in X(\mathbb{R}^d, \mathbb{R}^d)$, then $I_1\mu \in X$. Moreover, there exists a constant $C_X > 0$ only depending on X such that*

$$\left\| I_1\mu \right\|_X \le C_X \left\| F \right\|_X. \tag{0.41}$$

Here, $\underline{\alpha}_X$ and $\overline{\alpha}_X$ are the Boyd indexes of X; their definition will be recalled in Chapter 1. We mention that in the case $X = L^p$ both Boyd indexes of X are equal to $1/p$. On the other hand, I_1 is the 1-Riesz potential, whose action is given by

$$I_1\mu(x) := \int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^{d-1}}.$$

Note that (0.41) is a *lower* bound for F.

The latter result is obtained by following the ideas in Phuc and Torres in [**24**]. In our case, we rely on the Calderón-Zygmund theory for r. i. spaces. Formally, we have from (0.21) that

$$I_1\mu = R_1F_1 + \ldots + R_dF_d, \tag{0.42}$$

where $R_1, \ldots, R_d$ are the Riesz transforms on $\mathbb{R}^d$. The heart of the proof consists of justifying (0.42); this can be achieved under the assumptions on F and on the Boyd indexes.

We also establish a partial converse of Theorem 0.20.

THEOREM 0.21. *Let X be a r. i. space of functions on $\mathbb{R}^d$ with the property that whenever μ is a signed Radon measure on $\mathbb{R}^d$ with $\mu = \operatorname{div} F$ for a vector field $F \in X(\mathbb{R}^d, \mathbb{R}^d)$, we have that $I_1\mu^+$, $I_1\mu^-$ are finite a.e., $I_1\mu \in X$ and $\left\| I_1\mu \right\|_X \le C_X \left\| F \right\|_X$ for a positive constant C_X. Then $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$.*

The above theorem is quite technical and relies on the properties of the Calderón operator (for a definition see Chapter 1).

A common difficulty related to the proofs of the results in this chapter is the lack of explicit expressions for the r. i. norms.

Chapters 2 and 3. We consider the Hodge system

$$d\lambda = f, \text{ in } \mathbb{R}^d, \tag{0.43}$$

where f and λ are l and $(l-1)$-forms respectively, with f given and satisfying the compatibility condition $df = 0$. We focus on the case where f has L^1 coefficients.

In the case $l = d$, (0.43) becomes the divergence equation. It was first shown by Wojciechowski in [**31**] that there exists $g \in L^1_c(\mathbb{R}^d)$, with zero integral, such that the equation $\operatorname{div} Y = g$ has no solution $Y \in W^{1,1}_{loc}(\mathbb{R}^d;\mathbb{R}^d)$.

In Chapters 2 and 3, we prove a similar nonexistence result for all the Hodge systems for which $1 < l \le d$: there exists an l-form $f \in L^1_c$ with $df = 0$ such that there is no $(l-1)$-form $\lambda \in W^{1,1}_{loc}$ satisfying (0.43). Both proofs are based on reducing the problem to the case of the divergence equation. Roughly speaking, we deal in both proofs with assertions of the following form. There exists a subspace V of $\mathbb{R}^d$ such that any $g \in L^1_c(V)$ can be written as

$$\operatorname{div} Y = g + R, \text{ on } V \tag{0.44}$$

for some $Y \in W^{1,1}_{loc}(V)$ and a remainder term R. The idea is to show that this remainder is negligible in some sense and eventually can be eliminated. In other words, we reduce (0.44) to

$$\operatorname{div} Y' = g, \text{ on } V$$

for some $Y' \in W^{1,1}_{loc}$. After showing this, we can use the above negative result for the divergence equation in order to get a contradiction.

The proof in Chapter 2 is elementary. It uses, as a key tool, the hypoellipticity of the Laplacian. We show in this case that the remainder R is C^2_c. The proof in Chapter 3 is less elementary, however, it is more compact. It uses, as a key tool, the boundedness of the Calderón-Zygmund operators on the homogeneous Besov spaces. In this case we show that a Besov norm of R is small and by a limiting argument we conclude that R can be eliminated. (The Appendix of Chapter 3 contains another proof of the same nonexistence result, which is more elementary than the previous ones. The proof is based on a “compactness” argument which reduces the problem to its easier version on $\mathbb{T}^d$.)

On the other hand, we mention that Bourgain and Brezis proved in [**3**] the following stronger nonexistence result for the divergence equation: there exists $g \in L^1_c(\mathbb{R}^d)$ with zero integral, such that the equation $\operatorname{div} Y = g$ has no solution $Y \in L^{d/(d-1)}_{loc}(\mathbb{R}^d;\mathbb{R}^d)$. In view of the embedding $W^{1,1}_{loc} \hookrightarrow L^{d/(d-1)}_{loc}$, this improves the result of Wojciechowski ([**31**]).

We show in Chapter 2 that an analogous result in the case $1 \le l < d$ does not hold. More precisely, if $1 \le l < d$, then for any l-form $f \in L^1_c$ with $df = 0$ there exists an $(l-1)$-form $\lambda \in L^{d'}_{loc}$ satisfying (0.43). This result is a direct consequence of (0.35).

Chapter 4. In this chapter, we come back to the divergence equation. We give a new proof of the following classical result of Preiss and McMullen: there exists $g \in L^\infty(\mathbb{T}^d)$, with zero integral, such that the equation $\operatorname{div} Y = g$ has no solution $Y \in W^{1,\infty}(\mathbb{T}^d;\mathbb{R}^d)$. Our proof is based on the Riesz products technique introduced by Wojciechowski in [**31**] for the study of the divergence equation with L^1 sources. We show that his idea is also suitable, after minor modifications and simplifications, in the case of L^∞ sources. Our proof is short, more elementary than the one in [**31**] and yields a significant improvement of the above mentioned result of Preiss and McMullen, which does not seem to be attainable with their respective methods.

More specifically, we introduce the function spaces S_λ defined on $\mathbb{T}^2$ as follows:

$$S_\lambda(\mathbb{T}^2) := \left\{ f \in \mathscr{D}'(\mathbb{T}^2) \,\middle|\, \sup_{n \in \mathbb{Z}^2} \frac{|\hat{f}(n)|}{\lambda(|n|)} < \infty \right\},$$

where $\lambda : \mathbb{N} \to (0,\infty)$ is a given decreasing function such that $\lambda(k) \to 0$ when $k \to \infty$.

Our result is the following.

THEOREM 0.22. *Suppose $\lambda : \mathbb{N} \to (0,\infty)$ is decreasing to 0. There exists $g \in L^\infty(\mathbb{T}^2)$ such that there are no f_0, f_1, $f_2 \in S_\lambda(\mathbb{T}^2)$ with $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$ and*

$$g = f_0 + \partial_1 f_1 + \partial_2 f_2.$$

In order to give an idea about the meaning of this result, let us fix a small $\varepsilon > 0$. We see immediately that the Sobolev space $H^\varepsilon(\mathbb{T}^2)$ is embedded in $S_\lambda(\mathbb{T}^2)$ for $\lambda(|n|) = 1/(1+|n|)^\varepsilon$. Our result implies that even the weak regularity condition f_0, f_1, $f_2 \in H^\varepsilon(\mathbb{T}^2)$, $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$ rules out the existence of a solution for the above equation with general L^∞ source g. Intuitively speaking, we do not have, in general, a solution of this equation such that f_0, f_1 and f_2 are even "slightly better than L^1".

Part II. Hodge systems in critical function spaces. In this part we study underdetermined Hodge systems for which classical regularity theory provides solutions in Sobolev spaces which are "critical for the Sobolev embedding", in the sense that they "almost" embed into L^∞. We show that in this case it is possible to obtain solutions that are both bounded and with the expected Sobolev regularity. In the first chapter of this part, we study an approximation property of functions on stratified homogeneous groups. This property implies the above mentioned existence result for Hodge systems. In the second chapter, we prove the existence of bounded and critically smooth solutions to Hodge systems with Dirichlet boundary conditions on Euclidean domains.

Chapter 5. The purpose of this chapter is to find a common roof to Theorem 0.17 and Theorem 0.14 and to give an affirmative answer to Open question 1.4 in [**6**].

Following closely the ideas in [**30**], we define some natural homogeneous spaces of Triebel-Lizorkin type on stratified homogeneous groups. We mention that, in the non Euclidean setting, it is common to have different, non equivalent definitions of function spaces. Spaces similar to ours were already introduced in the literature (see, for example [**16**]). However, our proofs concerning the properties of these spaces are more elementary and also their construction is more flexible than the previous one, and well adapted to our purposes.

Following the proof structure in [**6**], we were able to prove an approximation result very similar to Theorem 0.14 in the context of stratified homogeneous groups and concerning the Triebel-Lizorkin spaces that we have introduced. This generalizes both results of Theorem 0.14 and Theorem 0.17.

The definition and some basic properties of stratified homogeneous groups will be given in Chapter 5. For the sake of simplicity, we specialize here to the "concrete" case of the Heisenberg group $\mathbb{H}^n$.

To start with, we sketch our definition of homogeneous Triebel-Lizorkin spaces on $\mathbb{H}^n$. Let $n \geq 1$ and let $Q := 2n+2$ be the homogeneous dimension of $\mathbb{H}^n$. If Λ is a Schwartz function on $\mathbb{H}^n$ and j is an integer, we write Λ_j for the function given by

$$\Lambda_j(x) := 2^{jQ}\Lambda(2^j x), \quad x \in \mathbb{H}^n.$$

Recall that here, $2^j x$ is the group dilation of x with the factor 2^j, i.e., if $x = [\zeta, t] \in \mathbb{H}^n$, then $2^j x = \left[2^j\zeta, 2^{2j}t\right]$ (see (0.40)).

Fix $s \geq 0$, $p, q \in (1,\infty)$. We define the space $\dot{F}^{s,p}_q(\mathbb{H}^n)$ as being formed by the tempered distributions f on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$ for which the following seminorm

$$\|f\|_{\dot{F}^{s,p}_q(\mathbb{H}^n)} := \left\| \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \left|\Lambda^1_j f\right|^q \right)^{1/q} \right\|_{L^p} \tag{0.45}$$

is finite. Here, $\Lambda^1 := \left(\Lambda^{1,a}\right)_{a\in A}$ is an adapted finite family of Schwartz functions on $\mathbb{R}^{2n+1}$; its construction is part of the theory (see Proposition 0.23 below). The quantity $\left|\Lambda^1_j f\right|$ is defined by

$$\left|\Lambda^1_j f\right| := \sum_{a\in A} \left|\Lambda^{1,a}_j f\right|,$$

where

$$\Lambda^{1,a}_j f(x) := \int_{\mathbb{R}^{2n+1}} f(x \circ y^{-1}) \Lambda^{1,a}_j(y)\,dy,$$

for all $a \in A$. In other words, $\Lambda_j^{1,a} f = f * \Lambda_j^{1,a}$, where "$*$" is the convolution on the group $\mathbb{H}^n$.

The definition (0.45) is very similar to the definition of classical homogeneous Triebel-Lizorkin spaces on $\mathbb{R}^d$ via Littlewood-Paley decomposition formula. However, in our situation, the existence of a Littlewood-Paley decomposition having all the expected properties is a delicate matter.

The existence of the above family $\Lambda^1$is a consequence of the following result.

PROPOSITION 0.23. *Given $m \in \mathbb{N}$, there exist finite Schwartz families $\Lambda^1 = (\Lambda^{1,a})_{a\in A}$, $\Lambda^2 = (\Lambda^{2,a})_{a\in A}$, $\Lambda^3 = (\Lambda^{3,a})_{a\in A}$ on $\mathbb{R}^{2n+1}$ such that, for all $a \in A$,*

$$\int_{\mathbb{R}^{2n+1}} P(x)\Lambda^{1,a}(x)dx = \int_{\mathbb{R}^{2n+1}} P(x)\Lambda^{2,a}(x)dx = \int_{\mathbb{R}^{2n+1}} P(x)\Lambda^{3,a}(x)dx = 0,$$

for all the polynomials P of degree $\leq m$ and such that for all Schwartz functions f we have

$$f = \sum_{j\in\mathbb{Z}} \sum_{a\in A} f * \Lambda_j^{1,a} * \Lambda_j^{2,a} * \Lambda_j^{3,a}$$

the convergence being in any $L^p(\mathbb{R}^{2n+1})$ for any $1 < p < \infty$.

When we define, via (0.45), spaces $\dot{F}_q^{s,p}(\mathbb{H}^n)$ of regularity $s \geq 0$, Λ^1 is as in the above proposition, and m is any integer $> s$.

This leaves the possibility that these spaces depend on Λ^1 and m. It turns out that *this is not the case*: any triple of families Λ^1, Λ^2, Λ^3 and any integer $m > s$ as in the above proposition will lead to the definition of the same space $\dot{F}_q^{s,p}(\mathbb{H}^n)$.

Also, we mention that, whenever k is a nonnegative integer and $1 < p < \infty$ we have the pleasant identity, reminiscent of the famous square function theorem in the Euclidean case:

$$\dot{F}_2^{k,p}(\mathbb{H}^n) = \dot{N}L^{k,p}(\mathbb{H}^n),$$

with equivalent seminorms. This is a key identity that permits us to view our approximation result as a generalization of Theorem 0.17.

The use of a decomposition formula with three convolutions, as the one above, turns out to be very convenient. It enables us to handle the estimates required in the proof of our approximation result.

On $\mathbb{H}^n$, our main result reads as follows.

THEOREM 0.24. *Let $Q := 2n+2$ and consider the parameters $1 < p, q < \infty$, $\alpha := Q/p$. Suppose $J_1, J_2 \subset \{1, ..., n\}$ are two nonempty sets such that $|J_1| + |J_2| < \min(p, 2n)$. Then, for any Schwartz function f on $\mathbb{H}^n$ and any $\delta > 0$ there exists a function F such that:*

$$\sum_{j\in J_1} \|X_j(f-F)\|_{\dot{F}_q^{\alpha-1,p}(\mathbb{H}^n)} + \sum_{j\in J_2} \|Y_j(f-F)\|_{\dot{F}_q^{\alpha-1,p}(\mathbb{H}^n)} \leq \delta \, \|\nabla_b f\|_{\dot{F}_q^{\alpha,p}(\mathbb{H}^n)}$$

and

$$\|F\|_{L^\infty(\mathbb{H}^n)} + \|F\|_{\dot{F}_q^{\alpha,p}(\mathbb{H}^n)} \leq C_\delta \, \|\nabla_b f\|_{\dot{F}_q^{\alpha,p}(\mathbb{H}^n)},$$

where C_δ is a constant depending only on δ.

Note that, if we let $\alpha := 1$ and $p := Q$ in the above theorem, we recover Theorem 0.17.

In general, when proving Theorem 0.24, apart from the difficulties that were already present in the Euclidean case, the problems that arise are related to the noncommutativity of convolutions and the noncommutativity of the vector fields.

Chapter 6. We prove a version of Theorem 0.15 for the exterior differential operator with Dirichlet boundary condition on smooth bounded domains. More precisely, we obtain the following result.

THEOREM 0.25. *Let Ω be a smooth bounded domain in $\mathbb{R}^d$. Let $1 \le l \le d-2$ be an integer and consider the parameters $d-l < p < \infty$, $1 < q < \infty$, $\alpha := d/p$. Suppose $\gamma \in C(\partial\Omega) \cap B_p^{\alpha-1/p,p}(\partial\Omega)$ is an l-form and $v \in F_q^{\alpha,p}(\Omega)$ is an l-form satisfying $\nu \wedge dv = \nu \wedge d\gamma$ on $\partial\Omega$. Then, there exists an l-form $u \in C(\overline{\Omega}) \cap F_q^{\alpha,p}(\Omega)$ such that*

$$\begin{cases} du = dv, & \text{on } \Omega \\ u = \gamma, & \text{on } \partial\Omega \end{cases}.$$

Moreover, u can be chosen such that

$$\|u\|_{L^\infty(\Omega)} + \|u\|_{F_q^{\alpha,p}(\Omega)} \lesssim \|\gamma\|_{L^\infty(\partial\Omega)} + \|\gamma\|_{B_p^{\alpha-1/p,p}(\partial\Omega)} + \|v\|_{F_q^{\alpha,p}(\Omega)}.$$

Here, ν is the 1-form on $\partial\Omega$ given by

$$\nu = \sum_{j=1}^{d} \nu_j dx_j,$$

where the vector $\nu = (\nu_1, ..., \nu_d)$ is the outward unit normal to $\partial\Omega$.

We note that the compatibility condition $\nu \wedge dv = \nu \wedge d\gamma$ in the sense of distributions on $\partial\Omega$ is meaningful.

Theorem 0.25 extends the result of Theorem 0.7 to other Hodge systems. The method we use is adapted after the method used in [**5**, Section 7].

Part III. Miscellaneous. In the first chapter, we investigate the uniqueness of minimal liftings of Sobolev mappings with values into the unit circle. We prove that, in 2D, minimal liftings are "generically" unique. In the second chapter, investigate the properties of the Fourier multipliers on the homogeneous Sobolev space $\dot{W}^{l,1}$.

Chapter 7. In this chapter we study the equation

$$u = e^{i\varphi} \text{ on } \Omega. \tag{0.46}$$

Here, Ω is a smooth, bounded and simply connected domain in $\mathbb{R}^2$ and $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ is given. It is known that there exists a BV-lifting of u on Ω, i.e., there exists $\varphi \in BV(\Omega, \mathbb{R})$ satisfying (0.46) (for example see [**8**]). A BV-lifting with minimal BV-seminorm is called a minimal BV-lifting.

We are going to answer the following question raised in [**8**]: is the set of functions $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ which admit a unique $(\bmod 2\pi)$ minimal BV-lifting, residual in $W^{1,1}(\Omega, \mathbb{S}^1)$?

We prove that more is true: roughly speaking, most of the functions $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ with a fixed number of singularities have unique $(\bmod 2\pi)$ minimal BV-lifting. More precisely, we establish the following result.

THEOREM 0.26. *Suppose Ω is a smooth, bounded and simply connected domain in $\mathbb{R}^2$. Let k be a positive integer. The set of vectors $a = (a_1, ..., a_k) \in \Omega^k$ for which each $u \in W^{1,1}(\Omega, \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, ..., a_k\})$ admits a unique minimal BV-lifting is of full measure in Ω^k.*

We prove this result by reducing the problem to a geometrical one. For each u as above we spot a set of "geometrical structures" that determine whether or not u has a minimal lifting. Thanks to a sort of rigidity of these "structures", they fail to be "good" only in few cases. In this way Theorem 0.26 reduces to the following elementary looking fact:

PROPOSITION 0.27. *Let $\Omega \subset \mathbb{R}^2$ be an open set such that $\Omega \neq \emptyset, \mathbb{R}^2$, and $k \in \mathbb{N}^*$. For almost all $X = (x_1, ..., x_k) \in \Omega^k$ we have that the numbers $\operatorname{dist}(x_i, \partial\Omega)$, $1 \le i \le k$ and $|x_i - x_j|$, $1 \le i < j \le k$ are linearly independent over $\mathbb{Z}$ and each x_i has a unique projection on $\partial\Omega$.*

The main idea that enters in the proof of this fact is to show that each nontrivial linear combination over $\mathbb{Z}$ of the elements $\operatorname{dist}(x_i, \partial\Omega)$ and $|x_i - x_j|$, as above, cannot vanish apart from a Lebesgue null set in Ω^k. Once we have this, we can obtain Proposition 0.27 by using the fact that $\mathbb{Z}^N$, where $N := k + \binom{k}{2}$, is a countable set.

Chapter 8. We study the continuity of multipliers of some "pathological" homogeneous Sobolev spaces. Suppose $l \geq 0$ is an integer and $1 \leq p \leq \infty$. A measurable function $m : \mathbb{R}^d \to \mathbb{C}$ is a Fourier multiplier on $\dot{W}^{l,p}(\mathbb{R}^d)$ if there exists a bounded operator

$$T_m : \dot{W}^{l,p}\left(\mathbb{R}^d\right) \to \dot{W}^{l,p}\left(\mathbb{R}^d\right),$$

such that

$$\widehat{T_m f} = m\widehat{f},$$

for any Schwartz function f on $\mathbb{R}^d$.

Bonami and Poornima [**25**] proved in 1982 that the only Fourier multipliers on $\dot{W}^{1,1}(\mathbb{R}^d)$ which are homogeneous functions of degree zero are the constant functions. More precisely, they proved the following (see [**25**, Theorem 2.9]):

THEOREM 0.28. *Suppose $d \geq 2$ and let Ω be a continuous function on $\mathbb{R}^d \setminus \{0\}$, homogeneous of degree zero. If Ω is a Fourier multiplier on $\dot{W}^{1,1}(\mathbb{R}^d)$, then Ω is a constant.*

This result was generalized by Kazaniecki and Wojciechowski in 2013 as follows (see [**17**, Theorem 1.1]):

THEOREM 0.29. *Suppose $d \geq 2$. If m is a Fourier multiplier on $\dot{W}^{1,1}(\mathbb{R}^d)$, then $m \in C_b(\mathbb{R}^d)$.*

We follow the ideas in [**17**] in order to prove a generalisation of this theorem for the case of $\dot{W}^{l,1}(\mathbb{R}^d)$, where $l \geq 1$. We also deal with multipliers on $\dot{W}^{l,\infty}(\mathbb{R}^d)$.

Our results are the following:

THEOREM 0.30. *Suppose $d \geq 2$ and $l \geq 1$ are some integers. If m is a Fourier multiplier on $\dot{W}^{l,\infty}(\mathbb{R}^d)$, then $m \in C_b(\mathbb{R}^d)$.*

THEOREM 0.31. *Suppose $d \geq 2$ and $l \geq 1$ are some integers. If m is a Fourier multiplier on $\dot{W}^{l,1}(\mathbb{R}^d)$, then $m \in C_b(\mathbb{R}^d)$.*

As in [**17**], our proof relies on Riesz products. However, our approach is more elementary than the one in [**17**].

The methods we use here are reminiscent of those in Chapter 4. For example, if m is a multiplier on $\dot{W}^{l,1}$, we prove that for any bounded continuous function f there exists functions $(g_\alpha)_{|\alpha|=l}$ in L^∞ such that

$$\partial_1^l T_m f = \sum_{|\alpha|=l} \nabla^\alpha g_\alpha,$$

in the sense of distributions. In order to prove Theorem 0.31, we show that, if m is not a continuous function, then there exists f as above for which this equation does not have solutions.

4. In short

In this manuscript, we study the existence of solutions to Hodge systems when the source term belongs to various function spaces. The Hodge systems are in general underdetermined and, in this framework, we mainly study two types of problems. First one is to decide whether or not the Hodge systems admit solutions with the expected regularity when the source term is "pathological" in some sense. By this we mean the situations in which standard Calderón-Zygmund theory cannot be applied. Secondly, we are interested in finding solutions more regular than the solutions provided by Calderón-Zygmund theory. Roughly speaking, when the standard theory provides a solution in a critical Sobolev space, we aim to obtain a solution which is simultaneously in this critical Sobolev space and bounded.

One particular case of Hodge system is the divergence equation on the Euclidean space. For this equation, the first problem addressed was the existence of solutions in $W^{1,p}_{loc}$, with $p = 1$ or ∞, when the source is in L^p_{loc}. The answer is negative both in L^1 (Wojciechowski, Bourgain-Brezis) and in L^∞ (Preiss, McMullen). We show similar results for more general Hodge systems. Also, we obtain a substantial improvement of the nonexistence result for the divergence equation with L^∞ sources.

Concerning the second type of problem, the answer was known to be positive, in the Euclidean case, for a large class of Hodge systems and critical Sobolev-type spaces. This was possible thanks to a new type of approximation result of functions in critical Sobolev spaces. This approximation result was proved by Bourgain and Brezis, for spaces of regularity one, and extended to higher regularity spaces by Bousquet, Russ, Wang and Yung. The regularity one case was also settled in the framework of stratified homogeneous groups by Wang and Yung. We prove a similar approximation result in the general setting of stratified homogeneous groups for functions in critical Sobolev-type spaces defined on these groups. For this purpose we define homogeneous spaces of Triebel-Lizorkin type on stratified homogeneous groups that are similar to the classical ones defined on the Euclidean space.

In a different direction, we study the existence of solutions F of $\operatorname{div} F = \mu$ in rearrangement-invariant spaces, when the source μ is a nonnegative Radon measure. Our results generalize previous ones of Phuc and Torres, obtained for L^p spaces.

We also investigate the uniqueness of minimal BV-liftings of $W^{1,1}(\Omega, \mathbb{S}^1)$ maps. Here, $\mathbb{S}^1$ is the unit circle and Ω is a 2-dimensional smooth, bounded, simply connected domain. It is well-known that each map in $W^{1,1}(\Omega, \mathbb{S}^1)$ has BV-liftings. We prove that "almost all" the maps in $W^{1,1}(\Omega, \mathbb{S}^1)$ have unique minimal BV-liftings.

Finally, we study some properties of the Fourier multipliers on the homogeneous Sobolev spaces $\dot{W}^{l,1}$ and $\dot{W}^{l,\infty}$.

5. Some notation concerning the function spaces used

Apart for some very common notation, we use also the following:

1. $C_b(\mathbb{R}^d)$ is the space of the continuous bounded functions on $\mathbb{R}^d$.
2. $C_c^\infty(\mathbb{R}^d)$ is the space of compactly supported C^∞ functions on $\mathbb{R}^d$.
3. $L^p_c(\mathbb{R}^d)$ (with $1 \le p \le \infty$) is the space of compactly supported L^p functions on $\mathbb{R}^d$.
4. $\dot{W}^{k,p}(\mathbb{R}^d)$ (with $k \ge 0$ is an integer and $1 \le p \le \infty$) is the homogeneous Sobolev space consisting of those distributions f on $\mathbb{R}^d$ for which $\nabla^k f \in L^p(\mathbb{R}^d)$. The space $\dot{W}^{k,p}(\mathbb{R}^d)$ is endowed with the following seminorm
$$\|f\|_{\dot{W}^{k,p}(\mathbb{R}^d)} := \left\|\nabla^k f\right\|_{L^p(\mathbb{R}^d)}.$$
5. Consider a radial function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}\varphi \subset B(0,2)$ and $\varphi \equiv 1$ on $B(0,1)$. For $j \in \mathbb{Z}$ we define the operators P_j, acting on the space of tempered distributions on $\mathbb{R}^d$, by the relation
$$\widehat{P_j f}(\xi) := \left(\varphi\left(\frac{\xi}{2^j}\right) - \varphi\left(\frac{\xi}{2^{j-1}}\right)\right)\widehat{f}(\xi),$$
for any Schwartz function f on $\mathbb{R}^d$. We will also consider the operator $P_{\le 0}$ defined by
$$\widehat{P_{\le 0} f}(\xi) := \varphi(\xi)\widehat{f}(\xi)$$
for any Schwartz function f on $\mathbb{R}^d$. The operators $P_{\le 0}$, P_j will be called *Littlewood-Paley "projections" adapted to* $\mathbb{R}^d$. For any Schwartz function f we have that
$$f = \sum_{j \in \mathbb{Z}} P_j f,$$

in the sense of tempered distributions.

6. $F_q^{s,p}\left(\mathbb{R}^d\right)$ (with $1 \le p,q < \infty$ and s a real number) is the inhomogeneous Triebel-Lizorkin space consisting of those tempered distributions f on $\mathbb{R}^d$ for which the following norm is finite.
$$\|f\|_{F_q^{s,p}(\mathbb{R}^d)} := \|P_{\le 0}f\|_{L^p(\mathbb{R}^d)} + \left\| \left(\sum_{j\ge 0} 2^{sjq} \left|P_j f\right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} .$$
A remarkable fact is that, if $k \ge 0$ is an integer and $1 < p < \infty$, then $F_2^{k,p}\left(\mathbb{R}^d\right) = W^{k,p}(\mathbb{R}^d)$ with equivalent norms.

7. $B_q^{s,p}\left(\mathbb{R}^d\right)$ (with $1 \le p,q \le \infty$ and s a real number) is the inhomogeneous Besov space consisting of those tempered distributions f on $\mathbb{R}^d$ for which the following seminorm is finite.
$$\|f\|_{B_q^{s,p}(\mathbb{R}^d)} := \|P_{\le 0}f\|_{L^p(\mathbb{R}^d)} + \left(\sum_{j\ge 0} 2^{sjq} \left\|P_j f\right\|^q_{L^p(\mathbb{R}^d)} \right)^{1/q} .$$

8. $\dot{F}_q^{s,p}\left(\mathbb{R}^d\right)$ (with $1 \le p,q < \infty$ and s a real number) is the homogeneous Triebel-Lizorkin space consisting of those tempered distributions f on $\mathbb{R}^d$ for which the following seminorm is finite.
$$\|f\|_{\dot{F}_q^{s,p}(\mathbb{R}^d)} := \left\| \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \left|P_j f\right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} .$$
A remarkable fact is that, if $k \ge 0$ is an integer and $1 < p < \infty$, then $\dot{F}_2^{k,p}\left(\mathbb{R}^d\right) = \dot{W}^{k,p}(\mathbb{R}^d)$ with equivalent norms.

9. $\dot{B}_q^{s,p}\left(\mathbb{R}^d\right)$ (with $1 \le p,q \le \infty$ and s a real number) is the homogeneous Besov space consisting of those tempered distributions f on $\mathbb{R}^d$ for which the following seminorm is finite.
$$\|f\|_{\dot{B}_q^{s,p}(\mathbb{R}^d)} := \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \left\|P_j f\right\|^q_{L^p(\mathbb{R}^d)} \right)^{1/q} .$$

10. One can also define Littlewood-Paley operators P_j in the case of $\mathbb{T}^d$. For each $j \in \mathbb{N}$ we define the operators P_j, given by
$$\widehat{P_j f}(n) := \mathbb{1}_{\{2^{j-1} \le |n|_\infty < 2^j\}}(n)\widehat{f}(n),$$
for any distribution f on $\mathbb{T}^d$. Here, $|n|_\infty := \max_{1\le i\le d}|n_i|$. The operators P_j will be called *Littlewood-Paley projections adapted to* $\mathbb{T}^d$. Notice that each P_j is a genuine projection: $P_j^2 = P_j$. We have the identity
$$f = \sum_{j\ge 0} P_j f,$$
for any distribution f on $\mathbb{T}^d$.

11. $F_q^{s,p}\left(\mathbb{T}^d\right)$ (with $1 \le p,q < \infty$ and s a real number) is the homogeneous Triebel-Lizorkin space consisting of those distributions f on $\mathbb{T}^d$ for which the following seminorm is finite.
$$\|f\|_{F_q^{s,p}(\mathbb{T}^d)} := \left\| \left(\sum_{j\ge 0} 2^{sjq} \left|P_j f\right|^q \right)^{1/q} \right\|_{L^p(\mathbb{T}^d)} .$$
A remarkable fact is that, if $k \ge 0$ is an integer and $1 < p < \infty$, then $F_2^{k,p}\left(\mathbb{T}^d\right) = W^{k,p}(\mathbb{T}^d)$.

12. $B_q^{s,p}\left(\mathbb{T}^d\right)$ (with $1 \le p,q \le \infty$ and s a real number) is the homogeneous Besov space consisting of those distributions f on $\mathbb{T}^d$ for which the following seminorm is finite.
$$\|f\|_{B_q^{s,p}(\mathbb{T}^d)} := \left(\sum_{j\ge 0} 2^{sjq} \left\|P_j f\right\|^q_{L^p(\mathbb{T}^d)} \right)^{1/q} .$$
In what follows, Ω is a domain in $\mathbb{R}^d$.

13. $C_c^\infty(\Omega)$ is the space of C^∞ functions which are compactly supported in the domain Ω.
14. $W_0^{k,p}(\Omega)$ (with $k \geq 0$ an integer and $1 \leq p \leq \infty$) is the closure of $C_c^\infty(\Omega)$ under the $W^{k,p}$-norm.
15. $F_q^{s,p}(\Omega)$ (with $1 \leq p, q < \infty$ and s a real number) is the space consisting of restrictions to Ω of elements from $F_q^{s,p}(\mathbb{R}^d)$, normed with

$$\|f\|_{F_q^{s,p}(\Omega)} := \inf\left\{\|g\|_{F_q^{s,p}(\mathbb{R}^d)} \mid g \in F_q^{s,p}(\mathbb{R}^d),\, g = f \text{ on } \Omega\right\}.$$

16. $B_q^{s,p}(\Omega)$ (with $1 \leq p, q \leq \infty$ and s a real number): is the space consisting of restrictions to Ω of elements from $B_q^{s,p}(\mathbb{R}^d)$, normed with

$$\|f\|_{B_q^{s,p}(\Omega)} := \inf\left\{\|g\|_{B_q^{s,p}(\mathbb{R}^d)} \mid g \in B_q^{s,p}(\mathbb{R}^d),\, g = f \text{ on } \Omega\right\}.$$

For more details, see [**29**].

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Part 1

Hodge systems with “pathological” source terms

CHAPTER 1

On the existence of vector fields with nonnegative divergence in r. i. spaces

We investigate the existence of solutions of

$$\operatorname{div} F = \mu, \quad \text{on } \mathbb{R}^d. \tag{$*$}$$

Here, $\mu \geq 0$ is a Radon measure, and we look for a solution $F \in X(\mathbb{R}^d, \mathbb{R}^d)$, where X is a rearrangement-invariant space.

We first prove the equivalence of the following assertions:
(i) $(*)$ has a solution for some nontrivial μ;
(ii) the function $x \longmapsto |x|^{1-d} \mathbb{1}_{B^c}(x)$ belongs to X.

Here, B is the unit ball in $\mathbb{R}^d$.

We next investigate the solvability of $(*)$ when μ is fixed. A *sufficient* condition is that $I_1\mu \in X$, where $I_1\mu$ is the 1-Riesz potential of μ. This condition turns out to be also *necessary* when the Boyd indexes of X belong to $(0,1)$.

Our analysis generalizes the one of Phuc and Torres (2008) when $X = L^p$.

1. Introduction

We will study the existence of solutions in different function spaces for the divergence equation

$$\operatorname{div} F = \mu, \quad \text{on } \mathbb{R}^d, \tag{1.1}$$

where μ is a nonnegative Radon measure. Here, $d \geq 2$.

Our work is motivated by the following result of Phuc and Torres (see [**4**, Theorem 3.1]):

THEOREM 1.1. *Let $1 \leq p \leq d/(d-1)$ and let μ be a nonnegative Radon measure on $\mathbb{R}^d$. If there exists a vector field $F \in L^p(\mathbb{R}^d, \mathbb{R}^d)$ such that* $\operatorname{div} F = \mu$ *on $\mathbb{R}^d$, then we have that $\mu \equiv 0$.*

The proof given in [**4**] uses the Calderón-Zygmund theory. More specifically, assume that (1.1) has a solution F in L^p. It is shown first that the 1-Riesz potential $I_1\mu$ of μ, defined by the formula

$$I_1\mu(x) = \int_{\mathbb{R}^d} |x-y|^{1-d} d\mu(y),$$

satisfies the relation

$$I_1\mu(x) = (1-d) \lim_{\varepsilon \to 0} \int_{|-|>\varepsilon} F(y) \frac{x-y}{|x-y|^{d+1}} d\mu(y) = c_d \sum_{j=1}^{d} R_j F_j(x), \quad \text{a.e. on } \mathbb{R}^d,$$

where R_j are the Riesz transforms and c_d is a constant only depending on d. Now, since $F \in L^p$, the Calderón-Zygmund theory ensures that $I_1\mu \in L^p$, whenever $p > 1$, and that $I_1\mu \in L^{1,\infty}$, if $p = 1$. However, since we have the trivial inequality

$$I_1\mu(x) \geq \frac{\mu(B(0,R))}{(|x|+R)^{d-1}}, \text{ for any } R > 0,$$

we must have $\mu(B(0,R)) = 0$ for all $R > 0$. Indeed, the function $(|x|+R)^{1-d}$ is never in $L^{1,\infty}$ or an L^p space for $p \in (1, d/(d-1)]$. Therefore $\mu \equiv 0$.

Also, using functional analytical methods, in [**4**] is proved (this follows easily from Theorem 3.2 and Theorem 3.3 from [**4**]) that the constant $d/(d-1)$ in the above theorem is sharp, in the sense that if $d/(d-1) < p \leq \infty$ then there exists an $F \in L^p$ such that $\operatorname{div} F = 1_B m$. Here, m is the Lebesgue measure and 1_B is the characteristic function of the unit ball.

Rewriting the condition on p in an integral form, we can express these facts by saying that if the divergence equation has a solution in L^p, then the measure μ is forced to be trivial if and only if the function $|x|^{1-d}\mathbb{1}_{B^c}$ is not in L^p (here, $\mathbb{1}_{B^c}$ is the characteristic function of the complement of the unit ball). As we will see, this phenomenon still occurs in a more general context where instead of the L^p spaces we consider rearrangement-invariant spaces (r. i. spaces for short). Our proof is quite elementary and does not use tools like the Calderón-Zygmund theory. It only makes use of basic properties of r. i. spaces whose definition is recalled below.

Following the presentation in [**1**, Chapters 1 and 2] we define first the notion of the *Banach function space*. Consider a measured space (Y, ν) and the set

$$M^+ := \{f : Y \to [0,\infty] \mid f \text{ is } \nu\text{-measurable}\}.$$

We call *function norm* a mapping $\rho : M^+ \to [0,\infty]$ with the following properties:

(P1) $\rho(f) = 0$ iff $f = 0$ $\nu - a.e.$, $\rho(af) = a\rho(f)$ and $\rho(f+g) \leq \rho(f) + \rho(g)$;
(P2) $0 \leq g \leq f$ $\nu - a.e.$ implies $\rho(g) \leq \rho(f)$;
(P3) $0 \leq f_n \uparrow f$ $\nu - a.e.$ implies $\rho(f_n) \uparrow \rho(f)$;
(This condition has an immediate important consequence called the *Fatou property: if* f_n, f are nonnegative measurable functions and $f_n \to f$ $\nu - a.e.$, then $\rho(f) \leq \underline{\lim}_{n\to\infty} \rho(f_n)$.)
(P4) $\nu(E) < \infty$ implies $\rho(1_E) < \infty$;
(P5) $\nu(E) < \infty$ implies $\int_E f\,d\nu \leq C_E \rho(f)$

whenever $f, f_n, g \in M^+$, $a \geq 0$ and E is a measurable subset of Y. Here, $C_E > 0$ is a constant only depending on E.

The set of measurable functions $f : Y \to \mathbb{R}$ for which $\rho(|f|) < \infty$ is called the *Banach function space* associated to ρ. It turns out that this space (in which we consider two functions equal when they are equal $\nu - a.e.$) with the norm $\|\cdot\| = \rho(|\cdot|)$ is a complete normed vector space (see [**1**, Theorem 1.6, p. 5]).

A r. i. *space* is a Banach function space associated to a function norm ρ with the property that $\rho(|f|) = \rho(|g|)$ for every pair of measurable functions on Y with the same distribution function $\lambda_f = \lambda_g$ (we recall that $\lambda_f(t) = \nu(\{x \in Y \mid |f(x)| > t\})$ for all $t \geq 0$). Here are few examples: the Lebesgue spaces L^p, the Lorentz spaces $L^{p,q}$ $(1 < p < \infty)$, the Orlicz spaces $\Phi(L)$.

In our case we will always have $Y = \mathbb{R}^d$ and $\nu = m$ will be the Lebesgue measure on $\mathbb{R}^d$. In this setting, we will use the following version of Theorem 4.8 in [**1**], p. 61:

LEMMA 1.2. *Let m be the Lebesgue measure on $\mathbb{R}^d$ and let $(E_j)_{j\geq 1}$ be a sequence of measurable pairwise disjoint subsets of $\mathbb{R}^d$, each with finite positive measure. Let $E = \mathbb{R}^d \setminus \bigcup_j E_j$. For each measurable nonnegative function f on $\mathbb{R}^d$, we define*

$$Af = f 1_E + \sum_{j=1}^{\infty} \left(\frac{1}{m(E_j)} \int_{E_j} f\,dx \right) 1_{E_j}.$$

Then A is a contraction on each r. i. space X over $(\mathbb{R}^d, m)$, that is,

$$\|Af\|_X \leq \|f\|_X, \text{ for all } f \in X.$$

2. The main nonexistence result

We can now state the first result:

THEOREM 1.3. *Let μ be a nonnegative Radon measure on $\mathbb{R}^d$, and X a r. i. space of functions on $\mathbb{R}^d$ such that $|x|^{1-d}\mathbb{1}_{B^c}$ does not belong to X. If the equation* $\operatorname{div}F = \mu$ *has a solution* $F \in X(\mathbb{R}^d,\mathbb{R}^d)$*, then* $\mu \equiv 0$.

PROOF. For each integer $j \geq 0$ we consider the set $U_j = B(0,2^{j+1})\setminus B(0,2^j)$ and the function $\varphi_j \in C_c^{0,1}(\mathbb{R}^d)$ defined by $\varphi_j(x) = 1$ if $|x| \in [0,2^j)$, $\varphi_j(x) = -2^{-j}|x| + 2$ if $|x| \in [2^j,2^{j+1})$ and $\varphi_j(x) = 0$ if $|x| \geq 2^{j+1}$. We consider also the weights $g_j := \mu(B(0,2^j))$.

Supposing that the equation $\operatorname{div}F = \mu$ has a solution in the space X, we estimate the weights g_j as follows:

$$g_j \leq \int_{\mathbb{R}^d} \varphi_j d\mu = -\int_{\mathbb{R}^d} F \cdot \nabla\varphi_j dx \leq \frac{1}{2^j}\int_{U_j} |F|dx \quad \text{for all } j \geq 0$$

so that

$$\frac{g_j}{2^{j(d-1)}} \leq c\frac{1}{m(U_j)}\int_{U_j} |F|dx \quad \text{for all } j \geq 0, \tag{1.2}$$

where c is a positive constant depending on d. Now if A is the operator defined in Lemma 1.2 corresponding to the sequence of sets U_0, U_1,..., we have

$$A|F| = |F|1_B + \sum_{j=0}^{\infty}\left(\frac{1}{m(U_j)}\int_{U_j}|F|dx\right)1_{U_j}$$

and, by Lemma 1.2 and axiom (P2), we obtain that

$$\left\|\sum_{j=0}^{\infty}\left(\frac{1}{m(U_j)}\int_{U_j}|F|dx\right)1_{U_j}\right\|_X \leq 2\,\|F\|_X < \infty. \tag{1.3}$$

Of course we always have $g_j \geq g_0$ and we can use (P2), (1.2) and (1.3) to write

$$g_0\rho(|x|^{1-d}\mathbb{1}_{B^c}) \leq \rho\left(\sum_{j=0}^{\infty}\frac{g_0}{2^{j(d-1)}}1_{U_j}\right) \leq \rho\left(\sum_{j=0}^{\infty}\frac{g_j}{2^{j(d-1)}}1_{U_j}\right) < \infty,$$

where ρ is the norm function which defines the norm on X.

However, since $\rho(|x|^{1-d}\mathbb{1}_{B^c}) = \infty$, the quantity g_0 must be zero. By a translation argument, the measure μ must be trivial. □

We saw that the condition

$$|x|^{1-d}\mathbb{1}_{B^c} \notin X \tag{1.4}$$

was used for proving the nonexistence of a solution F when $\mu \not\equiv 0$.

In order to obtain existence results we assume that condition (1.4) does not hold, that is

$$|x|^{1-d}\mathbb{1}_{B^c} \in X. \tag{1.5}$$

In this case we will prove the following

PROPOSITION 1.4. *Assume (1.5) and let μ be a measure such that $\mu = \phi m$ for a nonnegative function $\phi \in L_c^{\infty}(\mathbb{R}^d)$. Then (1.1) has a solution F in $X(\mathbb{R}^d,\mathbb{R}^d)$.*

The above result is an immediate consequence of the following two statements (which do not require (1.5)):

PROPOSITION 1.5. *Let $\phi \in L_c^\infty(\mathbb{R}^d)$ be such that $\phi \geq 0$ and let μ be the measure defined by $\mu = \phi m$. Then there exists a constant $C > 0$ only depending on μ and d such that*

$$I_1\mu(x) \leq C(1_B(x) + |x|^{1-d}\mathbb{1}_{B^c}(x)), \quad \text{on } \mathbb{R}^d.$$

(The proof of Proposition 1.5 is immediate.)

PROPOSITION 1.6. *Let μ be a nonnegative Radon measure on $\mathbb{R}^d$ and let X be a r. i. space of functions on $\mathbb{R}^d$. If $I_1\mu \in X$, then there exists a vector field $F \in X(\mathbb{R}^d, \mathbb{R}^d)$ such that $\operatorname{div} F = \mu$ in the distributional sense.*

PROOF OF PROPOSITION 1.6. If ρ is the norm function defining the norm on X, we have (using the property (P5)) that, for any $x_0 \in \mathbb{R}^d$, there exist a constant C_{x_0} such that

$$\int_{B(x_0,1)} I_1\mu dx \leq C_{x_0}\rho(I_1\mu) = C_{x_0}\left\|I_1\mu\right\|_X < \infty.$$

It follows that $I_1\mu$ must be a finite quantity a.e. on $\mathbb{R}^d$. Now we can fix a point $x_1 \in \mathbb{R}^d$, such that $I_1\mu(x_1) < \infty$. Using this property of x_1, we find that:

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{d\mu(y)}{\langle y\rangle^{d-1}} \leq & C_1\left(\int_{|x_1-y|<1} \frac{d\mu(y)}{\langle y\rangle^{d-1}} + \int_{|x_1-y|\geq 1} \frac{d\mu(y)}{|x_1-y|^{d-1}}\right) \\ \leq & C_2\left(\mu(B(x_1,1)) + I_1\mu(x_1)\right) < \infty\end{aligned} \tag{1.6}$$

for some positive constants C_1 and C_2. Here, $\langle y\rangle := (1+|y|^2)^{1/2}$.

If E is the standard fundamental solution of the Laplacian on $\mathbb{R}^d$, we define the vector field $F : \mathbb{R}^d \to \mathbb{R}^d$ by the formula

$$F_j(x) = \int_{\mathbb{R}^d} \partial_j E(x-y)d\mu(y), \ \ j \in \{1, \ldots, d\}.$$

We can easily see that F is a.e. well-defined. Indeed there exist a constant $C_3 > 0$ such that

$$\int_{\mathbb{R}^d} \left|\partial_j E(x-y)\right| d\mu(y) \leq C_3 \int_{\mathbb{R}^d} \left|\frac{x_j - y_j}{|x-y|^d}\right| d\mu(y) \leq C_3 I_1\mu(x),$$

and thus $|F| \leq C_4 I_1\mu < \infty$ a.e. In addition, since $I_1\mu$ is already in X, using the monotonicity of ρ we get $F \in X$. Choosing a test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ and using (1.6), we get that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\frac{x_j - y_j}{|x-y|^d}\right| d\mu(y)|\partial_j\varphi(x)|dx \leq \int_{\mathbb{R}^d} I_1|\partial_j\varphi|d\mu \leq C_\varphi \int_{\mathbb{R}^d} \frac{d\mu(y)}{\langle y\rangle^{d-1}} < \infty.$$

Here, we have used the straightforward estimate

$$I_1|\partial_j\varphi|(y) \leq C_\varphi \frac{1}{\langle y\rangle^{d-1}}.$$

We can now prove, using Fubini's theorem, that F solves (1.1):

$$-\sum_j < F_j, \partial_j\varphi >= -\sum_j \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_j E(x-y)\partial_j\varphi(x)dxd\mu(y) = \sum_j \int_{\mathbb{R}^d} (\partial_j E) * (\partial_j\varphi)d\mu$$

$$= \sum_j \int_{\mathbb{R}^d} \partial_j^2(E * \varphi)d\mu = \int_{\mathbb{R}^d} \varphi d\mu =< \varphi, \mu > .$$

So $\operatorname{div} F = \mu$ in the distributional sense on $\mathbb{R}^d$. ☐

REMARK 1.7. The above proof does not extend to the case of signed Radon measures. The existence problem in this case is more difficult and seems to be unsolved even in the L^p setting (see [**4**, p. 1575] and the references therein).

3. The rearrangement invariant norm of the 1-Riesz potential

As we saw, in the L^p case, the proof sketched after the statement of Theorem 1.1 gives a stronger conclusion when $1 < p < \infty$, namely if we can find a solution $F \in L^p$ of equation (1.1) then, not only that the condition (1.5) is satisfied, but the 1-Riesz potential of μ must be in L^p too. In what follows we prove that we have a similar situation in the case of r. i. spaces, giving a sufficient condition in terms of the Boyd indexes of the considered space. We recall some basic facts which will be useful and the definition of these indexes, again following the presentation in [**1**]:

Let X be a r. i. space over $\mathbb{R}^d$ whose function norm is ρ. We can define the *associate norm* ρ' of ρ by:

$$\rho'(g) = \sup\left\{\int_{\mathbb{R}^d} fg\,dx \mid f \in M^+,\ \rho(f) \le 1\right\},\ \text{for } g \in M^+.$$

It is known (see [**1**, Theorem 2.2, p. 8]) that ρ' is a norm function whose corresponding Banach function space, which is also an r.i space, will be denoted by X'. The following Hölder type inequality is a direct consequence of the definition:

$$\int_{\mathbb{R}^d} |fg|\,dx \le \|f\|_X \|g\|_{X'},\ \text{when } f \in X,\ g \in X'.$$

Let g^* denote the nonincreasing rearrangement of a measurable function $g : \mathbb{R}^d \to \mathbb{R}$:

$$g^*(s) := \inf\{t > 0 \mid \lambda_g(t) \le s\},\ s > 0.$$

We also recall the following inequality of Hardy and Littlewood (see [**1**, Theorem 2.2, p. 44]) that will be useful later. We have that

$$\int_{\mathbb{R}^d} |fg|\,dx \le \int_0^\infty f^*(s) g^*(s)\,ds$$

for all measurable functions f, g on $\mathbb{R}^d$.

The Luxemburg representation theorem (see [**1**, Theorem 4.10. p. 62]) provides a unique rearrangement-invariant function norm $\overline{\rho}$ defined on the nonnegative measurable functions on $(0,\infty)$, defined by

$$\overline{\rho}(h) = \sup\left\{\int_0^\infty h^* g^*\,dx \mid g \in M^+,\ \rho'(g) \le 1\right\},$$

with the property that $\rho(f) = \overline{\rho}(f^*)$. The corresponding r. i. space of $\overline{\rho}$ will be denoted by $\overline{X}$. For any $\theta > 0$ we can define the dilation operator $E_\theta : \overline{X} \to \overline{X}$ by the formula $E_\theta f(s) = f(\theta s)$ for all $f \in \overline{X}$. One may prove that each E_θ is a bounded operator. The *lower Boyd index* and *the upper Boyd index* are given by

$$\underline{\alpha}_X := \sup_{0<t<1} \frac{\log \|E_{1/t}\|}{\log t}, \qquad \overline{\alpha}_X := \inf_{1<t} \frac{\log \|E_{1/t}\|}{\log t}$$

respectively. Here $\|E_{1/t}\|$ is the norm of the operator $E_{1/t}$. It turns out (see [**1**, Proposition 5.13, p. 149]) that we can actually take limits in the definition:

$$\underline{\alpha}_X = \lim_{t\to 0} \frac{\log \|E_{1/t}\|}{\log t}, \qquad \overline{\alpha}_X = \lim_{t\to\infty} \frac{\log \|E_{1/t}\|}{\log t},$$

and that always $0 \leq \underline{\alpha}_X \leq \overline{\alpha}_X \leq 1$. As an important example consider the spaces L^p. In this case both indexes are equal to $1/p$. For the Lorentz spaces $L^{p,q}$ ($1 < p < \infty$, $1 \leq q \leq \infty$) the indexes are again both equal to $1/p$.

In order to obtain the necessity of the condition $I_1\mu \in X$, we adapt the proof of Theorem 1.1. To do so, we will need the following lemma which is just a rephrasing of some ideas presented in [**1**] and [**3**] (see, more specifically, the results of Calderón and Stein in section 3 in [**3**]).

Recall that a singular integral operator is an operator K of the form

$$Kf(x) = \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} k(y) f(x-y) dy.$$

The kernel k is *odd* if k is a function of the form $k(r\omega) = r^{-d}\Omega(\omega)$ for all $r > 0$ and all $\omega \in \mathbb{S}^{d-1}$, where $\Omega \in L^1(\mathbb{S}^{d-1})$ is odd.

LEMMA 1.8. *Let X be a r. i. space of functions on $\mathbb{R}^d$ such that $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$. Then any singular integral operator with odd kernel is well-defined and bounded from $L^2 \cap X$ into X. In particular, the Riesz transforms $R_1,\dots, R_d : L^2 \cap X \to X$ are well-defined and bounded.*

PROOF. Let K, k be as above. It is well-known that the operator K is well-defined and continuous on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. According to Theorem 3 in [**3**, p. 193], if $f \in L^2(\mathbb{R}^d)$ then, for all $s > 0$, we have that

$$(Kf)^*(s) \leq \frac{1}{s}\int_0^s (Kf)^*(t)dt \leq \|\Omega\|_{L^1(S(0,1))} \left(\frac{1}{s}\int_0^\infty f^*(t) \sinh^{-1}\left(\frac{s}{t}\right) dt\right).$$

Introducing the two operators

$$Pg(s) = \frac{1}{s}\int_0^s g(t)dt \quad \text{and} \quad Qg(s) = \int_s^\infty g(t)\frac{dt}{t},$$

for g measurable nonnegative, and integrating by parts, we can write for all $s > 0$,

$$\begin{aligned}\frac{1}{s}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt = \int_0^\infty \frac{Pf^*(t)}{\sqrt{s^2+t^2}}dt &= \int_0^s \frac{Pf^*(t)}{\sqrt{s^2+t^2}}dt + \int_s^\infty \frac{Pf^*(t)}{\sqrt{s^2+t^2}}dt \\ &\leq P^2 f^*(s) + QPf^*(s),\end{aligned}$$

concluding that there exist a constant $C_k > 0$ such that for all $s > 0$ we have

$$(Kf)^*(s) \leq C_k \left(P^2 + QP\right) f^*(s). \tag{1.7}$$

Theorem 5.15 in [**1**] guarantees that the operators P, Q are well-defined and continuous from $\overline{X}$ into $\overline{X}$ in the case where the Boyd indexes of X are in the interval $(0,1)$. Under this assumption, the inequality (1.7) implies that there exist a constant $C_{k,X} > 0$ only depending on k and X such that, for all $f \in L^2 \cap X(\mathbb{R}^d)$ we have

$$\|Kf\|_X \leq C_{k,X}\|f\|_X,$$

and we obtain the conclusion. □

THEOREM 1.9. *Let X be a r. i. space of functions on $\mathbb{R}^d$ such that $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$. If (1.1) has a solution $F \in X(\mathbb{R}^d, \mathbb{R}^d)$, then $I_1\mu \in X$. Moreover, there exists a constant $C_X > 0$ only depending on X such that $\left\|I_1\mu\right\|_X \leq C_X \|F\|_X$.*

REMARK 1.10. In particular, Theorem *1.9* applies to all L^p and, more generally, $L^{p,q}$ spaces with $1 < p < \infty$, $1 \le q \le \infty$. Also the theorem applies to all reflexive Orlicz spaces.

PROOF. First we observe that, using Fubini's theorem and the monotone convergence theorem, we can rewrite the 1-Riesz potential of a Radon measure ν on $\mathbb{R}^d$, for which $I_1|\nu| < \infty$ a.e.:

$$\begin{aligned} I_1\nu(x) &= \lim_{\delta\to 0^+}\int_{\mathbb{R}^d}\min(|x-y|^{1-d},\delta^{1-d})d\nu(y) = (d-1)\lim_{\delta\to 0^+}\int_{\mathbb{R}^d}\left(\int_\delta^\infty \frac{1_{B(x,r)}(y)}{r^d}dr\right)d\nu(y) \\ &=(d-1)\lim_{\delta\to 0^+}\int_\delta^\infty \frac{\nu(B(x,r))}{r^d}dr = (d-1)\int_0^\infty \frac{\nu(B(x,r))}{r^d}dr. \end{aligned} \tag{1.8}$$

Suppose (1.1) has a solution $F \in X$. Consider a standard radial bump function $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $0 \le \varphi \le 1$, $\operatorname{supp}\varphi \subseteq \overline{B}(0,1)$, $\|\varphi\|_{L^1(\mathbb{R}^d)} = 1$ and some $\phi \in C_c^\infty(\mathbb{R}^d)$ with $0 \le \phi \le 1$, $\phi = 1$ on $\overline{B}(0,1)$. For any $\theta, \varepsilon > 0$ we define φ_ε and ϕ_θ on $\mathbb{R}^d$ by the formula $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$ and $\phi_\theta(x) = \phi(\theta x)$. Fixing an $\varepsilon > 0$, the smooth functions $F_\varepsilon := F * \varphi_\varepsilon$ and $\mu_{\varepsilon,\theta} := (\mu * \varphi_\varepsilon)\phi_\theta + F_\varepsilon \cdot \nabla\phi_\theta$, clearly satisfy $\operatorname{div}(\phi_\theta F_\varepsilon) = \mu_{\varepsilon,\theta}$. As in [**4**], we can now use the Gauss-Ostrogradskii theorem and (1.8) to compute $I_1\mu_{\varepsilon,\theta}(x)$ for all x in $\mathbb{R}^d$, in terms of $\phi_\theta F_\varepsilon$:

$$\begin{aligned} I_1\mu_{\varepsilon,\theta}(x) &=(d-1)\lim_{\delta\to 0^+}\int_\delta^\infty \frac{1}{r^d}\int_{S(x,r)}(\phi_\theta F_\varepsilon)\cdot n\,d\sigma\,dr \\ &=(d-1)\lim_{\delta\to 0^+}\int_\delta^\infty\int_{S(x,r)}(\phi_\theta F_\varepsilon)(y)\cdot\frac{x-y}{|x-y|^{d+1}}d\sigma(y)dr \\ &=(d-1)\lim_{\delta\to 0^+}\int_{|x-y|>\delta}(\phi_\theta F_\varepsilon)(y)\cdot\frac{x-y}{|x-y|^{d+1}}dy. \end{aligned}$$

The last expression equals $c_d\sum_j R_j(\phi_\theta F_{\varepsilon,j})(x)$ a.e. in x. Thanks to Lemma 1.8 and noticing that $\phi_\theta F_{\varepsilon,j} \in C_c^\infty(\mathbb{R}^d) \subset L^2$, we have that there exists a constant $C_X > 0$, only depending on X, with

$$\|I_1\mu_{\varepsilon,\theta}\|_X \le c_d\left\|\sum_j R_j(\phi_\theta F_{\varepsilon,j})\right\|_X \le C_X\sum_j\|\phi_\theta F_{\varepsilon,j}\|_X \le C_X\sum_j\|F_{\varepsilon,j}\|_X, \quad \text{for all } \varepsilon > 0. \tag{1.9}$$

It is not hard to see that, if $f \in X$, we have

$$\begin{aligned} \rho(|f*\varphi_\varepsilon|) &=\rho\left(\left|\int_{\mathbb{R}^d}f(\cdot-\varepsilon y)\varphi(y)dy\right|\right) \le \int_{\mathbb{R}^d}\rho(|f(\cdot-\varepsilon y)|\varphi(y))\,dy \\ &= \int_{\mathbb{R}^d}\rho(|f(\cdot-\varepsilon y)|)\varphi(y)dy \end{aligned}$$

(we just consider an increasing sequence of nonnegative continuous functions converging pointwise to the function $|f|$ and then we apply (P3) to reduce the problem to the case of continuous functions, case which can be handled using Riemann sums and the property (P3) as before). Since f and $|f(\cdot-\varepsilon y)|$ have the same distribution function and X is a r. i. space, we get that $\int_{\mathbb{R}^d}f(\cdot-\varepsilon y)\varphi(y)dy$ belongs to X and its norm is bounded by $\|f\|_X$. This fact combined with (1.9) gives us

$$\|I_1\mu_{\varepsilon,\theta}\|_X \le C_X\|F\|_X < \infty, \text{ for all } \theta, \varepsilon > 0. \tag{1.10}$$

It remains to show that this implies $I_1\mu \in X$ and the expected estimate. We have that $I_1\mu_{\varepsilon,\theta} = I_1(\mu_\varepsilon\phi_\theta) + I_1(F_\varepsilon\cdot\nabla\phi_\theta)$, where $\mu_\varepsilon := \mu * \varphi_\varepsilon$. When $\theta \to 0$, for the second term we can write for each

$x \in \mathbb{R}^d$,

$$\begin{aligned}\left|I_1\left(F_\varepsilon \cdot \nabla\phi_\theta\right)(x)\right| \le& \theta \int_{\mathbb{R}^d} \frac{\left|F_\varepsilon(y)\cdot\nabla\phi(\theta y)\right|}{|x-y|^{d-1}}dy = \int_{\mathbb{R}^d} \frac{\left|F_\varepsilon(y/\theta)\cdot\nabla\phi(y)\right|}{|\theta x-y|^{d-1}}dy \\ \le& \|E_{1/\theta}F_\varepsilon\|_X \left\|\frac{\nabla\phi}{|\theta x-\cdot|^{d-1}}\right\|_{X'} \le \|E_{1/\theta}\|\,\|F_\varepsilon\|_X \left\|\frac{\nabla\phi}{(1-|\theta x|)^{d-1}}\right\|_{X'} \\ \le& 2\,\|E_{1/\theta}\|\,\|F_\varepsilon\|_X \left\|\nabla\phi\right\|_{X'} \le 2\theta^{\underline{\alpha}_X/2}\,\|F_\varepsilon\|_X \left\|\nabla\phi\right\|_{X'} \to 0.\end{aligned}$$

The dominated convergence theorem gives for the first term that $I_1\left(\phi_\theta\mu_\varepsilon\right) \to I_1\mu_\varepsilon$ pointwise when $\theta \to 0$. From these two observations, (1.10) and the Fatou property of ρ (which follows from (P3)) we conclude:

$$\left\|I_1\mu_\varepsilon\right\|_X \le \underline{\lim}_{\varepsilon\to 0} \left\|I_1\mu_{\varepsilon,\theta}\right\|_X \le C_X\,\|F\|_X < \infty, \text{ for all } \varepsilon > 0. \tag{1.11}$$

We now let $\varepsilon \to 0$ in (1.11). For each $x \in \mathbb{R}^d$ and $r > 0$ we can write:

$$\mu_\varepsilon(B(x,r)) = \int_{B(x,r)}\int_{\mathbb{R}^d} \varphi_\varepsilon(z-y)d\mu(y)dz = \int_{\mathbb{R}^d} \varphi_\varepsilon * 1_{B(x,r)}(y)d\mu(y).$$

It is not hard to see that, taking $\varepsilon \to 0$, $\varphi_\varepsilon * 1_{B(x,r)}(y) \to 1$ when $y \in B(x,r)$, $\varphi_\varepsilon * 1_{B(x,r)}(y) \to 0$ when $y \notin \overline{B}(x,r)$ and $\varphi_\varepsilon * 1_{B(x,r)}(y) \to 1/2$ when $y \in \partial B(x,r)$. Moreover the function $\varphi_\varepsilon * 1_{B(x,r)}$ is bounded by 1 and has its support contained in $B(x,r+1)$ when ε is small. The dominated convergence theorem yields

$$\mu_\varepsilon(B(x,r)) \to \mu(B(x,r)) + \frac{1}{2}\mu(\partial B(x,r)), \text{ when } \varepsilon \to 0$$

and hence, for any $l \ge 1$,

$$\begin{aligned}\int_{1/l}^{l} \frac{\mu(B(x,r))}{r^d}dr \le& \int_{1/l}^{l} \frac{\mu(B(x,r))}{r^d}dr + \frac{1}{2}\int_{1/l}^{l} \frac{\mu(\partial B(x,r))}{r^d}dr \\ =& \lim_{\varepsilon\to 0}\int_{1/l}^{l} \frac{\mu_\varepsilon(B(x,r))}{r^d}dr.\end{aligned}$$

The inequality from (1.11) and the Fatou property of ρ will give

$$\begin{aligned}(d-1)\left\|\int_{1/l}^{l} \frac{\mu(B(\cdot,r))}{r^d}dr\right\|_X \le& (d-1)\underline{\lim}_{\varepsilon\to 0}\left\|\int_{1/l}^{l} \frac{\mu_\varepsilon(B(\cdot,r))}{r^d}dr\right\|_X \\ \le& \underline{\lim}_{\varepsilon\to 0}\left\|I_1\mu_\varepsilon\right\|_X \le C_X\,\|F\|_X\end{aligned}$$

and we can finish the proof by using the Fatou property and (1.8), taking $l \to \infty$. □

The above result covers the case of L^p spaces when $1 < p < \infty$. However, even in the L^p setting, the fact that the equation (1.1) has a solution in X does not imply that the 1-Riesz potential of the measure belongs to X. More specifically, we have the following classical result (see [**4**]):

THEOREM 1.11 (Theorem 3.3 in [**4**]). *Let μ be a nonnegative Radon measure on $\mathbb{R}^d$. Then the equation (1.1) has a solution $F \in L^\infty(\mathbb{R}^d \to \mathbb{R}^d)$ if and only if the measure μ is $(d-1)$–Frostman, i.e., there exist a constant M only depending on μ such that*

$$\mu(B(x,r)) \le Mr^{d-1}, \textit{ for all } x \in \mathbb{R}^d \textit{ and } r > 0.$$

In order to prove Theorem 1.11, let μ be a Radon measure. The fact that there exists a solution $F \in L^\infty$ for the equation $\operatorname{div} F = \mu$ is equivalent to the fact that μ belongs to the dual of the space $w^{1,1}$. Here, $w^{1,1}$ is the closure of $C_c^\infty(\mathbb{R}^d)$ under the norm $\|\cdot\|_{w^{1,1}}$, where $\|u\|_{w^{1,1}} := \|\nabla u\|_{L^1}$, $u \in C_c^\infty(\mathbb{R}^d)$. Now, for nonnegative measures this condition is equivalent to the fact that μ is

$(d-1)-$Frostman. This result is due to Meyers and Ziemer (originally appearing in [**2**]; see also [**5**, Lemma 4.9.1, p. 209] for a proof of a more general statement).

Clearly, there exist $(d-1)-$Frostman nonnegative measures whose 1-Riesz potential is unbounded. Take for example the measure μ defined by $\mu(E) = m_{d-1}(E \cap \{x_1 = 0\})$ for all Borel sets $E \subset \mathbb{R}^d$. Here, m_{d-1} is the $(d-1)-$dimensional Lebesgue measure on the hyperplane $\{x_1 = 0\}$. For this μ, the quantity $\int_1^\infty r^{-d}\mu(B(x,r))dr$ is infinite for all $x \in \mathbb{R}^d$ and then, by (1.8), $I_1\mu$ is infinite everywhere.

Note that the Boyd indexes of L^∞ are 0 and thus the example obtained in the previous paragraph does not contradict Theorem 1.9.

The case of the space L^1 is also a pathological one, the Boyd indexes being equal to 1. However we cannot find a counterexample for the assertion of the Theorem 1.9 in the case of nonnegative measures. Indeed, by Theorem 1.1, the measure μ will be trivial and then $I_1\mu \equiv 0 \in L^1$. Nevertheless, we can give a simple example of a vector field $F \in L^1$ and of a *signed* Radon measure μ such that $\operatorname{div} F = \mu$, but the 1-Riesz potential, $I_1\mu$, does not belong to L^1. The construction of F and μ relies on the following observation. Consider $\psi \in C_c^\infty(B(0,1))$. When $|x|$ is large we can write, for $r = |x|$, $\omega = x/|x|$, that

$$I_1\psi(x) = \frac{1}{r^{d-1}}\int_{B(0,1)} \frac{\psi(y)}{|\omega - y/r|^{d-1}}dy = \frac{1}{r^{d-1}}\int_{B(0,1)} \frac{\psi(y)}{|1-2y\cdot\omega/r + |y|^2/r^2|^{(d-1)/2}}dy$$
$$= \frac{1}{r^{d-1}}\int_{B(0,1)} \psi(y)dy + \frac{d-1}{r^d}\omega\cdot\left(\int_{B(0,1)} y\psi(y)dy\right) + \frac{1}{r^{d+1}}\int_{B(0,1)} \psi(y)h(r,y)dy,$$

where h is a smooth bounded function on $(1,\infty)\times\mathbb{R}^d$. Thus,

$$I_1\psi(x) = \frac{A}{|x|^{d-1}} + \frac{b\cdot x}{|x|^{d+1}} + O\left(\frac{1}{|x|^{d+1}}\right) \quad \text{as } |x| \to \infty, \tag{1.12}$$

where

$$A := \int_{B(0,1)} \psi(y)dy \quad \text{and} \quad b := \int_{B(0,1)} y\psi(y)dy.$$

The right hand side of (1.12) belongs to L^1 if and only if $A = 0$ and $b = 0$. In conclusion, $\psi \in C_c^\infty(B(0,1))$ has the property that $I_1\psi \in L^1$ if and only if

$$\int_{B(0,1)} \psi(y)dy = \int_{B(0,1)} y_j\psi(y)dy = 0 \text{ for all } j \in \{1,\dots,d\}.$$

We can now construct our example. Let $\varphi \in C_c^\infty(B(0,1))$ be such that $\int_{B(0,1)}\varphi(y)dy \neq 0$, and set $F = (\varphi, 0, \dots, 0) \in L^1$ and $\mu = (\partial_1\varphi)\, m$. Clearly, we have

$$\int_{B(0,1)} y_1\partial_1\varphi(y)dy = -\int_{B(0,1)} \varphi(y)dy \neq 0$$

and, by the above observation (with $\psi = \partial_1\varphi$), $I_1\mu$ does not belong to L^1.

These examples show that, at least in the case where the measure is signed, we cannot expect for the pathological L^p spaces, namely L^1 and L^∞, to have the property stated in the above Theorem 1.9. This is also the case in the more general context of r. i. spaces: as long as at least one of the Boyd indexes of the space X is equal to 0 or 1, we can always find a signed Radon measure which is the divergence of a field F belonging to X, but whose 1-Riesz potential does not have the norm in X controlled by the norm of F. It is not hard to observe that, after minor modifications in the proof, the conclusion of Theorem 1.9 remains true in the case of signed Radon measures. With this in mind, Proposition 1.12 below is a sort of converse.

PROPOSITION 1.12. *Let X be a r. i. space of functions on $\mathbb{R}^d$ with the property that whenever μ is a signed Radon measure on $\mathbb{R}^d$ with $\mu = \operatorname{div} F$ for a vector field $F \in X(\mathbb{R}^d, \mathbb{R}^d)$, we have that $I_1\mu^+$, $I_1\mu^-$ are finite a.e., $I_1\mu \in X$ and $\|I_1\mu\|_X \le C_X \|F\|_X$ for a positive constant C_X. Then $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$.*

Proposition 1.12 is a consequence of Lemma 1.13 below, which is a d−dimensional version of Proposition 4.10 p. 140 in [**1**], with essentialy the same proof. To state Lemma 1.13, consider the operators P and Q defined in the proof of Lemma 1.8 and let S be the Calderón operator defined by the formula:

$$Sf(s) = Pf(s) + Qf(s) = \int_0^s \frac{f(t)}{s}dt + \int_s^\infty \frac{f(t)}{t}dt = \int_0^\infty f(t)\min\left(\frac{1}{t}, \frac{1}{s}\right)dt, \ s > 0,$$

initially for nonnegative measurable functions f on $(0,\infty)$ (see [**1**, p 133 and 142]).

LEMMA 1.13. *Let X be a r. i. space of functions on $\mathbb{R}^d$ and $f \in \overline{X}$. Consider the sets*

$$C^+ := (0,\infty)^d, \quad C^- := (-\infty, 0)^d$$

and the function $G : \mathbb{R}^d \to [0,\infty]$, $G(x) = f^*(v_d|x|^d)1_{C^-}(x)$, where $v_d := m(B(0,1))/2^d$. Then G and f are equimeasurable functions, the Riesz transforms R_1G,..., R_dG of G are well-defined as functions on C^+ and there exist a constant $c_d > 0$ such that:

$$((R_1G + ... + R_dG)1_{C^+})^*(s) \ge c_d S(f^*)(s), \text{ for all } s > 0. \tag{1.13}$$

Moreover, if $(R_1G + ... + R_dG)1_{C^+} \in X$ for all $f \in \overline{X}$ then $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$. In particular, the same conclusion holds if the sum of the Riesz transforms is a well-defined operator from X into X .

PROOF OF LEMMA 1.13. Consider for simplicity the function $g : (0,\infty) \to [0,\infty]$ with $g(s) = f^*(v_d s^d)$. Note that g is nonincreasing, and thus $g^* = g$. It is easy to see that, since $G(x) = g(|x|)1_{C^-}(x)$ and since g is nonincreasing, we have $\lambda_G(t) = v_d \lambda_g^d(t)$ for all $t > 0$. Hence, a simple computation gives us $G^*(s) = g^*(v_d^{-1/d}s^{1/d}) = g(v_d^{-1/d}s^{1/d}) = f^*(s)$ for all $s > 0$, which shows the equimeasurability of G and f. Taking now a $j \in \{1,...,d\}$ we can write, for $x \in C^+$:

$$\begin{aligned}
R_jG(x) =& c_d^1 \int_{C^+} \frac{x_j + y_j}{|x+y|^{d+1}} g(|y|)dy = c_d^1 \int_{C^+} \frac{x_j}{|x+y|^{d+1}} g(|y|)dy + c_d^1 \int_{C^+} \frac{y_j}{|x+y|^{d+1}} g(|y|)dy \\
\ge& c_d^2 x_j \int_{C^+} \min\left(\frac{1}{|x|^{d+1}}, \frac{1}{|y|^{d+1}}\right) g(|y|)dy + c_d^2 \int_{C^+} y_j \min\left(\frac{1}{|x|^{d+1}}, \frac{1}{|y|^{d+1}}\right) g(|y|)dy \\
\ge& c_d^3 \frac{x_j}{|x|^{d+1}} \int_{B(0,|x|)\cap C^+} g(|y|)dy + c_d^3 \int_{B^c(0,|x|)\cap C^+} \frac{y_j}{|y|^{d+1}} g(|y|)dy.
\end{aligned}$$

Summing up these inequalities yields

$$\begin{aligned}
\sum_{j=1}^d R_jG(x) \ge& c_d^3 \frac{x_1 + ... + x_d}{|x|^{d+1}} \int_{B(0,|x|)\cap C^+} g(|y|)dy + c_d^3 \int_{B^c(0,|x|)\cap C^+} \frac{y_1 + ... + y_d}{|y|^{d+1}} g(|y|)dy \\
\ge& c_d^4 \frac{1}{|x|^d} \int_{B(0,|x|)\cap C^+} g(|y|)dy + c_d^4 \int_{B^c(0,|x|)\cap C^+} \frac{1}{|y|^d} g(|y|)dy \\
\ge& c_d^5 \frac{1}{|x|^d} \int_0^{|x|} r^{d-1} g(r)dr + c_d^5 \int_{|x|}^\infty \frac{g(r)}{r}dr \\
\ge& c_d^6 \frac{1}{|x|^d} \int_0^{v_d|x|^d} f^*(t)dt + c_d^6 \int_{v_d|x|^d}^\infty \frac{f^*(t)}{t}dt \ge c_d S(f^*)(v_d|x|^d).
\end{aligned}$$

Since $S(f^*)$ is a nonincreasing function, we can see as above that the nonincreasing rearrangement of the function $x \to S(f^*)(v_d|x|^d)1_{C^+}(x)$ computed in $s > 0$ is equal to $S(f^*)(s)$. Hence, we have proved the inequality (1.13).

To prove the next claim, observe that the inequality (1.13) under the assumption that $(R_1G + ... + R_dG)1_{C^+} \in X$ gives us that, if $f \in \overline{X}$, then we must have $S(f^*) \in \overline{X}$.

Let us note that we have $S(|f|) \leq S(f^*)$. This can be easily seen by applying the Hardy-Littlewood inequality to $|f|$ and the nonincreasing function $t \to \min(1/s, 1/t)$ when $s > 0$ is fixed:

$$S(|f|)(s) = \int_0^\infty |f(t)| \min\left(\frac{1}{s}, \frac{1}{t}\right) dt \leq \int_0^\infty f^*(t) \min\left(\frac{1}{s}, \frac{1}{t}\right) dt = S(f^*)(s).$$

Up to now we have that, if $f \in \overline{X}$ and $(R_1G + ... + R_dG)1_{C^+} \in X$, then $S(|f|) \in \overline{X}$. As in the proof of Theorem 1.8, p. 7, [**1**] we suppose by contradiction that the operator $S : \overline{X} \to \overline{X}$ is not continuous. Then we can find a sequence $(f_n)_{n\geq 1}$ of nonnegative functions in $\overline{X}$ with $\|f_n\|_{\overline{X}} = 1$ and such that $\|S(f_n)\|_{\overline{X}} \geq n^3$ for all $n \geq 1$. The series $\sum_{n\geq 1} f_n/n^2$ being absolutely convergent in $\overline{X}$, it defines a function $f \in \overline{X}$, hence $S(f) \in \overline{X}$. However since all the functions f_n are nonnegative, we have $f \geq f_n/n^2$ and consequently $S(f) \geq S(f_n)/n^2$ which implies $\|S(f)\|_{\overline{X}} \geq \|S(f_n)\|_{\overline{X}}/n^2 \geq n$ for all $n \geq 1$, obtaining a contradiction.

Having that S is a continuous operator, we can use Theorem 5.15 in [**1**, p. 150] to obtain the statement about the Boyd indexes of X. ☐

PROOF OF PROPOSITION 1.12. To prove the Proposition 1.12, consider a function $f \in \overline{X}$ and the field $F = (G, ..., G)$, with G constructed from f as in Lemma 1.13. Suppose first that f^* is compactly supported. Note that $\mu := \operatorname{div} F$ is not always a Radon measure (we can compute explicitly μ^+ and μ^- to see that these measures are not always locally finite), but is of course a distribution. With the notation from the proof of Theorem 1.9, we have that $\mu * \varphi_\varepsilon$ and $F * \varphi_\varepsilon$ are smooth compactly supported functions. In particular, $\mu * \varphi_\varepsilon$ is a compactly supported signed Radon measure on $\mathbb{R}^d$. Since $\mu * \varphi_\varepsilon = \operatorname{div}\left(F * \varphi_\varepsilon\right)$ and, as in the proof of Theorem 1.9, $\left\|F * \varphi_\varepsilon\right\|_X \leq \|F\|_X$, we must have then, that $\left\|I_1\left(\mu * \varphi_\varepsilon\right)\right\|_X \leq C_X \left\|F * \varphi_\varepsilon\right\|_X \leq C_X \|F\|_X$ for all $\varepsilon > 0$. As above, the formula (1.8) and the Gauss-Ostrogradskii theorem give us that:

$$I_1\left(\mu * \varphi_\varepsilon\right)(x) = (d-1) \lim_{\delta \to 0^+} \int_{|x-y|>\delta} F * \varphi_\varepsilon(y) \cdot \frac{x-y}{|x-y|^{d+1}} dy \text{ for all } x \in \mathbb{R}^d. \tag{1.14}$$

Fix $r > 0$ and take $\varepsilon \in (0, r)$. The support of F is contained in the closure of the set C^-. Hence $F * \varphi_\varepsilon$ is supported in the closure of $C^- + \varepsilon B$. We can write, for all $x \in C_r^+ := (r, ..., r) + C^+$, that

$$\begin{aligned} \lim_{\delta \to 0^+} \int_{|x-y|>\delta} F * \varphi_\varepsilon(y) \cdot \frac{x-y}{|x-y|^{d+1}} dy &= \int_{C^- + \varepsilon B} F * \varphi_\varepsilon(y) \cdot \frac{x-y}{|x-y|^{d+1}} dy \\ &\geq \int_{C^-} F * \varphi_\varepsilon(y) \cdot \frac{x-y}{|x-y|^{d+1}} dy. \end{aligned} \tag{1.15}$$

Since the integrand of the last term is nonnegative, one can use Tonelli's theorem to change the order of integration, and find that

$$\begin{aligned} \int_{C^-} F * \varphi_\varepsilon(y) \cdot \frac{x-y}{|x-y|^{d+1}} dy &= \int_{\mathbb{R}^d} \left(\int_{C^-} F(y - \varepsilon\xi) \cdot \frac{x-y}{|x-y|^{d+1}} dy \right) \varphi(\xi) d\xi \\ &\geq \int_{C^+} \left(\int_{C^-} g(|y - \varepsilon\xi|) \cdot \frac{x_1 - y_1 + ... + x_d - y_d}{|x-y|^{d+1}} dy \right) \varphi(\xi) d\xi. \end{aligned}$$

The fact that the function g is nonincreasing enables us to see that for a fixed $\xi \in C^+$ we have $g(|y - \varepsilon\xi|) \leq g(|y - \varepsilon'\xi|)$ for all $y \in C^-$ and for all $0 < \varepsilon' < \varepsilon$. Hence the monotone convergence theorem gives us

$$\int_{C^-} g(|y - \varepsilon\xi|) \cdot \frac{x_1 - y_1 + ... + x_d - y_d}{|x-y|^{d+1}} dy \to \int_{C^-} g(|y|) \cdot \frac{x_1 - y_1 + ... + x_d - y_d}{|x-y|^{d+1}} dy,$$

when $\varepsilon \to 0$. With the help of Fatou's lemma and the above calculation we have

$$\underline{\lim}_{\varepsilon \to 0} \int_{C^-} F * \varphi_\varepsilon(y) \cdot \frac{x-y}{|x-y|^{d+1}} dy \geq A \int_{C^-} g(|y|) \cdot \frac{x_1 - y_1 + ... + x_d - y_d}{|x-y|^{d+1}} dy,$$

where $A = \int_{C^+} \varphi(\xi) d\xi > 0$. Using (1.14), (1.15) we obtain

$$\varliminf_{\varepsilon\to 0} I_1\left(\mu * \varphi_\varepsilon\right)(x) \geq (d-1) A \int_{C^-} g(|y|) \cdot \frac{x_1 - y_1 + \ldots + x_d - y_d}{|x-y|^{d+1}} dy \geq 0,$$

for all $x \in C_r^+$ and all $r > 0$. Since $(d-1)A$ do not depend on r, this inequality can be rewriten as

$$\varliminf_{\varepsilon\to 0} \left(I_1\left(\mu * \varphi_\varepsilon\right)\right) 1_{C^+} \geq c_d \left(R_1 G + \ldots + R_d G\right) 1_{C^+} \geq 0,$$

and consequently by the Fatou property of X, we get

$$\|(R_1 G + \ldots + R_d G) 1_{C^+}\|_X \leq C_X^1 \varliminf_{\varepsilon\to 0} \left\| I_1\left(\mu * \varphi_\varepsilon\right)\right\|_X$$

which implies $\|(R_1 G + \ldots + R_d G) 1_{C^+}\|_X \leq C_X^2 \|F\|_X$, and by using inequality (1.13) from Lemma 1.13, we get that

$$\left\|S(f^*)\right\|_{\overline{X}} \leq C_X^3 \left\|f^*\right\|_{\overline{X}} \tag{1.16}$$

whenever $f \in \overline{X}$ and f^* is compactly supported. Now, if $f \in \overline{X}$ and f^* is not necessarily compactly supported, from (1.16) we have $\left\|S(f^* 1_{(0,n)})\right\|_{\overline{X}} \leq C_X^3 \left\|f^* 1_{(0,n)}\right\|_{\overline{X}}$ for all $n \geq 1$. By the monotone convergence theorem and the Fatou property of $\overline{X}$ we get, taking $n \to \infty$, that (1.16) is true whenever $f \in \overline{X}$. Since as in the proof of Lemma 1.13 we have $S(|f|) \leq S(f^*)$, we get now that $\|S(|f|)\|_{\overline{X}} \leq C_X^3 \|f\|_{\overline{X}}$ for all $f \in \overline{X}$, obtaining that S is bounded from $\overline{X}$ into $\overline{X}$.

The conclusion follows now as in Lemma 1.13. □

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CHAPTER 2

On the representation as exterior differentials of closed forms with L^1-coefficients

Let $N \geq 2$. If $g \in L^1_c(\mathbb{R}^N)$ has zero integral, then the equation $\operatorname{div} X = g$ need not have a solution $X \in W^{1,1}_{loc}(\mathbb{R}^N;\mathbb{R}^N)$ (Wojciechowski 1999) or even $X \in L^{N/(N-1)}_{loc}(\mathbb{R}^N;\mathbb{R}^N)$ (Bourgain and Brezis 2003). Using these results, we prove that, whenever $N \geq 3$ and $2 \leq \ell \leq N-1$, there exists some ℓ-form $f \in L^1_c(\mathbb{R}^N;\Lambda^\ell)$ such that $df = 0$ and the equation $d\lambda = f$ has no solution $\lambda \in W^{1,1}_{loc}(\mathbb{R}^N;\Lambda^{\ell-1})$. This provides a negative answer to a question raised by Baldi, Franchi and Pansu (2019).

1. Introduction

We consider the Hodge system

$$d\lambda = f \text{ in } \mathbb{R}^N, \tag{2.1}$$

where f and λ are ℓ and $(\ell-1)$-forms respectively, f being given and satisfying the compatibility condition $df = 0$. We focus on the case where f has L^1 coefficients.

To start with, let us recall some known facts about the cases $\ell = N$ and $\ell = 1$.

In the case $\ell = N$, (2.1) reduces to the divergence equation. It was first shown by Wojciechowski [**6**] that there exists $g \in L^1_c(\mathbb{R}^N)$, with zero integral, such that the equation $\operatorname{div} X = g$ has no solution $X \in W^{1,1}_{loc}(\mathbb{R}^N;\mathbb{R}^N)$. On the other hand, Bourgain and Brezis [**2**] proved, using a different method, the following: there exists $g \in L^1_c(\mathbb{R}^N)$ with zero integral, such that the equation $\operatorname{div} X = g$ has no solution $X \in L^{N/(N-1)}_{loc}(\mathbb{R}^N;\mathbb{R}^N)$. In view of the embedding $W^{1,1}_{loc} \hookrightarrow L^{N/(N-1)}_{loc}$, this improves [**6**].

In the case $\ell = 1$, (2.1) reduces to the following "gradient" equation

$$\nabla\lambda = f, \tag{2.2}$$

where f is a vector field satisfying the compatibility condition $\nabla \times f = 0$ and λ is a function. Unlike the case $\ell = N$, this time (2.2) has a solution $\lambda \in W^{1,1}_{loc}(\mathbb{R}^N)$. Actually, any solution of (2.2) belongs to $W^{1,1}_{loc}$ and, moreover, if f is compactly supported then we may choose $\lambda \in W^{1,1}$.

The question of the solvability in $W^{1,1}_{loc}$ of the system (2.1) with datum in L^1 in the remaining cases, i.e., $2 \leq \ell \leq N-1$, has been recently raised by Baldi, Franchi and Pansu [**1**]. Our main result settles this problem.

THEOREM 2.1. *Let $N \geq 3$. Let $2 \leq \ell \leq N-1$. Then there exists some $f \in L^1_c(\mathbb{R}^N;\Lambda^\ell)$ such that $df = 0$ and the equation $d\lambda = f$ has no solution $\lambda \in W^{1,1}_{loc}(\mathbb{R}^N;\Lambda^{\ell-1})$.*

The proof of Theorem 2.1 we present is a simplification, communicated to the author by P. Mironescu, of the original one. This simplified version has the advantage of being relatively self-contained and elementary.

2. Proof of Theorem 2.1

We start with some auxiliary results.

LEMMA 2.2. *Let $1 \leq \kappa \leq N-1$ and $f \in L^1_c(\mathbb{R}^N;\Lambda^\kappa)$ be such that $df = 0$. Then there exists some $\omega \in L^q_{loc}(\mathbb{R}^N;\Lambda^{\kappa-1})$, for all $1 \leq q < N/(N-1)$, such that $d\omega = f$.*

PROOF. Let E be "the" fundamental solution of Δ and set $\eta := E * f$. Let $\omega := d^*\eta$. First, $\eta \in W^{1,q}_{loc}(\mathbb{R}^N)$ (by elliptic regularity) and thus $\omega \in L^q_{loc}(\mathbb{R}^N)$, $1 \le q < N/(N-1)$. Next, $d\eta = E * df = 0$. Finally,

$$d\omega = dd^*\eta = (dd^* + d^*d)\eta = \Delta\eta = f.$$

Hence, ω has the required properties. □

A similar argument leads to the following.

LEMMA 2.3. *Let $1 < r < \infty$, $k \in \mathbb{N}$. Let $1 \le \kappa \le N-1$. Let $f \in W^{k,r}_c(\mathbb{R}^N;\Lambda^\kappa)$ be such that $df = 0$. Then there exists some $\omega \in W^{k+1,r}_{loc}(\mathbb{R}^N;\Lambda^{\kappa-1})$ such that $d\omega = f$.*

We next recall the following "inversion of d with loss of regularity". It is folklore, and one possible proof consists of using Bogovskiĭ's formula (see for example [**4**, Corollary 3.3 and Corollary 3.4] for related arguments).

LEMMA 2.4. *Let $1 \le \kappa \le N-1$. Let Q be an open cube in $\mathbb{R}^N$. Then there exists some integer $m = m(N,\kappa)$ such that if $f \in C^k_c(Q;\Lambda^\kappa)$ (whose coefficients have zero integral), with $k \in \{m, m+1, \ldots\} \cup \{\infty\}$, satisfies $df = 0$, then there exists some $\omega \in C^{k-m}_c(Q;\Lambda^{\kappa-1})$ such that $d\omega = f$.*

Combining Lemmas 2.2–2.4, we obtain the following

PROPOSITION 2.5. *Let $1 \le \kappa \le N-1$. Let Q be an open cube in $\mathbb{R}^N$. Let $f \in L^1_c(Q;\Lambda^\kappa)$ (whose coefficients have zero integral) be such that $df = 0$. Then there exists some $\omega \in L^q_c(Q;\Lambda^{\kappa-1})$, for all $1 \le q < N/(N-1)$, such that $d\omega = f$.*

PROOF. Set $f_0 := f$. We consider a sequence $(\zeta_j)_{j\ge 0}$ in $C^\infty_c(Q;\mathbb{R})$ such that $\zeta_0 = 1$ on $\operatorname{supp} f_0$ and, for $j \ge 1$, $\zeta_j = 1$ on $\operatorname{supp}\zeta_{j-1}$. We let η_0 be a solution of $d\eta_0 = f_0$, constructed as in Lemma 2.2. We set $\omega_0 := \zeta_0\eta_0$, so that $\omega_0 \in L^q_c(Q;\Lambda^{\kappa-1})$, $1 \le q < N/(N-1)$ and

$$d\omega_0 = d\zeta_0 \wedge \eta_0 + \zeta_0 d\eta_0 = d\zeta_0 \wedge \eta_0 + \zeta_0 f_0 = \underbrace{d\zeta_0 \wedge \eta_0}_{-f_1} + f_0.$$

Let us note that $df_1 = -d^2\omega_0 + df_0 = 0$ and that $f_1 \in L^q_c(Q;\Lambda^\kappa)$, $1 \le q < N/(N-1)$.

Fix some $1 < r < N/(N-1)$. By Lemma 2.3, there exists some $\eta_1 \in W^{1,r}_{loc}(\mathbb{R}^N;\Lambda^{\kappa-1})$ such that $d\eta_1 = f_1$. Set $\omega_1 := \zeta_1\eta_1$. Then $\omega_1 \in W^{1,r}_c(Q;\Lambda^{\kappa-1})$ and, as above, $f_2 := f_1 - d\omega_1$ satisfies $df_2 = 0$ and $f_2 \in W^{1,r}_c(Q;\Lambda^\kappa)$. Applying again Lemma 2.3, we may find $\eta_2 \in W^{2,r}_{loc}(\mathbb{R}^N)$ such that $d\eta_2 = f_2$.

Iterating the above, we have

$$\omega_0 + \cdots + \omega_j \in L^q_c(Q;\Lambda^{\kappa-1}),\ 1 \le q < N/(N-1),$$

$$d(\omega_0 + \cdots + \omega_j) = f_0 - f_j,\ \text{with } df_j = 0 \text{ and } f_j \in W^{j,r}_c(Q;\Lambda^\kappa).$$

Let now j be such that $W^{j,q}(Q) \hookrightarrow C^m(Q)$, with m as in Lemma 2.4. Let $\xi \in C^0_c(Q;\Lambda^{\kappa-1})$ be such that $d\xi = -f_j$. Set $\omega := \omega_0 + \cdots + \omega_j + \xi$. Then ω has all the required properties. □

Let us note the following consequence of hypoellipticity of Δ and of the proofs of Proposition 2.5 and Lemmas 2.2 and 2.3 (but *not* of their statements).

COROLLARY 2.6. *Assume, in addition to the hypotheses of Proposition 2.5, that $f \in C^\infty(U)$ for some open set $U \subset Q$. Let $s \in \mathbb{N}$. Then we may choose ω such that, in addition, $\omega \in C^s(U)$.*

PROOF OF THEOREM 2.1. We write the variables in $\mathbb{R}^N$ as follows: $x = (x', x'')$, with $x' \in \mathbb{R}^\ell$ and $x'' \in \mathbb{R}^{N-\ell}$.

Pick some $g \in L^1_c((0,1)^\ell;\mathbb{R})$ with zero integral, such that the equation $\operatorname{div} X = g$ has no solution $X \in L^{\ell/(\ell-1)}_{loc}(\mathbb{R}^\ell;\mathbb{R}^\ell)$ (see [**6**], [**2**]). Clearly, for any $G \in C^2_c((0,1)^\ell;\mathbb{R})$,

$$\text{the equation } \operatorname{div} Y = g + G \text{ has no solution } Y \in L^{\ell/(\ell-1)}_{loc}(\mathbb{R}^\ell;\mathbb{R}^\ell). \tag{2.3}$$

Let $\psi \in C_c^\infty((0,1)^{N-\ell})$ be such that $\psi \equiv 1$ in some nonempty open set $V \subset (0,1)^{N-\ell}$. Set $Q := (0,1)^N$ and $\eta := g(x')\psi(x'')dx' \in L_c^1(Q;\Lambda^\ell)$. We note that $d\eta = g(x')d\psi(x'') \wedge dx' \in L_c^1(Q;\Lambda^{\ell+1})$. Let us also note that $d\eta = 0$ in $\mathbb{R}^\ell \times V$. By Corollary 2.6 with $U = (0,1)^\ell \times V$, there exists some $\omega \in L_c^q(Q;\Lambda^\ell)$, $1 \le q < N/(N-1)$, such that $d\omega = d\eta$ and $\omega \in C^2((0,1)^\ell \times V)$.

Consider now the closed form $f := \eta - \omega \in L_c^1(Q;\Lambda^\ell)$. We claim that there exists no $\lambda \in W_{loc}^{1,1}(\mathbb{R}^N;\Lambda^{\ell-1})$ such that $d\lambda = f$. Argue by contradiction and let λ_i denote the coefficient, in λ, of $dx_1 \wedge dx_2 \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_\ell$, $1 \le i \le \ell$. Let ω_0 denote the coefficient of dx' in ω. Then, in $\mathbb{R}^\ell \times V$, we have

$$\sum_{i=1}^{\ell}(-1)^{i+1}\partial_i\lambda_i(x',x'') = g(x')\psi(x'') - \omega_0(x',x'') = g(x') - \omega_0(x',x''). \tag{2.4}$$

Hence, for a.e. $x'' \in V$, the following equation is satisfied in $\mathscr{D}'(\mathbb{R}^\ell)$:

$$\sum_{i=1}^{\ell}(-1)^{i+1}\partial_i\lambda_i' = g - \omega_0', \tag{2.5}$$

with

$$\lambda_i' := \lambda_i(\cdot,x'') \in W_{loc}^{1,1}(\mathbb{R}^\ell) \text{ and } \omega_0' = \omega_0(\cdot,x'') \in C_c^2((0,1)^\ell). \tag{2.6}$$

The above properties (2.5) and (2.6), combined with the embedding $W_{loc}^{1,1}(\mathbb{R}^\ell) \hookrightarrow L^{\ell/(\ell-1)}(\mathbb{R}^\ell)$, contradict (2.3). $\square$

REMARK 2.7. We have actually proved the following improvement of Theorem 2.1. Let $N \ge 3$ and $2 \le \ell \le N-1$. Then there exists some $f \in L_c^1(\mathbb{R}^d;\Lambda^\ell)$ satisfying $df = 0$ and such that the system $d\lambda = f$ has no solution

$$\lambda \in L_{loc}^1(\mathbb{R}^{(N-\ell)}; L_{loc}^{\ell/(\ell-1)}(\mathbb{R}^\ell;\Lambda^{\ell-1})).$$

REMARK 2.8. A similar question can be raised in L^∞. We have the following analogue of Theorem 2.1.

THEOREM 2.9. *Let $N \ge 3$. Let $2 \le \ell \le N-1$. Then there exists some $f \in L_c^\infty(\mathbb{R}^N;\Lambda^\ell)$ such that $df = 0$ and the equation $d\lambda = f$ has no solution $\lambda \in W_{loc}^{1,\infty}(\mathbb{R}^N;\Lambda^{\ell-1})$.*

The proof of Theorem 2.9 is very similar to the one of Theorem 3. The main difference is the starting point, in dimension ℓ. Here, we use the fact that there exists some $g \in L_c^\infty(\mathbb{R}^\ell)$, with zero integral, such that the equation $\operatorname{div} X = g$ has no solution $X \in W_{loc}^{1,\infty}(\mathbb{R}^\ell;\mathbb{R}^\ell)$ (see [**5**]).

3. Solution in $L^{N/(N-1)}$ when $1 \le \ell \le N-1$

As mentioned in the introduction, when $\ell = N$, the system (2.1) with right-hand side $f \in L^1$ need not have a solution $\lambda \in L_{loc}^{N/(N-1)}$. In view of Theorem 2.1 and of Proposition 2.5, it is natural to ask whether, in the remaining cases $1 \le \ell \le N-1$, given a closed ℓ-form $f \in L_c^1$, it is possible to solve (2.1) with $\lambda \in L_{loc}^{N/(N-1)}$. This is clearly the case when $\ell = 1$ (by the Sobolev embedding $W_{loc}^{1,1} \hookrightarrow L_{loc}^{N/(N-1)}$). Moreover, we may pick $\lambda \in W^{1,1}$. The remaining cases are settled by our next result. In what follows, we do not make any support assumption on f, and therefore the case where $\ell = 1$ is also of interest.

PROPOSITION 2.10. *Let $N \ge 2$ and $1 \le \ell \le N-1$. Then, for every $f \in L^1(\mathbb{R}^N;\Lambda^\ell)$ with $df = 0$, there exists some $\lambda \in L^{N/(N-1)}(\mathbb{R}^N;\Lambda^{\ell-1})$ such that $f = d\lambda$.*

PROOF. Suppose $f \in L^1(\mathbb{R}^N;\Lambda^{\ell-1})$ with $df = 0$ as above. According to Bourgain and Brezis [**3**] (see [**3**, Corollary 20] for a very similar statement; see also [**7**, Theorem 3]), we have

$$\left|\int_{\mathbb{R}^d}\langle\psi,f\rangle\right| \lesssim \|f\|_{L^1}\,\|d^*\psi\|_{L^N},\ \forall\psi \in C_c^\infty(\mathbb{R}^N;\Lambda^\ell). \tag{2.7}$$

Consider the functional

$$L_f : S = \{d^*\psi;\ \ \psi \in C_c^\infty(\mathbb{R}^N;\Lambda^\ell)\} \to \mathbb{R},\ L_f\left(d^*\psi\right) := \int_{\mathbb{R}^d} \langle \psi, f \rangle.$$

Here, S is endowed with the L^N-norm. The inequality (2.7) shows that L_f is well-defined and bounded. By the Hahn-Banach theorem, there exists an extension $\widetilde{L}_f : L^N(\mathbb{R}^N;\Lambda^{\ell+1}) \to \mathbb{R}$ of L_f with $\|\widetilde{L}_f\| = \|L_f\|$. Hence, there exists an $(\ell-1)$-form $\lambda \in L^{N/(N-1)}(\mathbb{R}^N;\Lambda^{\ell-1})$ such that

$$\int_{\mathbb{R}^N} \langle \psi, f \rangle = L_f\left(d^*\psi\right) = \widetilde{L}_f\left(d^*\psi\right) = \int_{\mathbb{R}^N} \langle d^*\psi, \lambda \rangle = \int_{\mathbb{R}^N} \langle \psi, d\lambda \rangle$$

for all ℓ -forms $\psi \in C_c^\infty(\mathbb{R}^N;\Lambda^\ell)$.

This implies that $\lambda \in L^{N/(N-1)}(\mathbb{R}^N;\Lambda^{\ell-1})$ satisfies $d\lambda = f$. □

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CHAPTER 3

Hodge systems with L^1 sources

Let $d \geq 2$. In [**3**], Bourgain and Brezis proved that there exists $g \in L^1_c(\mathbb{R}^d)$ with zero integral, such that the equation $\operatorname{div} X = g$ has no solution $X \in W^{1,1}_{loc}(\mathbb{R}^d)$, and actually not even in $L^{d'}(\mathbb{R}^d)$. Using this result, we prove that whenever $d \geq 3$ and $2 \leq l \leq d$, there exists some $G \in L^1_c(\mathbb{R}^d;\Lambda^l)$ such that $dG = 0$ and the equation $dF = G$ has no solution $F \in \dot{W}^{1,1}(\mathbb{R}^d;\Lambda^{l-1})$. This was originally proved in [**6**] by completely different methods, and answers negatively a question in [**1**].

1. Short introduction

Our goal is to prove the following:

THEOREM 3.1. *Suppose $d \geq 2$ and $l \in \{2,\dots,d\}$. There exists an l-form $G \in L^1_c$ on $\mathbb{R}^d$ with $dG = 0$, whose coefficients have zero integral, and such that there is no $(l-1)$-form $F \in \dot{W}^{1,1}$ on $\mathbb{R}^d$ with $G = dF$.*

This was essentially proved in [**6**] (providing a negative answer to a question in [**1**, p. 6]) by reducing the problem to the study of the divergence equation on a lower dimensional subspace of $\mathbb{R}^d$, and then using the nonexistence result from [**3**]. More exactly, in [**6**] (see also Chapter 2) it was shown that, if Theorem 3.1 fails to be true, then, for any $g \in L^1_c\left(\mathbb{R}^l\right)$ with zero integral, there exists a vector field $Y \in L^{l/(l-1)}_{loc}\left(\mathbb{R}^l,\mathbb{R}^l\right)$ and a sufficiently smooth (C^2 and compactly supported) remainder G, such that $\operatorname{div} Y = g + G$. (The smoothness of G was established by using the hypoellipticity of the operator $\triangle$.) However, the existence, *for each* $g \in L^1$, of such Y and G contradicts the nonexistence result in [**3**, Section 2.1].

In this chapter, we prove the above result via a completely different approach, that may be useful in more general problems. More specifically, we reduce the problem to the study of the divergence equation in $\mathbb{R}^2$ (here, the dimension is two for every value of l). On the other hand, instead of proving the smoothness of the remainder G (which will be different and with a different role) we prove its smallness in some appropriate Besov norm. The key property that we will use is the boundedness of the Calderón-Zygmund operators on the homogeneous Besov spaces, even for the "limit" parameter $p = 1$ (see Lemma 3.4).

2. Proof of Theorem 3.1

We will need several lemmas.

LEMMA 3.2. *We have* [1]

$$\left(L^1(\mathbb{R}^d), \dot{W}^{-2,1}(\mathbb{R}^d)\right)_{1/2,1} = \dot{B}^{-1,1}_1(\mathbb{R}^d).$$

Here, for a positive integer m, $\dot{W}^{-m,1}(\mathbb{R}^d)$ is defined as being the space of those distributions f on $\mathbb{R}^d$ for which there exists a family of functions $(F_\alpha)_{|\alpha|=m}$ in $L^1(\mathbb{R}^d)$ such that

$$f = \sum_{|\alpha|=m} \nabla^\alpha F_\alpha.$$

[1]Here, $\dot{B}^{-1,1}_1$ is the Banach space obtained as the closure of the space of Schwartz functions in the norm of $\dot{B}^{-1,1}_1$. This definition does not coincide with the definition of the homogeneous Besov spaces given in the Section 5 of the Introduction Chapter. However, for simplicity we keep this notation throughout this chapter.

The space $\dot{W}^{-m,1}(\mathbb{R}^d)$ is endowed with the following norm:

$$\|f\|_{\dot{W}^{-m,1}} = \inf\left\{ \sum_{|\alpha|=m} \|F_\alpha\|_{L^1} \;\middle|\; f = \sum_{|\alpha|=m} \nabla^\alpha F_\alpha \right\}.$$

In particular, the elements of $\dot{W}^{-m,1}(\mathbb{R}^d)$ are distributions of order $\le m$.

PROOF. We adapt the method in [**2**, p. 143]. To obtain the embedding

$$\left(L^1, \dot{W}^{-2,1}\right)_{1/2,1} \hookrightarrow \dot{B}_1^{-1,1}, \tag{3.1}$$

we use the K-method.

We recall that for a compatible couple (A_0, A_1) of Banach spaces, the K functional is defined as follows (see [**2**, Chapter 3]):

$$K(f,t,A_0,A_1) := \inf_{f=f_0+f_1} \left(\|f_0\|_{A_0} + t\,\|f_1\|_{A_1}\right),$$

for any $f \in A_0 + A_1$ and $t > 0$. For any constant $\mu > 0$, the norm of the space $(A_0, A_1)_{\theta,q,K}$ (where $\theta \in (0,1)$ and $1 \le q \le \infty$) satisfies the equivalence (see [**2**, Lemma 3.1.3])

$$\|f\|_{(A_0,A_1)_{\theta,q,K}} \sim \left(\sum_{\nu\in\mathbb{Z}} 2^{-\mu\nu\theta q} K^q\left(f, 2^{\mu\nu}, A_0, A_1\right) \right)^{1/q}.$$

In what follows, we let $\mu = 2$.

We now return to (3.1). Consider a distribution $f \in L^1 + \dot{W}^{-2,1}$ and a decomposition $f = f_0 + f_1$ with $f_0 \in L^1$ and $f_1 \in \dot{W}^{-2,1}$.

There exists a family of functions $f_\alpha \in L^1$ such that

$$f_1 = \sum_{|\alpha|=2} \nabla^\alpha f_\alpha \quad \text{with} \quad \sum_{|\alpha|=2} \|f_\alpha\|_{L^1} \le 2\,\|f_1\|_{\dot{W}^{-2,1}},$$

and therefore we have

$$\left\|P_j f_1\right\|_{L^1} \le \sum_{|\alpha|=2} \left\|\nabla^\alpha P_j f_\alpha\right\|_{L^1} \lesssim 2^{2j} \sum_{|\alpha|=2} \left\|P_j f_\alpha\right\|_{L^1} \lesssim 2^{2j} \sum_{|\alpha|=2} \|f_\alpha\|_{L^1} \lesssim 2^{2j}\,\|f_1\|_{\dot{W}^{-2,1}}, \tag{3.2}$$

where P_j are the homogeneous Paley-Littlewood "projections". For the second inequality in (3.2), we have used the "direct" Nikolskiĭ's inequality

$$\|D^k f\|_{L^p} \lesssim R^k\,\|f\|_{L^p} \text{ if } \operatorname{supp}\widehat{f} \subset \{|\xi| \le R\} \tag{3.3}$$

for any $1 \le p \le \infty$.

For further use, let us also note the "inverse" Nikolskiĭ's inequality

$$\|D^k f\|_{L^p} \approx R^k\,\|f\|_{L^p} \text{ if } \operatorname{supp}\widehat{f} \subset \{R/2 \le |\xi| \le 2R\} \tag{3.4}$$

For (3.3) and (3.4), see e.g. [**4**, Lemma 2.1.1].

From (3.2) we have, for all $j \in \mathbb{Z}$,

$$2^{-j}\left\|P_j f\right\|_{L^1} \le 2^{-j}\left(\left\|P_j f_0\right\|_{L^1} + \left\|P_j f_1\right\|_{L^1}\right) \lesssim 2^{-j}\left(\left\|P_j f_0\right\|_{L^1} + 2^{2j}\,\|f_1\|_{\dot{W}^{-2,1}}\right),$$

which gives

$$2^{-j}\left\|P_j f\right\|_{L^1} \lesssim 2^{-j} K(f, 2^{2j}, L^1, \dot{W}^{-2,1})$$

and hence, by summing up,

$$\|f\|_{\dot{B}_1^{-1,1}} \lesssim \sum_j 2^{-2(1/2)j} K(f, 2^{2j}, L^1, \dot{W}^{-2,1}) \lesssim \|f\|_{1/2,1,K}.$$

To obtain the embedding

$$\dot{B}_1^{-1,1} \hookrightarrow \left(L^1, \dot{W}^{-2,1}\right)_{1/2,1} \tag{3.5}$$

we use the J-method.

We recall that for a compatible couple (A_0, A_1) of Banach spaces, the J functional is defined as follows (see [**2**, Chapter 3]):

$$J(f,t,A_0,A_1) := \max\left(\|f\|_{A_0}, t\|f\|_{A_1}\right),$$

for any $f \in A_0 \cap A_1$ and $t > 0$. For any constant $\mu > 0$, the norm of the space $(A_0,A_1)_{\theta,q,J}$ (where $\theta \in (0,1)$ and $1 \le q \le \infty$) satisfies the equivalence (see [**2**, Lemma 3.2.3])

$$\|f\|_{(A_0,A_1)_{\theta,q,J}} \sim \inf\left(\sum_{\nu\in\mathbb{Z}} 2^{-\mu\nu\theta q} J^q\left(u_\nu, 2^{\mu\nu}, A_0, A_1\right)\right)^{1/q}.$$

Here, the infimum is taken over all the representations

$$f = \sum_{\nu\in\mathbb{Z}} u_\nu, \text{ with convergence in } A_0 + A_1,$$

where $u_\nu \in A_0 \cap A_1$ for all $\nu \in \mathbb{Z}$. In what follows, we let $\mu = 2$.

We now return to (3.5). For $f \in \dot{B}_1^{-1,1}$, we have

$$\|f\|_{1/2,1,J} \lesssim \sum_j 2^{-2(1/2)j} J(P_j f, 2^{2j}, L^1, \dot{W}^{-2,1}). \tag{3.6}$$

Let ψ_0 be the Schwartz function satisfying $P_0 g = g * \psi_0$, $\forall g \in \mathscr{S}'$. Consider a Schwartz function Φ with $0 \notin \operatorname{supp}\widehat{\Phi}$ and such that $\widehat{\Phi} = 1$ on $\operatorname{supp}\psi_0$. Then, clearly, we have $P_j g * \Phi_j = P_j g$ for all j and any Schwartz function g. Here, $\Phi_j(x) := 2^{jd}\Phi(2^j x)$. Therefore, the function $\phi := \triangle^{-1}\Phi$ (noticing that $\widehat{\phi}(\xi) = -|\xi|^{-2}\widehat{\Phi}(\xi)$) is Schwartz and we have

$$\begin{aligned}\left\|P_j f\right\|_{\dot{W}^{-2,1}} &= \left\|P_j f * \Phi_j\right\|_{\dot{W}^{-2,1}} = \left\|P_j f * \left(\triangle\phi\right)_j\right\|_{\dot{W}^{-2,1}} = 2^{-2j}\left\|P_j f * \triangle\phi_j\right\|_{\dot{W}^{-2,1}} \\ &= 2^{-2j}\left\|\triangle P_j f * \phi_j\right\|_{\dot{W}^{-2,1}} \lesssim 2^{-2j}\left\|\triangle(P_j f)\right\|_{\dot{W}^{-2,1}} \le 2^{-2j}\left\|P_j f\right\|_{L^1}.\end{aligned}$$

Therefore, we have $J(P_j f, 2^{2j}, L^1, \dot{W}^{-2,1}) \lesssim \left\|P_j f\right\|_{L^1}$. Combining this with (3.6), we obtain

$$\|f\|_{1/2,1,J} \lesssim \sum_j 2^{-j}\left\|P_j f\right\|_{L^1} \lesssim \|f\|_{\dot{B}_1^{-1,1}},$$

whence the conclusion of the lemma. □

The following consequence of the above lemma will be used in the proof of Theorem 3.1.

LEMMA 3.3. *Let $\varepsilon \in (0,1)$ and define the operator T_ε by $T_\varepsilon\varphi(x) := \varepsilon^{d-2}\varphi(x', \varepsilon x'')$, for $\varphi \in C_c^\infty\left(\mathbb{R}^d\right)$, where $x' \in \mathbb{R}^2$, $x'' \in \mathbb{R}^{d-2}$ and $x = (x', x'') \in \mathbb{R}^d$. Then,*

$$\|T_\varepsilon\|_{\dot{B}_1^{-1,1}\to\dot{B}_1^{-1,1}} \le \frac{C}{\varepsilon}$$

for some $0 < C < \infty$.

PROOF. We see that

$$\|T_\varepsilon\|_{L^1\to L^1} = 1. \tag{3.7}$$

Consider now $\varphi \in \dot{W}^{-2,1}$ and write

$$\varphi = \sum_{\substack{\alpha\in\mathbb{N}^2,\beta\in\mathbb{N}^{d-2}\\ |\alpha|+|\beta|=2}} \nabla_{x'}^\alpha \nabla_{x''}^\beta F_{\alpha\beta}, \tag{3.8}$$

where the functions $F_{\alpha\beta} \in L^1$ satisfy the inequality

$$\sum_{\substack{\alpha\in\mathbb{N}^2,\beta\in\mathbb{N}^{d-2}\\ |\alpha|+|\beta|=2}} \left\|F_{\alpha\beta}\right\|_{L^1} \le 2\left\|\varphi\right\|_{\dot{W}^{-2,1}}.$$

From (3.8), we have

$$T_\varepsilon\varphi(x) = \varepsilon^{d-2}\varphi(x', \varepsilon x'') = \sum_{\substack{\alpha\in\mathbb{N}^2,\beta\in\mathbb{N}^{d-2}\\ |\alpha|+|\beta|=2}} \nabla_{x'}^\alpha \nabla_{x''}^\beta \left(\varepsilon^{d-2-|\beta|} F_{\alpha\beta}(x', \varepsilon x'')\right),$$

and thus

$$\left\|T_\varepsilon\varphi\right\|_{\dot W^{-2,1}} \le \sum_{\substack{\alpha\in\mathbb{N}^2,\beta\in\mathbb{N}^{d-2}\\|\alpha|+|\beta|=2}} \varepsilon^{-|\beta|}\left\|F_{\alpha\beta}\right\|_{L^1} \le \frac{1}{\varepsilon^2}\sum_{\substack{\alpha\in\mathbb{N}^2,\beta\in\mathbb{N}^{d-2}\\|\alpha|+|\beta|=2}}\left\|F_{\alpha\beta}\right\|_{L^1} \le \frac{2}{\varepsilon^2}\left\|\varphi\right\|_{\dot W^{-2,1}}.$$

We obtain

$$\|T_\varepsilon\|_{\dot W^{-2,1}\to\dot W^{-2,1}} \le \frac{2}{\varepsilon^2}. \tag{3.9}$$

From this, (3.7) and Lemma 3.2 we get

$$\begin{aligned}\|T_\varepsilon\|_{\dot B_1^{-1,1}\to\dot B_1^{-1,1}} &\le C\,\|T_\varepsilon\|_{(L^1(\mathbb{R}^d),\dot W^{-2,1}(\mathbb{R}^d))_{1/2,1}\to(L^1(\mathbb{R}^d),\dot W^{-2,1}(\mathbb{R}^d))_{1/2,1}}\\ &\le C\,\|T_\varepsilon\|_{L^1\to L^1}^{1/2}\,\|T_\varepsilon\|_{\dot W^{-2,1}\to\dot W^{-2,1}}^{1/2} \le C\frac{1}{\varepsilon},\end{aligned}$$

whence the claim. □

We will also need the following well-known result concerning the boundedness of the Calderón-Zygmund operators (see Section 1 of the Introduction Chapter for a definition) on the homogeneous Besov spaces (see [**7**, Corollary 6.7.2, p. 96]):

LEMMA 3.4. *Suppose K is a Calderón-Zygmund operator and $s\in\mathbb{R}$, $1\le p,q\le\infty$ are given. Then, for any Schwartz function f on $\mathbb{R}^d$ we have*

$$\|Kf\|_{\dot B_q^{s,p}} \lesssim \|f\|_{\dot B_q^{s,p}}.$$

Combining the "lifting property"

$$\|f\|_{\dot B_q^{s,p}} \sim \|\nabla f\|_{\dot B_q^{s-1,p}}$$

(which is a straightforward consequence of (3.4)) with the above lemma applied with $s=0$, $p=q=1$ for the Calderón-Zygmund operator $\nabla\triangle^{-1}d^*$ (again, see see Section 1 of the Introduction Chapter), we obtain that for any $(l-1)$-form v with Schwartz coefficients, the following inequality holds

$$\left\|\triangle^{-1}d^*v\right\|_{\dot B_1^{0,1}} \sim \left\|\left(\nabla\triangle^{-1}d^*\right)v\right\|_{\dot B_1^{-1,1}} \lesssim \|v\|_{\dot B_1^{-1,1}}. \tag{3.10}$$

In order to prove Theorem 3.1 it suffices to prove the following fact:

PROPOSITION 3.5. *Let $r>0$. Suppose $d\ge 2$ and $l\in\{2,\dots,d\}$. There exists an l-form $G\in L^1_c(B(0,r))$ with $dG=0$ and whose coefficients have zero integral, such that there is no $(l-1)$-form $F\in W_c^{1,1}(B(0,3r))$ on $\mathbb{R}^d$ with $G=dF$.*

(Here $L^1_c(B(0,r))$ is the space of L^1-functions which are supported in $B(0,r)$ and $W_c^{1,1}(B(0,3r))$ is the space of $W^{1,1}$ functions which are supported in $B(0,3r)$. Note that the main difference between Theorem 3.1 and Proposition 3.5 is that Proposition 3.5 involves the inhomogeneous Sobolev space, while Theorem 3.1 involves the homogenous space.)

PROOF THAT PROPOSITION 3.5 IMPLIES THEOREM 3.1. We prove that, assuming Theorem 3.1 is false, then Proposition 3.5 must be false too.

Suppose Theorem 3.1 is false. Then, for any closed l-form $G\in L^1_c(B(0,r))$ whose coefficients have zero integral, one can find an $(l-1)$-form $F\in\dot W^{1,1}$ such that $dF=G$. By using the open mapping theorem, we can choose F such that

$$dF=G \text{ and } \|\nabla F\|_{L^1}\lesssim\|G\|_{L^1}. \tag{3.11}$$

Fix a closed l-form $G\in L^1$ as above and with $\operatorname{supp}G\subset B(0,r)$ (for some $r>0$) and let $\eta\in C_c^\infty(B(0,2r))$ be a function such that $\eta\equiv 1$ on $B(0,r)$. Decompose the form F given in (3.11) as

$$F=\sum_{|I|=l-1}F_I dx_I.$$

By considering regularizations with convolutions, we can assume without loss of generality that G and F are smooth. We define a new $(l-1)$-form $F^1 := \eta(F-c)$ where the multiplication is considered component-wise and $c=(c_I)_{|I|=l-1}$ is the vector with $\begin{pmatrix} d \\ l-1 \end{pmatrix}$ components defined by

$$c_I := \int_{B(0,2r)} F_I.$$

From Poincaré's inequality, (3.11) and the properties of η, we find that F^1 satisfies

$$\|F^1\|_{L^1} + \|\nabla F^1\|_{L^1} \lesssim \|G\|_{L^1}, \tag{3.12}$$

$$\operatorname{supp} F^1 \subseteq B(0,2r), \tag{3.13}$$

$$dF^1 = d\left(\eta(F-c)\right) = \eta\, d(F-c) + G^1 = G + G^1, \tag{3.14}$$

where G^1 is an l-form whose coefficients are linear combinations of products between the coefficients of $(F-c)$ and the derivatives of η. As in (3.12), we have

$$\|G^1\|_{L^1} + \|\nabla G^1\|_{L^1} \lesssim \|G\|_{L^1}. \tag{3.15}$$

Note that $G^1 \in W^{1,1}_c(B(0,2r))$. Thanks to Gagliardo's embedding we get $G^1 \in L^{d'}_c(B(0,2r))$ and, using (3.15),

$$\|G^1\|_{L^{d'}} \lesssim \|G\|_{L^1}.$$

By an inspection of Bogovskiĭ's formula (see [**5**, Corollary 3.3 and Corollary 3.4]) one can find a compactly supported $(l-1)$-form $F^2 \in W^{1,d'}_c(B(0,3r))$ satisfying $dF^2 = G^1$ and such that

$$\|F^2\|_{W^{1,d'}} \lesssim \|G^1\|_{L^{d'}} \lesssim \|G\|_{L^1}.$$

Now, if $F' := F^1 - F^2$, then, from (3.14), we have $dF' = G$. Since $W^{1,d'}_c(B(0,3r)) \hookrightarrow W^{1,1}_c(B(0,3r))$, we have

$$\|F'\|_{W^{1,1}} \lesssim \|F^1\|_{W^{1,1}} + \|F^2\|_{W^{1,d'}} \lesssim \|G\|_{L^1}.$$

To summarize, as claimed, we have proved that for each compactly supported l-form $G \in L^1$, whose coefficients have zero integral, there exists an $(l-1)$-form $F' \in W^{1,1}_c(B(0,3r))$ such that $dF' = G$. This completes the proof. □

Now, we are going to prove Proposition 3.5. We argue by contradiction. If Proposition 3.5 *does not hold*, then we have the following consequence (that we will later disprove, in order to obtain a contradiction).

LEMMA 3.6. *Assume that Proposition 3.5 does not hold for some $d \geq 2$ and $l \in \{2,\dots,d\}$. Let $r > 0$. Then for any l-form $G \in L^1_c(B(0,r))$ with $dG = 0$ and whose coefficients have zero integral, there exists an $(l-1)$-form $F \in W^{1,1}_c(B(0,3r))$ on $\mathbb{R}^d$ with $G = dF$. Moreover, we can choose F such that*

$$\|F\|_{\dot W^{1,1}} \leq C\,\|G\|_{L^1}, \tag{3.16}$$

where C is a constant independent of r.

Indeed, this follows from a scaling argument. Suppose e.g. that Proposition 3.5 does not hold for $r = 1$. Than one can use the open mapping theorem in order to chose F satisfying (3.16), with a constant independent of G. To see that the constant C remains the same when r changes, it suffices to note that, for any $r > 0$, we have $d(rF(x/r)) = G(x/r)$ and

$$\left\| G\left(\frac{\cdot}{r}\right)\right\|_{L^1} = r^d\,\|G\|_{L^1}$$

and

$$\left\| rF\left(\frac{\cdot}{r}\right)\right\|_{\dot W^{1,1}} = r^d\,\|F\|_{\dot W^{1,1}}.$$

Hence (3.16) is dilation-invariant.

PROOF (BY CONTRADICTION) OF PROPOSITION 3.5. We observe that it suffices to consider only the case where $d > 2$. Indeed, when $d = 2$ the only possible Hodge system to which Proposition 3.5 applies is the divergence equation; as we mentioned above this case was already settled in [**3**].

Let $g \in C_c^\infty(\mathbb{R}^2)$ and a nonnegative $\psi \in C_c^\infty(\mathbb{R}^{d-2})$ satisfying the conditions

$$\int_{\mathbb{R}^2} g = 0, \quad \int_{\mathbb{R}^{d-2}} \psi = 1.$$

Via a standard explicit construction, we may find $h, k \in C_c^\infty(\mathbb{R}^2)$ such that

$$g = \partial_1 h + \partial_2 k \text{ on } \mathbb{R}^2. \tag{3.17}$$

Consider now the l-form on $\mathbb{R}^d$:

$$\begin{aligned} G_\varepsilon :=& \left(g \otimes \psi_\varepsilon\right) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_l + (-1)^{l+1} \sum_{i=l+1}^{d} \left(h \otimes \partial_i \psi_\varepsilon\right) dx_2 \wedge dx_3 \wedge \ldots \wedge dx_l \wedge dx_i \\ &+ (-1)^l \sum_{i=l+1}^{d} \left(k \otimes \partial_i \psi_\varepsilon\right) dx_1 \wedge dx_3 \wedge \ldots \wedge dx_l \wedge dx_i, \end{aligned} \tag{3.18}$$

where $\psi^\varepsilon(x'') := \varepsilon^{d-2}\psi(\varepsilon x'')$. Here, if g_1 is a function on $\mathbb{R}^2$ and g_2 is a function on $\mathbb{R}^{d-2}$, we write $g_1 \otimes g_2$ for the function

$$(g_1 \otimes g_2)(x) := g_1\left(x'\right) g_2\left(x''\right), \quad x = \left(x', x''\right) \in \mathbb{R}^2 \times \mathbb{R}^{d-2}.$$

Computing dG_ε by using (3.17), one obtains:

$$\begin{aligned} dG_\varepsilon =& (-1)^l \sum_{l+1 \le i \le d} \left(g \otimes \partial_i \psi^\varepsilon\right) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_l \wedge dx_i \\ &+ (-1)^{l+1} \sum_{i=l+1}^{d} \left(\partial_1 h \otimes \partial_i \psi^\varepsilon\right) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_l \wedge dx_i \\ &+ (-1)^{l+1} \sum_{i=l+1}^{d} \left(\partial_2 k \otimes \partial_i \psi^\varepsilon\right) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_l \wedge dx_i + R_\varepsilon \\ =& (-1)^l \sum_{l+1 \le i \le d} (g - \partial_1 h - \partial_2 k) \otimes \partial_i \psi^\varepsilon \; dx_1 \wedge dx_2 \wedge \ldots \wedge dx_l \wedge dx_i + R_\varepsilon = R_\varepsilon, \end{aligned} \tag{3.19}$$

where R_ε is an $(l+1)$-form whose coefficients are linear combinations of terms of the form $h \otimes \partial_{j_1}\partial_{i_1}\psi^\varepsilon$ and $k \otimes \partial_{j_2}\partial_{i_2}\psi^\varepsilon$, with $i_1, j_1, i_2, j_2 \in \{l+1, \ldots, d\}$.

We next note that $R_\varepsilon \in \dot{B}_1^{-1,1}(\mathbb{R}^d)$ and we have, using Lemma 3.3,

$$\begin{aligned} \|R_\varepsilon\|_{\dot{B}_1^{-1,1}} \le& C \left\|h \otimes \nabla_{x''}^2 \psi_\varepsilon\right\|_{\dot{B}_1^{-1,1}} + C \left\|k \otimes \nabla_{x''}^2 \psi_\varepsilon\right\|_{\dot{B}_1^{-1,1}} \\ =& C\varepsilon^2 \left\|h \otimes \left(\nabla_{x''}^2 \psi\right)^\varepsilon\right\|_{\dot{B}_1^{-1,1}} + C\varepsilon^2 \left\|k \otimes \left(\nabla_{x''}^2 \psi\right)^\varepsilon\right\|_{\dot{B}_1^{-1,1}} \\ =& C\varepsilon^2 \left\|T_\varepsilon\left(h \otimes \nabla_{x''}^2 \psi\right)\right\|_{\dot{B}_1^{-1,1}} + C\varepsilon^2 \left\|T_\varepsilon\left(k \otimes \nabla_{x''}^2 \psi\right)\right\|_{\dot{B}_1^{-1,1}} \\ \le& C'\varepsilon \left\|h \otimes \nabla_{x''}^2 \psi\right\|_{\dot{B}_1^{-1,1}} + C'\varepsilon \left\|k \otimes \nabla_{x''}^2 \psi\right\|_{\dot{B}_1^{-1,1}} = C_{h,k,\psi}\varepsilon. \end{aligned} \tag{3.20}$$

Since $dR_\varepsilon = 0$, the l-form $\omega_\varepsilon := \triangle^{-1} d^* R_\varepsilon$ satisfies (see (3.10))

$$\omega_\varepsilon \in \dot{B}_1^{0,1}(\mathbb{R}^d), \; d\omega_\varepsilon = R_\varepsilon \; \text{ and } \; \|\omega_\varepsilon\|_{\dot{B}_1^{0,1}} \le C \|R_\varepsilon\|_{\dot{B}_1^{-1,1}} \le C_{h,k,\psi}\varepsilon,$$

from which we get

$$\|\omega_\varepsilon\|_{L^1} \le \|\omega_\varepsilon\|_{\dot{B}_1^{0,1}} \le C_{h,k,\psi}\varepsilon. \tag{3.21}$$

To justify (3.21), we first observe that R_ε is a smooth compactly supported function. Therefore, we have

$$R_\varepsilon = \sum_{j \in \mathbb{Z}} P_j R_\varepsilon, \tag{3.22}$$

in the sense of distributions. By applying on both sides of (3.22) the operator $\triangle^{-1}d^*$, we find that

$$\omega_\varepsilon = \sum_{j\in\mathbb{Z}} P_j\omega_\varepsilon, \tag{3.23}$$

in the sense of distributions. Using (3.10) we obtain that, for any $j \in \mathbb{Z}$,

$$\left\|P_j\omega_\varepsilon\right\|_{L^1} \lesssim 2^{-j}\left\|P_jR_\varepsilon\right\|_{L^1},$$

and, since $R_\varepsilon \in \dot{B}_1^{-1,1}\left(\mathbb{R}^d\right)$, we get that the sum in (3.23) is absolutely convergent in L^1. Therefore, we have $\omega_\varepsilon \in L^1\left(\mathbb{R}^d\right)$. From the triangle inequality and (3.23), we obtain (3.21).

Suppose Proposition 3.5 is false. We show that, in the above construction, ω_ε can be replaced by a compactly supported form still satisfying an estimate of the form (3.21). For this purpose, we use the following lemma that follows from Lemma 3.6.

LEMMA 3.7. *Assume that Proposition 3.5 does not hold. Let $r>0$ and R be an $(l+1)$-form in $C_c^\infty(B(0,r))$ whose coefficients have zero integral and such that there exists a smooth l-form $\omega \in L^1\left(\mathbb{R}^d\right)$ with $d\omega = R$ and*

$$\|\omega\|_{L^1} \le C\,\|R\|_{\dot{B}_1^{-1,1}}$$

for some fixed constant C.

Then there exists an l-form $\omega' \in L_c^1(B(0,6r))$ such that

$$d\omega' = R \text{ and } \left\|\omega'\right\|_{L^1} \le C'\,\|R\|_{\dot{B}_1^{-1,1}}, \tag{3.24}$$

where C' depends only on C, but not on ω.

PROOF OF LEMMA 3.7. Suppose first that $r=1$. Consider a function $\eta \in C_c^\infty(B(0,2))$ such that $|\eta| \le 1$, $|\nabla\eta| \le 1$ and $\eta \equiv 1$ on $B(0,1)$. Setting $\omega^1 := \eta\omega$, we have that

$$\left\|\omega^1\right\|_{L^1} \le \|\omega\|_{L^1} \le C\,\|R\|_{\dot{B}_1^{-1,1}}. \tag{3.25}$$

The l-form ω^1 is supported on $B(0,2)$ and

$$d\omega^1 = \eta\, d\omega + R^1 = R + R^1, \tag{3.26}$$

where R^1 is an $(l+1)$-form whose coefficients are linear combinations of products between the coefficients of ω and the derivatives of η. Hence,

$$\left\|R^1\right\|_{L^1} \lesssim \|\nabla\eta\|_{L^\infty}\left\|\omega^1\right\|_{L^1} \lesssim \left\|\omega^1\right\|_{L^1} \le C\,\|R\|_{\dot{B}_1^{-1,1}}.$$

Clearly, the form R^1 is closed and (by (3.26), the fact that ω^1 is compactly supported, and the assumption that the coefficients of R have zero integral), its coefficients have zero integral.

On the other hand, R^1 is compactly supported in $B(0,2)$. Lemma 3.6 implies the existence of some $\omega^2 \in W_c^{1,1}(B(0,6))$ satisfying $d\omega^2 = R^1$ and

$$\left\|\omega^2\right\|_{L^1} \le \left\|\omega^2\right\|_{W^{1,1}} \lesssim \left\|R^1\right\|_{L^1} \lesssim C\,\|R\|_{\dot{B}_1^{-1,1}}. \tag{3.27}$$

Hence, if we set $\omega' := \omega^1 - \omega^2$, then we have $\omega' \in L_c^1(B(0,6))$, $d\omega' = R$ and thanks to (3.25), (3.27),

$$\left\|\omega'\right\|_{L^1} \lesssim C\,\|R\|_{\dot{B}_1^{-1,1}} \le C'\,\|R\|_{\dot{B}_1^{-1,1}}.$$

To obtain the statement for a general r, we use the same dilation argument as above: for any $r>0$, we have $d\left(r\omega'(x/r)\right) = R(x/r)$,

$$\left\|R\left(\frac{\cdot}{r}\right)\right\|_{\dot{B}_1^{-1,1}} = r^{d+1}\,\|R\|_{\dot{B}_1^{-1,1}}$$

and

$$\left\|r\omega'\left(\frac{\cdot}{r}\right)\right\|_{L^1} = r^{d+1}\left\|\omega'\right\|_{L^1}.$$

Hence (3.24) is dilation-invariant. □

PROOF OF PROPOSITION 3.5 COMPLETED. As we observed above, the coefficients of R_ε are sums of second derivatives of compactly supported smooth functions (see (3.19)), and one can easily check that $R = R_\varepsilon$ satisfies the hypotheses of Lemma 3.7. Thanks to the Lemma 3.7 and (3.21), we can find a compactly supported l-form ω'_ε such that $d\omega'_\varepsilon = R_\varepsilon$ and

$$\left\|\omega'_\varepsilon\right\|_{L^1} \le C_{h,k,\psi}\varepsilon, \tag{3.28}$$

where the constant $C_{h,k,\psi}$ does not depend on ε.

We have $d\left(G_\varepsilon - \omega'_\varepsilon\right) = dG_\varepsilon - R_\varepsilon = 0$ and hence, from (3.18) and (3.28), we can find (using Lemma 3.6) an $(l-1)$-form $F_\varepsilon \in W^{1,1}_c(\mathbb{R}^d)$ such that

$$G_\varepsilon = \omega'_\varepsilon + dF_\varepsilon, \tag{3.29}$$

$$\|F_\varepsilon\|_{\dot W^{1,1}} \le C\left\|G_\varepsilon - \omega'_\varepsilon\right\|_{L^1} \le C\|G_\varepsilon\|_{L^1} + C_{h,k,\psi}\varepsilon \le C\|g\|_{L^1} + C_{h,k,\psi}\varepsilon. \tag{3.30}$$

By identification of the coefficient $dx_I := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_l$ in (3.29), we see that

$$g \otimes \psi^\varepsilon = (G_\varepsilon)_I = \left(\omega'_\varepsilon\right)_I + (dF_\varepsilon)_I. \tag{3.31}$$

Consider now a nonnegative function $\varphi \in C^\infty_c(\mathbb{R}^2)$ with the integral equal to 1. Set

$$\varphi^\varepsilon(x') := \varepsilon^2 \varphi(x'),\ \forall x' \in \mathbb{R}^2.$$

Taking in (3.31) the convolution with

$$\varphi^\varepsilon \otimes \psi^\varepsilon = \varphi^\varepsilon \otimes \psi^\varepsilon(x', x'') = \varepsilon^d \left(\varphi \otimes \psi\right)(\varepsilon x', \varepsilon x'') = \varepsilon^d \left(\varphi \otimes \psi\right)(\varepsilon x)$$

and integrating in x'' on $\mathbb{R}^{d-2}$, we obtain that

$$g * \varphi^\varepsilon = \int_{\mathbb{R}^{d-2}} \left(\omega'_\varepsilon\right)_I * \left(\varphi^\varepsilon \otimes \psi^\varepsilon\right)(\cdot, x'')\,dx'' + \int_{\mathbb{R}^{d-2}} (dF_\varepsilon)_I * \left(\varphi^\varepsilon \otimes \psi^\varepsilon\right)(\cdot, x'')\,dx''. \tag{3.32}$$

(Here, we have used the fact that $\int_{\mathbb{R}^{d-2}} \psi^\varepsilon = 1$.)
Setting

$$f_{0\varepsilon} := \int_{\mathbb{R}^{d-2}} \left(\omega'_\varepsilon\right)_I * \left(\varphi^\varepsilon \otimes \psi^\varepsilon\right)(\cdot, x'')\,dx'',$$

we find, using (3.28), that

$$\|f_{0\varepsilon}\|_{L^1(\mathbb{R}^2)} \le \left\|\omega'_\varepsilon\right\|_{L^1(\mathbb{R}^d)} \le C_{h,k,\psi}\varepsilon. \tag{3.33}$$

On the other hand, we note that the second term on the right hand side of (3.32) can be rewritten as

$$\int_{\mathbb{R}^{d-2}} (dF_\varepsilon)_I * \left(\varphi^\varepsilon \otimes \psi^\varepsilon\right)(\cdot, x'')\,dx'' = \partial_1 f_{1\varepsilon} + \partial_2 f_{2\varepsilon},$$

for some $f_{1\varepsilon}, f_{2\varepsilon} \in W^{1,1}_c(\mathbb{R}^2)$ such that

$$\|f_{1\varepsilon}\|_{\dot W^{1,1}(\mathbb{R}^2)} + \|f_{2\varepsilon}\|_{\dot W^{1,1}(\mathbb{R}^2)} \le \|F_\varepsilon\|_{\dot W^{1,1}(\mathbb{R}^d)} \le C\|g\|_{L^1} + C_{h,k,\psi}\varepsilon. \tag{3.34}$$

Since $f_{1\varepsilon}$ and $f_{2\varepsilon}$ are compactly supported, we get by (3.34) and Gagliardo's embedding that

$$\|f_{1\varepsilon}\|_{L^2(\mathbb{R}^2)} + \|f_{2\varepsilon}\|_{L^2(\mathbb{R}^2)} \le C\|g\|_{L^1} + C_{h,k,\psi}\varepsilon. \tag{3.35}$$

(Note that the above constants *do not depend on* ε.)

Using the estimates (3.33), (3.35) and the Banach-Alaoglu theorem in $L^2(\mathbb{R}^2)$, we can pass to the limit $\varepsilon \to 0$ (possibly up to a subsequence) in the identity

$$g * \varphi^\varepsilon = f_{0\varepsilon} + \partial_1 f_{1\varepsilon} + \partial_2 f_{2\varepsilon} \tag{3.36}$$

and obtain the existence of some $f_1, f_2 \in L^2(\mathbb{R}^2)$ satisfying

$$g = \partial_1 f_1 + \partial_2 f_2 \text{ and } \|f_1\|_{L^2} + \|f_2\|_{L^2} \le C\|g\|_{L^1}. \tag{3.37}$$

This implies the existence of a solution in $L^2(\mathbb{R}^2)$ for the divergence equation on $\mathbb{R}^2$ with $L^1(\mathbb{R}^2)$ source terms. However, this was disproved in [**3**]. This contradiction achieves the proof of Proposition 3.5. □

REMARK 3.8. Warning: at the end of the day, we know that Lemma 3.6 and Lemma 3.7 are *wrong*.

REMARK 3.9. By following the above proof, with minor modifications, one can prove:

THEOREM 3.10. *Suppose $d \geq 2$ and $l \in \{2,...,d\}$. There exists an l-form $G \in L^\infty_c$ on $\mathbb{R}^d$ with $dG = 0$, whose coefficients have zero integral and such that there is no $(l-1)$-form $F \in \dot{W}^{1,\infty}$ on $\mathbb{R}^d$ with $G = dF$.*

Theorem 3.10 can also be obtained by adapting the argument in [**6**].

3. Appendix: Yet another proof

We give here a sketch for another proof of the main result. This proof is mainly based on a "compactness argument" (see Lemma 3.11 below) which enables us to reduce the problem on $\mathbb{R}^d$ to its analogue on $\mathbb{T}^d$. (Here we identify $\mathbb{T}^d$ with $[0,1)^d$.) The problem in this last case can be immediately solved by using the nonexistence result for the divergence equation in [**3**].

According to [**3**, Section 2] one can choose a function $g \in L^1(\mathbb{T}^2)$ with zero integral which is not the divergence of a $W^{1,1}$ vector field. Now, consider on $\mathbb{T}^d$ (assuming $d > 2$) the following l-form:

$$G := (g \otimes 1)\, dx_1 \wedge ... \wedge dx_l.$$

It is easy to see that $G \in L^1(\mathbb{T}^d)$ is closed and has the coefficients of integral zero, and still is not the exterior derivative of a $W^{1,1}$ form on $\mathbb{T}^d$. In fact, it is not possible to write G as $G = dF' - R'$ for some $(l-1)$-form $F' \in W^{1,1}(\mathbb{T}^d)$ and an l-form $R' \in L^{d'}(\mathbb{T}^d)$. Indeed, by looking at the coefficient corresponding to $I = \{1,...,l\}$ and integrating in $x_3,...,x_d$ (on $\mathbb{T}^{d-2}$), we get that $g = \operatorname{div} f - r$ on $\mathbb{T}^2$ for some vector field $f \in W^{1,1}(\mathbb{T}^2)$ and a function $r \in L^{d'}(\mathbb{T}^2)$. Since the integral of r is zero, r is the divergence of a vector field in $W^{1,d'}(\mathbb{T}^2) \hookrightarrow W^{1,1}(\mathbb{T}^2)$. Hence, g is the divergence of a $W^{1,1}$ vector field. This contradicts our choice of g.

We now explain how the problem on $\mathbb{R}^d$ reduces to its analogue on $\mathbb{T}^d$.

Let $k_1, k_2, ...$ be an enumeration of the elements of $\mathbb{Z}^d$ and consider the family of cubes $(Q_j)_{j\geq 1}$ defined by $Q_j := k_j + (0, 3/2)^d$, for $j \geq 1$. There exists a family $(\eta_j)_{j \geq 1}$ of functions with $\eta_j \in C^\infty_c(Q_j)$, $|\nabla \eta_j| \leq 1$ for each $j \geq 1$, and such that $\eta_1 + \eta_2 + ... = 1$ on $\mathbb{R}^d$.

We extend the above G by periodicity to $\mathbb{R}^d$ and we observe that

$$\sup_{j\geq 1} \|G\|_{L^1(Q_j)} =: C_G < \infty.$$

For each $j \geq 1$ we choose a vector $c_j \in \mathbb{R}^N$ (with $N = \binom{d}{l}$) such that the coefficients of $(G - c_j)\eta_j$ have zero integral. It is easy to see that $|c_j| \lesssim C_G$. We have

$$d\left((G - c_j)\eta_j\right) = (dG)\eta_j + G \wedge d\eta_j = G \wedge d\eta_j \in L^1_c(Q_j).$$

There exists $G^1_j \in L^{d'}_c(Q_j)$, whose coefficients have zero integral, such that $dG^1_j = G \wedge d\eta_j$ and $\left\|G^1_j\right\|_{L^{d'}} \lesssim C_G$ (see the proof of Lemma 3.6). Hence, $(G - c_j)\eta_j - G^1_j$ is closed and its coefficients have zero integral.

If Theorem 3.1 is false, then (see Lemma 3.6) there exists an $(l-1)$-form $F_j \in W^{1,1}_c(3Q_j)$ such that $dF_j = (G - c_j)\eta_j - G^1_j$ and

$$\|F_j\|_{W^{1,1}} \lesssim \left\|(G - c_j)\eta_j - G^1_j\right\|_{L^1} \lesssim C_G. \tag{3.38}$$

Let X_1 be the completion of the space of Schwartz functions on $\mathbb{R}^d$ under the following norm:

$$\|f\|_{X_1} := \sup_{j\geq 1} \|f\|_{L^1(Q_j)} + \|\nabla f\|_{L^1(\mathbb{R}^d)}.$$

In a similar way we define the space X_2 of the $L^{d'}_{loc}$ functions for which the following norm is finite:

$$\|f\|_{X_2} := \sup_{j\geq 1} \|f\|_{L^{d'}(Q_j)}.$$

Define $F := F_1 + F_2 + \ldots$. Thanks to (3.38), we easily see that $\|F\|_{X_1} \lesssim C_G$. Also,

$$dF = G + R, \tag{3.39}$$

where

$$R := -\sum_{j=1}^{\infty} \left(c_j \eta_j + G_j^1\right) \text{ with } \|R\|_{X_2} \lesssim C_G.$$

In order to transfer the problem on $\mathbb{T}^d$ we need the following lemma.

LEMMA 3.11. *Let $u \in X_1$ ($u \in X_2$) and consider the sequence*

$$u_n := \frac{1}{|B_n|} \sum_{\chi \in \mathbb{Z}^d} u\left(x + \chi\right),$$

where $B_n := B(0,n) \cap \mathbb{Z}^d$. Then, there exists a function $u' \in BV_{loc}\left(\mathbb{R}^d\right)$ ($u' \in X_2$) which is component-wise 1-periodic and $u_n \to u'$, up to a subsequence, in the sense of distributions.

This lemma can be proved on the same lines as Lemma 8.17 from Chapter 8.

From (3.39), we have $dF_n = G + R_n$, for all $n \geq 1$. Letting $n \to \infty$, and applying Lemma 3.11 we get that $dF' = G + R'$ for some component-wise 1-periodic forms $F' \in BV_{loc}\left(\mathbb{R}^d\right)$ and $R' \in X_2$ with

$$\left\|F'\right\|_{BV(\mathbb{T}^d)} \lesssim \|G\|_{L^1(\mathbb{T}^d)} \text{ and } \left\|R'\right\|_{L^{d'}(\mathbb{T}^d)} \lesssim \|G\|_{L^1(\mathbb{T}^d)}.$$

(Note that, since G is component-wise 1-periodic, we have $C_G \sim \|G\|_{L^1(\mathbb{T}^d)}$.)

By a standard regularization with convolution and a limiting argument, we can replace the space $BV(\mathbb{T}^d)$ with $W^{1,1}(\mathbb{T}^d)$. Now Theorem 3.1 follows from the discussion at the beginning of the Appendix. Theorem 3.10 can also be proved on the same lines.

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CHAPTER 4

The divergence equation with L^∞ source

A well-known fact is that there exists $g \in L^\infty(\mathbb{T}^2)$ with zero integral, such that the equation

$$\operatorname{div} f = g \tag{$*$}$$

has no solution $f = (f_1, f_2) \in W^{1,\infty}(\mathbb{T}^2)$. This was proved by Preiss ([**4**]), using an involved geometric argument, and, independently, by McMullen ([**2**]), via Ornstein's non-inequality. We improve this result: roughly speaking, we prove that, there exists $g \in L^\infty$ for which $(*)$ has no solution such that $\partial_2 f_2 \in L^\infty$ and f is "slightly better" than L^1. Our proof relies on Riesz products in the spirit of the approach of Wojciechowski ([**6**]) for the study of $(*)$ with source $g \in L^1$. The proof we give is elementary, self-contained and completely avoids the use of Ornstein's non-inequality.

1. Introduction

In this chapter, we improve the following result of Preiss ([**4**]) and McMullen (Theorem 2.1 in [**2**]):

THEOREM 4.1. *There exists $g \in L^\infty(\mathbb{T}^2)$ with zero integral, such that there are no f_1, $f_2 \in W^{1,\infty}(\mathbb{T}^2)$ with*

$$g = \partial_1 f_1 + \partial_2 f_2.$$

The proof in [**4**] is "geometric", the one in [**2**] relies essentially on Ornstein's non-inequality ([**3**]).

Note that, in the above statement, the conditions on f_1, f_2 are isotropic, i.e., we require $\partial_l f_j \in L^\infty(\mathbb{T}^2)$ for all $l, j = 1,2$. In what follows, we will prove that, under some mild regularity assumptions on f_1, f_2, the above requirements can be weakened to anisotropic conditions. Namely, it is enough to impose $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$. In order to state this more precisely, we introduce the following spaces of distributions.

Suppose $\lambda : \mathbb{N} \to (0,\infty)$ is a decreasing function such that $\lambda(k) \to 0$ when $k \to \infty$. To such a function we associate the Banach space of those distributions whose Fourier transform decays at the rate at least λ. More precisely, consider the space

$$S_\lambda\left(\mathbb{T}^2\right) := \left\{ f \in \mathscr{D}'(\mathbb{T}^2) \;\middle|\; \sup_{n \in \mathbb{Z}^2} \frac{|\hat{f}(n)|}{\lambda(|n|)} < \infty \right\},$$

endowed with the norm given by

$$\|f\|_{S_\lambda} := \sup_{n \in \mathbb{Z}^2} \frac{|\widehat{f}(n)|}{\lambda(|n|)}, \quad f \in S_\lambda\left(\mathbb{T}^2\right).$$

To mention only few examples, we note that, for any $m \in \mathbb{N}^*$, $W^{m,1}(\mathbb{T}^2) \hookrightarrow S_\lambda\left(\mathbb{T}^2\right)$, with $\lambda(|n|) = 1/(1+|n|)^m$ and, if $s > 0$, the fractional Sobolev space $H^s(\mathbb{T}^2)$ is embedded in $S_\lambda\left(\mathbb{T}^2\right)$ for $\lambda(|n|) = 1/(1+|n|)^s$.

With this notation, we can formulate our result.

THEOREM 4.2. *Suppose* $\lambda : \mathbb{N} \to (0,\infty)$ *is decreasing to* 0. *There exists* $g \in L^\infty(\mathbb{T}^2)$ *such that there are no* f_0, f_1, $f_2 \in S_\lambda(\mathbb{T}^2)$ *with* $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$ *and*

$$g = f_0 + \partial_1 f_1 + \partial_2 f_2.$$

We can easily observe that Theorem 4.2 implies Theorem 4.1. Indeed, if f_1, $f_2 \in W^{1,\infty}(\mathbb{T}^2)$ then $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$ and, as we mentioned above, we have f_1, $f_2 \in S_\lambda(\mathbb{T}^2)$ for $\lambda(|n|) = 1/(1+|n|)$. Also, even the weaker regularity condition f_0, f_1, $f_2 \in H^\varepsilon(\mathbb{T}^2)$, $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$ ($\varepsilon > 0$, a small fixed number) rules out the existence of a solution. Intuitively, $f \in S_\lambda(\mathbb{T}^2)$, with λ slowly decaying, means that f is "slightly better" than L^1. The above result asserts that solutions with such regularity satisfying $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$ need not exist.

Finally, we discuss the most important aspect, which is the proof of Theorem 4.2. Our proof completely avoids the use of Ornstein's non-inequality. It is an adaptation of the Riesz products based proof, given by Wojciechowski in [**6**], of the fact that there exist L^1 functions which are not divergences of $W^{1,1}$ vector fields. We follow the general structure of his proof making the needed modifications in order to handle the L^∞ case. While the proof in [**6**] relies on a relatively difficult lemma (Lemma 1, in [**6**]), in our case, the role of this lemma will be played by Lemma 4.3 below, which is elementary and easy. Another aspect of our proof is the presence of the function λ. This allows us to quantify the regularity that we impose to the solution and to improve the result described by Theorem 4.1. The approach based on Ornstein's non-inequality does not seem to be suited for obtaining this improvement.

We also mention that the proof given below of Theorem 4.2 is self-contained and elementary.

2. Proof of Theorem 4.2

Before starting the proof, we recall first the following well-known elementary fact (see [**5**, Lemma 6.3, p. 118]):

LEMMA 4.3. *Suppose* $z_1,\dots, z_N$ *are some complex numbers. Then, there exist* $\sigma_1,\dots, \sigma_N \in \{0,1\}$ *such that*

$$\left|\sum_{k=1}^{N} \sigma_k z_k\right| \geq \frac{1}{\pi}\sum_{k=1}^{N} |z_k|.$$

PROOF. We follow [**5**]. View $z_1,\dots, z_N$ as vectors in $\mathbb{R}^2$. For a given $\theta \in [0,2\pi]$, let $r_\theta := (\cos\theta, \sin\theta)$. If H_θ is the half-plane given by

$$H_\theta := \left\{z \in \mathbb{R}^2 |\ \langle z, r_\theta\rangle \geq 0\right\},$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1, z_k\in H_\theta}^{N} z_k\right| d\theta \geq \frac{1}{2\pi}\int_0^{2\pi}\sum_{j=1}^{N}\langle z_j, r_\theta\rangle^+ d\theta = \sum_{j=1}^{N}\frac{1}{2\pi}\int_0^{2\pi}\langle z_j, r_\theta\rangle^+ d\theta,$$

and we easily see that, for all j,

$$\frac{1}{2\pi}\int_0^{2\pi}\langle z_j, r_\theta\rangle^+ d\theta = |z_j|\frac{1}{2\pi}\int_0^{2\pi}(\cos\theta)^+ d\theta = \frac{1}{\pi}|z_j|.$$

Using the above inequality, we complete the proof of Lemma 4.3 via a mean value argument. □

We will also need few facts concerning the trigonometric polynomials.

Fix a finite sequence $(a_k)_{k=1,N}$ in $\mathbb{Z}^2$. For each finite sequence $(\alpha_1,...,\alpha_N)$ of complex numbers we have the following expansion rule:

$$\prod_{k=1}^{N}(1+\alpha_k\cos\langle t,a_k\rangle)=1+\sum_{k=1}^{N}\sum_{\substack{\varepsilon_1,...,\varepsilon_k\in\{-1,0,1\}\\ \varepsilon_k\neq 0}}\left(\prod_{\varepsilon_j\neq 0}\frac{\alpha_j}{2}\right)e^{i\langle t,\varepsilon_1a_1+...+\varepsilon_ka_k\rangle}.$$

Suppose, moreover, that $(a_k)_{k=1,N}$ is component-wise lacunary, i.e., there exists a constant $M>3$ such that $|a_{k+1}(1)|/|a_k(1)|>M$ and $|a_{k+1}(2)|/|a_k(2)|>M$ for all $1\le k\le N-1$. Then, all the expressions $\varepsilon_1a_1+...+\varepsilon_ka_k$ in the above formula are distinct and nonzero. Hence, if $\alpha_1,...,\alpha_N$ and $\beta_1,...,\beta_N$ are complex numbers, by using the above formula and the relation between convolution and the Fourier transform, we obtain

$$\prod_{k=1}^{N}(1+\alpha_k\cos\langle\cdot,a_k\rangle)*\prod_{k=1}^{N}\left(1+\beta_k\cos\langle\cdot,a_k\rangle\right)=\prod_{k=1}^{N}\left(1+\frac{\alpha_k\beta_k}{2}\cos\langle\cdot,a_k\rangle\right). \tag{4.1}$$

We will also use the following standard algebraic identity:

$$\prod_{k=1}^{N}(1+c_k)=1+\sum_{k=1}^{N}c_k\prod_{j=1}^{k-1}\left(1+c_j\right) \tag{4.2}$$

for any complex numbers $c_1,..., c_N$.

Proof of Theorem 4.2. Suppose that the assertion of Theorem 4.2 is false and fix a function λ as in the statement. Then, by the open mapping principle, there exists a constant $C>0$ such that for any $g\in L^\infty(\mathbb{T}^2)$ there exist distributions f_0, f_1, $f_2\in S_\lambda(\mathbb{T}^2)$, satisfying $g=f_0+\partial_1f_1+\partial_2f_2$, with the properties that $\partial_2f_2\in L^\infty(\mathbb{T}^2)$ and

$$\|f_0\|_{S_\lambda}+\|f_1\|_{S_\lambda}+\|f_2\|_{S_\lambda}+\|\partial_2f_2\|_{L^\infty}\le C\,\|g\|_{L^\infty}. \tag{4.3}$$

Let N be a large positive integer such that $\ln N>25\pi C$ and consider the functions on $\mathbb{T}^2$

$$g_N(t):=\prod_{k=1}^{N}\left(1+\frac{i}{k}\cos\langle t,a_k\rangle\right)\quad\text{and}\quad G_N(t):=\prod_{k=1}^{N}(1+\cos\langle t,a_k\rangle),$$

where the finite sequence $(a_k)_{k=1,N}$ in $(\mathbb{N}^*)^2$ is defined below.

Using Lemma 4.3, applied to the sequence of complex numbers

$$z_k:=\frac{1}{k}\prod_{j=1}^{k-1}\left(1+\frac{i}{2j}\right)\ \text{for } k=1,...,N,$$

(here and after the product over an empty set is by convention equal to 1), we can find a sequence $\sigma_1,..., \sigma_N\in\{0,1\}$ such that

$$\left|\sum_{k=1}^{N}\frac{\sigma_k}{k}\prod_{j=1}^{k-1}\left(1+\frac{i}{2j}\right)\right|\ge\frac{1}{\pi}\sum_{k=1}^{N}\frac{1}{k}\prod_{j=1}^{k-1}\left(1+\frac{1}{4j^2}\right)^{1/2}\ge\frac{1}{\pi}\sum_{k=1}^{N}\frac{1}{k}\ge\frac{1}{\pi}\ln N. \tag{4.4}$$

Now we impose the sequence $(a_k)_{k=1,N}$ to satisfy the following properties:

(i) $(a_k)_{k=1,N}$ is component-wise lacunary;

(ii) If $\sigma_k=1$, then

$$\left|a_k(1)+\sum_{1\le j\le k-1}\varepsilon_ja_j(1)\right|\lambda\left(\left|a_k(2)+\sum_{1\le j\le k-1}\varepsilon_ja_j(2)\right|\right)<\frac{1}{4^N}\ \text{ for all } \varepsilon_1,...,\varepsilon_{k-1}\in\{-1,0,1\};$$

(iii) If $\sigma_k=0$, then

$$\left|a_k(2)+\sum_{1\le j\le k-1}\varepsilon_ja_j(2)\right|\lambda\left(\left|a_k(1)+\sum_{1\le j\le k-1}\varepsilon_ja_j(1)\right|\right)<\frac{1}{4^N}\ \text{ for all } \varepsilon_1,...,\varepsilon_{k-1}\in\{-1,0,1\}.$$

(By convention the sum over an empty set is equal to 0.)

Such a sequence can be easily constructed by induction on k: if $a_1,\ldots, a_{k-1}$ are chosen, then we choose $a_k(2)$ much larger than $a_k(1)$, or $a_k(1)$ much larger than $a_k(2)$, depending on whether $\sigma_k = 1$ or $\sigma_k = 0$ respectively. Since λ is decreasing to 0, we can satisfy in this way the conditions (ii), respectively (iii). Also, the condition (i) can be easily satisfied.

We now return to the proof of Theorem 4.2. Note that

$$\|g_N\|_{L^\infty} = \prod_{k=1}^{N}\left(1+\frac{1}{k^2}\right)^{1/2} \le e^{\pi^2/12} < 3, \text{ and also } G_N \ge 0 \text{ and } \|G_N\|_{L^1} = 1. \tag{4.5}$$

Using (4.1) and (4.2), we get

$$G_N * g_N(t) = \prod_{k=1}^{N}\left(1+\frac{i}{2k}\cos\langle t, a_k\rangle\right) = 1 + \sum_{k=1}^{N}\frac{i}{2k}\cos\langle t, a_k\rangle \prod_{j=1}^{k-1}\left(1+\frac{i}{2j}\cos\langle t, a_j\rangle\right). \tag{4.6}$$

Consider the sets

$$A := \bigcup_{\substack{k=1\\ \sigma_k=1}}^{N} \{\varepsilon_1 a_1 + \ldots + \varepsilon_k a_k \mid \varepsilon_1,\ldots,\varepsilon_k \in \{-1,0,1\},\ \varepsilon_k \neq 0\},$$

$$B := \bigcup_{\substack{k=1\\ \sigma_k=0}}^{N} \{\varepsilon_1 a_1 + \ldots + \varepsilon_k a_k \mid \varepsilon_1,\ldots,\varepsilon_k \in \{-1,0,1\},\ \varepsilon_k \neq 0\}.$$

Since the sequence $(a_k)_{k=1,N}$ is component-wise lacunary, we have $(\{0\}\times\mathbb{Z})\cap(A\cup B) = \varnothing$, $(\mathbb{Z}\times\{0\})\cap(A\cup B) = \varnothing$ and $A\cap B = \varnothing$, while clearly $|A\cup B| \le 3^N$. In particular, $|A| \le 3^N$, $|B| \le 3^N$.

Using now (4.6) and (4.4), we obtain

$$|P_A G_N * g_N(0)| = \left|\sum_{k=1}^{N}\frac{i\sigma_k}{2k}\prod_{j=1}^{k-1}\left(1+\frac{i}{2j}\right)\right| \ge \frac{1}{2\pi}\ln N, \tag{4.7}$$

where P_A is the linear operator on trigonometric polynomials, satisfying $P_A e^{i\langle t,n\rangle} = e^{i\langle t,n\rangle}$ if $n\in A$ and $P_A e^{i\langle t,n\rangle} = 0$ otherwise.

On the other hand, according to our assumption and (4.5), we can find $f_0,\ f_1,\ f_2 \in S_\lambda(\mathbb{T}^2)$, satisfying $g_N = f_0 + \partial_1 f_1 + \partial_2 f_2$, with the properties that $\partial_2 f_2 \in L^\infty(\mathbb{T}^2)$ and

$$\|f_0\|_{S_\lambda} + \|f_1\|_{S_\lambda} + \|f_2\|_{S_\lambda} + \|\partial_2 f_2\|_{L^\infty} \le 3C.$$

Let us note that

$$P_A G_N * g_N = P_A G_N * f_0 + P_A G_N * \partial_1 f_1 + P_A G_N * \partial_2 f_2. \tag{4.8}$$

We next estimate each term on the right hand side of (4.8).

For the second term, we have:

$$\begin{aligned}
\|P_A G_N * \partial_1 f_1\|_{L^\infty} &= \|G_N * P_A\partial_1 f_1\|_{L^\infty} \le \|G_N\|_{L^1}\|P_A\partial_1 f_1\|_{L^\infty} = \|P_A\partial_1 f_1\|_{L^\infty}\\
&\le |A|\max_{n\in A}\left|\widehat{\partial_1 f_1}(n)\right| = |A|\max_{n\in A}|n(1)|\left|\widehat{f_1}(n)\right|\\
&\le |A|\max_{n\in A}|n(1)|\,\lambda(|n|)\,\|f_1\|_{S_\lambda} \le |A|\max_{n\in A}|n(1)|\,\lambda(|n(2)|)\,\|f_1\|_{S_\lambda}\\
&\le 3^N 4^{-N} 3C < 3C,
\end{aligned}$$

where we have used (ii).

For the third term, we have:

$$\begin{aligned}\|P_A G_N * \partial_2 f_2\|_{L^\infty} &= \|G_N * \partial_2 f_2 - P_B G_N * \partial_2 f_2\|_{L^\infty} \le \|G_N * \partial_2 f_2\|_{L^\infty} + \|P_B G_N * \partial_2 f_2\|_{L^\infty} \\ &\le \|G_N\|_{L^1} \|\partial_2 f_2\|_{L^\infty} + \|G_N\|_{L^1} \|P_B \partial_2 f_2\|_{L^\infty} = \|\partial_2 f_2\|_{L^\infty} + \|P_B \partial_2 f_2\|_{L^\infty} \\ &\le 3C + |B| \max_{n\in B} \left|\widehat{\partial_2 f_2}(n)\right| = 3C + |B| \max_{n\in B} |n(2)| \left|\widehat{f_2}(n)\right| \\ &\le 3C + 3^N \max_{n\in B} |n(2)|\, \lambda(|n|)\, \|f_2\|_{S_\lambda} \le 3C + 3^N \max_{n\in B} |n(2)|\, \lambda(|n(1)|)\, \|f_2\|_{S_\lambda} \\ &\le 3C + 3^N 4^{-N} 3C < 6C,\end{aligned}$$

where we have used the identity $G_N = P_A G_N + P_B G_N + 1$ and (iii).

Finally, the first term is easier to handle. We have:

$$\begin{aligned}\|P_A G_N * f_0\|_{L^\infty} &= \|G_N * P_A f_0\|_{L^\infty} \le \|P_A f_0\|_{L^\infty} \le |A| \max_{n\in A} \left|\widehat{f_0}(n)\right| \\ &\le |A| \max_{n\in A} \lambda(|n|)\, \|f_0\|_{S_\lambda} \le |A| \max_{n\in A} |n(1)|\, \lambda(|n(2)|)\, \|f_0\|_{S_\lambda} \\ &\le 3^N 4^{-N} 3C < 3C.\end{aligned}$$

These estimates together with (4.8) give us

$$\|P_A G_N * g_N\|_{L^\infty} \le 3C + 6C + 3C = 12C,$$

which contradicts (4.7), since $\ln N > 25\pi C$. □

REMARK 4.4. (1) Similarly, a closer look to the proof in [**6**] gives the following analogue of Theorem 4.2 in the case of L^1.

THEOREM 4.5. *Suppose $\lambda : \mathbb{N} \to (0,\infty)$ is decreasing to 0. There exists $g \in L^1(\mathbb{T}^2)$ such that there are no f_0, f_1, $f_2 \in S_\lambda(\mathbb{T}^2)$ with $\partial_2 f_2 \in L^1(\mathbb{T}^2)$ and*

$$g = f_0 + \partial_1 f_1 + \partial_2 f_2.$$

(2) The d-dimensional case, with $d \ge 3$, can be easily obtained from Theorem 4.2. More precisely, we have

THEOREM 4.6. *Let $d \ge 2$. Suppose $\lambda : \mathbb{N} \to (0,\infty)$ is decreasing to 0. There exists $g \in L^\infty(\mathbb{T}^d)$ such that there are no f_0, f_1, f_2,..., $f_d \in \mathscr{D}'(\mathbb{T}^d)$ with f_0, f_1, $f_2 \in S_\lambda(\mathbb{T}^d)$, $\partial_2 f_2 \in L^\infty(\mathbb{T}^d)$ and*

$$g = f_0 + \partial_1 f_1 + \partial_2 f_2 + \dots + \partial_d f_d.$$

Indeed, consider a $g' \in C^\infty(\mathbb{T}^2)$ and $\psi \in C^\infty(\mathbb{T}^{d-2})$ such that $0 \le \psi \le 1$ and $\int_{\mathbb{T}^{d-2}} \psi = 1$. If the above result were not true, we could find f_0, f_1, f_2,..., $f_d \in \mathscr{D}'(\mathbb{T}^d)$ such that

$$g' \otimes \psi = f_0 + \partial_1 f_1 + \partial_2 f_2 + \dots + \partial_d f_d$$

and

$$\|f_0\|_{S_\lambda(\mathbb{T}^d)} + \|f_1\|_{S_\lambda(\mathbb{T}^d)} + \|f_2\|_{S_\lambda(\mathbb{T}^d)} + \|\partial_2 f_2\|_{L^\infty(\mathbb{T}^d)} \le C \left\|g'\right\|_{L^\infty(\mathbb{T}^2)}.$$

Without loss of generality, we can suppose that f_0, f_1, f_2,..., f_d are smooth. Integrating this equation in the last $d-2$ coordinates, we reduce the problem to the 2-dimensional case: $g' = f_0' + \partial_1 f_1' + \partial_2 f_2'$ where

$$f_j'(t) := \int_{\mathbb{T}^{d-2}} f_j(t,\tau)\, d\tau, \text{ for } j = 0,1,2,$$

satisfy

$$\left\|f_0'\right\|_{S_\lambda(\mathbb{T}^2)} + \left\|f_1'\right\|_{S_\lambda(\mathbb{T}^2)} + \left\|f_2'\right\|_{S_\lambda(\mathbb{T}^2)} + \left\|\partial_2 f_2'\right\|_{L^\infty(\mathbb{T}^2)} \le C \left\|g'\right\|_{L^\infty(\mathbb{T}^2)}.$$

Here, we have used the fact that, for all $n' \in \mathbb{Z}^2$,

$$\left|\widehat{f'_j}(n')\right| = \left|\widehat{f_j}(n',0)\right| \leq \lambda\left(\left|(n',0)\right|\right) \|f_j\|_{S_\lambda(\mathbb{T}^d)} = \lambda\left(\left|n'\right|\right) \|f_j\|_{S_\lambda(\mathbb{T}^d)}.$$

(3) Using Lemma 4.3, and adapting the technique in [**1**], we can obtain similar anisotropic Ornstein type inequalities adapted to the L^∞ case. We give below an example. For any $\varepsilon > 0$, there exists a trigonometric polynomial f on $\mathbb{T}^2$, depending on ε, such that

$$\varepsilon \left\|\partial_1^3\partial_2^2 f\right\|_{L^\infty} \geq \left\|\partial_1^4 f\right\|_{L^\infty} + \left\|\partial_1^2\partial_2^4 f\right\|_{L^\infty} + \left\|\partial_1\partial_2^6 f\right\|_{L^\infty} + \left\|\partial_2^8 f\right\|_{L^\infty}.$$

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Part 2

Hodge systems in critical function spaces

CHAPTER 5

Approximation of critical regularity functions on stratified homogeneous groups

Let G be a stratified homogeneous group with homogeneous dimension Q and whose Lie algebra is generated by the left-invariant vector fields $X_1,...,X_{d_1}$. Let $1 < p,q < \infty$, $\alpha := Q/p$ and $\delta > 0$. We prove that for any function $f \in \dot{F}^{\alpha,p}_q(G)$ there exists a function $F \in L^\infty(G) \cap \dot{F}^{\alpha,p}_q(G)$ such that

$$\sum_{i=1}^{\Bbbk} \|X_i(f-F)\|_{\dot{F}^{\alpha-1,p}_q(G)} \leq \delta \|f\|_{\dot{F}^{\alpha,p}_q(G)},$$
$$\|F\|_{L^\infty(G)} + \|F\|_{\dot{F}^{\alpha,p}_q(G)} \leq C_\delta \|f\|_{\dot{F}^{\alpha,p}_q(G)}$$

where $\Bbbk$ is the largest integer smaller than $min(p,d_1)$ and C_δ is a positive constant depending only on δ. Here, $\dot{F}^{\alpha,p}_q(G)$ is a homogeneous Triebel-Lizorkin type space adapted to G.

This generalizes earlier results of Bourgain, Brezis [**4**] and of Bousquet, Russ, Wang, Yung [**6**] in the Euclidean case and answers an open problem in [**6**].

1. Introduction

Let $B \subset \mathbb{R}^d$ ($d \geq 2$) be a Euclidean ball. It is well-known that, if $f \in L^p_{loc}(B,\mathbb{R})$ with $1 < p < \infty$, then the equation $\operatorname{div} Y = f$ has a solution $Y \in W^{1,p}_{loc}(B,\mathbb{R}^d)$. When $p = d$, this Y "almost" belongs to $L^\infty_{loc}(B,\mathbb{R}^d)$. A striking result obtained by Bourgain and Brezis (in [**3**]) asserts that is possible to find $Y \in W^{1,d}_{loc}(B,\mathbb{R}^d) \cap L^\infty_{loc}(B,\mathbb{R}^d)$, solving $\operatorname{div} Y = f$. Their argument relies on a new type of approximation results.

This seminal work has been followed by a number of approximation results of similar type [**4**], [**5**], [**13**], [**6**]. Our work is primarily motivated by two types of developments of the results in [**13**], [**6**] concerning functions in critical Sobolev spaces that barely fail the embedding in L^∞.

The first of these results ([**13**, Lemma 1.7]) deals with the extension of the approximation result given in [**4**, Theorem 11] in the Euclidean case, to the more general case of stratified homogeneous groups. Somewhat informally this reads (see Section 2 for definitions):

THEOREM 5.1. *Suppose G is a stratified homogeneous group whose homogeneous dimension is Q and let $X_1,...,X_{n_1}$ be a minimal family of vector fields generating the Lie algebra of G. Then, for any Schwartz function f on G and any $\delta > 0$ there exists a function F such that:*

$$\sum_{i=1}^{n_1-1} \|X_i(f-F)\|_{L^Q(G)} \leq \delta \|\nabla_b f\|_{L^Q(G)},$$
$$\|F\|_{L^\infty(G)} + \|\nabla_b F\|_{L^Q(G)} \leq C_\delta \|\nabla_b f\|_{L^Q(G)},$$

where C_δ is a constant depending only on δ.

Here, $\nabla_b f = (X_1 f,...,X_{n_1} f)$. Theorem 11 in [**4**] corresponds to the Euclidean case.

On the other hand, it was proved in [**6**, Theorem 1.1] that Theorem 11 in [**4**] remains true, in the Euclidean case if we replace the critical Sobolev space $\dot{W}^{1,d}(\mathbb{R}^d)$ by more general critical spaces such as $\dot{F}^{d/p,p}_q(\mathbb{R}^d)$. More precisely, we have the following:

THEOREM 5.2. *Consider the parameters* $1<p,q<\infty$, $\alpha := d/p$ *and let* $\Bbbk$ *be the largest positive integer with* $\Bbbk < \min(p,d)$. *Then, for every* $\delta>0$ *there exists a constant* $C_\delta>0$ *depending only on* δ, *such that for every function* $f \in \dot{F}^{\alpha,p}_q(\mathbb{R}^d)$ *there exists* $F \in L^\infty(\mathbb{R}^d)\cap \dot{F}^{\alpha,p}_q(\mathbb{R}^d)$ *satisfying the following estimates:*

$$\sum_{i=1}^{\Bbbk} \|\partial_i(f-F)\|_{\dot{F}^{\alpha-1,p}_q(\mathbb{R}^d)} \le \delta \|f\|_{\dot{F}^{\alpha,p}_q(\mathbb{R}^d)},$$
$$\|F\|_{L^\infty(\mathbb{R}^d)} + \|F\|_{\dot{F}^{\alpha,p}_q(\mathbb{R}^d)} \le C_\delta \|f\|_{\dot{F}^{\alpha,p}_q(\mathbb{R}^d)}.$$

Note that here we have a somewhat unnatural technical condition on $\Bbbk$, which does not seem to be optimal. Namely, we impose $\Bbbk < \min(p,d)$ instead of only imposing $\Bbbk < d$. (See **[6]** for a discussion on this assumption.)

The purpose of this chapter is to find a common roof to Theorem 5.1 and Theorem 5.2 and to give an affirmative answer to Open question 1.4 in **[6]**. Our generalisation is an adaptation of Theorem 5.2 above to the stratified homogeneous groups context of Theorem 5.1. In this case the role of the Euclidean dimension is played by the homogeneous dimension Q of the group and the critical regularity becomes, in this case, $\alpha = Q/p$. The role of the derivatives is played by the vector fields that generate the full Lie algebra of G.

The statement of our main result is:

THEOREM 5.3. *Consider the parameters* $1<p,q<\infty$, $\alpha := Q/p$ *and let* $\Bbbk$ *be the largest positive integer with* $\Bbbk < \min(p,d_1)$. *Then, for every* $\delta>0$ *there exists a constant* $C_\delta>0$ *depending only on* δ, *such that, for every function* $f \in \dot{F}^{\alpha,p}_q(G)$, *there exists* $F \in L^\infty(G)\cap \dot{F}^{\alpha,p}_q(G)$ *satisfying the following estimates:*

$$\sum_{i=1}^{\Bbbk} \|X_i(f-F)\|_{\dot{F}^{\alpha-1,p}_q(G)} \le \delta \|f\|_{\dot{F}^{\alpha,p}_q(G)},$$
$$\|F\|_{L^\infty(G)} + \|F\|_{\dot{F}^{\alpha,p}_q(G)} \le C_\delta \|f\|_{\dot{F}^{\alpha,p}_q(G)}.$$

We will give in Section 2 precise definition of the function spaces we consider on G. For the time being, let us mention that we cover the case of the more familiar anisotropic homogeneous Sobolev spaces $\dot{N}L^{m,p}$, defined informally as containing the functions f on G for which $\nabla_b^m f \in L^p$.

Despite the fact that we also have the unnatural restriction $\Bbbk < \min(p,d_1)$, as in the Euclidean case, this suffices for some applications to divergence-like systems. Basically, all the applications to such systems presented in **[4]** can be easily adapted to the stratified homogeneous group setting and higher order Sobolev spaces. We give one example, formulated for simplicity for spaces of integer regularity.

THEOREM 5.4. *Let* $m<Q$ *be a positive integer. Suppose* $f \in \dot{N}L^{m-1,Q/m}(G)$ *and there exist functions* $v_1,\dots,v_{d_1} \in \dot{N}L^{m,Q/m}(G)$ *such that*

$$X_1 v_1 + \dots + X_{d_1} v_{d_1} = f.$$

Then, there exist $u_1,\dots,u_{d_1} \in L^\infty(G)\cap \dot{N}L^{m,Q/m}(G)$ *such that*

$$X_1 u_1 + \dots + X_{d_1} u_{d_1} = f.$$

The chapter is divided into two parts. The first one (Section 2) deals with the construction of the Triebel-Lizorkin spaces on stratified homogeneous groups. We mention that the Euclidean analogues of these spaces coincide with the classical ones and that in the general stratified homogeneous group setting, they also satisfy similar interpolation and duality properties as their classical analogues.

Spaces of a similar kind were already defined and studied for example in **[1]**, **[10]** and other works (see also **[9]** for a construction of inhomogeneous spaces in the more general context of Lie groups of polynomial volume growth). Our construction is very similar to the one given in **[10]** (it turns out that our spaces essentially coincide with the ones introduced in [**10**], as a consequence of our Proposition 5.18). While the construction in [**10**] is based on spectral decomposition of sublaplacians, our construction is based only on the relatively elementary technique developed in [**13**] for obtaining a Littlewood-Paley decomposition for functions defined on the group. (We also notice that our purpose is *not* to explore the properties of these spaces, but rather to prove a minimal number of their properties, required in the proof of Theorem 5.3.)

While in [**13**] Littlewood-Paley decomposition is obtained by a Calderón reproducing formula with two convolutions, we will also need similar reproducing formulas with three convolutions (we will prove that all the definitions of the spaces with two or more convolutions coincide). This allows us to prove the full analogue of the Littlewood-Paley inequality as well as other inequalities needed in the proof of Theorem 5.3.

The second part (Sections 3 and 4) is devoted to the proof of Theorem 5.3. We follow closely the proof in [**6**]. Several relatively minor modifications were made in order to simplify the exposition. Some more substantial adaptations were required in order to bypass the lack of commutativity of the vector fields. In some cases the arguments are easily adapted to the group setting, and in these situations we only sketch the arguments or refer to the proofs in [**6**]. In the Appendix we recall the Calderón-Zygmund theory on stratified homogeneous groups in order to give a direct proof of an inequality (Proposition 5.48) whose Euclidean analogue was proved in the Appendix of [**6**] by similar but more complicated means.

2. Function spaces on stratified homogeneous groups

Basic facts on stratified homogeneous groups. Here, we follow mainly Folland and Stein [**8**] and Stein [**12**]. We also present some auxiliary results, possibly known to experts, that we will need in order to develop the Littlewood-Paley theory of function spaces on stratified homogeneous groups. We will consider homogeneous groups as defined in [**12**, p. 618]. For such a group G, we write the following decomposition of its Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \ldots \oplus V_\ell, \tag{5.1}$$

where $V_1, \ldots, V_\ell$ are vector spaces of left-invariant vector fields such that

(i) $[V_i, V_j] \subseteq V_{i+j}$ (making the convention that V_ℓ is not trivial and any V_j with $j > \ell$ is trivial),

(ii) V_1 generates the whole algebra $\mathfrak{g}$ (this is the so called Hörmander condition).

Dimension. We let $d_j := \dim V_j$ and set $d := d_1 + \ldots + d_\ell$; the number $Q := d_1 + 2d_2 + \ldots + \ell d_\ell$ is called the homogeneous dimension of G. As sets, we identify G with $\mathbb{R}^d$. In view of this identification, we consider the following dilation rule: if $x = (x_1, \ldots, x_d) \in G$ and $\lambda > 0$, then $\lambda x := (\lambda^{a_1} x_1, \ldots, \lambda^{a_d} x_d)$, where

$$a := (a_1, \ldots, a_d) = (1, \ldots, 1, 2, \ldots, 2, \ldots, \ell, \ldots, \ell) \tag{5.2}$$

is the vector of the homogeneities, each $j \in \{1, \ldots, \ell\}$ appearing d_j times. The dilations are known to be automorphisms of G and, with respect to them, the following "norm" on G is homogeneous:

$$\|x\|_G := \left(\sum_{j=1}^{\ell} \sum_{d_1+\ldots+d_{j-1} < i \le d_1+\ldots+d_j} |x_i|^{\frac{2\ell!}{j}} \right)^{\frac{1}{2\ell!}}. \tag{5.3}$$

We have also the quasi-triangle inequality

$$\|x \cdot y\|_G \lesssim \|x\|_G + \|y\|_G, \text{ for } x, y \in G.$$

Subgradient. We write $X_1, X_2, ..., X_d$ for the left-invariant vector fields forming the standard basis of $\mathfrak{g}$, with $X_1, X_2, ..., X_{d_1}$ forming a basis of V_1. We will call full gradient and subgradient respectively the following operators

$$\nabla := (X_1, X_2, ..., X_d), \quad \nabla_b := \left(X_1, X_2, ..., X_{d_1}\right).$$

Note that, whenever f is a sufficiently smooth function on $\mathbb{R}^d$ with $\nabla_b f \equiv 0$ then, thanks to the Hörmander condition, we get $\nabla f \equiv 0$. Hence, in a sense, the subgradient encodes all the differential information about f. We will always be concerned with the subgradient of functions rather than with the full gradient. We will consider for example the Sobolev-type space $\dot{N}L^{1,Q}$, which informally is a space of functions on G whose subgradient is in L^Q. Note that this space is not the same as $\dot{W}^{1,Q}$ on G seen as a manifold.

Similar considerations hold for right-invariant vector fields. We will write X_j^R for the right-invariant analogue of X_j.

An important aspect is that, with the identification $G = \mathbb{R}^d$, we have that $x \cdot y$ is a polynomial in x, y and $(x \cdot y)_k = x_k + y_k$ for any $x, y \in G$ as long as $1 \le k \le d_1$. Also we have $x^{-1} = -x$ for all $x \in G$ (see for example [**13**, Section 2]).

Balls and the maximal function. We consider balls on G defined by the quasimetric ρ on G, given by

$$\rho(x, y) := \left\| y^{-1} \cdot x \right\|_G$$

for $x, y \in G$. The open ball centred at x and of radius $\delta > 0$ is the set

$$B(x, \delta) := \left\{ y \in G \mid \rho(y, x) < \delta \right\},$$

whose Lebesgue measure is $|B(x,\delta)| \sim \delta^Q$. For all balls $B = B(x,\delta)$ and $\lambda > 0$ we will write $\lambda B := B(x, \lambda\delta)$.

We also consider the Hardy-Littlewood maximal function M on G, defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy,$$

for all functions $f \in L^1_{loc}(G)$, where the supremum is taken over all balls $B \subset G$ containing x.

Often, the maximal operator will be used to bound convolutions. For two functions $f \in L^1(G)$ and $g \in L^\infty(G)$, their convolution on G is defined by

$$f * \phi(x) = \int_{\mathbb{R}^d} f(y)\phi(y^{-1} \cdot x) dy = \int_{\mathbb{R}^d} f(x \cdot y^{-1})\phi(y) dy.$$

REMARK 5.5. Throughout the chapter, we will often use the simbol "$\lesssim$" in order to compare two nonnegatve quantities. Namely, if $A_1, A_2 \ge 0$ are some variable quantities, "$A_1 \lesssim A_2$" will mean that there exists a constant $C > 0$ such that "$A_1 \le C A_2$". In the case where the constant C will depend on some parameters $s_1, s_2, ..., s_n$, we will sometimes write "$A_1 \lesssim_{s_1, s_2, ..., s_n} A_2$".

We recall the following classical facts (for proofs see [**12**, Chapter 2]):

PROPOSITION 5.6. *(i) If φ is a nonnegative decreasing function on $[0,\infty)$, such that $C_\varphi := \int_G \varphi(\|y\|_G) dy < \infty$ and ϕ is a measurable function on G such that $\left|\phi(y)\right| \le \varphi(\|y\|_G)$ on G, then*

$$\left| f * \phi \right| \lesssim C_\varphi M f \quad \text{on } G,$$

for any Schwartz f.

(ii) M is of weak type $(1,1)$ and of strong type (p,p) for all $1 < p \le \infty$.

(iii) (the Fefferman-Stein inequality) Consider a sequence of Schwartz functions $\left(f_j\right)_{j\in\mathbb{Z}}$. Then, for $1 < p, q < \infty$, we have

$$\left\| \|Mf_j\|_{l^q_j} \right\|_{L^p} \lesssim_{p,q} \left\| \|f_j\|_{l^q_j} \right\|_{L^p}.$$

Vector fields and polynomials. We remind the following elementary formula (see [**12**, p. 621]):

$$X_j f(x) := \left.\frac{\partial f(x\cdot y)}{\partial y_j}\right|_{y=0} = \partial_j f(x) + \sum_{k>j} q_{j,k}(x)\partial_k f(x) \tag{5.4}$$

where $y := (0,...0, y_j, 0,...0)$ and $q_{j,k}$ are homogeneous polynomials of degree $a_k - a_j$.

Another elementary fact is that the integral of the functions of the form $X_j f$, where f is a Schwartz function is, as in the Euclidean case, equal to 0. Here is a proof of this fact. For any $y = (0,...0, y_j, 0,...0) \in G$, with $y_j \neq 0$, using the fact that the Lebesgue measure on $\mathbb{R}^d$ is a bi-invariant Haar measure on G ([**8**, Proposition (1.2), p. 3]), we have

$$\int_{\mathbb{R}^d} \frac{f(x\cdot y) - f(x)}{y_j} dx = \frac{1}{y_j}\left(\int_{\mathbb{R}^d} f(x\cdot y)dx - \int_{\mathbb{R}^d} f(x)dx\right) = 0.$$

Using now the formula (5.4), the classical mean value theorem in the (Euclidean) $\mathbb{R}^d$ and the dominated convergence theorem, we can pass to the limit when $y_j \to 0$ in the above formula to obtain

$$\int_{\mathbb{R}^d} X_j f(x)dx = 0.$$

A similar formula holds for right-invariant vector fields. As an immediate consequence of this and the Leibniz rule we get the formula (see [**8**, p. 21])

$$\int_{\mathbb{R}^d} \left(X_j f\right) g dx = -\int_{\mathbb{R}^d} f\left(X_j g\right) dx \tag{5.5}$$

whenever f and g are Schwartz functions or one of them is Schwartz and the other one is polynomial.

Before going to the next step let us fix some notation. For a real valued function f sufficiently smooth on G and a positive integer m, we write $\nabla_b^m f$ for the vector valued function whose components are

$$\nabla_b^\gamma f := \left(X_1^{\gamma_1^1} X_2^{\gamma_2^1} ... X_{d_1}^{\gamma_{d_1}^1}\right)\left(X_1^{\gamma_1^2} X_2^{\gamma_2^2} ... X_{d_1}^{\gamma_{d_1}^2}\right) ... \left(X_1^{\gamma_1^m} X_2^{\gamma_2^m} ... X_{d_1}^{\gamma_{d_1}^m}\right) f \tag{5.6}$$

listed in the lexicographic order given by $\gamma = \left(\gamma_1^1, ..., \gamma_{d_1}^1, ..., \gamma_1^m, ..., \gamma_{d_1}^m\right) \in \mathbb{N}^{d_1} \times ... \times \mathbb{N}^{d_1}$ (m times) with $|\gamma| = \sum_{i,j}\gamma_j^i = m$. Note that by the embedding $\mathbb{N}^{d_1} \times ... \times \mathbb{N}^{d_1}$ in $\left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$, we can define $\nabla_b^\gamma f$ by the above formula whenever $|\gamma| < \infty$.

We will use many times the notation $\nabla_b^m \cdot \varphi$ where $\varphi := \left(\varphi_\gamma\right)_{|\gamma|=m}$ is a finite family of Schwartz functions. This has the following meaning

$$\nabla_b^m \cdot \varphi := \sum_{|\gamma|=m} \nabla_b^\gamma \varphi_\gamma. \tag{5.7}$$

Also, we will often deal with vectors of Schwartz families. If $\phi := \left(\varphi^1, ..., \varphi^N\right)$ is a vector of Schwartz families (where $\varphi^j := \left(\varphi_\gamma^j\right)_{|\gamma|=m}$), we write, with an abuse of notation,

$$\nabla_b^m \cdot \phi = \left(\nabla_b^m \cdot \varphi^1, ..., \nabla_b^m \cdot \varphi^N\right). \tag{5.8}$$

Sometimes the situation can be more complex and we need to apply gradients to expressions like $\nabla_b^m \cdot \phi$ above. In this case we write, again with an abuse of notation,

$$\nabla_b^{m_1}\left(\nabla_b^{m_2} \cdot \phi\right) = \nabla_b^{m_1+m_2}\phi. \tag{5.9}$$

Since only the number of derivatives and their nature (left or right invariant) will be important for us, such conventions (which will be clear in the context) will be harmless.

Let us see that high powers of the subgradient are able to annihilate low degree polynomials. More specifically,

PROPOSITION 5.7. *Suppose $p \in \mathbb{R}[x_1,...,x_d]$ is a polynomial and consider $m \in \mathbb{N}^*$. Then $\nabla_b^m p$ is a vector valued polynomial with $\deg \nabla_b^m p \le \ell \deg p - m$. In particular, if m is such that $m > \ell \deg p$, then we have that $\nabla_b^m p \equiv 0$.*

The similar assertion for the right-invariant subgradient also holds.

(Here, we recall that ℓ is defined by (5.1).)

PROOF. It suffices to prove the statement when p is a monomial. Suppose $p(x) = x^\alpha = x_1^{\alpha_1}...x_d^{\alpha_d}$ for some $\alpha = (\alpha_1,...,\alpha_d) \in \mathbb{N}^d$ and consider the function $q := \nabla_b^m p$. We can see from the formula (5.4) that q is a vector valued polynomial on $\mathbb{R}^d$. Writing λx for the group dilation of $x \in G$ with the parameter $\lambda > 0$, we immediately see from the definition of the subgradient that $\nabla_b^m(p(\lambda x)) = \lambda^m \nabla_b^m p(\lambda x)$. Also, we have

$$p(\lambda x) = \left(\lambda^{a_1}x\right)_1^{\alpha_1} \dots \left(\lambda^{a_d}x\right)_d^{\alpha_d} = \lambda^{\langle a,\alpha\rangle} p(x),$$

where $a = (a_1,...,a_d)$ is given by (5.2).

From this we conclude that, for all $x \in G$,

$$q(\lambda x) = \left(\nabla_b^m p\right)(\lambda x) = \lambda^{-m}\nabla_b^m(p(\lambda x)) = \lambda^{\langle a,\alpha\rangle - m}\nabla_b^m p(x) = \lambda^{\langle a,\alpha\rangle - m} q(x).$$

If cx^β is a monomial ($c \neq 0$) of maximum degree in q, as before we get $(\lambda x)^\beta = \lambda^{\langle a,\beta\rangle} x^\beta$ for all $\lambda > 0$. Choosing from these monomials one for which $\langle a,\beta\rangle$ is maximum, we get by the above formula that $\langle a,\beta\rangle = \langle a,\alpha\rangle - m$ and hence $\deg q = |\beta| \le \langle a,\beta\rangle \le \ell\,|\alpha| - m$. □

Let us next recall a fundamental formula that makes a connection between the derivatives on $\mathbb{R}^d$ and the vector fields from $\mathfrak{g}$. More specifically, for any $1 \le i \le d$ we have ([**8**, p. 25])

$$\partial_i = \sum_{k=1}^{d} P_{k,i} X_k, \tag{5.10}$$

where $P_{k,i}$ are homogeneous polynomials of degree $a_k - a_i$.

We will also need the following.

PROPOSITION 5.8. *We have that*

$$\partial_i = \sum_{k=1}^{d_1} X_k D_{k,i}^*, \tag{5.11}$$

where the operators $D_{k,i}^*$ are the adjoints of some operators of the form $\sum_\gamma p_\gamma \nabla_{\mathbb{R}^d}^\gamma$ for appropriate polynomials p_γ and multi-indexes γ in a finite subset of $\mathbb{N}^d$.

PROOF. Since the vector fields $X_1, X_2, ..., X_{d_1}$ are generating the full Lie algebra of the group, we can write each X_j in terms of $X_1, X_2, ..., X_{d_1}$ using commutators, which are linear combinations of expressions of the form $\nabla_b^\gamma = \nabla_b^{\gamma'} X_k$ for some $1 \le k \le d_1$ and some indexes γ, $\gamma' \in \left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$. Keeping the last vector field from such an expression and using (5.4) to express $\nabla_b^{\gamma'}$ in terms of derivatives on $\mathbb{R}^d$ and polynomials, we can rewrite (5.10) as

$$\partial_i = \sum_{k=1}^{d_1} D_{k,i} X_k, \tag{5.12}$$

where each operator $D_{k,i}$ is of the form $\sum_\gamma p_\gamma \nabla_{\mathbb{R}^d}^\gamma$ for some polynomials p_γ and γ in a finite subset of $\mathbb{N}^d$.

Now, if f and g are arbitrary Schwartz functions we can write (see (5.5)):

$$\int_{\mathbb{R}^d} f \partial_i g dx = -\int_{\mathbb{R}^d} (\partial_i f) g dx = -\sum_{k=1}^{d_1} \int_{\mathbb{R}^d} \left(D_{k,i} X_k f\right) g dx = \sum_{k=1}^{d_1} \int_{\mathbb{R}^d} f X_k \left(D_{k,i}^* g\right) dx$$

and hence, by identification,

$$\partial_i = \sum_{k=1}^{d_1} X_k D_{k,i}^*,$$

which proves the Proposition 5.8. □

PROPOSITION 5.9. *Let $m \in \mathbb{N}$ and f be a Schwartz function.*

(i) If $f = \nabla_b^m \cdot \varphi$ for a family of Schwartz functions φ, then for any polynomial p with $\deg p < m/\ell$ *we have $\int_G p f dx = 0$.*

(ii) There exists an $m' \in \mathbb{N}$ depending only on m and G such that if we have $\int_G p f dx = 0$ for any polynomial p with $\deg p \le m'$*, then there exists a family of Schwartz functions φ such that $f = \nabla_b^m \cdot \varphi$.*

The same is also true in the case of the right-invariant subgradient.

REMARK 5.10. (1) Since the assertion of *(ii)* in the above proposition remains true for any integer larger than m', when applying this part of the proposition, we will assume for technical reasons that $m' > m\ell$.

(2) In particular, Proposition 5.9 gives the following (informally speaking): if φ_1 is a Schwartz family, then there exists another Schwartz family φ_2 such that:

$$\left(\nabla_b^R\right)^{m'} \cdot \varphi_1 = \nabla_b^m \cdot \varphi_2.$$

This property will be used several times.

PROOF. Part *(i)* follows from Proposition 5.7 and by a repeated application of the formula (5.5). Part *(ii)* will be proved by induction on m. The case $m = 0$ is trivial (we take by convention $m' = 0$). Fix $m \ge 1$ and suppose we have the statement of *(ii)* for $m-1$. Consider the number $m' := (m-1)' + M + 2$, where M is the maximum degree reached by a polynomial p_γ entering in the expression of the operators $D_{k,i}$ that occur in (5.12). If $\int_G p f dx = 0$ for any polynomial p of degree at most m', then we can use the well-known fact that in the Euclidean case there exists a collection of Schwartz families $(\phi_i)_{1 \le i \le d}$ such that

$$f = \sum_{i=1}^{d} \partial_i \left(\nabla_{\mathbb{R}^d}^{m'-1} \cdot \phi_i\right).$$

Using now formula (5.11) we can write:

$$\begin{aligned} f &= \sum_{i=1}^{d} \partial_i \left(\nabla_{\mathbb{R}^d}^{m'-1} \cdot \phi_i\right) = \sum_{i=1}^{d} \sum_{k=1}^{d_1} X_k D_{k,i}^* \left(\nabla_{\mathbb{R}^d}^{m'-1} \cdot \phi_i\right) \\ &= \sum_{k=1}^{d_1} X_k \left(\sum_{i=1}^{d} D_{k,i}^* \nabla_{\mathbb{R}^d}^{m'-1} \cdot \phi_i\right) = \sum_{k=1}^{d_1} X_k \tilde{\phi}_k, \end{aligned} \tag{5.13}$$

where $\tilde{\phi}_k$ are the Schwartz functions $\tilde{\phi}_k := \sum_{i=1}^{d} D_{k,i}^* \nabla_{\mathbb{R}^d}^{m'-1} \cdot \phi_i$. It is easy to see that $\int_G p\tilde{\phi}_k dx = 0$ for all polynomials p of degree at most $(m-1)'$. By the induction hypothesis, we get that for each k there exists a family of Schwartz functions φ_k such that $\tilde{\phi}_k = \nabla_b^{m-1} \cdot \varphi_k$. From this and formula (5.13), we get the conclusion. □

Convolutions. We recall that, for two Schwartz functions f, g their convolution is defined by the formula:

$$f * g(x) = \int_{\mathbb{R}^d} f(y) g(y^{-1} \cdot x) dy = \int_{\mathbb{R}^d} f(x \cdot y^{-1}) g(y) dy.$$

It can be verified directly that the convolution is associative.

Concerning the interaction of vector fields with the convolution, it is known that (see [**8**, p. 22]):

PROPOSITION 5.11. *For all Schwartz functions f, g we have:*

$$X_j(f * g) = f * \left(X_j g\right), \quad X_j^R(f * g) = \left(X_j^R f\right) * g,$$

and

$$\left(X_j f\right) * g = f * \left(X_j^R g\right). \tag{5.14}$$

We have also the following elementary fact.

PROPOSITION 5.12. *If Φ_1, Φ_2 are two Schwartz functions, then $\Phi_1 * \Phi_2$ is also Schwartz.*

PROOF. We can easily observe that, since each component of $x \cdot y$ is a polynomial in x and y, we can find a large number $n_G \in \mathbb{N}^*$ such that

$$1 + |x \cdot y| \lesssim (1+|x|)^{n_G} (1+|y|)^{n_G}, \tag{5.15}$$

for all $x, y \in \mathbb{R}^d$. This implies that, for example, we have

$$\begin{aligned} \sup_x (1+|x|)^N |\Phi_1 * \Phi_2(x)| &\le \sup_x \int_{\mathbb{R}^d} \left(1 + \left|x \cdot y^{-1} \cdot y\right|\right)^N \left|\Phi_1(x \cdot y^{-1})\right| |\Phi_2(y)| dy \\ &\lesssim \sup_x \int_{\mathbb{R}^d} \left(1 + \left|x \cdot y^{-1}\right|\right)^{N n_G} \left|\Phi_1(x \cdot y^{-1})\right| (1+|y|)^{N n_G} |\Phi_2(y)| dy \\ &\lesssim \int_{\mathbb{R}^d} (1+|y|)^{N n_G} |\Phi_2(y)| dy < \infty. \end{aligned}$$

More generally, the estimate of $\sup_x (1+|x|)^N \left|\partial^\beta (\Phi_1 * \Phi_2)(x)\right|$ is reduced to the above calculation using the connection between the derivatives and the vector fields on G via (5.10) and (5.4). □

The Littlewood-Paley decomposition. We introduce the following notation. Whenever Λ is a Schwartz function on G and j is an integer, we write Λ_j for the function defined by $\Lambda_j(x) := 2^{jQ}\Lambda(2^j x)$. Also, if f is another Schwartz function, we write $\Lambda_j f := f * \Lambda_j$.

PROPOSITION 5.13. *Given $m \in \mathbb{N}$, there exist Schwartz families Λ^1, Λ^2, Λ^3 on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} P(x)\Lambda^1(x)dx = \int_{\mathbb{R}^d} P(x)\Lambda^2(x)dx = \int_{\mathbb{R}^d} P(x)\Lambda^3(x)dx = 0$ for all the polynomials P of degree $\le m'$ (with m' as in Proposition 5.9) and such that for all Schwartz functions f we have*

$$f = \sum_{j\in\mathbb{Z}} f * \Lambda_j^1 * \Lambda_j^2 * \Lambda_j^3 = \sum_{j\in\mathbb{Z}} \Lambda_j^3\Lambda_j^2\Lambda_j^1 f, \tag{5.16}$$

the convergence being in any $L^p(\mathbb{R}^d)$ for $1 < p < \infty$.

In particular, according to Proposition 5.9 (ii), there exist families of Schwartz families φ_i, ϕ_i ($i = 1,2,3$) such that $\Lambda^i = \nabla_b^m \cdot \varphi_i = \left(\nabla_b^R\right)^m \cdot \phi_i$ for each $i = 1,2,3$.

REMARK 5.14. Some explanations are in order. The proposition literally states that there exist three finite Schwartz families $\Lambda^i = \left(\Lambda^{i,a}\right)_{a\in A}$ (A is a finite set), $i = 1,2,3$, such that all the moments of order up to m' of each $\Lambda^{i,a}$ are zero and

$$f = \sum_{j\in\mathbb{Z}}\sum_{a\in A} f * \Lambda_j^{1,a} * \Lambda_j^{2,a} * \Lambda_j^{3,a} = \sum_{j\in\mathbb{Z}}\sum_{a\in A} \Lambda_j^{3,a}\Lambda_j^{2,a}\Lambda_j^{1,a} f.$$

The last assertion means that there exists $6|A|$ Schwartz families $\varphi_{i,a}$, $\phi_{i,a}$ such that

$$\Lambda^{i,a} = \nabla_b^m \cdot \varphi_{i,a} = \left(\nabla_b^R\right)^m \cdot \phi_{i,a}$$

for all $a \in A$ and $i = 1,2,3$ (see (5.7)). Since the use of the family A leads to heavy notation, we prefer the form of the above proposition which turns out to be more convenient in the calculations that follow. This can be compared with the summation convention in geometry. We also note that the absolute value of expressions like $\Lambda_j f$, where $\Lambda = (\Lambda_a)_{a\in A}$ is a Schwartz family, will have the following meaning:

$$\left|\Lambda_j^1 f\right| := \sum_{a\in A}\left|\Lambda_j^{1,a} f\right|.$$

Similarly, we set

$$\left|\Lambda_j^2\Lambda_j^1 f\right| := \sum_{a\in A}\left|\Lambda_j^{2,a}\Lambda_j^{1,a} f\right|,$$

and so on.

These conventions, together with (5.7), (5.8) and (5.9), will enable us to estimate expressions involving Schwartz families as if they were *functions*. We will also abuse the notation in other situations, where the distinction between functions and finite families of functions will be clearly irrelevant (see also the conventions in [**13**]).

PROOF. This proof follows the lines of Proposition 5.5 in [**13**]. We consider a radial Schwartz function Ψ with $\hat{\Psi} \equiv 1$ on $B_{\mathbb{R}^d}(0,1)$ and $\operatorname{supp}\hat{\Psi} \subseteq B_{\mathbb{R}^d}(0,2)$ (here $B_{\mathbb{R}^d}(0,1)$ and $B_{\mathbb{R}^d}(0,2)$ are Euclidean balls). We need now the easy argument used in the proof of Proposition 5.1 from [**13**] which we reproduce below for the convenience of the reader.

LEMMA 5.15. *Let Φ be a Schwartz function on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d}\Phi dx = 1$ and fix some $1 < p < \infty$. Then, for any Schwartz function f, we have*

$$f = \sum_{j\in\mathbb{Z}} f * (\Phi_j - \Phi_{j-1}),$$

the convergence being in L^p.

PROOF. We have, for any $N \in \mathbb{N}^*$,

$$\sum_{|j|\le N} f * (\Phi_j - \Phi_{j-1}) = f * \Phi_N - f * \Phi_{-N-1}.$$

Hence it remains to see that $f * \Phi_N \to f$ and $f * \Phi_{-N} \to 0$ in L^p when $N \to \infty$. In order to prove the first claim we write, using Minkowski's integral inequality,

$$\begin{aligned}\|f * \Phi_N - f\|_{L^p} &= \left\| \int_{\mathbb{R}^d} \left(f(x\cdot(2^{-N}y)^{-1}) - f(x) \right) \Phi(y) dy \right\|_{L^p_x} \\ &\le \int_{\mathbb{R}^d} \left\| f(x\cdot(2^{-N}y)^{-1}) - f(x) \right\|_{L^p_x} |\Phi(y)|\, dy \to 0.\end{aligned}$$

This can be seen by using the dominated convergence theorem, since $\left\| f(x\cdot(2^{-N}y)^{-1}) - f(x) \right\|_{L^p_x}$ is uniformly bounded and converges to 0 when $N \to \infty$. Indeed, fix $y \in G$. We have

$$\left\| f(x\cdot(2^{-N}y)^{-1}) - f(x) \right\|_{L^p_x} \le \left\| f(x\cdot(2^{-N}y)^{-1}) \right\|_{L^p_x} + \|f(x)\|_{L^p_x} = 2\,\|f\|_{L^p},$$

hence $\left\| f(x\cdot(2^{-N}y)^{-1}) - f(x) \right\|_{L^p_x}$ is uniformly bounded.

Using (5.15) we have, for all $x \in G$,

$$1 + |x| \gtrsim \frac{\left(1 + \left|x\cdot(2^{-N}y)\right|\right)^{1/n_G}}{1 + \left|(2^{-N}y)\right|},$$

and we get $(x \to x\cdot(2^{-N}y)^{-1})$

$$1 + \left|x\cdot(2^{-N}y)^{-1}\right| \gtrsim \frac{(1+|x|)^{1/n_G}}{1 + \left|(2^{-N}y)\right|}.$$

Using this inequality and the fact that f is Schwartz, we get

$$\begin{aligned}\left| f(x\cdot(2^{-N}y)^{-1}) - f(x) \right| &\le \left| f(x\cdot(2^{-N}y)^{-1}) \right| + |f(x)| \\ &\lesssim \left(1 + \left|x\cdot(2^{-N}y)^{-1}\right|\right)^{-(d+1)n_G} + (1+|x|)^{-(d+1)} \\ &\lesssim \left(1 + \left|(2^{-N}y)\right|\right)^{(d+1)n_G} (1+|x|)^{-(d+1)} + (1+|x|)^{-(d+1)} \lesssim_y (1+|x|)^{-(d+1)}.\end{aligned}$$

Hence, for any fixed $y \in G$, $\left| f(x\cdot(2^{-N}y)^{-1}) - f(x) \right|^p$ is dominated by an L^1 function. Also, $\left| f(x\cdot(2^{-N}y)^{-1}) - f(x) \right| \to 0$, when $N \to \infty$. Using the dominated convergence theorem, we get $\left\| f(x\cdot(2^{-N}y)^{-1}) - f(x) \right\|_{L^p_x} \to 0$, when $N \to \infty$.

In order to prove the second claim, again by Minkowski's integral inequality we have

$$\|f * \Phi_{-N}\|_{L^p} \le \|f\|_{L^1}\, \|\Phi_{-N}\|_{L^p} = 2^{-NQ(1-1/p)}\, \|f\|_{L^1} \to 0,$$

proving the lemma. □

Proof of Proposition 5.13 continued.

The above Lemma applied to $\Phi = \Psi * \Psi * \Psi$ (see Proposition 5.12) yields

$$\begin{aligned} f &= \sum_{j\in\mathbb{Z}} f * \left((\Psi * \Psi * \Psi)_j - (\Psi_{-1} * \Psi_{-1} * \Psi_{-1})_j \right) = \sum_{j\in\mathbb{Z}} f * (\Psi * \Psi * \Psi - \Psi_{-1} * \Psi_{-1} * \Psi_{-1})_j \\ &= \sum_{j\in\mathbb{Z}} f * (\Psi * \Psi * (\Psi - \Psi_{-1}) + \Psi * (\Psi - \Psi_{-1}) * \Psi_{-1} + (\Psi - \Psi_{-1}) * \Psi_{-1} * \Psi_{-1})_j,\end{aligned}$$

(5.17)

the convergence being in $L^p(\mathbb{R}^d)$ with $1 < p < \infty$. Since we have $\hat{\Psi} - \hat{\Psi}_{-1} \equiv 0$ in a neighborhood of 0, the function $\Psi - \Psi_{-1}$ is orthogonal to all polynomials. By applying Proposition 5.9 *(ii)* we can find a Schwartz family φ such that $\Psi - \Psi_{-1} = \left(\nabla_b^R\right)^{2n_2} \cdot \varphi$, with $n_2 := (2n_1)'$ where $n_1 := \left(m'\right)'$. Using (5.14) we can write in short, abusing the notation,

$$\begin{aligned}\Psi * \Psi * (\Psi - \Psi_{-1}) =& \Psi * \Psi * \left(\nabla_b^R\right)^{2n_2} \cdot \varphi = \Psi * \nabla_b^{n_2} \Psi * \left(\nabla_b^R\right)^{n_2} \varphi \\ =& \Psi * \left(\nabla_b^R\right)^{2n_1} \tilde{\Psi} * \left(\nabla_b^R\right)^{n_2} \varphi = \nabla_b^{n_1} \Psi * \left(\nabla_b^R\right)^{n_1} \tilde{\Psi} * \left(\nabla_b^R\right)^{n_2} \varphi,\end{aligned}$$

where $\tilde{\Psi}$ is a Schwartz family such that $\left(\nabla_b^R\right)^{2n_1} \tilde{\Psi} = \nabla_b^{n_2} \Psi$; this can be seen to exist thanks to Proposition 5.9 (see Remark (2)). The other terms in (5.17), namely $\Psi * (\Psi - \Psi_{-1}) * \Psi_{-1}$ and $(\Psi - \Psi_{-1}) * \Psi_{-1} * \Psi_{-1}$ can be handled in a similar way. We find that each one of them is a finite sum in which each term is of the form $Y_1^{(m')'} \cdot \phi_1 * Y_2^{(m')'} \cdot \phi_2 * Y_3^{(m')'} \cdot \phi_3$ where ϕ_i are Schwartz families and Y_i is ∇_b or ∇_b^R. This implies (5.16) via Proposition 5.9 *(i)*, once we note that $\left(m'\right)' > m'\ell$ (see the Remark (1) after Proposition 5.9). □

REMARK 5.16. (1) We will use sometimes the function $\Delta := \Psi * \Psi * \Psi - \Psi_{-1} * \Psi_{-1} * \Psi_{-1}$ for which, as we can see in the above proof, we have the estimate $\left|\Delta_j f\right| \le \left|\Lambda_j^3 \Lambda_j^2 \Lambda_j^1 f\right|$ for all integers j and all Schwartz functions f. From (5.17), we have

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f \quad \text{in } L^p,\ 1 < p < \infty.$$

In short we write $\Delta = \Lambda^3 \Lambda^2 \Lambda^1$. We will also consider its weaker analogue,

$$\Delta^1 := \Lambda^2 \Lambda^1. \tag{5.18}$$

(2) It is easy to see that we can obtain decompositions of the form

$$f = \sum_{j \in \mathbb{Z}} \Lambda_j^k \ldots \Lambda_j^3 \Lambda_j^2 \Lambda_j^1 f,$$

with arbitrary $k \ge 1$ and $\Lambda^1, \ldots, \Lambda^k$ as in Proposition 5.13. It turns out that, for the estimates we need in this work, convolutions involving $k \ge 3$ terms are in some cases *very convenient*. Note that a decomposition formula as above with $k \ge 2$ convolutions implies a decomposition with $k-1$ convolutions. In this respect we note that even if in most cases a decomposition formula with two convolutions suffices (to define Triebel-Lizorkin spaces and to prove several of their properties), the proof of Theorem 5.3 relies on decomposition formulas with three convolutions (this will be used, for example to prove the Bernstein type inequalities (5.25)).

Definition of function spaces on stratified homogeneous groups. Let $s \in \mathbb{R}$, $p, q \in (1, \infty)$ and fix $m > |s|$ and some Schwartz families Λ^1, Λ^2 whose moments up to order m' are zero (see Proposition 5.13 and the Remarks after) and such that we have the following decomposition formula with two convolutions:

$$f = \sum_{j \in \mathbb{Z}} \Lambda_j^2 \Lambda_j^1 f,$$

for any Schwartz function f.

We define the spaces $\dot{F}_q^{s,p}$ and $\dot{B}_q^{s,p}$ as being the spaces of tempered distributions f on $\mathbb{R}^d$ whose (semi)norms, respectively defined as:

$$\|f\|_{\dot{F}_q^{s,p}} := \left\| \left(\sum_{j \in \mathbb{Z}} 2^{sjq} \left|\Lambda_j^1 f\right|^q \right)^{1/q} \right\|_{L^p}$$

and

$$\|f\|_{\dot{B}^{s,p}_q} := \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \left\|\Lambda^1_j f\right\|^q_{L^p}\right)^{1/q},$$

are finite.

We notice that at first sight these definitions seem to depend on the families Λ^1,Λ^2. We will show however (Proposition 5.18), that the definition of $\dot{F}^{s,p}_q$ (and of $\dot{B}^{s,p}_q$) does not depend on Λ^1, Λ^2. We will also show (Proposition 5.23) that, as expected, the space $\dot{F}^{n,p}_2$ with n a nonnegative integer, is the same as the more “classical" Sobolev space $\dot{N}L^{n,p}$.

Independence of the definition. We will need the following simple lemma:

LEMMA 5.17. *Consider a sequence $(f_k)_{k\in\mathbb{Z}}$ of Schwartz functions such that all but a finite number of them are zero. Consider also an $s\in\mathbb{R}$, an integer $m>|s|$ and two finite Schwartz families Λ and Θ for which all the moments up to the order m' are zero. Then, for $1<p,q<\infty$, we have:*

$$\left\|\left(\sum_k 2^{skq}\left|\Lambda_k\sum_j\Theta_j f_j\right|^q\right)^{1/q}\right\|_{L^p} \lesssim \left\|\left(\sum_k 2^{skq}\,|f_k|^q\right)^{1/q}\right\|_{L^p}. \tag{5.19}$$

PROOF. From the assumptions on Θ and Λ, and Proposition 5.9, we know there are some Schwartz families ϕ and φ such that $\Theta=\nabla^m_b\cdot\phi$ and $\Lambda=\left(\nabla^R_b\right)^m\cdot\varphi$. With compact notation (using (5.14)),

$$\Theta_j*\Lambda_k=\left(\Theta*\Lambda_{k-j}\right)_j=\left(\nabla^m_b\cdot\phi*\Lambda_{k-j}\right)_j=2^{m(k-j)}\left(\phi*\left(\left(\nabla^R_b\right)^m\Lambda\right)_{k-j}\right)_j,$$

hence,

$$\Theta_j*\Lambda_k=2^{m(k-j)}\phi_j*\left(\left(\nabla^R_b\right)^m\Lambda\right)_k. \tag{5.20}$$

In a similar way, we get

$$\Theta_j*\Lambda_k=2^{m(j-k)}\left((\nabla_b)^m\Theta\right)_j*\varphi_k. \tag{5.21}$$

Note that, if g, ϕ and ψ are Schwartz and j, k are two integers, then

$$\left|g*\phi_j*\psi_k\right|\lesssim M(g*\phi_j)\lesssim MMg,$$

where the implicit multiplicative constants only depend on ϕ and ψ. Using this observation and (5.20), (5.21), we can write

$$\left|\Lambda_k\Theta_j f_j\right|\lesssim 2^{-m|k-j|}MMf_j.$$

Choosing $\beta\in(0,1)$ such that $\beta m>|s|$, and using Hölder's inequality, we can write:

$$\begin{aligned}\sum_k 2^{skq}\left|\sum_j\Lambda_k\Theta_j f_j\right|^q &\lesssim \sum_k 2^{skq}\left(\sum_j 2^{-m|k-j|}MMf_j\right)^q\\ &=\sum_k 2^{skq}\left(\sum_j 2^{-(1-\beta)m|k-j|}2^{-\beta m|k-j|}MMf_j\right)^q\\ &\lesssim \sum_k 2^{skq}\sum_j 2^{-q\beta m|k-j|}\left|MMf_j\right|^q=\sum_j\left(\sum_k 2^{skq}2^{-q\beta m|k-j|}\right)\left(MMf_j\right)^q,\end{aligned}$$

where we had used, in the third line, the fact that

$$\left(\sum_j 2^{-q'(1-\beta)m|k-j|}\right)^{q/q'} \lesssim 1.$$

We have now, for all $j \in \mathbb{Z}$,

$$\sum_k 2^{skq}2^{-q\beta m|k-j|} = \sum_{k\geq j} \ldots + \sum_{k<j} \ldots = \sum_{k\geq 0} 2^{sqj}2^{(s-\beta m)qk} + \sum_{k<0} 2^{sqj}2^{(s+\beta m)qk} \sim 2^{sjq}$$

and, as a consequence of the above inequality,

$$\left(\sum_k 2^{skq}\left|\sum_j \Lambda_k\Theta_j f_j\right|^q\right)^{1/q} \lesssim \left(\sum_j 2^{sjq}\left(MMf_j\right)^q\right)^{1/q}.$$

Applying twice the Fefferman-Stein inequality (Proposition 5.6, *(iii)*) we get (5.19). □

Now we can see that the above lemma implies the independence of the definition of the spaces of Triebel-Lizorkin type with respect to the choice of Λ^1, Λ^2. (The following statement is similar to Theorem 7 in [**10**].)

PROPOSITION 5.18. *Given the parameters $s \in \mathbb{R}$, $p,q \in (1,\infty)$, the space $\dot{F}^{s,p}_q$ does not depend on the auxiliary functions Λ^1, Λ^2.*

PROOF. Indeed, let $s \in \mathbb{R}$, $p,q \in (1,\infty)$, and $m_1, m_2 > |s|$. Consider, as in the definition of the Triebel-Lizorkin spaces, two couples of functions Λ^1, Λ^2 and Θ^1, Θ^2 corresponding to m_1, m_2 respectively. We can construct, using the first and the second couples of functions, the spaces $\left(\dot{F}^{s,p}_q\right)_\Lambda$ and $\left(\dot{F}^{s,p}_q\right)_\Theta$ respectively. Using Proposition 5.13 and Lemma 5.17 for $\Lambda = \Lambda^1$, $\Theta = \Theta^2$ and $f_j = \Theta^1_j f$ for a Schwartz function f, we get, after a limiting argument that:

$$\|f\|_{(\dot{F}^{s,p}_q)_\Lambda} = \left\|\left(\sum_k 2^{skq}\left|\Lambda^1_k \sum_j \Theta^2_j\left(\Theta^1_j f\right)\right|^q\right)^{1/q}\right\|_{L^p} \lesssim \left\|\left(\sum_k 2^{skq}\left|\Theta^1_k f\right|^q\right)^{1/q}\right\|_{L^p} = \|f\|_{(\dot{F}^{s,p}_q)_\Theta}.$$

Note that in a similar way we can obtain the converse inequality. Hence, by density, we have that $\left(\dot{F}^{s,p}_q\right)_\Lambda = \left(\dot{F}^{s,p}_q\right)_\Theta$ with equivalent norms. □

REMARK 5.19. (1) The same type of independence can be proved, in a very similar way, for the Besov spaces $\dot{B}^{s,p}_q$. In this case the analogue of Lemma 5.17 is

LEMMA 5.20. *Consider a sequence $(f_k)_{k\in\mathbb{Z}}$ of Schwartz functions such that all but a finite number of them are zero. Consider also an $s \in \mathbb{R}$, an integer $m > |s|$ and two finite Schwartz families Λ and Θ for which all the moments up to the order m' are zero. Then, for $1 < p,q \leq \infty$, we have:*

$$\left(\sum_k 2^{skq}\left\|\Lambda_k \sum_j \Theta_j f_j\right\|^q_{L^p}\right)^{1/q} \lesssim \left(\sum_k 2^{skq}\|f_k\|^q_{L^p}\right)^{1/q}.$$

Note that here we allow the values $p = \infty$, $q = \infty$. This is due to the fact that the Fefferman-Stein inequality is no longer needed.

(2) Lemma 5.17 can also be used to prove real and complex interpolation results for the Triebel-Lizorkin spaces with the same retraction method as for the classical spaces. In this case, the extension and retract operators $E : \dot{F}^{s,p}_q \to L^p(\dot{l}^q_s)$ and $R : L^p(\dot{l}^q_s) \to \dot{F}^{s,p}_q$ are defined by $Ef := \left(\Lambda^1_k f\right)_{k\in\mathbb{Z}}$ and $R(f_k)_{k\in\mathbb{Z}} := \sum_{j\in\mathbb{Z}} \Lambda^2_j f_j$. Lemma 5.17 is used to prove that R is well-defined and bounded, while these properties are obvious for E. Similarly for Besov spaces, relying on Lemma 5.20.

Inspecting the above proof of Proposition 5.18, we can see immediately that, by a very similar reasoning, we get the following:

COROLLARY 5.21. *Consider some parameters $1 < p,q < \infty$, $s \in \mathbb{R}$. Also consider an integer $m > |s|$ and a Schwartz family $\tilde{\Lambda}$ such that all its moments of order up to m' are zero. Then, for any Schwartz function f, we have:*

$$\left\| \left(\sum_k 2^{skq} \left| \tilde{\Lambda}_k f \right|^q \right)^{1/q} \right\|_{L^p} \lesssim \|f\|_{\dot{F}^{s,p}_q}.$$

The lifting property. Let us now see how Corollary 5.21 implies the lifting property for the spaces $\dot{F}^{s,p}_q$ (the following statement is similar to Corollary 21 in [**10**]).

PROPOSITION 5.22. *For any Schwartz function f, we have*

$$\|\nabla_b f\|_{\dot{F}^{s,p}_q} \sim \|f\|_{\dot{F}^{s+1,p}_q}.$$

PROOF. Consider some Schwartz functions Λ^1_j, Λ^2_j for which all the moments of order up to m' are zero ($s \in \mathbb{R}$ and the integer $m > |s|$ being fixed) and such that

$$f = \sum_{j\in\mathbb{Z}} \Lambda^2_j \Lambda^1_j f,$$

for any Schwartz function f. Combining the definition of the Triebel-Lizorkin spaces, Proposition 5.11 and Corollary 5.21, we have

$$\begin{aligned}
\|\nabla_b f\|_{\dot{F}^{s,p}_q} &\sim \left\| \left(\sum_{j\in\mathbb{Z}} 2^{sqj} \left| (\nabla_b f) * \Lambda^1_j \right|^q \right)^{1/q} \right\|_{L^p} = \left\| \left(\sum_{j\in\mathbb{Z}} 2^{sqj} \left| f * \left(\nabla^R_b \Lambda^1_j \right) \right|^q \right)^{1/q} \right\|_{L^p} \\
&= \left\| \left(\sum_{j\in\mathbb{Z}} 2^{(s+1)qj} \left| f * \left(\nabla^R_b \Lambda^1 \right)_j \right|^q \right)^{1/q} \right\|_{L^p} = \left\| \left(\sum_{j\in\mathbb{Z}} 2^{(s+1)qj} \left| \tilde{\Lambda}^1_j f \right|^q \right)^{1/q} \right\|_{L^p} \\
&\lesssim \|f\|_{\dot{F}^{s+1,p}_q},
\end{aligned}$$

where $\tilde{\Lambda}^1 := \nabla^R_b \Lambda^1$.

For the opposite inequality, using Proposition 5.9 and the independence of the definition (Proposition 5.18), we can assume that $\Lambda^1 = \nabla^R_b \phi$ where $\phi := \left(\nabla^R_b\right)^{m'} \psi$ for some Schwartz function ψ, and then we have:

$$\begin{aligned}
\|f\|_{\dot{F}^{s+1,p}_q} &\sim \left\| \left(\sum_{j\in\mathbb{Z}} 2^{(s+1)qj} \left| f * \Lambda^1_j \right|^q \right)^{1/q} \right\|_{L^p} = \left\| \left(\sum_{j\in\mathbb{Z}} 2^{(s+1)qj} \left| f * \left(\nabla^R_b \phi \right)_j \right|^q \right)^{1/q} \right\|_{L^p} \\
&= \left\| \left(\sum_{j\in\mathbb{Z}} 2^{sqj} \left| f * \nabla^R_b \phi_j \right|^q \right)^{1/q} \right\|_{L^p} = \left\| \left(\sum_{j\in\mathbb{Z}} 2^{sqj} \left| \nabla_b f * \phi_j \right|^q \right)^{1/q} \right\|_{L^p} \\
&= \left\| \left(\sum_{j\in\mathbb{Z}} 2^{sqj} \left| \phi_j (\nabla_b f) \right|^q \right)^{1/q} \right\|_{L^p} \lesssim \|\nabla_b f\|_{\dot{F}^{s,p}_q}.
\end{aligned}$$

Hence, for all Schwartz functions f we have $\|\nabla_b f\|_{\dot F^{s,p}_q} \sim \|f\|_{\dot F^{s+1,p}_q}$. □

The identification $\dot F^{n,p}_2 = \dot NL^{n,p}$. The following statement is a generalisation of Proposition 5.7 in [**13**].

PROPOSITION 5.23. *Fix an $m \in \mathbb{N}^*$ and consider Schwartz families Λ^1, Λ^2 corresponding to m as in Proposition 5.13. Then, for any Schwartz function f we have*

$$\left\| \left(\sum_{j\in\mathbb{Z}} 2^{2nj} \left|\Lambda^1_j f\right|^2 \right)^{1/2} \right\|_{L^p} \sim \|\nabla^n_b f\|_{L^p},$$

for all $n \in \mathbb{N}$ with $n \le m-1$ and $1<p<\infty$. In other words, we have $\dot F^{n,p}_2 = \dot NL^{n,p}$ with equivalent norms.

PROOF. We follow the lines of Proposition 5.7 in [**13**], which proves a similar statement in the case $n=1$. The estimate "$\lesssim$" easily follows by writing $\Lambda^1 = \left(\nabla^R_b\right)^{n+1} \cdot \varphi$ for a Schwartz family φ and then applying Proposition 5.4 in [**13**], whose statement is reproduced below in a simplified form (see also [**12**, Chapter 13, section 5.3]):

LEMMA 5.24. *If D is a Schwartz function such that $\int_G D\,dx = 0$, then for a fixed $1<p<\infty$ and any Schwartz function f we have:*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |D_j f|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}.$$

Using this we immediately obtain (using also (5.14)):

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left|2^{nj}\Lambda^1_j f\right|^2 \right)^{1/2} \right\|_{L^p} = \left\| \left(\sum_{j\in\mathbb{Z}} \left|\nabla^n_b f * \left(\nabla^R_b \cdot \varphi\right)_j\right|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|\nabla^n_b f\|_{L^p}.$$

For the reverse estimate we need to observe that, according to the proof of Proposition 5.5 in [**13**], whenever we have a decomposition of the form $f = \sum_j f * \Lambda_j * \Theta_j$ with Λ and Θ Schwartz and having zero integral, we get for any Schwartz function f that

$$\|f\|_{L^p} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |\Lambda_j f|^2 \right)^{1/2} \right\|_{L^p}. \tag{5.22}$$

Before going further, we sketch, for the convenience of the reader the standard duality argument to prove (5.22). For all Schwartz functions g write, using Fubini's theorem and the above Lemma 5.24,

$$\begin{aligned}\langle f,g\rangle &= \sum_j \langle \Theta_j\Lambda_j f, g\rangle = \sum_j \left\langle \Lambda_j f, \Theta^*_j g\right\rangle \le \int_G \left(\sum_j |\Lambda_j f|^2\right)^{1/2} \left(\sum_j \left|\Theta^*_j g\right|^2\right)^{1/2} dx \\ &\le \left\| \left(\sum_{j\in\mathbb{Z}} |\Lambda_j f|^2 \right)^{1/2} \right\|_{L^p} \left\| \left(\sum_{j\in\mathbb{Z}} \left|\Theta^*_j g\right|^2 \right)^{1/2} \right\|_{L^{p'}} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |\Lambda_j f|^2 \right)^{1/2} \right\|_{L^p} \|g\|_{L^{p'}}.\end{aligned}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the standard L^2 scalar product and $\Theta^*_j(x) := \Theta_j(x^{-1})$ on G.
We obtain (5.22) by taking, in (4.2), the supremum over g such that $\|g\|_{L^{p'}} \le 1$.

Using (5.22) with $\Lambda := \Lambda^1$ and $\Theta := \Lambda^2$, replacing f by $\nabla_b^n f$ and using (5.14) together with Corollary 5.21, we obtain:

$$\begin{aligned}\|\nabla_b^n f\|_{L^p} &\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} \left|\nabla_b^n f * \Lambda_j^1\right|^2 \right)^{1/2} \right\|_{L^p} = \left\| \left(\sum_{j\in\mathbb{Z}} \left| f * \left(\left(\nabla_b^R\right)^n \Lambda_j^1 \right) \right|^2 \right)^{1/2} \right\|_{L^p} \\ &= \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{jn} f * \left(\left(\nabla_b^R\right)^n \Lambda^1 \right)_j \right|^2 \right)^{1/2} \right\|_{L^p} = \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{jn} \tilde{\Lambda}_j^1 f \right|^2 \right)^{1/2} \right\|_{L^p} \\ &\lesssim \|f\|_{\dot{F}_2^{n,p}} \sim \left\| \left(\sum_{j\in\mathbb{Z}} \left| 2^{jn} \Lambda_j^1 f \right|^2 \right)^{1/2} \right\|_{L^p},\end{aligned}$$

where $\tilde{\Lambda}^1 := \left(\nabla_b^R\right)^n \Lambda^1$. This proves the proposition. □

3. Estimates of the auxiliary functions

Remark concerning the approximations. Following [**6**], our purpose is to prove the approximation property stated in Theorem 5.3. In the remaining part of the chapter we will use decomposition formulas with three convolutions, as in Proposition 5.13.

It suffices to prove this approximation property for functions of a special form:

$$f_J := \sum_{|j|\le J} \Lambda_j^3 \Lambda_j^2 \Lambda_j^1 f = \sum_{|j|\le J} \Delta_j f,$$

where Λ_j^1, Λ_j^2, Λ_j^3 and $m > \alpha$ are fixed. (This particular form of the functions f_J will ensure, as we will see, that some expressions involving infinite sums and products are well-defined.) Indeed, suppose that f is a fixed Schwartz function and for each positive integer J we can find an F_J satisfying the estimates:

$$\begin{aligned}\sum_{i=1}^{\Bbbk} \|X_i(f_J - F_J)\|_{\dot{F}_q^{\alpha-1,p}} &\le \delta \|f_J\|_{\dot{F}_q^{\alpha,p}}, \\ \|F_J\|_{L^\infty} + \|F_J\|_{\dot{F}_q^{\alpha,p}} &\le C_\delta \|f_J\|_{\dot{F}_q^{\alpha,p}}.\end{aligned}$$

Note that Lemma 5.17 immediately implies that $\|f - f_J\|_{\dot{F}_q^{\alpha,p}} \to 0$ when $J \to \infty$. By the sequential Banach-Alaoglu theorem, we can choose a subsequence $(J_k)_{k\ge1}$ such that F_{J_k} converges weakly star in L^∞ to a function $F \in L^\infty$. Together with the last estimate and the above observation, this easily implies that $F \in \dot{F}_q^{\alpha,p}$ as follows. For any positive integer N and any compact set $K \subset G$ we have

$$\left\| \left(\sum_{|j|\le N} 2^{\alpha q j} \left| \Lambda_j^1 F_{J_k} \right|^q \right)^{1/q} \right\|_{L^p(K)} \le C_\delta \left\| f_{J_k} \right\|_{\dot{F}_q^{\alpha,p}} \lesssim_\delta \|f\|_{\dot{F}_q^{\alpha,p}},$$

where by $\lesssim_\delta$ we indicate that the implicit multiplicative constant may depend on δ.

Since, $\left\|F_{J_k}\right\|_{L^\infty} \lesssim_\delta \|f\|_{\dot{F}_q^{\alpha,p}}$ we get $\left\|\Lambda_j^1 F_{J_k}\right\|_{L^\infty} \lesssim_\delta \|f\|_{\dot{F}_q^{\alpha,p}}$ for all j. We also can see that $\Lambda_j^1 F_{J_k}(x) \to \Lambda_j^1 F(x)$ for every $x \in G$. Hence, the above inequality and the dominated convergence theorem imply that

$$\left\| \left(\sum_{|j|\le N} 2^{\alpha q j} \left| \Lambda_j^1 F \right|^q \right)^{1/q} \right\|_{L^p(K)} \lesssim_\delta \|f\|_{\dot{F}_q^{\alpha,p}},$$

and from this we get the claim. Also we obtain that

$$\|F\|_{L^\infty} + \|F\|_{\dot{F}_q^{\alpha,p}} \lesssim_\delta \|f\|_{\dot{F}_q^{\alpha,p}}$$

and, in a similar way,

$$\sum_{i=1}^{\Bbbk} \|X_i(f-F)\|_{\dot F_q^{\alpha-1,p}} \le \delta \|f\|_{\dot F_q^{\alpha,p}}.$$

From now, we consider J is a fixed positive integer.

Definitions and properties of some auxiliary functions. This subsection is inspired by the approach in [**4**], and its variants in [**13**], [**6**].

For a real number σ and $x \in G$ we will write $x_\sigma := (2^{-\sigma}x_1, ..., 2^{-\sigma}x_{\Bbbk}, x_{\Bbbk+1}, ..., x_d)$. Consider the functions $S, E : G \to \mathbb{R}$ defined by:

$$S(x) := \min(1, \|x\|_G^{-Q-1}) \text{ and } E(x) := \exp\Big(-(1+\|x_\sigma\|_G^{2\ell!})^{1/2\ell!}\Big).$$

We will also consider the functions

$$S_j(x) := 2^{jQ} S(2^j x), \quad E_j(x) := 2^{jQ} E(2^j x)$$

and set $S_j f := f * S_j$. With this notation we introduce the new functions (where Δ^1 was defined in (5.18)):

$$\omega_j(x) := \left(\int_{\mathbb{R}^d} \Big[\Big(S_j \Big|\Delta_j^1 f\Big|\Big)(2^{-j}r) E(r^{-1}\cdot(2^j x))\Big]^p dr\right)^{1/p}, \text{ if } |j| \le J \text{ and } 0 \text{ otherwise.}$$

Consider a smooth function $\zeta : [0,\infty) \to [0,1]$ such that $\zeta \equiv 1$ on $[0, 1/2]$ and $\zeta \equiv 0$ on $[1,\infty)$. Following [**13**], we define the functions ζ_j as follows:

$$\zeta_j := \begin{cases} \zeta\Big(\frac{2^{\alpha j}\omega_j}{\sum_{k<j, k\equiv j (\mathrm{mod} R)} 2^{\alpha k}\omega_k}\Big), & \text{if } \sum_{k<j, k\equiv j(\mathrm{mod} R)} 2^{\alpha k}\omega_k \ne 0, \\ 0, & \text{otherwise,} \end{cases}$$

where R is a large positive integer that will be chosen later.

Using the ζ_j's, we decompose a finite sum $f_J = \sum_{|j|\le J} \Delta_j f$ as follows:

$$f_J = \sum_{|j|\le J} \Delta_j f = \sum_{|j|\le J} (1-\zeta_j)\Delta_j f + \sum_{|j|\le J} \zeta_j \Delta_j f = \sum_j h_j + \sum_j g_j = h + g$$

where

$$h := \sum_j h_j, \text{ with } h_j := (1-\zeta_j)\Delta_j f \ \text{ if } |j| \le J \text{ and } 0 \text{ otherwise,}$$

$$g := \sum_j g_j, \text{ with } g_j := \zeta_j \Delta_j f \ \text{ if } |j| \le J \text{ and } 0 \text{ otherwise.}$$

Then we let

$$\tilde h := \sum_j h_j \prod_{j'>j} (1 - U_{j'}), \quad \text{with} \quad U_j := (1-\zeta_j)\omega_j,$$

$$\tilde g := \sum_{c=0}^{R-1} \sum_{j\equiv c(\mathrm{mod} R)} g_j \prod_{\substack{j'>j \\ j'\equiv c(\mathrm{mod} R)}} (1 - G_{j'}), \text{ with } G_j := \sum_{\substack{t>0 \\ t\equiv 0(\mathrm{mod} R)}} 2^{-\alpha t}\omega_{j-t}.$$

The heart of the proof of Theorem 5.3 consists in establishing the fact that $F_J := \tilde h + \tilde g$ is a "good approximation" of $f_J = h + g$.

Point-wise and integral estimates on ω_j. Here we collect several useful estimates on ω_j in which we will see an instance of the role played by the critical condition on the exponents: $\alpha p = Q$.

In what follows we will need the following elementary approximation property proved in **[13]** (Proposition 3.6):

PROPOSITION 5.25. *For any $\sigma \in \mathbb{R}$ and $x,\theta \in G$ we have:*

$$\left|\|(x\cdot\theta)_\sigma\|_G - \|x_\sigma\|_G\right| \le C\,\|\theta\|_G \quad \text{and} \quad \left|\|(\theta\cdot x)_\sigma\|_G - \|x_\sigma\|_G\right| \le C\,\|\theta\|_G.$$

In particular,

$$|\|x\cdot\theta\|_G - \|x\|_G| \le C\,\|\theta\|_G \quad \text{and} \quad |\|\theta\cdot x\|_G - \|x\|_G| \le C\,\|\theta\|_G.$$

PROPOSITION 5.26. *Let $\sigma > 0$. With the above notation we have:*

(i) $\omega_j \lesssim E_j S_j \left|\Delta_j^1 f\right| \lesssim 2^{Q\sigma} MM\left(\Delta_j^1 f\right)$ *for all $j \in \mathbb{Z}$;*

(ii) $\left|\Delta_j f\right| \lesssim \omega_j$ *for all $j \in \mathbb{Z}$;*

(iii) $\left\|\omega_j\right\|_{L^\infty} \lesssim 2^{\Bbbk\sigma}\|f\|_{\dot F^{\alpha,p}_q}$ *for all $j \in \mathbb{Z}$;*

(iv) $\left\|U_j\right\|_{L^\infty} \lesssim 2^{\Bbbk\sigma}\|f\|_{\dot F^{\alpha,p}_q}$ *for all $j \in \mathbb{Z}$;*

(v) $\left\|\left\|2^{\alpha j}\omega_j\right\|_{l^q_j}\right\|_{L^p} \lesssim 2^{Q\sigma}\|f\|_{\dot F^{\alpha,p}_q}$.

PROOF. It is not hard to see that there exist measurable pairwise disjoint sets $M_1, M_2, \ldots$ covering G, such that we have $B_i \subseteq M_i \subseteq (3C)\cdot B_i$ for some balls B_i of radius 1/3 in G, where $(3C)\cdot B_i$ is the ball of the same center as B_i and of radius $3C$. (Here $C > 1$ is a constant such that $\rho(x,y) \le C\left(\rho(x,z)+\rho(z,y)\right)$ for all $x,y,z \in G$.) Indeed, let $(x_n)_{n\ge1}$ be a C-net in G. That is, the balls $(B(x_n,C))_{n\ge1}$ cover G, and $\rho\left(x_i,x_j\right) \ge C$ for all $i \ne j$. We note that, if $i \ne j$, then the balls $B(x_i,1/3)$ and $B\left(x_j,1/3\right)$ are disjoint. Now we put $B_i := B(x_i,1/3)$ and $M_1 := B(x_1,C)\setminus\left(\cup_{j\ne1}B_j\right)$, and $M_k := (B(x_k,C)\setminus(M_1\cup\ldots\cup M_{k-1}))\setminus\left(\cup_{j\ne k}B_j\right)$ for all $k \ge 2$.

We observe that Proposition 5.25 implies that, for each $x,\theta \in G$ with $\|\theta\|_G \lesssim 1$ we have $E(x\cdot\theta) \sim E(\theta\cdot x) \sim E(x)$ and $S(x\cdot\theta) \sim S(\theta\cdot x) \sim S(x)$. It follows, that

$$\begin{aligned}
S_j\left|\Delta_j^1 f\right|(x\cdot\theta) &= 2^{jQ}\int_{\mathbb{R}^d}\left|\Delta_j^1 f\right|(y) S\left(\left(2^j y^{-1}\right)\cdot\left(2^j x\right)\cdot\left(2^j\theta\right)\right)dy \\
&\sim 2^{jQ}\int_{\mathbb{R}^d}\left|\Delta_j^1 f\right|(y) S\left(\left(2^j y^{-1}\right)\cdot\left(2^j x\right)\right)dy \\
&= S_j\left|\Delta_j^1 f\right|(x),
\end{aligned}$$

for all $x \in G$, provided $\|\theta\|_G \lesssim 2^{-j}$.

If r_i is the center of B_i, then for all r in $2B_i$, and hence for all r in M_i, we can write $r = r_i\cdot\theta$ for some θ depending on r with $\|\theta\|_G \le 2$. Now, considering the above estimates and the decomposition $G = \bigcup_i M_i$ we can write, since $|M_i| \sim 1$,

$$\begin{aligned}
\omega_j(x) &= \left(\sum_{i=1}^{\infty}\int_{M_i}\left(S_j\left|\Delta_j^1 f\right|(2^{-j}r)E(r^{-1}\cdot(2^j x))\right)^p dr\right)^{1/p} \\
&\sim \left(\sum_{i=1}^{\infty}\left(S_j\left|\Delta_j^1 f\right|(2^{-j}r_i)E(r_i^{-1}\cdot(2^j x))\right)^p\right)^{1/p} \\
&\le \sum_{i=1}^{\infty} S_j\left|\Delta_j^1 f\right|(2^{-j}r_i)E(r_i^{-1}\cdot(2^j x)) \\
&\sim \sum_{i=1}^{\infty}\int_{M_i} S_j\left|\Delta_j^1 f\right|(2^{-j}r)E(r^{-1}\cdot(2^j x))dr \\
&= \int_G S_j\left|\Delta_j^1 f\right|(2^{-j}r)E(r^{-1}\cdot(2^j x))dr = E_j S_j\left|\Delta_j^1 f\right|(x).
\end{aligned} \tag{5.23}$$

Next we note that $E(x) \le \tilde{E}(x) := \exp(-\|2^{-\sigma}x\|_G)$ and therefore (using Proposition 5.6)

$$E_j S_j \left|\Delta_j^1 f\right| \le \tilde{E}_j S_j \left|\Delta_j^1 f\right| \lesssim \tilde{E}_j M \left|\Delta_j^1 f\right| \lesssim \|\tilde{E}_j\|_{L^1} MM \left|\Delta_j^1 f\right| \lesssim 2^{Q\sigma} MM \left|\Delta_j^1 f\right|. \tag{5.24}$$

We obtain *(i)*, from (5.23) and (5.24).

Now we prove *(ii)*. By the change of variables $s^{-1} = r^{-1} \cdot (2^j x)$ we can write, as above,

$$\begin{aligned} \omega_j(x) &= \left(\sum_{i=1}^{\infty} \int_{M_i} \left(S_j \left|\Delta_j^1 f\right| \left(x \cdot (2^{-j}s)\right) E(s^{-1}) \right)^p ds \right)^{1/p} \\ &\ge \left(\int_{M_1} \left(S_j \left|\Delta_j^1 f\right| \left(x \cdot (2^{-j}s)\right) E(s^{-1}) \right)^p ds \right)^{1/p} \sim S_j \left|\Delta_j^1 f\right|(x). \end{aligned}$$

To conclude we observe that, for all $j \in \mathbb{Z}$,

$$\begin{aligned} \left|\Delta_j f\right| &\le \left|\Lambda_j^3 \Lambda_j^2 \Lambda_j^1 f\right| = \left|\left(\Lambda_j^2 \Lambda_j^1 f\right) * \Lambda_j^3\right| \le \left|\Lambda_j^2 \Lambda_j^1 f\right| * \left|\Lambda_j^3\right| \\ &= \left|\Delta_j^1 f\right| * \left|\Lambda_j^3\right| \lesssim \left|\Delta_j^1 f\right| * S_j = S_j \left|\Delta_j^1 f\right|, \end{aligned}$$

where we used the fact that, since Λ^3 is Schwartz, we have $\left|\Lambda^3\right| \lesssim S$ and hence $\left|\Lambda_j^3\right| \lesssim S_j$.

In order to prove *(iii)* we observe that, since $\alpha p = Q$,

$$\begin{aligned} \left\|\Delta_j^1 f\right\|_{L^\infty} &\lesssim \left\|\Lambda_j^1 f * \Lambda_j^2\right\|_{L^\infty} \lesssim \left\|\Lambda_j^1 f\right\|_{L^p} \left\|\Lambda_j^2\right\|_{L^{p'}} \\ &\lesssim 2^{\alpha j} \left\|\Lambda_j^1 f\right\|_{L^p} \lesssim \|f\|_{\dot{F}_q^{\alpha,p}}, \end{aligned} \tag{5.25}$$

which together with *(i)*, the fact that $\|E_j\|_{L^1} \lesssim 2^{\Bbbk\sigma}$ and the Young inequality (see [**8**, Proposition 1.18]) gives the estimate.

Item *(iii)* and the definition of U_j immediately imply *(iv)*.

In order to prove *(v)*, we observe that

$$\begin{aligned} \left\|\left\|2^{\alpha j}\Delta_j^1 f\right\|_{l_j^q}\right\|_{L^p} &= \left\|\left\|2^{\alpha j}\Lambda_j^2\Lambda_j^1 f\right\|_{l_j^q}\right\|_{L^p} \lesssim \left\|\left\|2^{\alpha j} M \Lambda_j^1 f\right\|_{l_j^q}\right\|_{L^p} \\ &\lesssim \left\|\left\|2^{\alpha j}\Lambda_j^1 f\right\|_{l_j^q}\right\|_{L^p} = \|f\|_{\dot{F}_q^{\alpha,p}}, \end{aligned} \tag{5.26}$$

which, again, together with *(i)* and the Fefferman-Stein inequality, gives the estimate. □

REMARK 5.27. Items *(i)*, *(ii)* and *(v)* do not use the fact that $\alpha = Q/p$. In contrast, *(iii)* and *(iv)* require $\alpha = Q/p$.

PROPOSITION 5.28. *For σ large, we have $\left\|\sup_{j\in\mathbb{Z}} 2^{\alpha j}\omega_j\right\|_{L^p} \lesssim \sigma 2^{\frac{\Bbbk\sigma}{p}} \|f\|_{\dot{F}_q^{\alpha,p}}$.*

PROOF. We follow the proof in [**6**] of Proposition 4.7. We have (with the change of variables $r^{-1} \cdot (2^j x) \to r^{-1}$):

$$\begin{aligned} \left|\sup_{j\in\mathbb{Z}} 2^{\alpha j}\omega_j(x)\right|^p &= \sup_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} \left(2^{\alpha j} S_j \left|\Delta_j^1 f\right| (2^{-j}r) E(r^{-1} \cdot (2^j x))\right)^p dr \\ &= \sup_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} \left(2^{\alpha j} S_j \left|\Delta_j^1 f\right| (x \cdot \left(2^{-j}r\right)) E(r^{-1})\right)^p dr \\ &\le \int_{\mathbb{R}^d} E^p(r^{-1}) \left(\sup_{j\in\mathbb{Z}} 2^{\alpha j} S_j \left|\Delta_j^1 f\right| (x \cdot \left(2^{-j}r\right))\right)^p dr \\ &\le \int_{\mathbb{R}^d} E^p(r^{-1}) \left\|2^{\alpha j} S_j \left|\Delta_j^1 f\right| (x \cdot \left(2^{-j}r\right))\right\|_{l_j^q}^p dr. \end{aligned}$$

We note that, according to (5.26), $\left\|\left\|2^{\alpha j}\Delta_j^1 f\right\|_{l_j^q}\right\|_{L^p} \lesssim \|f\|_{\dot F_q^{\alpha,p}}$, and hence, using Proposition 5.48 (see the Appendix) we get

$$\left\|\sup_{j\in\mathbb{Z}} 2^{\alpha j}\omega_j\right\|_{L^p}^p \le \int_{\mathbb{R}^d} E^p(r^{-1})\left\|\left\|2^{\alpha j}S_j\left|\Delta_j^1 f\right|(x\cdot\left(2^{-j}r\right))\right\|_{l_j^q}\right\|_{L_x^p}^p dr$$
$$\lesssim \left(\int_{\mathbb{R}^d} E^p(r^{-1})\ln^p(2+\|r\|_G)dr\right)\|f\|_{\dot F_q^{\alpha,p}}^p .$$

By a change of variables, we can write

$$\int_{\mathbb{R}^d} E^p(r^{-1})\ln^p(2+\|r\|_G)dr = 2^{\Bbbk\sigma}\int_{\mathbb{R}^d}\exp\left(-p(1+\|y\|_G^{2\ell!})^{1/2\ell!}\right)\ln^p(2+\|y_{-\sigma}\|_G)dy.$$

We can estimate this as follows. We have, for all $y\in G$,

$$\ln^p(2+\|y_{-\sigma}\|_G) \lesssim 1+\ln^p(2+2^\sigma|y_1|+\ldots+2^\sigma|y_{\Bbbk}|+|y_{\Bbbk+\mathbb{1}}|+\ldots+|y_d|)$$
$$\le 1+\ln^p(2^\sigma(2+|y_1|+\ldots+|y_d|)) \lesssim \sigma^p+\ln^p(2+|y_1|+\ldots+|y_d|).$$

Now, clearly

$$\int_{\mathbb{R}^d} E^p(r^{-1})\ln^p(2+\|r\|_G)dr \lesssim \left(\sigma^p+1\right)2^{\Bbbk\sigma} \lesssim \sigma^p 2^{\Bbbk\sigma},$$

and we get the claim. □

To make the notation more compact we introduce the functions $\mathbb{l}_m(x) = 1_{A_m}(x)$, where

$$A_m := \left\{ y\in\mathbb{R}^d \;\middle|\; 2^{\alpha m}\omega_m(y) > \frac{1}{2}\sum_{k<m,\; k\equiv m(\mathrm{mod} R)} 2^{\alpha k}\omega_k(y)\right\},\quad m\in\mathbb{Z}.$$

With this we have:

PROPOSITION 5.29. *For σ large, we have* $\left\|\|2^{\alpha m}\omega_m\mathbb{l}_m\|_{l_m^q}\right\|_{L^p} \lesssim R\sigma 2^{\frac{\Bbbk\sigma}{p}}\|f\|_{\dot F_q^{\alpha,p}}$.

PROOF. Fix a $j\in\{0,1,\ldots,R-1\}$. Since $\omega_m \equiv 0$ for all but a finite number of $m\in\mathbb{Z}$, we can choose for each $x\in G$, the largest integer $m_x\equiv j(\mathrm{mod} R)$ with the property that $x\in A_{m_x}$, in particular, $2^{\alpha m_x}\omega_{m_x}(x) > \frac{1}{2}\sum_{k<m_x,\; k\equiv m_x(\mathrm{mod} R)} 2^{\alpha k}\omega_k(x)$. Using this, we can write

$$\sum_{m\equiv j(\mathrm{mod} R)} 2^{\alpha m}\omega_m(x)\mathbb{l}_m(x) \le 2^{\alpha m_x}\omega_{m_x}(x) + \sum_{k<m_x,\; k\equiv j(\mathrm{mod} R)} 2^{\alpha k}\omega_k(x)$$
$$\le 3\cdot 2^{\alpha m_x}\omega_{m_x}(x) \le 3\sup_m 2^{\alpha m}\omega_m(x)$$

and hence,

$$\left\|\|2^{\alpha m}\omega_m\mathbb{l}_m\|_{l_m^q}\right\|_{L^p} \le \left\|\sum_m 2^{\alpha m}\omega_m\mathbb{l}_m\right\|_{L^p} \le \sum_{j=0}^{R-1}\left\|\sum_{m\equiv j(\mathrm{mod} R)} 2^{\alpha m}\omega_m\mathbb{l}_m\right\|_{L^p} \le 3R\left\|\sup_m 2^{\alpha m}\omega_m\right\|_{L^p}.$$

By using Proposition 5.28, we get the claim. □

REMARK 5.30. Proposition 5.28 and Proposition 5.29 do not use the fact that $\alpha = Q/p$.

Estimates involving derivatives. Consider a function u on G, smooth on $\mathbb{R}^d\setminus\{0\}$ and homogeneous of degree 1. If $a=(1,\dots 1,2,\dots,2,\dots\ell,\dots,\ell)$ is the vector of the homogeneities of G defined in (5.2) and γ is a multi-index, then we easily see that

$$\nabla^{\gamma}(u(\lambda x))=\lambda^{\langle\gamma,a\rangle}\left(\nabla^{\gamma}u\right)(\lambda x)$$

for any $\lambda>0$ (where $x\to\lambda x$ is the group dilation). Since u is homogeneous of degree 1, we have

$$\lambda\left(\nabla_b^{\gamma}u\right)(x)=\nabla_b^{\gamma}(\lambda u(x))=\nabla_b^{\gamma}(u(\lambda x))=\lambda^{|\gamma|}\left(\nabla_b^{\gamma}u\right)(\lambda x)$$

and hence

$$\left(\nabla_b^{\gamma}u\right)(\lambda x)=\lambda^{1-|\gamma|}\left(\nabla_b^{\gamma}u\right)(x),\text{ for all } x\in G \text{ and } \lambda>0. \tag{5.27}$$

Thus, for all $x\neq 0$, writing $x=\lambda v$ where $\lambda:=\|x\|_G$, $v:=x/\|x\|_G$, we get by (5.27) that

$$\left(\nabla_b^{\gamma}u\right)(x)=\lambda^{1-|\gamma|}\left(\nabla_b^{\gamma}u\right)(v),$$

which implies in particular that if $\|x\|_G\geq 1$ and $|\gamma|\geq 1$ then

$$\left|\nabla_b^{\gamma}u(x)\right|\lesssim_{\gamma}\|x\|_G^{1-|\gamma|}\lesssim_{\gamma}1. \tag{5.28}$$

Let us also note that if $\tau:\mathbb{R}\to\mathbb{R}$ and $v:G\to\mathbb{R}$ are some smooth functions, then

$$X_j(\tau(v(x)))=\tau'(v(x))X_jv(x) \text{ for all } 1\leq j\leq d.$$

Iterating this, we get

$$\left|\nabla_b^{\gamma}(\tau(v(x)))\right|\lesssim_{\gamma}\sum_{k=1}^{|\gamma|}\left|\tau^{(k)}(v(x))\right|\sum_{\gamma_1+\dots+\gamma_k=\gamma}\prod_{i=1}^{k}\left|\nabla_b^{\gamma_i}v(x)\right|, \tag{5.29}$$

for all multi-indexes $\gamma\in\left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$ with $\left|\gamma\right|<\infty$.

These observations are the basis for proving the following proposition.

PROPOSITION 5.31. *For every $\gamma'\in\left(\mathbb{N}^{\Bbbk}\times\{0\}^{d_1-\Bbbk}\right)^{\mathbb{N}}$ and $\gamma\in\left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$ with $\left|\gamma\right|+\left|\gamma'\right|<\infty$ (see (5.6)), we have*

$$\left|\nabla_b^{\gamma+\gamma'}\omega_j\right|\lesssim_{\gamma,\gamma'}2^{j|\gamma|}2^{(j-\sigma)|\gamma'|}\omega_j.$$

PROOF. Replacing G with $\mathbb{R}\times G$, and considering

$$u(t,x):=\|(t,x)\|_{\mathbb{R}\times G}=(|t|^{2\ell!}+\|x\|_G^{2\ell!})^{1/2\ell!},$$

we get by observation (5.28) above, with $t=1$, that $\left|\nabla_b^{\gamma}(1+\|x\|_G^{2\ell!})^{1/2\ell!}\right|\lesssim 1$ for all finite $\gamma\in\left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$, $\gamma\neq 0$ as above.

By (5.29) we obtain

$$\left|\nabla_b^{\gamma}\exp\left(-p(1+\|x\|_G^{2\ell!})^{1/2\ell!}\right)\right|\lesssim\exp\left(-p(1+\|x\|_G^{2\ell!})^{1/2\ell!}\right)$$

and as in [**6**, Proposition 4.4], we get from this that,

$$\left|\nabla_b^{\gamma+\gamma'}E^p(r^{-1}\cdot\left(2^jx\right))\right|\lesssim 2^{j|\gamma|}2^{(j-\sigma)|\gamma'|}E^p(r^{-1}\cdot\left(2^jx\right)).$$

Consequently we have

$$\left|\nabla_b^{\gamma+\gamma'}\omega_j^p\right|\lesssim 2^{j|\gamma|}2^{(j-\sigma)|\gamma'|}\omega_j^p \tag{5.30}$$

and by writing $\omega_j=\tau\left(\omega_j^p\right)$, where $\tau(t):=t^{1/p}$, we can conclude the proof of Proposition 5.31 by using (5.29). We give below the argument. Firstly, we can suppose without loss of generality that $\gamma\in\left(\{0\}^{\Bbbk}\times\mathbb{N}^{d_1-\Bbbk}\right)^{\mathbb{N}}$. Clearly, if $\overline{\gamma}_1+\dots+\overline{\gamma}_k=\gamma+\gamma'$ for some multi-indexes $\overline{\gamma}_1,\dots,\overline{\gamma}_k\in\left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$ then, we can write $\overline{\gamma}_i=\gamma_i+\gamma_i'$ for each $1\leq i\leq k$, where $\gamma_1,..,\gamma_k\in\left(\{0\}^{\Bbbk}\times\mathbb{N}^{d_1-\Bbbk}\right)^{\mathbb{N}}$ and $\gamma_1',..,\gamma_k'\in\left(\mathbb{N}^{\Bbbk}\times\{0\}^{d_1-\Bbbk}\right)^{\mathbb{N}}$ are such that $\gamma_1+..+\gamma_k=\gamma$ and $\gamma_1'+..+\gamma_k'=\gamma'$. From the definition of ω_j we see that if $\omega_j(x) = 0$ for some $x \in G$, then $\omega_j \equiv 0$ on G. Suppose this is not the case, i.e., $\omega_j > 0$ on G. Using (5.29) and (5.30), we get

$$\begin{aligned}\left|\nabla_b^{\gamma+\gamma'}\omega_j\right| &\lesssim \sum_{k=1}^{|\gamma|+|\gamma'|}\left|\tau^{(k)}\left(\omega_j^p\right)\right| \sum_{\overline{\gamma}_1+\ldots+\overline{\gamma}_k=\gamma+\gamma'}\prod_{i=1}^{k}\left|\nabla_b^{\gamma_j+\gamma'_j}\omega_j^p\right| \\ &\lesssim \sum_{k=1}^{|\gamma|+|\gamma'|}\omega_j^{1-pk}\left(\sum_{\overline{\gamma}_1+\ldots+\overline{\gamma}_k=\gamma+\gamma'}\prod_{i=1}^{k}2^{j|\gamma_i|}2^{(j-\sigma)|\gamma'_i|}\right)\omega_j^{pk} \\ &= \left(\sum_{k=1}^{|\gamma|+|\gamma'|}2^{j|\gamma|}2^{(j-\sigma)|\gamma'|}\right)\omega_j \lesssim 2^{j|\gamma|}2^{(j-\sigma)|\gamma'|}\omega_j.\end{aligned}$$

This concludes the proof. □

PROPOSITION 5.32. *For every* $\gamma' \in \left(\mathbb{N}^{\Bbbk} \times \{0\}^{d_1-\Bbbk}\right)^{\mathbb{N}}$ *and* $\gamma \in \left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$ *with* $|\gamma| + |\gamma'| < \infty$, *we have*

$$\left|\nabla_b^{\gamma+\gamma'}\zeta_j\right| \lesssim_{\gamma,\gamma'} 2^{j|\gamma|}2^{(j-\sigma)|\gamma'|}. \tag{5.31}$$

PROOF. Since the proof of (5.31) follows very closely the similar estimate in [**6**, (Proposition 4.5)], we only sketch the argument.

We suppose $\zeta_j \neq 0$ and write $\zeta_j = \zeta(2^{\alpha j}\omega_j/v_j)$, where $v_j := \sum_{k<j,k\equiv j(\mathrm{mod}R)} 2^{\alpha k}\omega_k$. From Proposition 5.31 we get

$$\left|\nabla_b^{\gamma+\gamma'}v_j\right| \lesssim 2^{j|\gamma|}2^{(j-\sigma)|\gamma'|}v_j. \tag{5.32}$$

Since $\nabla_b^{\gamma+\gamma'}\left(v_j/v_j\right) = 0$, the Leibniz rule gives us,

$$\begin{aligned}\left|v_j\nabla_b^{\gamma+\gamma'}\left(\frac{1}{v_j}\right)\right| &\lesssim \sum_{\substack{\beta\le\gamma+\gamma'\\|\beta|<|\gamma|+|\gamma'|}}\left|\nabla_b^{\gamma+\gamma'-\beta}v_j\right|\left|\nabla_b^{\beta}\left(\frac{1}{v_j}\right)\right| \\ &= \sum_{\substack{\beta_1\le\gamma,\beta_2\le\gamma'\\|\beta_1|+|\beta_2|<|\gamma|+|\gamma'|}}\left|\nabla_b^{(\gamma-\beta_1)+(\gamma'-\beta_2)}v_j\right|\left|\nabla_b^{\beta_1+\beta_2}\left(\frac{1}{v_j}\right)\right|.\end{aligned} \tag{5.33}$$

The inequality (5.33) used in conjunction with (5.32) leads by a straightforward induction on $|\gamma| + |\gamma'|$, to

$$\left|\nabla_b^{\gamma+\gamma'}\left(\frac{1}{v_j}\right)\right| \lesssim 2^{j|\gamma|}2^{(j-\sigma)|\gamma'|}\frac{1}{v_j}.$$

Using this, Proposition 5.31 and (5.29) for the functions ζ and $2^{\alpha j}\omega_j/v_j$, we can conclude as in [**6**, Proposition 4.5]. □

4. Estimates of the approximation function

Estimates of the L^∞ norm. In this subsection we are going to verify that the functions $\tilde{h}$ and $\tilde{g}$ are well-defined and, under a smallness condition on $\|f\|_{\dot{F}_q^{\alpha,p}}$ ((5.35) below), obey the L^∞ estimates:

$$\|\tilde{h}\|_{L^\infty} \lesssim 1, \quad \|\tilde{g}\|_{L^\infty} \lesssim R. \tag{5.34}$$

In the remaining part of the chapter we assume that f satisfies

$$\|f\|_{\dot{F}_q^{\alpha,p}} \le \eta, \tag{5.35}$$

where η is a sufficiently small number (depending only on σ, R and δ) that will be chosen later. We also assume that $R > 1/\alpha$.

In order to obtain the bounds (5.34), we will need the following observation. If $(a_k)_{k\in\mathbb{Z}}$ is a sequence with finite support, then we have the identity (see [**6**, Lemma 3.2]):

$$\sum_{j'>j} a_{j'} \prod_{j<j''<j'} (1-a_{j''}) + \prod_{j'>j}(1-a_{j'}) = 1. \tag{5.36}$$

An immediate consequence of this equality is that, whenever $a_k \in [0,1]$, we must have, for all j,

$$\sum_{j'>j} a_{j'} \prod_{j<j''<j'} (1-a_{j''}) \le 1. \tag{5.37}$$

The boundedness of $\tilde{h}$. First of all we easily see that $\tilde{h}$ is well-defined (as a consequence of the fact that only a finite number of functions h_j, ω_j and U_j are nonzero). Recalling the definition of h_j and using Proposition 5.26 *(ii)*, we can write:

$$\left|h_j\right| = \left(1-\zeta_j\right)\left|\Delta_j f\right| \lesssim \left(1-\zeta_j\right)\omega_j = U_j.$$

If f satisfies (5.35) with small η then, by Proposition 5.26 *(iv)*, we get $U_j \in [0,1]$ for all $j \in \mathbb{Z}$ and hence, by using (5.37) and the definition of $\tilde{h}$, we get the estimate:

$$|\tilde{h}| \le \sum_j \left|h_j\right| \prod_{j'>j}(1-U_{j'}) \lesssim \sum_j U_j \prod_{j'>j}(1-U_{j'}) \lesssim 1.$$

The boundedness of $\tilde{g}$. Let us see first that $\tilde{g}$ is well-defined. We have that all but a finite number of the functions g_j are identically zero, hence it remains to discuss the nature of the products of the form

$$\prod_{j'>j}(1-G_{j'}). \tag{5.38}$$

Following [**6**], we show that these products converge uniformly. Indeed, we have $\omega_j \equiv 0$ for all $j > J$. For small η in (5.35), by Proposition 5.26 *(iii)*, we have $\left|\omega_j\right| < 1$ and thus we can write:

$$0 \le G_j < \sum_{\substack{t>0,\ t\ge j-J \\ t\equiv 0 (\mathrm{mod} R)}} 2^{-\alpha t} \le \frac{\min\left(2^{-\alpha R}, 2^{-\alpha(j-J)}\right)}{1-2^{-\alpha R}}.$$

If j is large, then we have $G_j \lesssim_R 2^{-\alpha(j-J)}$ which proves the uniform convergence of (5.38).

Now we estimate the L^∞ norm of $\tilde{g}$. When $R > 1/\alpha$, from the above inequality we get $G_j \in [0,1]$ for all j. By the definition of ζ_j, we see that $\zeta_j(x) \ne 0$ only if

$$2^{\alpha j}\omega_j(x) \le \sum_{\substack{k<j \\ k\equiv j (\mathrm{mod} R)}} 2^{\alpha k}\omega_k(x).$$

Hence,

$$\left|g_j(x)\right| \lesssim \zeta_j(x)\omega_j(x) \lesssim \sum_{\substack{k<j \\ k\equiv j (\mathrm{mod} R)}} 2^{\alpha(k-j)}\omega_k(x) = G_j,$$

and by using (5.37) and the definition of $\tilde{g}$ we obtain,

$$|\tilde{g}| \le \sum_{c=0}^{R-1} \sum_{j\equiv c (\mathrm{mod} R)} \left|g_j\right| \prod_{\substack{j'>j \\ j'\equiv c (\mathrm{mod} R)}} (1-G_{j'}) \lesssim \sum_{c=0}^{R-1} \sum_{j\equiv c (\mathrm{mod} R)} G_j \prod_{\substack{j'>j \\ j'\equiv c (\mathrm{mod} R)}} (1-G_{j'}) \le R.$$

Estimating $h-\tilde{h}$. Our goal in this subsection is to prove the following estimates:

PROPOSITION 5.33. *Suppose $1<p,q<\infty$, $\alpha=Q/p$ and $\Bbbk$ are as in Theorem 5.3. Then, we have*

$$\text{(i)}\ \sum_{i=1}^{\Bbbk}\left\|X_i(h-\tilde{h})\right\|_{\dot{F}^{\alpha-1,p}_q}\lesssim R\sigma^2 2^{-\sigma\min(1,\alpha)+\frac{\Bbbk\sigma}{p}}\|f\|_{\dot{F}^{\alpha,p}_q}+R\sigma^2 2^{\sigma\max(1-\alpha,0)+\left(1+[\alpha]+\frac{1}{p}\right)\Bbbk\sigma}\|f\|^2_{\dot{F}^{\alpha,p}_q};$$

$$\text{(ii)}\ \sum_{i=1}^{d_1}\left\|X_i(h-\tilde{h})\right\|_{\dot{F}^{\alpha-1,p}_q}\lesssim R\sigma^2 2^{\frac{\Bbbk\sigma}{p}}\|f\|_{\dot{F}^{\alpha,p}_q}+R\sigma^2 2^{\sigma\max(1-\alpha,0)+\left(1+[\alpha]+\frac{1}{p}\right)\Bbbk\sigma}\|f\|^2_{\dot{F}^{\alpha,p}_q}.$$

(Here, $[\alpha]$ stands for the integer part of α.)

Before starting the proof, we note that, writing:

$$V_j:=\sum_{j'<j}h_{j'}\prod_{j'<j''<j}(1-U_{j''}),$$

and by using the definition of $\tilde{h}$ together with the identity (5.36) (as in [**13**, p. 19]), one obtains

$$h-\tilde{h}=\sum_j V_jU_j. \tag{5.39}$$

In order to obtain Proposition 5.33, we first collect some estimates satisfied by U_j and V_j.

LEMMA 5.34. *For every $\gamma'\in\left(\mathbb{N}^{\Bbbk}\times\{0\}^{d_1-\Bbbk}\right)^{\mathbb{N}}$ and $\gamma\in\left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$ with $\left|\gamma\right|+\left|\gamma'\right|<\infty$, we have*
(i) $\left|\nabla_b^{\gamma+\gamma'}U_m\right|\lesssim 2^{m|\gamma|}2^{(m-\sigma)|\gamma'|}\omega_m\mathbb{l}_m$;
(ii) $\left\|\nabla_b^{\gamma}U_m\right\|_{L^\infty}\lesssim 2^{m|\gamma|}2^{\Bbbk\sigma}\|f\|_{\dot{F}^{\alpha,p}_q}$.

PROOF. As in [**6**, Lemma 5.2], this follows from Propositions 5.26, 5.31 and 5.32. ☐

LEMMA 5.35. *For all $m\in\mathbb{Z}$, $\gamma\in\left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$ with $\left|\gamma\right|<\infty$ we have*

$$\left\|\nabla_b^{\gamma}h_m\right\|_{L^\infty}\lesssim 2^{m|\gamma|}\|f\|_{\dot{F}^{\alpha,p}_q}.$$

PROOF. This is a direct consequence of the definition of h_m, (5.31) and of the Bernstein type inequality (5.25), since we have

$$\left\|\Delta_jf\right\|_{L^\infty}=\left\|\Lambda_j^3\Delta_j^1f\right\|_{L^\infty}\lesssim\left\|\Delta_j^1f\right\|_{L^\infty}\lesssim\|f\|_{\dot{F}^{\alpha,p}_q}, \tag{5.40}$$

for all j. ☐

LEMMA 5.36. *Under the smallness assumption (5.35), we have*

(i) $|V_m|\lesssim 1$,
(ii) for all $\gamma\in\left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$ *with* $\left|\gamma\right|<\infty$, $\left\|\nabla_b^{\gamma}V_m\right\|_{L^\infty}\lesssim 2^{m|\gamma|}2^{\sigma|\gamma|\Bbbk}\|f\|_{\dot{F}^{\alpha,p}_q}$.

PROOF. We just follow the proof in [**6**, Lemma 5.4]. Item *(i)* follows directly from the construction and by using (5.37). The arguments are very similar to the ones used to prove (5.34). Item *(i)* is also proved in [**13**] (the inequality (6.6)).

We prove now item *(ii)*. By induction we can write (see [**6**] or [**13**, Section 6])

$$\nabla_b^{\gamma}V_m=\sum_{m'<m}\left(\nabla_b^{\gamma}h_{m'}-\sum_{0<\beta\leq\gamma}c_{\gamma',\gamma}\nabla_b^{\beta}U_{m'}\nabla_b^{\gamma-\beta}V_{m'}\right)\prod_{m'<m''<m}(1-U_{m''}). \tag{5.41}$$

This can be seen as follows. Suppose $(A_m)_{m\in\mathbb{Z}}$ and $(B_m)_{m\in\mathbb{Z}}$ are two sequences of smooth functions on G, such that for all integers m we have

$$A_m=\sum_{m'<m}B_{m'}\prod_{m'<m''<m}(1-U_{m''}) \tag{5.42}$$

(also we assume "good" convergence properties for all the derivatives).

Then, if X is a left-invariant vector field from the Lie algebra of G, we can write

$$\begin{aligned}
XA_m &= \sum_{m'<m} (XB_{m'}) \prod_{m'<m''<m} (1-U_{m''}) \\
&\quad - \sum_{\substack{m' \\ m'<m}} B_{m'} \sum_{\substack{v \\ m'<v<m}} (XU_v) \prod_{m'<m''<v} (1-U_{m''}) \prod_{v<m''<m} (1-U_{m''}) \\
&= \sum_{m'<m} (XB_{m'}) \prod_{m'<m''<m} (1-U_{m''}) \\
&\quad - \sum_{\substack{v \\ v<m}} (XU_v) \sum_{\substack{m' \\ m'<v}} B_{m'} \prod_{m'<m''<m} (1-U_{m''}) \prod_{v<m''<m} (1-U_{m''}) \\
&= \sum_{m'<m} (XB_{m'}) \prod_{m'<m''<m} (1-U_{m''}) - \sum_{\substack{v \\ v<m}} (XU_v) A_v \prod_{v<m''<m} (1-U_{m''}) \\
&= \sum_{m'<m} (XB_{m'}) \prod_{m'<m''<m} (1-U_{m''}) - \sum_{m'<m} (XU_{m'}) A_{m'} \prod_{m'<m''<m} (1-U_{m''}),
\end{aligned}$$

and hence, we get

$$XA_m = \sum_{m'<m} ((XB_{m'}) - (XU_{m'})A_{m'}) \prod_{m'<m''<m} (1-U_{m''}).$$

We observe that this equality is of the same form as (5.42); in the sense that, if we now define

$$A^1_m := XA_m \quad \text{and} \quad B^1_m := (XB_m) - (XU_m)A_m,$$

then

$$A^1_m = \sum_{m'<m} B^1_{m'} \prod_{m'<m''<m} (1-U_{m''}).$$

Applying this iteratively, using the definition of V_m, we get (5.41).

By Lemmas 5.35 and 5.34, we have

$$\begin{aligned}
\|\nabla_b^\gamma V_m\|_{L^\infty} &\lesssim \sum_{m'<m} \left(\|\nabla_b^\gamma h_{m'}\|_{L^\infty} + \sum_{0<\gamma'\leq\gamma} \left\|\nabla_b^\beta U_{m'}\right\|_{L^\infty} \left\|\nabla_b^{\gamma-\beta} V_{m'}\right\|_{L^\infty} \right) \\
&\lesssim \sum_{m'<m} \left(2^{m'|\gamma|} + \sum_{0<\beta\leq\gamma} 2^{m'|\beta|} 2^{\Bbbk\sigma} \left\|\nabla_b^{\gamma-\beta} V_{m'}\right\|_{L^\infty} \right) \|f\|_{\dot F_q^{\alpha,p}}
\end{aligned}$$

and by induction on $|\gamma|$ we get the inequality in item *(ii)*. (Recall that we work under the smallness assumption (5.35).) □

We are now in position to complete the proof of Proposition 5.33.

PROOF OF PROPOSITION 5.33. We prove *(i)* in detail, following closely [**6**, Section 5]. As in [**6**], for all $1 \leq k \leq \Bbbk$, we write

$$\begin{aligned}
\|X_k(h-\tilde h)\|_{\dot F_q^{\alpha-1,p}} &= \left\| \left\| 2^{(\alpha-1)m} \Lambda^1_m X_k (h-\tilde h) \right\|_{l^q_m} \right\|_{L^p} \\
&= \left\| \left\| 2^{(\alpha-1)m} \Lambda^1_m X_k \left(\sum_{j\in\mathbb{Z}} V_j U_j \right) \right\|_{l^q_m} \right\|_{L^p} \\
&= \left\| \left\| 2^{(\alpha-1)m} \Lambda^1_m X_k \left(\sum_{r\in\mathbb{Z}} V_{r+m} U_{r+m} \right) \right\|_{l^q_m} \right\|_{L^p} \\
&\leq \sum_{r\in\mathbb{Z}} \left\| \left\| 2^{(\alpha-1)m} \Lambda^1_m X_k (U_{r+m} V_{r+m}) \right\|_{l^q_m} \right\|_{L^p}.
\end{aligned}$$

We split this last sum in three terms $\sum_{r>\sigma}$, $\sum_{r<0}$, $\sum_{0\leq r\leq\sigma}$.

(I) Estimate of $\sum_{r>\sigma}$. Following [**6**, Subsection 5.1] and using (5.14), we have:

$$\begin{aligned}
\left\| \left\| 2^{(\alpha-1)m} \Lambda^1_m X_k (U_{r+m} V_{r+m}) \right\|_{l^q_m} \right\|_{L^p} &= \left\| \left\| 2^{(\alpha-1)m} (U_{r+m} V_{r+m}) * X^R_k \Lambda^1_m \right\|_{l^q_m} \right\|_{L^p} \\
&= \left\| \left\| 2^{\alpha m} (U_{r+m} V_{r+m}) * \left(X^R_k \Lambda^1 \right)_m \right\|_{l^q_m} \right\|_{L^p} \\
&\lesssim \left\| \left\| 2^{\alpha m} M (U_{r+m} V_{r+m}) \right\|_{l^q_m} \right\|_{L^p} \\
&\lesssim \left\| \left\| 2^{\alpha m} (U_{r+m} V_{r+m}) \right\|_{l^q_m} \right\|_{L^p} \lesssim \left\| \left\| 2^{\alpha m} U_{r+m} \right\|_{l^q_m} \right\|_{L^p} \\
&= 2^{-\alpha r} \left\| \left\| 2^{\alpha m} U_m \right\|_{l^q_m} \right\|_{L^p} .
\end{aligned} \tag{5.43}$$

Recalling that $U_j = (1-\zeta_j)\omega_j$ and using Proposition 5.29 we get

$$\left\| \left\| 2^{\alpha m} U_m \right\|_{l^q_m} \right\|_{L^p} \lesssim \left\| \left\| 2^{\alpha m} \omega_m \mathbb{1}_m \right\|_{l^q_m} \right\|_{L^p} \lesssim R\sigma 2^{\Bbbk\sigma/p} \| f \|_{\dot F^{\alpha,p}_q} ,$$

and summing up,

$$\sum_{r>\sigma} \dots \lesssim \sum_{r>\sigma} \left(2^{-\alpha r} R\sigma 2^{\Bbbk\sigma/p} \| f \|_{\dot F^{\alpha,p}_q} \right) \lesssim R\sigma 2^{-\alpha\sigma + \Bbbk\sigma/p} \| f \|_{\dot F^{\alpha,p}_q} .$$

(II) Estimate of $\sum_{r<0}$. If $a := [\alpha]$ then, as we have already seen, we can write $\Lambda^1 = \left(\nabla^R_b\right)^a \cdot \varphi$ for a Schwartz family φ, and then $\Lambda^1_m = 2^{-ma} \left(\nabla^R_b\right)^a \cdot \varphi_m$. Hence, if X_k is a vector field in a "good" direction, i.e. $1 \le k \le \Bbbk$, we have

$$\begin{aligned}
\left\| \left\| 2^{(\alpha-1)m} \Lambda^1_m X_k (U_{m+r} V_{m+r}) \right\|_{l^q_m} \right\|_{L^p} &= \left\| \left\| 2^{(\alpha-1)m} 2^{-ma} X_k (U_{m+r} V_{m+r}) * \left(\nabla^R_b \right)^a \cdot \varphi_m \right\|_{l^q_m} \right\|_{L^p} \\
&= \left\| \left\| 2^{(\alpha-1)m} 2^{-ma} \left[\nabla^a_b X_k (U_{m+r} V_{m+r}) \right] * \varphi_m \right\|_{l^q_m} \right\|_{L^p} \\
&\lesssim \left\| \left\| 2^{(\alpha-1)m} 2^{-ma} M \nabla^a_b X_k (U_{m+r} V_{m+r}) \right\|_{l^q_m} \right\|_{L^p} \\
&\lesssim \left\| \left\| 2^{(\alpha-1)m} 2^{-ma} \nabla^a_b X_k (U_{m+r} V_{m+r}) \right\|_{l^q_m} \right\|_{L^p} \\
&\lesssim 2^{-(\alpha-1-a)r} \left\| \left\| 2^{(\alpha-1-a)m} \nabla^a_b X_k (U_m V_m) \right\|_{l^q_m} \right\|_{L^p} ,
\end{aligned}$$

where we have used the Fefferman-Stein inequality in the third line.

As in [**6**], using the Leibniz rule and Lemmas 5.34 and 5.36, we obtain

$$\begin{aligned}
\left| \nabla^a_b X_k (U_m V_m) \right| &\lesssim \left| V_m \left(\nabla^a_b X_k U_m \right) \right| + \sum_{l=0}^{a} \left| \nabla^l_b U_m \right| \left| \nabla^{a+1-l}_b V_m \right| \\
&\lesssim 2^{ma} 2^{m-\sigma} \omega_m \mathbb{1}_m + \sum_{l=0}^{a} \left(2^{ml} \omega_m \mathbb{1}_m \right) \left(2^{m(a+1-l)} 2^{\Bbbk(a+1-l)\sigma} \| f \|_{\dot F^{\alpha,p}_q} \right) \\
&\lesssim 2^{m(a+1)} \left(2^{-\sigma} + 2^{\Bbbk(a+1)\sigma} \| f \|_{\dot F^{\alpha,p}_q} \right) \omega_m \mathbb{1}_m .
\end{aligned}$$

Now we get, via Proposition 5.29,

$$\begin{aligned}
\left\| \left\| 2^{(\alpha-1)m} \Lambda^1_m X_k (U_{m+r} V_{m+r}) \right\|_{l^q_m} \right\|_{L^p} &\lesssim 2^{-(\alpha-1-a)r} \left(2^{-\sigma} + 2^{\Bbbk(a+1)\sigma} \| f \|_{\dot F^{\alpha,p}_q} \right) \left\| \left\| 2^{\alpha m} \omega_m \mathbb{1}_m \right\|_{l^q_m} \right\|_{L^p} \\
&\lesssim R\sigma 2^{\frac{\Bbbk\sigma}{p}} \| f \|_{\dot F^{\alpha,p}_q} 2^{-(\alpha-1-a)r} \left(2^{-\sigma} + 2^{\Bbbk(a+1)\sigma} \| f \|_{\dot F^{\alpha,p}_q} \right)
\end{aligned}$$

and, summing up,

$$\sum_{r<0}\cdots \lesssim \left(\sum_{r<0} 2^{-(\alpha-1-a)r}\right) R\sigma 2^{\frac{\Bbbk\sigma}{p}} \|f\|_{\dot F_q^{\alpha,p}} \left(2^{-\sigma}+2^{\Bbbk(a+1)\sigma}\|f\|_{\dot F_q^{\alpha,p}}\right)$$
$$\lesssim R\sigma \left(2^{-\sigma+\frac{\Bbbk\sigma}{p}} \|f\|_{\dot F_q^{\alpha,p}} + 2^{\Bbbk\left(a+1+\frac{1}{p}\right)\sigma} \|f\|^2_{\dot F_q^{\alpha,p}}\right).$$

(III) Estimate of $\sum_{0\le r\le\sigma}$. This is similar to the preceding estimate. Here, instead of taking a to be the integer part of α, we consider $a=0$. As above we conclude that

$$\left\| \left\| 2^{(\alpha-1)m} \Lambda^1_m X_k (U_{m+r}V_{m+r})\right\|_{l^q_m}\right\|_{L^p} \lesssim R\sigma 2^{\frac{\Bbbk\sigma}{p}} \|f\|_{\dot F_q^{\alpha,p}} 2^{-(\alpha-1)r}\left(2^{-\sigma}+2^{\Bbbk\sigma}\|f\|_{\dot F_q^{\alpha,p}}\right),$$

and by summing up,

$$\sum_{0\le r\le\sigma}\cdots \lesssim C_\alpha(\sigma) R\sigma\left(2^{-\sigma+\frac{\Bbbk\sigma}{p}}\|f\|_{\dot F_q^{\alpha,p}} + 2^{\Bbbk\left(1+\frac{1}{p}\right)\sigma}\|f\|^2_{\dot F_q^{\alpha,p}}\right)$$

where $C_\alpha(\sigma)\sim 1$ if $\alpha>1$, $C_\alpha(\sigma)\sim\sigma$ if $\alpha=1$ and $C_\alpha(\sigma)\sim 2^{(1-\alpha)\sigma}$ if $\alpha<1$.

With this we have proved *(i)*. The proof of *(ii)* follows the same lines as the one of *(i)*. The main difference is that since we are no longer restricted to the case of derivatives in "good" directions, we have to use, instead of Lemma 5.34 *(i)* applied with $|\gamma'|=1$ (as in (II) and implicitly in (III) above), the weaker statement for the case $|\gamma'|=0$. This will produce almost the same estimates, the difference being that the coefficient $2^{-\sigma+\frac{\Bbbk\sigma}{p}}$ of $\|f\|_{\dot F_q^{\alpha,p}}$ in the corresponding parts (I), (II) becomes $2^{\frac{\Bbbk\sigma}{p}}$. □

Estimating $g-\tilde g$. Our goal in this subsection is to prove the following counterpart of Proposition 5.33.

PROPOSITION 5.37. *Consider $1<p,q<\infty$ and $\alpha=Q/p$. Also consider $a_\alpha\in(0,\alpha]$ such that $a_\alpha=1$ if $\alpha\ge 1$. We have*

$$\|\nabla_b(g-\tilde g)\|_{\dot F_q^{\alpha-1,p}} \lesssim 2^{Q\sigma}R2^{-\min(1,\alpha a_\alpha)R}\|f\|_{\dot F_q^{\alpha,p}} + 2^{([\alpha]+1)Q\sigma}R^2 2^{-\min(1,\alpha a_\alpha)R}\|f\|^2_{\dot F_q^{\alpha,p}}.$$

We recall the definition of G_j:

$$G_j := \sum_{\substack{t>0\\ t\equiv 0 (\mathrm{mod} R)}} 2^{-\alpha t}\omega_{j-t}.$$

The starting point is the identity (similar to (5.39))

$$g-\tilde g = \sum_j G_j H_j,$$

where

$$H_j := \sum_{\substack{j'<j\\ j'\equiv j(\ mod R)}} g_{j'} \prod_{\substack{j'<j''<j\\ j''\equiv j(\mathrm{mod} R)}} (1-G_{j''})$$

and $g_j=\zeta_j\Delta_j f$.

LEMMA 5.38. *For all $m\in\mathbb{Z}$, $\gamma\in\left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$ with $|\gamma|<\infty$,*

$$\left|\nabla_b^\gamma G_m\right| \lesssim_\gamma 2^{Q\sigma} \sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod} R)}} 2^{-\alpha t} 2^{|\gamma|(m-t)} MM\left|\Delta^1_{m-t} f\right|.$$

PROOF. By the definition of G_m and Proposition 5.31,

$$\left|\nabla_b^\gamma G_m\right| \le \sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod} R)}} 2^{-\alpha t}\left|\nabla_b^\gamma \omega_{m-t}\right| \lesssim \sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod} R)}} 2^{-\alpha t} 2^{|\gamma|(m-t)}\omega_{m-t}.$$

Note now that, according to Proposition 5.26,

$$\omega_{m-t} \lesssim 2^{Q\sigma} MM\left|\Delta^1_{m-t} f\right|,$$

whence the estimate. □

LEMMA 5.39. *For all $m \in \mathbb{Z}$, $\gamma \in \left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$ with $|\gamma| < \infty$,*

$$\left|\nabla_b^\gamma g_m\right| \lesssim_\gamma 2^{|\gamma| m} M\left|\Delta^1_m f\right|.$$

PROOF. By Proposition 5.32 and the Leibniz rule, recalling the definition of g_m, we have

$$\begin{aligned}\left|\nabla_b^\gamma g_m\right| &\lesssim \sum_{0\le\gamma'\le\gamma} 2^{|\gamma-\gamma'|m}\left|\nabla_b^{\gamma'}\left(\Lambda^3_m\left(\Delta^1_m f\right)\right)\right| \lesssim \sum_{0\le\gamma'\le\gamma} 2^{|\gamma-\gamma'|m}\left|\left(\Delta^1_m f\right) * \nabla_b^{\gamma'}\Lambda^3_m\right| \\ &\lesssim \sum_{0\le\gamma'\le\gamma} 2^{|\gamma-\gamma'|m} 2^{|\gamma'|m} M\left|\Delta^1_m f\right| \lesssim_\gamma 2^{|\gamma|m} M\left|\Delta^1_m f\right|\end{aligned}$$

(since $|\gamma-\gamma'| = |\gamma| - |\gamma'|$ when $0 \le \gamma' \le \gamma$). □

LEMMA 5.40. *For all $m \in \mathbb{Z}$, $\gamma \in \left(\mathbb{N}^{d_1}\right)^{\mathbb{N}}$ with $|\gamma| < \infty$, and under the smallness condition (5.35) on f, we have*
(i) $|H_m| \lesssim 1$,
(ii) $\left|\nabla_b^\gamma H_m\right| \lesssim 2^{|\gamma|Q\sigma} \sum_{t>0,\, t\equiv 0(\ \mathrm{mod} R)} 2^{|\gamma|(m-t)} MM\left|\Delta^1_{m-t} f\right|$.

PROOF. Item *(i)* follows directly from the construction. Also, it is proved in [**13**, Section 11].
Item *(ii)* is obtained following the strategy in [**6**, Lemma 6.5]. The proof is similar to the one of Lemma 5.36. It is done by induction on $|\gamma|$ and using Lemmas 5.38, 5.39. □

PROOF OF PROPOSITION 5.37. As in the estimate of $h - \tilde{h}$, we can write

$$\|\nabla_b(g-\tilde{g})\|_{\dot{F}^{\alpha-1,p}_q} \le \sum_{r\in\mathbb{Z}}\left\|\left\|2^{(\alpha-1)m}\Lambda^1_m\nabla_b(G_{r+m}H_{r+m})\right\|_{l^q_m}\right\|_{L^p}.$$

Recalling that

$$G_{r+m} := \sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod} R)}} 2^{-\alpha t}\omega_{r+m-t},$$

we get

$$\begin{aligned}\|\nabla_b(g-\tilde{g})\|_{\dot{F}^{\alpha-1,p}_q} &\le \sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod} R)}} 2^{-\alpha t}\sum_{r\in\mathbb{Z}}\left\|\left\|2^{(\alpha-1)m}\Lambda^1_m\nabla_b(\omega_{r+m-t}H_{r+m})\right\|_{l^q_m}\right\|_{L^p} \\ &= \sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod} R)}} 2^{-\alpha t}\sum_{r>a_\alpha t}\ldots + \sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod} R)}} 2^{-\alpha t}\sum_{r\le 0}\ldots + \sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod} R)}} 2^{-\alpha t}\sum_{0<r\le a_\alpha t}\ldots.\end{aligned}$$

(I) Estimate of $\sum_{r>a_\alpha t}$. Using the fact that $\|H_m\|_{L^\infty} \lesssim 1$ and Proposition 5.26 we have (as in (5.43)):

$$\left\|\left\|2^{(\alpha-1)m}\Lambda_m^1\nabla_b(\omega_{r+m-t}H_{r+m})\right\|_{l_m^q}\right\|_{L^p} \lesssim 2^{-\alpha(r-t)}\left\|\left\|2^{\alpha m}\omega_m\right\|_{l_m^q}\right\|_{L^p} \lesssim 2^{-\alpha(r-t)}2^{Q\sigma}\|f\|_{\dot F_q^{\alpha,p}}.$$

Summing up we get:

$$\begin{aligned}\sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod}R)}} 2^{-\alpha t}\sum_{r>a_\alpha t}\dots &\lesssim \left(\sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod}R)}} 2^{-\alpha t}\sum_{r>a_\alpha t} 2^{-\alpha(r-t)}\right)2^{Q\sigma}\|f\|_{\dot F_q^{\alpha,p}}\\ &= \left(\sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod}R)}}\sum_{r>a_\alpha t}2^{-\alpha r}\right)2^{Q\sigma}\|f\|_{\dot F_q^{\alpha,p}}\\ &\lesssim \sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod}R)}} 2^{-\alpha a_\alpha t}2^{Q\sigma}\|f\|_{\dot F_q^{\alpha,p}} \lesssim 2^{-\alpha a_\alpha R}2^{Q\sigma}\|f\|_{\dot F_q^{\alpha,p}}.\end{aligned}$$

(II) Estimate of $\sum_{r\le 0}$. Let $a \ge 0$ be an integer. As in the estimate (II) for $h-\tilde h$ we obtain

$$\left\|\left\|2^{(\alpha-1)m}\Lambda_m^1\nabla_b(\omega_{r+m-t}H_{r+m})\right\|_{l_m^q}\right\|_{L^p} \lesssim 2^{-(\alpha-1-a)r}\left\|\left\|2^{(\alpha-1-a)m}\nabla_b^{a+1}(\omega_{m-t}H_m)\right\|_{l_m^q}\right\|_{L^p}. \tag{5.44}$$

In order to estimate the right hand side we recall that the following estimates hold (see Proposition 5.26, Proposition 5.31 and Lemma 5.40):

$$\begin{aligned}\omega_{m-t} &\lesssim 2^{Q\sigma}MM(\Delta_{m-t}^1 f),\\ \left|\nabla_b^l\omega_{m-t}\right| &\lesssim 2^{(m-t)l}\omega_{m-t},\\ |H_m| &\lesssim 1,\\ \left|\nabla_b^l H_m\right| &\lesssim 2^{lQ\sigma}\sum_{t>0}2^{(m-t)l}MM\left|\Delta_{m-t}^1 f\right|,\end{aligned}$$

for all $l \in \mathbb{N}$. Using the Leibniz rule, we get:

$$\begin{aligned}\left|\nabla_b^{a+1}(\omega_{m-t}H_m)\right| \lesssim\, & 2^{(m-t)(a+1)}\omega_{m-t}\\ &+2^{(a+1)Q\sigma}\sum_{t'>0}\sum_{l=0}^{a}2^{(t'-t)l}2^{(a+1)(m-t')}MM\left(\Delta_{m-t}^1 f\right)MM\left(\Delta_{m-t'}^1 f\right).\end{aligned} \tag{5.45}$$

Using (5.25), we estimate the double sum from the right hand side as follows:

$$\begin{aligned}\sum_{t'>0}\sum_{l=0}^{a}\dots &\lesssim \|f\|_{\dot F_q^{\alpha,p}}\left(\sum_{0<t'\le t}2^{(a+1)(m-t')}MM\left(\Delta_{m-t'}^1 f\right)+\sum_{t'>t}2^{(t'-t)a}2^{(a+1)(m-t')}MM\left(\Delta_{m-t}^1 f\right)\right)\\ &\lesssim \|f\|_{\dot F_q^{\alpha,p}}\sum_{0<t'\le t}2^{(a+1)(m-t')}MM\left(\Delta_{m-t'}^1 f\right).\end{aligned}$$

Going back to (5.45), we obtain

$$\left|\nabla_b^{a+1}(\omega_{m-t}H_m)\right| \lesssim 2^{(m-t)(a+1)}\omega_{m-t}+2^{(a+1)Q\sigma}\|f\|_{\dot F_q^{\alpha,p}}\sum_{0<t'\le t}2^{(a+1)(m-t')}MM\left(\Delta_{m-t'}^1 f\right)$$

and hence, the term $\left\| \left\| 2^{(\alpha-1-a)m} \nabla_b^{a+1}(\omega_{m-t} H_m) \right\|_{l^q_m} \right\|_{L^p}$ is bounded by

$$\begin{aligned}
&2^{(\alpha-1-a)t} \left\| \left\| 2^{\alpha(m-t)} \omega_{m-t} \right\|_{l^q_m} \right\|_{L^p} + 2^{(a+1)Q\sigma} \|f\|_{\dot F^{\alpha,p}_q} B_{a,\alpha}(t) \left\| \left\| 2^{\alpha m} MM\left(\Delta^1_m f\right) \right\|_{l^q_m} \right\|_{L^p} \\
\lesssim &2^{(\alpha-1-a)t} \left\| \left\| 2^{\alpha m} \omega_m \right\|_{l^q_m} \right\|_{L^p} + 2^{(a+1)Q\sigma} \|f\|_{\dot F^{\alpha,p}_q} B_{a,\alpha}(t) \left\| \left\| 2^{\alpha m} \Delta^1_m f \right\|_{l^q_m} \right\|_{L^p} \\
\lesssim &2^{Q\sigma} 2^{(\alpha-1-a)t} \|f\|_{\dot F^{\alpha,p}_q} + 2^{(a+1)Q\sigma} \|f\|_{\dot F^{\alpha,p}_q} B_{a,\alpha}(t) \left\| \left\| 2^{\alpha m} \Lambda^1_m f \right\|_{l^q_m} \right\|_{L^p} \\
\lesssim &2^{Q\sigma} 2^{(\alpha-1-a)t} \|f\|_{\dot F^{\alpha,p}_q} + 2^{(a+1)Q\sigma} B_{a,\alpha}(t) \|f\|^2_{\dot F^{\alpha,p}_q},
\end{aligned}$$

where $B_{a,\alpha}(t) := \sum_{0<t'\le t} 2^{(\alpha-1-a)t'}$. Here, we have used the Feffereman-Stein inequality to pass from the first to the second line, Proposition 5.26 *(v)* and (5.26) to pass from the second to the third line. Hence,

$$\left\| \left\| 2^{(\alpha-1-a)m} \nabla_b^{a+1}(\omega_{m-t} H_m) \right\|_{l^q_m} \right\|_{L^p} \lesssim 2^{Q\sigma} 2^{(\alpha-1-a)t} \|f\|_{\dot F^{\alpha,p}_q} + 2^{(a+1)Q\sigma} B_{a,\alpha}(t) \|f\|^2_{\dot F^{\alpha,p}_q}. \tag{5.46}$$

Finally, from (5.44) and (5.46) we obtain

$$\begin{aligned}
\left\| \left\| 2^{(\alpha-1)m} \Lambda^1_m \nabla_b(\omega_{r+m-t} H_{r+m}) \right\|_{l^q_m} \right\|_{L^p} \lesssim &2^{-(\alpha-1-a)r} 2^{Q\sigma} 2^{(\alpha-1-a)t} \|f\|_{\dot F^{\alpha,p}_q} \\
&+ 2^{-(\alpha-1-a)r} 2^{(a+1)Q\sigma} B_{a,\alpha}(t) \|f\|^2_{\dot F^{\alpha,p}_q}.
\end{aligned} \tag{5.47}$$

If we choose now $a = [\alpha]$ and we observe that in this case we have $B_{a,\alpha}(t) \lesssim 1$, then, using (5.47) we can bound the term

$$\sum_{\substack{t>0 \\ t\equiv 0 (\mathrm{mod} R)}} 2^{-\alpha t} \sum_{r\le 0} \left\| \left\| 2^{(\alpha-1)m} \Lambda^1_m \nabla_b(\omega_{r+m-t} H_{r+m}) \right\|_{l^q_m} \right\|_{L^p}$$

by

$$2^{Q\sigma} \left(\sum_{\substack{t>0 \\ t\equiv 0 (\mathrm{mod} R)}} 2^{-\alpha t} \sum_{r\le 0} 2^{-(\alpha-1-a)r} 2^{(\alpha-1-a)t} \right) \|f\|_{\dot F^{\alpha,p}_q} + 2^{(a+1)Q\sigma} \left(\sum_{\substack{t>0 \\ t\equiv 0 (\mathrm{mod} R)}} 2^{-\alpha t} \sum_{r\le 0} 2^{-(\alpha-1-a)r} \right) \|f\|^2_{\dot F^{\alpha,p}_q}.$$

Since

$$\sum_{\substack{t>0 \\ t\equiv 0 (\mathrm{mod} R)}} 2^{-\alpha t} \sum_{r\le 0} 2^{-(\alpha-1-a)r} 2^{(\alpha-1-a)t} = \sum_{\substack{t>0 \\ t\equiv 0 (\mathrm{mod} R)}} 2^{-(a+1)t} \sum_{r\le 0} 2^{(a+1-\alpha)r} \lesssim 2^{-(a+1)R}$$

and

$$\sum_{\substack{t>0 \\ t\equiv 0 (\mathrm{mod} R)}} 2^{-\alpha t} \sum_{r\le 0} 2^{-(\alpha-1-a)r} \lesssim 2^{-\alpha R},$$

we obtain

$$\sum_{\substack{t>0 \\ t\equiv 0 (\mathrm{mod} R)}} 2^{-\alpha t} \sum_{r\le 0} \ldots \lesssim 2^{Q\sigma} 2^{-(a+1)R} \|f\|_{\dot F^{\alpha,p}_q} + 2^{(a+1)Q\sigma} 2^{-\alpha R} \|f\|^2_{\dot F^{\alpha,p}_q}.$$

(III) Estimate of $\sum_{0\le r\le a_\alpha t}$. Using the estimate (5.47) above with $a=0$, we get

$$\sum_{0\le r\le a_\alpha t}\left\|\left\|2^{(\alpha-1)m}\Lambda_m^1\nabla_b(\omega_{r+m-t}H_{r+m})\right\|_{l^q_m}\right\|_{L^p}\lesssim A_\alpha(t)\left(2^{Q\sigma}2^{(\alpha-1)t}\|f\|_{\dot F^{\alpha,p}_q}+2^{Q\sigma}B_{0,\alpha}(t)\|f\|^2_{\dot F^{\alpha,p}_q}\right),$$

where

$$A_\alpha(t)\lesssim\begin{cases}2^{(1-\alpha)a_\alpha t} & \text{if } \alpha<1,\\ a_\alpha t & \text{if } \alpha=1,\\ 1 & \text{if } \alpha>1,\end{cases}\quad\text{and } B_{0,\alpha}(t)\lesssim\begin{cases}1 & \text{if } \alpha<1,\\ t & \text{if } \alpha=1,\\ 2^{(1-\alpha)t} & \text{if } \alpha>1.\end{cases}$$

Now summing up we get three possible bounds:

(1) if $\alpha<1$, we have $a_\alpha<\frac{\alpha}{1-\alpha}$ and

$$\sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod}R)}}2^{-\alpha t}\sum_{0\le r\le a_\alpha t}\dots\lesssim 2^{Q\sigma}2^{-R(1-(1-\alpha)a_\alpha)}\|f\|_{\dot F^{\alpha,p}_q}+2^{Q\sigma}2^{-R(\alpha-(1-\alpha)a_\alpha)}\|f\|^2_{\dot F^{\alpha,p}_q};$$

(2) if $\alpha=1$, we have $a_\alpha=1$ and

$$\sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod}R)}}2^{-\alpha t}\sum_{0\le r\le a_\alpha t}\dots\lesssim 2^{Q\sigma}R2^{-R}\|f\|_{\dot F^{\alpha,p}_q}+2^{Q\sigma}R^2 2^{-R}\|f\|^2_{\dot F^{\alpha,p}_q};$$

(3) if $\alpha>1$, we have $a_\alpha=1$ and

$$\sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod}R)}}2^{-\alpha t}\sum_{0\le r\le a_\alpha t}\dots\lesssim 2^{Q\sigma}2^{-R}\|f\|_{\dot F^{\alpha,p}_q}+2^{Q\sigma}2^{-R}\|f\|^2_{\dot F^{\alpha,p}_q}.$$

Now from the above estimates, since $0<a_\alpha\le\alpha$, we have

$$\sum_{\substack{t>0\\ t\equiv 0(\mathrm{mod}R)}}2^{-\alpha t}\sum_{0\le r\le a_\alpha t}\dots\lesssim 2^{Q\sigma}R2^{-R\min(1,\alpha a_\alpha)}\|f\|_{\dot F^{\alpha,p}_q}+2^{Q\sigma}R^2 2^{-R\min(1,\alpha a_\alpha)}\|f\|^2_{\dot F^{\alpha,p}_q}.$$

Toghether with (I) and (II), this gives Proposition 5.37. □

Proof of Theorem 5.3. Now we can estimate the Triebel-Lizorkin norm of $f_J-F_J=(h-\tilde h)+(g-\tilde g)$. By Proposition 5.33 *(i)* and Proposition 5.37, we have

$$\begin{aligned}\sum_{i=1}^{\Bbbk}\|X_i(f_J-F_J)\|_{\dot F^{\alpha-1,p}_q}&\lesssim\sum_{i=1}^{\Bbbk}\left\|X_i(h-\tilde h)\right\|_{\dot F^{\alpha-1,p}_q}+\sum_{i=1}^{\Bbbk}\|X_i(g-\tilde g)\|_{\dot F^{\alpha-1,p}_q}\\ &\lesssim\left(R\sigma^2 2^{-\sigma\min(1,\alpha)-\frac{\Bbbk\sigma}{p}}+2^{Q\sigma}R2^{-\min(1,\alpha a_\alpha)R}\right)\|f\|_{\dot F^{\alpha,p}_q}+D_{R,\sigma}\|f\|^2_{\dot F^{\alpha,p}_q},\end{aligned}$$

where $D_{R,\sigma}$ is a large constant depending on R and σ.

As in [**6**, Section 7], for $\sigma\in\mathbb{N}$, we set

$$R=R_\sigma:=\left[\frac{100Q}{\min(1,\alpha a_\alpha)}\right]\sigma.$$

If $\delta>0$ is fixed, then it is easy to see (using the fact that $\Bbbk/p<\min(1,\alpha)$) that for a σ large enough, we have

$$R\sigma^2 2^{-\sigma\min(1,\alpha)+\frac{\Bbbk\sigma}{p}}\le\delta/4\quad\text{and}\quad 2^{Q\sigma}R2^{-\min(1,\alpha a_\alpha)R}\le\delta/4.$$

Hence, for a large D_δ we have

$$\sum_{i=1}^{\Bbbk}\|X_i(f_J-F_J)\|_{\dot F^{\alpha-1,p}_q}\le\frac{\delta}{2}\|f\|_{\dot F^{\alpha,p}_q}+D_\delta\|f\|^2_{\dot F^{\alpha,p}_q},$$

and since we assumed that $\|f\|_{\dot F^{\alpha,p}_q}$ is small (see (5.35)), then we may take $D_\delta\|f\|_{\dot F^{\alpha,p}_q}\le\delta/2$ obtaining

$$\sum_{i=1}^{\Bbbk}\|X_i(f_J-F_J)\|_{\dot F^{\alpha-1,p}_q}\le\delta\|f\|_{\dot F^{\alpha,p}_q}.\tag{5.48}$$

In a similar way, using Proposition 5.33 *(ii)* and Proposition 5.37 we get

$$\sum_{i=1}^{d_1} \|X_i(f_J - F_J)\|_{\dot F_q^{\alpha-1,p}} \lesssim \sum_{i=1}^{d_1} \left\|X_i(h-\tilde h)\right\|_{\dot F_q^{\alpha-1,p}} + \sum_{i=1}^{d_1} \|X_i(g-\tilde g)\|_{\dot F_q^{\alpha-1,p}}$$
$$\lesssim D'_{R,\sigma}\|f\|_{\dot F_q^{\alpha,p}} + D''_{R,\sigma}\|f\|^2_{\dot F_q^{\alpha,p}},$$

and hence, as above,

$$\sum_{i=1}^{d_1} \|X_i(f_J - F_J)\|_{\dot F_q^{\alpha-1,p}} \lesssim_\delta \|f\|_{\dot F_q^{\alpha,p}}, \tag{5.49}$$

provided that $\|f\|_{\dot F_q^{\alpha,p}}$ is small enough. From (5.49) and the lifting property (Proposition 5.22) of the Triebel-Lizorkin norm, we get

$$\|F_J\|_{\dot F_q^{\alpha,p}} \le \|(f_J - F_J)\|_{\dot F_q^{\alpha,p}} + \|f_J\|_{\dot F_q^{\alpha,p}} \sim \sum_{i=1}^{d_1} \|X_i(f-F)\|_{\dot F_q^{\alpha-1,p}} + \|f\|_{\dot F_q^{\alpha,p}} \lesssim_\delta \|f\|_{\dot F_q^{\alpha,p}}. \tag{5.50}$$

Now (5.48) and (5.50) together with the L^∞ estimates (5.34) give Theorem 5.3 under the smallness assumption on $\|f\|_{\dot F_q^{\alpha,p}}$ (observing that the bounds proved do not depend on J and taking $J \to \infty$). We complete the proof of Theorem 5.3 via the homogeneity of the norms.

REMARK 5.41. (1) Following the same lines, it is also possible (and easier) to prove a version of Theorem 5.3 for the Besov spaces introduced in Subsection 2.3:

THEOREM 5.42. *Consider the parameters $1 < p < \infty$, $1 < q \le \infty$, $\alpha = Q/p$ and let $\Bbbk$ be the largest positive integer with $\Bbbk < \min(p, d_1)$. Then, for every $\delta > 0$ there exists a constant $C_\delta > 0$ depending only on δ, such that for every function $f \in \dot B_q^{\alpha,p}(G)$ there exists $F \in L^\infty(G) \cap \dot B_q^{\alpha,p}(G)$ satisfying the following estimates:*

$$\sum_{i=1}^{\Bbbk} \|X_i(f-F)\|_{\dot B_q^{\alpha-1,p}(G)} \le \delta \|f\|_{\dot B_q^{\alpha,p}(G)},$$
$$\|F\|_{L^\infty(G)} + \|F\|_{\dot B_q^{\alpha,p}(G)} \le C_\delta \|f\|_{\dot B_q^{\alpha,p}(G)}.$$

(2) To mention one application of Theorem 5.3, we state the following generalisation of Theorem 1.8 in [**13**] concerning the Hodge systems on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$. Note that in this case $d = 2n+1$, $d_1 = 2n$ and $Q = 2n+2$.

THEOREM 5.43. *Suppose $n \ge 3$ is an integer. Consider $1 < p, q < \infty$, $\alpha := (2n+2)/p$ and let r be an integer with $1 \le r < \min(p/2, n-1)$. For any $(0,r)$-form φ in $\dot F_q^{\alpha,p}(\mathbb{H}^n)$, there exists a $(0,r)$-form Y in $L^\infty(\mathbb{H}^n) \cap \dot F_q^{\alpha,p}(\mathbb{H}^n)$ such that*

$$\bar\partial_b^* Y = \bar\partial_b^* \varphi$$

and

$$\|Y\|_{L^\infty(\mathbb{H}^n)} + \|Y\|_{\dot F_q^{\alpha,p}(\mathbb{H}^n)} \lesssim \left\|\bar\partial_b^* \varphi\right\|_{\dot F_q^{\alpha-1,p}(\mathbb{H}^n)}.$$

We recall here the meaning of $\bar\partial_b$ and $\bar\partial_b^*$ following [**12**, p. 594-595]. Let $d\bar z_1, \dots, d\bar z_n$ be the basic $(0,1)$-forms on $\mathbb{H}^n$, where $z_j = x_j + i y_j$. If $I = \{j_1, \dots, j_q\}$, with $1 \le j_1 < \dots < j_q \le n$, we write

$$d\bar z_I := d\bar z_{j_1} \wedge \dots \wedge d\bar z_{j_q}.$$

Suppose $1 \le q \le n$ is given and for each I with $|I| = q$, some smooth complex-valued functions f_I are given on $\mathbb{H}^n$. Then,

$$\bar\partial_b \left(\sum_{|I|=q} f_I d\bar z_I \right) := \sum_{j=1}^{n} \sum_{|I|=q} \bar Z_j(f_I) d\bar z_j \wedge d\bar z_I,$$

where $\overline{Z}_j$ are the left-invariant Cauchy-Riemann operators

$$\overline{Z}_j := \frac{\partial}{\partial \overline{z}_j} - i z_j \frac{\partial}{\partial t}.$$

An expression like

$$\sum_{|I|=q} f_I d\overline{z}_I,$$

will be called $(0,q)$-form on $\mathbb{H}^n$.

The operator $\overline{\partial}_b^*$ is the formal adjoint of $\overline{\partial}_b$. We have $\left\langle \overline{\partial}_b^* f, g \right\rangle = \left\langle f, \overline{\partial}_b g \right\rangle$ for any $(0,q)$-form f and any $(0,(q-1))$-form g on $\mathbb{H}^n$.

Theorem 5.43 is proved by using Theorem 5.3 to approximate in an efficient way the coefficients of the form φ and then to conclude by using an iteration argument. Since the proof is very similar to the one given in [**13**] and its Euclidean analogue in [**6**, Theorem 1.2], we omit it. Theorem 5.4 can be proved following the same lines.

5. Appendix

We collect here some facts related to the Calderón-Zygmund theory on stratified homogeneous groups for vector-valued functions. These results (Lemma 5.44 and Theorem 5.45) are well-known. However, since it is hard to find the exact statements in the literature (see for example [**2**] for a Euclidean version, or [**7**] for similar considerations on spaces of homogeneous type), we have chosen to present them here.

Consider a Banach space A. In what follows we deal with functions from the space $L^p_A := L^p(G,A)$ where $1 \le p \le \infty$.

A first result is a Calderón-Zygmund decomposition of fuctions on G (see also [**7**, Théorème 2.2, Chapitre 3]), obtained via the weak $(1,1)$ estimate for the maximal operator:

LEMMA 5.44. *Consider a function $f \in L^1_A$ and a number $\lambda > 0$. Then there exist a countable family of measurable sets $(\Omega_n)_{n\ge 1}$ which are pairwise disjoint and a decomposition $f = g + b = g + \sum_n b_n$ where $g, b, b_n \in L^1_A$ for all $n \ge 1$, and such that:*
(i) $\|g\|_{L^\infty_A} \lesssim \lambda$;
(ii) $\operatorname{supp} b_n \subseteq \Omega_n$, $\int b_n(x)dx = 0$ and $\|b_n\|_{L^1_A} \lesssim \lambda |\Omega_n|$ for all n;
(iii) $\sum_n |\Omega_n| \lesssim \frac{1}{\lambda}\|f\|_{L^1_A}$.

PROOF. We adapt the standard proof in the Euclidean case. Consider the open set $\tilde{\Omega} := \{x \in G |\ M\|f\|_A(x) > \lambda\}$. For each $x \in \tilde{\Omega}$ we consider a ball $B(x,r_x)$ centered in x and such that $B(x,r_x) \subset \tilde{\Omega}$, but $2\cdot B(x,r_x) \subsetneq \tilde{\Omega}$ (recall that, if $c > 0$ and B is a ball in G centered in x_B of radius R_B, then $c \cdot B$ is the ball in G of center x_B and of radius cR_B). Notice that, by Proposition 5.6,

$$|B(x,r_x)| \le \left|\tilde{\Omega}\right| \le \frac{1}{\lambda}\|f\|_{L^1_A}$$

and hence, the balls $B(x,r_x)$ have uniformly bounded radii. Using the Vitali covering lemma (which has the same proof in G as in the Euclidean case), we can find a countable subfamily of balls $(B_k)_{k\ge1}$ of the family $(B(x,r_x))_{x\in\tilde{\Omega}}$, which are pairwise disjoint and such that $\tilde{\Omega} = \bigcup_{x\in\tilde{\Omega}} B(x,r_x) \subseteq \bigcup_{k\ge1} C\cdot B_k$, where $C > 2$ is an absolute constant depending only on G.

We set

$$\Omega_1 := (\tilde{\Omega} \cap C\cdot B_1) \setminus \left(\bigcup_{j\ne1} B_j\right)$$

and inductively we define

$$\Omega_k := \left((\tilde{\Omega} \cap C\cdot B_k) \setminus \bigcup_{1\le i\le k-1} \Omega_i\right) \setminus \left(\bigcup_{j\ne k} B_j\right)$$

for all $k \geq 2$. We see immediately that for all $k \geq 1$ we have $B_k \subseteq \Omega_k \subseteq C \cdot B_k$ and this also give us that $|\Omega_k| \sim |B_k| \sim |C \cdot B_k|$. By definition the sets Ω_k are pairwise disjoint and $\tilde{\Omega} = \bigcup_{k \geq 1} \Omega_k$. We can define the functions:

$$g(x) := \begin{cases} f(x), & \text{if } x \notin \tilde{\Omega} \\ f_{\Omega_k}, & \text{if } x \in \Omega_k \end{cases} \tag{5.51}$$

and $b_k := \left(f - f_{\Omega_k}\right) 1_{\Omega_k}$ for all $k \geq 1$. Here, $f_{\Omega_k} := |\Omega_k|^{-1} \int_{\Omega_k} f \, dx$.

To prove *(i)*, we see that if $x \in \Omega_k$ we have

$$\|g(x)\|_A = \left\|f_{\Omega_k}\right\|_A \leq \frac{1}{|\Omega_k|} \int_{\Omega_k} \|f(y)\|_A \, dy \lesssim \frac{1}{|C \cdot B_k|} \int_{C \cdot B_k} \|f(y)\|_A \, dy \leq M \|f\|_A (x_0) \leq \lambda,$$

where x_0 is a point in $C \cdot B_k \backslash \tilde{\Omega}$. (Such a point exists since $2 \cdot B_k \subsetneq \tilde{\Omega}$ and $2 \cdot B_k \subset C \cdot B_k$.) For a.e. $x \notin \tilde{\Omega}$, by the Lebesgue differentiation theorem (which is a consequence of the weak estimate for the operator M), we have $\|g(x)\|_A \leq M \|f\|_A (x) \leq \lambda$.

To prove *(ii)* and *(iii)*, observe that by the above inequality we have

$$\left\| \|b_k\|_A \right\|_{L^1} \leq |\Omega_k| \left(\frac{1}{|\Omega_k|} \int_{\Omega_k} \|f(y)\|_A \, dy + \left\|f_{\Omega_k}\right\|_A \right) = 2|\Omega_k| \left\|f_{\Omega_k}\right\|_A \lesssim |\Omega_k| \lambda, \text{ for all } k,$$

and, using the weak estimate for M,

$$\sum_k |\Omega_k| \lesssim \sum_{k=1}^{\infty} |B_k| = \left| \bigcup_{k=1}^{\infty} B_k \right| \leq \left|\tilde{\Omega}\right| \lesssim \frac{1}{\lambda} \left\| \|f\|_A \right\|_{L^1}.$$

We can also see from these inequalities that

$$\sum_k \|b_k\|_{L^1_A} \lesssim \lambda \sum_k |\Omega_k| \lesssim \|f\|_{L^1_A}. \tag{5.52}$$

This proves in particular that the series defining b is absolutely convergent in L^1_A and that $b, g \in L^1_A$ satisfy $\|g\|_{L^1_A} + \|b\|_{L^1_A} \lesssim \|f\|_{L^1_A}$. □

THEOREM 5.45. *Suppose A_1 and A_2 are two Banach spaces and $K \in L^1_{loc}(G \setminus \{0\} \to \mathfrak{L}(A_1, A_2))$ has the following properties:*

(i) there exists a constant $c > 0$ such that $\int_{\|x\|_G \geq c\|y\|_G} \left\|K(x) - K(y^{-1} \cdot x)\right\| dx \leq 1$ for all $y \in G$;

(ii) the operator $Tf := f * K$ is well-defined and bounded from $L^q_{A_1}$ to $L^q_{A_2}$ for some $q \in (1, \infty)$.

Then, $T : L^1_{A_1} \to L^{1,\infty}_{A_2}$ is well-defined and bounded. By real interpolation and duality we get that $T : L^p_{A_1} \to L^p_{A_2}$ is well-defined and bounded for any $p \in (1, \infty)$.

(Here $\mathfrak{L}(A_1, A_2)$ stands for the space of the bounded linear operators from A_1 to A_2.)

PROOF. We adapt again the proof in the Euclidean case. Using Lemma 5.44 we can write, for a given $f \in L^1(A_1)$ and $\lambda > 0$, the decomposition at height λ: $f = g + b$. We next note that $\left|\left\{\|Tf\|_{A_2}(x) > 2\lambda\right\}\right| \leq \left|\left\{\|Tg\|_{A_2}(x) > \lambda\right\}\right| + \left|\left\{\|Tb\|_{A_2}(x) > \lambda\right\}\right|$. The size of the set $\left\{\|Tg\|_{A_2} > \lambda\right\}$ can be bounded using *(ii)* above and the Markov inequality:

$$\begin{aligned} \left|\left\{\|Tg\|_{A_2}(x) > \lambda\right\}\right| \leq & \lambda^{-q} \left\| \|Tg\|_{A_2} \right\|^q_{L^q} \lesssim \lambda^{-q} \left\| \|g\|_{A_1} \right\|^q_{L^q} = \lambda^{-q} \left\| \|g\|^q_{A_1} \right\|_{L^1} \\ \leq & \lambda^{-q} \lambda^{q-1} \left\| \|g\|_{A_1} \right\|_{L^1} = \lambda^{-1} \|g\|_{L^1_{A_1}} \lesssim \lambda^{-1} \|f\|_{L^1_{A_1}}. \end{aligned}$$

To estimate the size of the set $\left\{\|Tb\|_{A_2}(x) > \lambda\right\}$ we proceed as follows. Consider the sets Ω_k from the proof of Lemma 5.44; for each such Ω_k we denote by y_{B_k} the center of the ball $B_k \subset \Omega_k$ and we set $\Omega^*_k := (C_1 + C) \cdot B_k \supset \Omega_k$ where $C_1 > 0$ is a large constant depending only on G and c. We write now

$$\left|\{\|Tb\|_{A_2}(x)>\lambda\}\right| \le \left|\bigcup_k \Omega_k^*\right| + \left|\left\{x\in G\setminus\bigcup_k \Omega_k^* \mid \|Tb\|_{A_2}(x)>\lambda\right\}\right|$$
$$\lesssim \lambda^{-1}\|f\|_{L^1_{A_1}} + \lambda^{-1}\int_{G\setminus\bigcup_k\Omega_k^*}\|Tb\|_{A_2}(x)dx,$$

and it remains to estimate the last term. For this purpose, we note that if $x \in G\setminus\Omega_k^*$ and $y\in\Omega_k$, then $\rho(x,y_{B_k}) = \left\|y_{B_k}^{-1}\cdot x\right\|_G \ge (C_1+C)R_{B_k} \ge C^{-1}(C_1+C)\rho(y,y_{B_k}) \ge C^{-1}C_1\left\|y_{B_k}^{-1}\cdot y\right\|_G$ (with R_{B_k} the radius of B_k) and thanks to the quasinorm property of $\|\cdot\|_G$, we find a constant $C_2>0$ depending on G only, such that $\|y^{-1}\cdot x\|_G = \left\|y^{-1}\cdot y_{B_k}\cdot y_{B_k}^{-1}\cdot x\right\|_G \ge C_2\left\|y_{B_k}^{-1}\cdot x\right\|_G - \|y^{-1}\cdot y_{B_k}\|_G \ge \left(C^{-1}C_1C_2-1\right)\left\|y_{\Omega_k}^{-1}\cdot y\right\|_G$, where we used the equality $a^{-1}=-a$ on G. If C_1 is sufficiently large, we deduce $\|y^{-1}\cdot x\|_G \ge c\,\|y^{-1}\cdot y_{B_k}\|_G = c\left\|\left(y^{-1}\cdot x\right)\left(y_{B_k}^{-1}\cdot x\right)^{-1}\right\|_G$. As a consequence,

$$\int_{G\setminus\bigcup_k\Omega_k^*}\|Tb\|_{A_2}(x)dx \le \sum_n\int_{G\setminus\bigcup_k\Omega_k^*}\left\|\int_{\Omega_n}K(y^{-1}\cdot x)b_n(y)dy\right\|_{A_2}dx$$
$$=\sum_n\int_{G\setminus\bigcup_k\Omega_k^*}\left\|\int_{\Omega_n}\left(K(y^{-1}\cdot x)-K(y_{B_n}^{-1}\cdot x)\right)b_n(y)dy\right\|_{A_2}dx$$
$$\le\sum_n\int_{\Omega_n}\left(\int_{G\setminus\bigcup_k\Omega_k^*}\left\|K(y^{-1}\cdot x)-K(y_{B_n}^{-1}\cdot x)\right\|dx\right)\|b_n(y)\|_{A_1}dy$$
$$\le\sum_n\int_{\Omega_n}\|b_n(y)\|_{A_1}dy \lesssim \|f\|_{L^1_{A_1}},$$

where we have used the condition *(i)* above and (5.52). □

REMARK 5.46. We see from the proof that if $\|T\|_{L^q_{A_1}\to L^q_{A_2}} \le 1$ then we have $\|T\|_{L^p_{A_1}\to L^p_{A_2}} \lesssim_p 1$. Hence if the quantity in *(i)* is bounded by a number $\beta>0$ (instead of 1) and also $\|T\|_{L^q_{A_1}\to L^q_{A_2}} \le \beta$, then we have $\|T\|_{L^p_{A_1}\to L^p_{A_2}} \lesssim_p \beta$.

LEMMA 5.47. *Suppose* $\varphi\in L^1(G)$ *and:*
(i) $\int_{\|y\|_G\ge R}\left|\varphi(y)\right|dy \lesssim R^{-1}$ *for any* $R\ge 1$*;*
(ii) $\int_{\mathbb{R}^d}\left|\varphi(x^{-1}\cdot y)-\varphi(y)\right|dy \lesssim \|x\|_G$ *for all* $x\in G$ *with* $\|x\|_G\le 1$.

If for $r\in G$ *we define* $k_j(x) := \varphi_j(x\cdot 2^{-j}r)$, *where* $\varphi_j(x) := 2^{jQ}\varphi(2^jx)$ *for all* $j\in\mathbb{Z}$, *then, there exists a constant* $c>0$ *depending only on G, such that we have*

$$\int_{\|y\|_G\ge c\|x\|_G}\sum_{j\in\mathbb{Z}}\left|k_j(x^{-1}\cdot y)-k_j(y)\right|dy \lesssim_\varphi \ln(2+\|r\|_G).$$

PROOF. We follow the proof in [**6**]. We decompose the sum under the integral as follows:

$$\sum_{j\in\mathbb{Z}}\left|k_j(x^{-1}\cdot y)-k_j(y)\right| = \sum_{2^j\|x\|_G\le 1}\ldots + \sum_{1<2^j\|x\|_G<2+\|r\|_G}\ldots + \sum_{2^j\|x\|_G\ge 2+\|r\|_G}\ldots =: I+II+III.$$

We now estimate each term. Using *(ii)*, we can estimate the first term as follows

$$\int_{\|y\|_G\ge c\|x\|_G} I \le \int_G\sum_{2^j\|x\|_G\le 1}2^{jQ}\left|\varphi(\left(2^jx^{-1}\right)\cdot\left(2^jy\right)\cdot r)-\varphi(\left(2^jy\right)\cdot r)\right|dy$$
$$\le\int_G\sum_{2^j\|x\|_G\le 1}\left|\varphi(\left(2^jx^{-1}\right)\cdot y)-\varphi(y)\right|dy \lesssim \sum_{2^j\|x\|_G\le 1}2^j\|x\|_G \lesssim 1.$$

For the second term we have:

$$\int_{\|y\|_G\ge c\|x\|_G} II \le \sum_{1<2^j\|x\|_G<2+\|r\|_G} 2\int_{\mathbb{R}^d}\left|k_j(y)\right|dy = 2\sum_{1<2^j\|x\|_G<2+\|r\|_G}\int_{\mathbb{R}^d}\left|\varphi(y)\right|dy$$

and this is bounded by

$$\left|\{\log_2 1/\|x\|_G < j < \log_2((2+\|r\|_G)/\|x\|_G)\}\right| \lesssim \ln(2+\|r\|_G).$$

Using the quasinorm property of $\|\cdot\|_G$ we can find a constant c_1 such that $\|y_1\cdot y_2\|_G \le c_1\|y_1\|_G + c_1\|y_2\|_G$ for all $y_1, y_2 \in G$. Assuming that c is sufficiently large we can estimate the third term as follows.

$$\begin{aligned}
\int_{\|y\|_G\ge c\|x\|_G} III \le & 2\sum_{2^j\|x\|_G\ge 2+\|r\|_G}\int_{\|y\|_G\ge c_2\|x\|_G}\left|k_j(y)\right|dy\\
=& 2\sum_{2^j\|x\|_G\ge 2+\|r\|_G}\int_{\|y\|_G\ge 2^j c_2\|x\|_G}\left|\varphi(y\cdot r)\right|dy\\
=& 2\sum_{2^j\|x\|_G\ge 2+\|r\|_G}\int_{\|y\cdot r^{-1}\|_G\ge 2^j c_2\|x\|_G}\left|\varphi(y)\right|dy\\
\lesssim & \sum_{2^j\|x\|_G\ge 2+\|r\|_G}\int_{\|y\|_G\ge 2^j c_3\|x\|_G}\left|\varphi(y)\right|dy\\
\lesssim & \sum_{2^j\|x\|_G\ge 2+\|r\|_G}\frac{1}{2^j\|x\|_G}\lesssim\frac{1}{2+\|r\|_G}\lesssim 1,
\end{aligned}$$

where $c_2 := (c-c_1)/c_1$ and $c_3 := (c_2-c_1)/c_1$. Here, we have used *(i)* to pass from the fourth to the last line.

Summing up these estimates we get the claim. □

In what follows we will need to apply the above lemma to the function $\varphi := S$, where we recall that $S(x) = \min(1, \|x\|_G^{-Q-1})$. It is easy to verify that the function S satisfies the conditions *(i)* and *(ii)* required by Lemma 5.47. Indeed, by a change of variables, we can write for all $R \ge 1$,

$$\int_{\|y\|_G\ge R}|S(y)|\,dy = R^{-1}\int_{\|y\|_G\ge 1}\|y\|_G^{-Q-1}\,dy \sim R^{-1},$$

which proves that *(i)* is satisfied. To verify *(ii)*, we recall that $|\|b\cdot a\|_G - \|a\|_G| \le C\|b\|_G$ for all $a, b \in G$ (see Proposition 5.25) and note that if $\|y\|_G \le 1 - C\|x\|_G \le 1$, then $\left\|x^{-1}\cdot y\right\|_G \le \|y\|_G + C\|x\|_G \le 1$. In this case $S(x^{-1}\cdot y) = S(y) = 1$. Also, if $\|y\|_G \ge 1 + C\|x\|_G$, then $\left\|x^{-1}\cdot y\right\|_G \ge \|y\|_G - C\|x\|_G \ge 1$. In this case $S(x^{-1}\cdot y) = \left\|x^{-1}\cdot y\right\|_G^{-Q-1}$ and $S(y) = \|y\|_G^{-Q-1}$. Hence, if $\|x\|_G \le 1$, we can write

$$\begin{aligned}
\int_{\mathbb{R}^d}\left|S(x^{-1}\cdot y) - S(y)\right|dy =& \int_{1-C\|x\|_G\le\|y\|_G\le 1+C\|x\|_G}\left|S(x^{-1}\cdot y)-S(y)\right|dy\\
&+\int_{\|y\|_G\ge 1+C\|x\|_G}\left|S(x^{-1}\cdot y)-S(y)\right|dy\\
&\lesssim \|x\|_G + \int_{\|y\|_G\ge 1+C\|x\|_G}\left|\frac{1}{\left\|x^{-1}\cdot y\right\|_G^{Q+1}} - \frac{1}{\|y\|_G^{Q+1}}\right|dy\\
&= \|x\|_G + \int_{\|y\|_G\ge 1+C\|x\|_G}\left|\frac{\left\|x^{-1}\cdot y\right\|_G^{Q+1} - \|y\|_G^{Q+1}}{\left\|x^{-1}\cdot y\right\|_G^{Q+1}\|y\|_G^{Q+1}}\right|dy\\
&\lesssim \|x\|_G + \|x\|_G\int_{\|y\|_G\ge 1+C\|x\|_G}\frac{1}{\|y\|_G^{Q+2}}dy \lesssim \|x\|_G.
\end{aligned}$$

PROPOSITION 5.48. *Suppose $p,q \in (1,\infty)$. Then, for every sequence $\left(f_j\right)_{j\in\mathbb{Z}}$ in $L^p(G,l^q(\mathbb{Z}))$ and for every $r \in G$ we have*

$$\left\| \left\| S_j f_j(x\cdot\left(2^{-j}r\right)) \right\|_{l^q_j} \right\|_{L^p_x} \lesssim_{p,q} \ln(2+\|r\|_G) \left\| \left\| f_j \right\|_{l^q_j} \right\|_{L^p}.$$

PROOF. As we already saw, the function S satisfies the requirements of Lemma 5.47. Let k_j as in Lemma 5.47 with $\varphi = S$. We see directly that, for any Schwartz function f, we have

$$f * k_j(x) = \int_{\mathbb{R}^d} f(y) S_j(y^{-1}\cdot x\cdot(2^{-j}r))dy = S_j f(x\cdot(2^{-j}r)).$$

Let K be the kernel given by $K := (k_j)_{j\in\mathbb{Z}}$. We consider

$$T(f_j)_{j\in\mathbb{Z}}(x) := (f_j)_{j\in\mathbb{Z}} * K(x) = (f_j * k_j)_{j\in\mathbb{Z}}(x) = (S_j f(x\cdot(2^{-j}r)))_{j\in\mathbb{Z}},$$

the operator T being initially defined for a sequence of Schwartz functions $(f_j)_{j\in\mathbb{Z}}$. Considering the Banach spaces $A_1 = A_2 = l^q(\mathbb{Z})$ we can see that the statement of the Proposition 5.48 is equivalent to the fact that the operator $T : L^p_{A_1} \to L^p_{A_2}$ is continuous, with its norm bounded by $\ln(2+\|r\|_G)$. This can be obtained as follows. Consider a sequence a in the unit sphere of $l^q(\mathbb{Z})$. We have that:

$$\begin{aligned}\left\langle K(x) - K(y^{-1}\cdot x), a\right\rangle &= \sum_{j\in\mathbb{Z}} \left(k_j(x) - k_j(y^{-1}\cdot x)\right) a_j \le \left(\sum_{j\in\mathbb{Z}} \left|k_j(x) - k_j(y^{-1}\cdot x)\right|^{q'}\right)^{1/q'} \\ &\le \sum_{j\in\mathbb{Z}} \left|k_j(x) - k_j(y^{-1}\cdot x)\right|,\end{aligned}$$

for all $x,y \in G$. Hence $\left\|K(x) - K(y^{-1}\cdot x)\right\| \le \sum_{j\in\mathbb{Z}} \left|k_j(x) - k_j(y^{-1}\cdot x)\right|$ and thanks to Lemma 5.47 we get (using the same notation):

$$\int_{\|x\|_G \ge c\|y\|_G} \left\|K(x) - K(y^{-1}\cdot x)\right\| dx \le \int_{\|x\|_G \ge c\|y\|_G} \sum_{j\in\mathbb{Z}} \left|k_j(x) - k_j(y^{-1}\cdot x)\right| dx \lesssim \ln(2+\|r\|_G).$$

Also we can easily see that $T : L^q_{A_1} \to L^q_{A_2}$ is bounded and of norm 1. These two last observations together with Theorem A1 and the Remark after, give us the claim. □

REMARK 5.49. Proposition 5.48 is reminiscent of an inequality due to Bourgain (see for example [**11**, Section 5]).

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CHAPTER 6

Hodge systems on smooth bounded domains

We consider the Hodge system

$$\begin{cases} du = dv, & \text{on } \Omega \\ u = \gamma, & \text{on } \partial\Omega \end{cases}. \tag{$*$}$$

Here, Ω is a smooth bounded domain in $\mathbb{R}^d$ and

$$v \in F_q^{d/p,p}(\Omega),\ \gamma \in C(\partial\Omega) \cap B_p^{d/p-1/p,p}(\partial\Omega)$$

are given l-forms satisfying a natural compatibility condition. When $1 \le l \le d-2$, $d-l < p < \infty$ and $1 < q < \infty$, we prove that $(*)$ admits a solution

$$u \in C(\overline{\Omega}) \cap F_q^{d/p,p}(\Omega).$$

1. Introduction

We start by recalling the following existence result for Hodge systems on $\mathbb{R}^d$ (see [**3**, Theorem 1.2], also Theorem 0.15 in the Introduction Chapter):

THEOREM 6.1. *Let $1 \le l \le d-2$ be an integer and consider the parameters $d-l < p < \infty$, $1 < q < \infty$, $\alpha := d/p$. If $\varphi \in \dot{F}_q^{\alpha,p}(\mathbb{R}^d)$ is an l-form, then there exists an l-form $u \in L^\infty(\mathbb{R}^d) \cap \dot{F}_q^{\alpha,p}(\mathbb{R}^d)$ such that*

$$du = d\varphi, \text{ on } \mathbb{R}^d.$$

Also, one can choose u such that

$$\|u\|_{L^\infty(\mathbb{R}^d)} + \|u\|_{\dot{F}_q^{\alpha,p}(\mathbb{R}^d)} \lesssim \|\varphi\|_{\dot{F}_q^{\alpha,p}(\mathbb{R}^d)}.$$

In what follows we focus mainly on the case $1 \le l \le d-2$ and we thus assume assume $d \ge 3$. We establish an analogue of the above theorem on smooth bounded domains when a Dirichlet condition is prescribed.

First, let us fix some notation and mention some conventions. Suppose that u is an l-form on some smooth bounded domain Ω. For simplicity, let us assume that u is smooth on $\overline{\Omega}$. If

$$u = \sum_{|I|=l} u_I dx_I,$$

then, on $\partial\Omega$, we have

$$u|_{\partial\Omega} = \sum_{|I|=l} u_I|_{\partial\Omega} dx_I,$$

where $u|_{\partial\Omega}$ is the restriction of u on $\partial\Omega$. Hence, even if expressions of the form

$$\gamma = \sum_{|I|=l} \gamma_I dx_I,$$

where γ_I are functions on $\partial\Omega$, are not differential forms on $\partial\Omega$, such expressions naturally appear as traces on $\partial\Omega$ of forms on Ω. In this chapter, by an abuse of terminology, expressions like γ's above will be denoted as l-forms on $\partial\Omega$.

We introduce an 1-form on $\partial\Omega$ given by

$$\nu = \sum_{j=1}^{d} \nu_j dx_j,$$

where the vector $(\nu_1,...,\nu_d)$ is the outward unit normal to $\partial\Omega$. We will often use the notation $\nu\wedge\phi$ where ϕ is a smooth l-form defined on $\overline{\Omega}$. By convention $\nu\wedge\phi$ is an $(l+1)$-form on $\partial\Omega$, defined by the formula

$$\nu\wedge\phi := \sum_{j=1}^{d}\sum_{|I|=l}\nu_j u_I|_{\partial\Omega} dx_j\wedge dx_I.$$

The compatibility conditions that we will impose are in the style of the following theorem.

THEOREM 6.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^d$. Consider the parameter $1<p<\infty$ and let $r\geq 2$ and $1\leq l\leq d-2$ be two integers. For any l-form $v\in W^{r,p}(\Omega)$ satisfying $\nu\wedge dv=0$ on $\partial\Omega$, there exists an l-form $u\in W^{r,p}(\Omega)$ such that*

$$\begin{cases} du = dv, & \text{on } \Omega\\ \operatorname{tr} u = 0, & \text{on } \partial\Omega\end{cases}.$$

Moreover, u can be chosen such that

$$\|u\|_{W^{r,p}(\Omega)}\lesssim \|v\|_{W^{r,p}(\Omega)}.$$

Theorem 6.2 can be easily deduced from the global regularity results of Dacorogna [**4**] (see [**4**, Theorem 11] for a Hölder spaces version of the above theorem, and [**4**, Introduction] for the arguments leading to Sobolev spaces versions).

Let us discuss the compatibility condition "$\nu\wedge dv=0$ on $\partial\Omega$" in the above theorem. Suppose for simplicity u and v are smooth up to the boundary and $\Omega = Q := (-1,1)^{d-1}\times(0,1)$. The "lower face" of Ω is $\partial_d Q := (-1,1)^{d-1}\times\{0\}$. Note that, on this lower face, we have $\nu=-dx_d$. Hence,

$$\nu\wedge dv = -dx_d\wedge\sum_{|I|=l}\sum_{1\leq i\leq d}(\partial_i v_I)|_{\partial\Omega}dx_i\wedge dx_I = -dx_d\wedge\sum_{\substack{|I|=l\\ d\notin I}}\sum_{1\leq i\leq d-1}(\partial_i v_I)|_{\partial\Omega}dx_i\wedge dx_I$$

and the condition "$\nu\wedge dv=0$" becomes, on the lower face of Ω,

$$\sum_{\substack{|I|=l\\ d\notin I}}\sum_{1\leq i\leq d-1}\partial_i v_I\left(x',0\right)dx_i\wedge dx_I = 0 \tag{6.1}$$

for any $x'\in(-1,1)^{d-1}$. One can see now that, if we define the "genuine" l-form

$$v'\left(x'\right) := \sum_{\substack{|I|=l\\ d\notin I}}\sum_{1\leq i\leq d-1}v_I\left(x',0\right)dx_I$$

in $(-1,1)^{d-1}$, then (6.1) reads $dv'=0$ in $(-1,1)^{d-1}$, i.e., v' is closed in $(-1,1)^{d-1}$. In general, using a pullback (see Section 2 below), one can interpret the condition "$\nu\wedge dv=0$ on $\partial\Omega$" as follows.

We will use the following notation. For an l-form

$$u=\sum_{|I|=l}u_I dx_I,$$

on Q we write

$$\mathfrak{T}u := \sum_{\substack{|I|=l\\ d\notin I}}u_I dx_I \quad\text{and}\quad \mathfrak{N}u := \sum_{\substack{|I|=l\\ d\in I}}u_I dx_I,$$

for the "tangential" and the "normal" component respectively. The same conventions apply to l-forms that are traces on $\partial_d Q$. Note that,

$$u=\mathfrak{T}u+\mathfrak{N}u.$$

Let $x_0\in\partial\Omega$ and let $F: W\to\Omega\cap B$ (here, B is a ball centred at x_0 and $W\subset\mathbb{R}^d$) be a diffeomorphism such that $F^{-1}(\partial\Omega\cap B)\subset\mathbb{R}^{d-1}\times\{0\}$. Then "$\nu\wedge dv=0$ on $\partial\Omega\cap B$" has the meaning that the trace on $\mathbb{R}^{d-1}$ of $\mathfrak{T}F^*v$ is a closed form in $F^{-1}(\partial\Omega\cap B)$. (In what follows forms like $\mathfrak{T}F^*v$ will always be enough regular in order to consider such traces in the sense of distributions.) It is easy to see that this property of v is local and does not depend on the parametrization. In order to see this, consider two diffeomorphisms $F_1 : W_1 \to \Omega \cap B$ and $F_2 : W_2 \to \Omega \cap B$ as above together with a diffeomorphism $G : W_2 \to W_1$ such that $F_2 = F_1 \circ G$. We have $tr\ \mathfrak{T}F_2^* v = tr\ \mathfrak{T}G^* F_1^* v = G|_{\mathbb{R}^{d-1}}^* tr\ \mathfrak{T}F_1^* v$. Hence, if $tr\ \mathfrak{T}F_1^* v$ is closed, then $tr\ \mathfrak{T}F_2^* v$ is closed. In general, the condition "$\nu \wedge dv = 0$ on $\partial\Omega$" means that after covering $\partial\Omega$ with a finite number of balls $B_1, ..., B_n$ as above, we have "$\nu \wedge dv = 0$ on $\partial\Omega \cap B_j$" for each j. This property of v does not depend on the covering of $\partial\Omega$.

REMARK 6.3. If v is smooth, then the condition "$\nu \wedge dv = 0$ on $\partial\Omega$" can be interpreted in the classical sense.

We prefer to work with source terms of the form dv, i.e., we work with exact forms, instead considering closed forms, in order to avoid imposing additional compatibility conditions which are related to the topology of the domain Ω (see [**4**, Remark 16]). Recall that on contractible domains, e.g. on balls, exact is equivalent with closed.

Our result is the following:

THEOREM 6.4. *Let Ω be a smooth bounded domain in $\mathbb{R}^d$. Let $1 \le l \le d-2$ be an integer and consider the parameters $d - l < p < \infty$, $1 < q < \infty$, $\alpha := d/p$. Suppose $\gamma \in C(\partial\Omega) \cap B_p^{\alpha-1/p,p}(\partial\Omega)$ is an l-form and $v \in F_q^{\alpha,p}(\Omega)$ is an l-form satisfying $\nu \wedge dv = \nu \wedge d\gamma$ on $\partial\Omega$. Then, there exists an l-form $u \in C(\overline{\Omega}) \cap F_q^{\alpha,p}(\Omega)$ such that*

$$\begin{cases} du = dv, & \text{on } \Omega \\ u = \gamma, & \text{on } \partial\Omega \end{cases}. \tag{6.2}$$

Moreover, u can be chosen such that

$$\|u\|_{L^\infty(\Omega)} + \|u\|_{F_q^{\alpha,p}(\Omega)} \lesssim \|\gamma\|_{L^\infty(\partial\Omega)} + \|\gamma\|_{B_p^{\alpha-1/p,p}(\partial\Omega)} + \|v\|_{F_q^{\alpha,p}(\Omega)}.$$

REMARK 6.5. The condition $d - l < p$ in Theorem 6.4 is a relic of the use of Theorem 6.1.

Here, the condition "$\nu \wedge dv = \nu \wedge d\gamma$ on $\partial\Omega$" means "$\nu \wedge d(v - \tilde{\gamma}) = 0$ on $\partial\Omega$", where $\tilde{\gamma}$ is any continuous extension of γ to Ω. It turns out that this condition do not depend on the extension $\tilde{\gamma}$ neither v, but only on γ and $tr\ v$. This is apparent from the preceding discussion on the compatibility condition. We will write this condition as $\nu \wedge d(v - \gamma) = 0$ or even as $\nu \wedge d(tr\ v - \gamma) = 0$.

A statement similar to Theorem 6.4 in the case $l = d - 1$ (for $2 \le q \le p < \infty$) which corresponds to the divergence equation, was already treated in [**2**, Section 7] (see also Theorem 0.7). (The full statement[1] corresponding to the case $l = d - 1$ can be obtained by following the strategy in [**2**, Section 7] and using Theorem 6.1 instead of [**2**, Theorem 1.1].) Note that if v is a sufficiently regular $(d-1)$-form, we have $\nu \wedge dv = 0$ on $\partial\Omega$ regardless the choice of γ. Here, the compatibility condition we have to impose is of a different type:

$$\int_\Omega dv = \int_{\partial\Omega} \langle \gamma, \nu \rangle d\sigma.$$

For simplicity, in what follows we do not treat the case $l = d - 1$.

Few words concerning the proof of Theorem 6.4. We prove our result via a sophistication of the techniques in [**2**, Section 7] and an application of Theorem 6.2. More specifically, we use the methods in [**2**, Section 7] in order to obtain the conclusion Theorem 6.4 up to a higher regularity "error term". Then we use Theorem 6.2 in order to deal with this "error term".

2. Some useful facts

We briefly recall below some facts from [**2**, Section 7] that will be useful later.

Reflection operator. Consider $\sigma > 0$. Let k be the integer part of σ and consider the Vandermonde matrix

$$A := \left((-1/j)^{i-1}\right)_{1 \le i,j \le k+1}.$$

[1] i.e., for $1 < p, q < \infty$.

Since A is invertible, we can introduce a vector $a \in \mathbb{R}^{k+1}$ by the formula

$$a = (a_1, ..., a_{k+1})^t := A^{-1}(1, ..., 1)^t.$$

Let $Q := (-1,1)^{d-1} \times (0,1)$. For a function $f \in F_q^{\sigma,p}(Q)$, we define its "reflection" $\mathfrak{R}f$ on $Q' := (-1,1)^{d-1} \times (-1,1)$ by

$$\mathfrak{R}f(x', x_d) := \begin{cases} f(x', x_d), & \text{if } x_d \geq 0 \\ \sum_{1 \leq j \leq k+1} a_j f(x', -\frac{x_d}{j}), & \text{if } x_d < 0 \end{cases}.$$

As shown in [**2**, Section 7], we have

$$\|\mathfrak{R}f\|_{F_q^{\sigma,p}(Q')} \lesssim \|f\|_{F_q^{\sigma,p}(Q)}. \tag{6.3}$$

(We note that it is easy to see that (6.3) holds when $q = 2$ and σ is an integer. In this case, the space $F_q^{\sigma,p}$ reduces to a classical Sobolev space, and then (6.3) is well-known, see e.g. [**1**, Theorem 5.19].)

Extensions and traces. Consider $\sigma > 1/p$ and a function $\rho \in C_c^\infty(\mathbb{R}^{d-1})$. We write ρ_t for the function defined by $\rho_t(x') := t^{1-d}\rho(x'/t)$, with $t > 0$. Given a function $f \in B_p^{\sigma-1/p,p}(\mathbb{R}^{d-1})$, one can "extend" it to $\mathbb{R}^{d-1} \times (0,\infty)$ by setting

$$F(x', x_d) := f * \rho_{x_d}(x'),\ \forall (x', x_d) \in \mathbb{R}^{d-1} \times (0,\infty).$$

In addition, assume in what follows that $\operatorname{supp} f \subseteq (-1,1)^{d-1}$. Then, we have (see [**2**, Lemma 7.3])

$$\|F\|_{F_q^{\sigma,p}(Q)} \lesssim \|f\|_{B_p^{\sigma-1/p,p}((-1,1)^{d-1})}. \tag{6.4}$$

When the integral of ρ on $\mathbb{R}^{d-1}$ is 1, the function F extends f, in the sense that $\operatorname{tr} F = f$ on $\mathbb{R}^{d-1} \times \{0\}$ that we identify with $\mathbb{R}^{d-1}$. When the integral of ρ on $\mathbb{R}^{d-1}$ is 0, we have $\operatorname{tr} F = 0$ on $\mathbb{R}^{d-1}$ (see [**2**, proof of Lemma 7.2]).

One can also see directly that, if f is continuous on $[-1,1]^{d-1}$, then F is continuous on $\mathbb{R}^{d-1} \times [0,\infty)$, and we have

$$\|F\|_{L^\infty(Q)} \lesssim \|f\|_{L^\infty([-1,1]^{d-1})}. \tag{6.5}$$

Besov spaces on $\partial\Omega$. Let $\sigma > 0$, $1 < p, q < \infty$ be some parameters and suppose Ω is a smooth bounded domain in $\mathbb{R}^d$. Then, there exist open sets V_1, ..., V_n covering $\partial\Omega$ such that each $V_j \cap \Omega$ is isometric with a smooth epigraph. Hence, for each j there exists a bounded domain $\Gamma_j \subset \mathbb{R}^{d-1}$ and a diffeomorphism $\psi_j : \Gamma_j \to V_j \cap \partial\Omega$. Let f be a smooth function defined on $\partial\Omega$. We define

$$\|f\|_{B_q^{\sigma,p}(\partial\Omega)} := \sum_{j=1}^n \|f \circ \psi_j\|_{B_q^{\sigma,p}(\Gamma_j)};$$

Different coverings yield equivalent norms (see [**7**, Section 3.3] for details). The space $B_q^{\sigma,p}(\partial\Omega)$ is the completion of $C^\infty(\partial\Omega)$ with the respect to the above norm. It is well-known that if $F \in F_q^{\sigma,p}(\Omega)$ and $\sigma > 1/p$, then $tr\ F \in B_p^{\sigma-1/p,p}(\partial\Omega)$. Conversely, given $f \in B_p^{\sigma-1/p,p}(\partial\Omega)$, there exists $F \in F_q^{\sigma,p}(\Omega)$ such that $tr\ F = f$ on $\partial\Omega$ (see e.g. [**7**, Section 3.3]).

Pullback. We quickly recall here some basic properties of pullbacks following [**8**]. Suppose $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ are two smooth bounded domains and let $F : \Omega_1 \to \Omega_2$ be a smooth function. The pullback $F^*\phi$ of an k-form ϕ on Ω_2 is the k-form on Ω_1 defined by

$$F^*\phi := \sum_{|I|=k} (\phi_I \circ F)(dF_{i_1}) \wedge ... \wedge (dF_{i_k}),$$

where $\{i_1 < ... < i_k\} = I$ in the above sum. We have the following properties:

$$F^*(\phi_1 \wedge \phi_2) = (F^*\phi_1) \wedge (F^*\phi_2),$$

for any two forms ϕ_1, ϕ_2 on Ω_2 (see [**8**, p. 75]);

$$F^*(d\phi) = dF^*(\phi),$$

for any form ϕ on Ω_2 (see [**8**, p. 76]).
If $\Omega_3 \subset \mathbb{R}^{d_3}$ is another smooth bounded domain and $F' : \Omega_2 \to \Omega_3$ is smooth, then

$$(F' \circ F)^* \phi = F^* F'^* \phi,$$

for any form ϕ on Ω_3 (see [**8**, Exercise 1, p. 81]). As a consequence, when F is a diffeomorphism,

$$(F^{-1})^* F^* \phi = \phi,$$

for any form ϕ on Ω_2.

Similar properties will be used also in the case where the forms are defined only on boundaries of domains.

Before preceding to the proof of Theorem 6.4, let us note that, the condition " $\nu \wedge dv = \nu \wedge d\gamma$ on $\partial\Omega$" is necessary for the solvability of (6.2). This can be easily deduced from the following proposition, by using some of the above facts.

PROPOSITION 6.6. *Suppose $1 < p, q < \infty$ and $s > 1/p$ are given and let $1 \leq l \leq d-2$ be an integer. If $u \in F_q^{s,p}(\Omega)$ is an l-form such that $du = 0$ in the distributions sense in Ω, then $\nu \wedge du = 0$ on $\partial\Omega$.*

PROOF. It suffices to prove that, for each $x \in \partial\Omega$, there exists a number $r_x > 0$ such that $\nu \wedge du = 0$ on $\partial\Omega \cap B(x, r_x)$. Fix $x_0 \in \partial\Omega$, and consider a number $r_{x_0} > 0$ such that for $B := B(x_0, r_{x_0})$ the set $\Omega \cap B$ is a piecewise smooth, simply connected domain. By standard regularity theory there exists an $(l-1)$-form $\phi \in F_q^{s+1,p}(\Omega \cap B)$ such that $u = d\phi$ on $\Omega \cap B$. Consider now a diffeomorphism $F : V \to B$ (here, $V \subset \mathbb{R}^d$ is an open set) such that $\Gamma := F^{-1}(\partial\Omega \cap B) \subset \mathbb{R}^{d-1} \times \{0\}$. $V_1 := F^{-1}(\Omega \cap B) \subset V$. Since $F^* u \in F_q^{s,p}(V_1)$ (see [**6**, Proposition 6, p. 16]), we easily see that the trace on Γ of $\mathfrak{T} F^* u$ is a genuine form on Γ whose coefficients are in $B_p^{s-1/p,p}(\Gamma)$. It remains to show that the trace on Γ of $\mathfrak{T} F^* u$ is closed on Γ.

We have

$$F^* u = F^* d\phi = d\phi', \text{ on } V_1,$$

where $\phi' := F^* \phi \in F_q^{s+1,p}(V_1)$. Writing

$$\phi' = \sum_{|I| = l-1} \phi'_I dx_I,$$

we get

$$tr\ \mathfrak{T} F^* u = tr \sum_{\substack{|I|=l-1 \\ d \notin I}} \sum_{1 \leq i \leq d-1} \partial_i \phi'_I dx_I = d_{\mathbb{R}^{d-1}} \left(tr\ \mathfrak{T} \phi' \right) \text{ on } \Gamma,$$

where $d_{\mathbb{R}^{d-1}}$ is the exterior derivative considered only in the coordinates $x_1, \ldots, x_{d-1}$.
This concludes the proof of Proposition 6.6. □

In what follows we will use several times Proposition 6.6.

3. Proof of the main result

From now on, we let $\alpha := d/p$. Also, l is an integer such that $1 \leq l \leq d-2$ and p is such that $d - l < p < \infty$, unless otherwise mentioned.

First, we can easily see that Theorem 6.1 implies the following local version.

THEOREM 6.7. *Suppose Ω is a bounded piecewise smooth domain in $\mathbb{R}^d$. Let $1 \leq l \leq d-2$ be an integer and consider the parameters $d - l < p < \infty$, $1 < q < \infty$, $\alpha := d/p$. If $\varphi \in F_q^{\alpha,p}(\Omega)$ is an l-form, then there exists an l-form $u \in C(\overline{\Omega}) \cap F_q^{\alpha,p}(\Omega)$ such that*

$$du = d\varphi, \text{ on } \Omega.$$

Also, one can choose u such that

$$\|u\|_{L^\infty(\Omega)} + \|u\|_{F_q^{\alpha,p}(\Omega)} \lesssim \|\varphi\|_{F_q^{\alpha,p}(\Omega)}.$$

Note that the above statement does not involve any boundary conditions, and is obtained by extending φ to $\mathbb{R}^d$. In order to handle the boundary conditions, we will use the following result:

LEMMA 6.8. *Let Ω be a smooth bounded domain in $\mathbb{R}^d$. Let $1 \le l \le d-2$ be an integer and consider the parameters $d-l < p < \infty$, $1 < q < \infty$, $\alpha := d/p$. For any l-form $\gamma \in C(\partial\Omega)\cap B_p^{\alpha-1/p,p}(\partial\Omega)$ satisfying $\nu \wedge d\gamma = 0$ on $\partial\Omega$, there exists an l-form $u \in C(\overline{\Omega})\cap F_q^{\alpha,p}(\Omega)$ such that*

$$\begin{cases} du = 0, & \text{on } \Omega \\ u = \gamma, & \text{on } \partial\Omega \end{cases}.$$

Moreover, u can be chosen such that

$$\|u\|_{L^\infty(\Omega)} + \|u\|_{F_q^{\alpha,p}(\Omega)} \lesssim \|\gamma\|_{L^\infty(\partial\Omega)} + \|\gamma\|_{B_p^{\alpha-1/p,p}(\partial\Omega)}.$$

Theorem 6.4 is a direct consequence of Theorem 6.7 and Lemma 6.8. Indeed, suppose γ and v are given as in the statement of Theorem 6.4. According to Theorem 6.7, there exists an l-form $u' \in C(\overline{\Omega})\cap F_q^{\alpha,p}(\Omega)$ such that $du' = dv$ in Ω. Since $\operatorname{tr} u' \in C(\partial\Omega)\cap B_p^{\alpha-1/p,p}(\partial\Omega)$ and $\nu\wedge d\left(\gamma - \operatorname{tr} u'\right) = 0$ on $\partial\Omega$ (see Proposition 6.6), Lemma 6.8 implies the existence of some $u'' \in C(\overline{\Omega})\cap F_q^{\alpha,p}(\Omega)$ such that $du'' = 0$ in Ω and $\operatorname{tr} u'' = \gamma - \operatorname{tr} u'$ on $\partial\Omega$. We find that $u := u' + u''$ satisfies the conclusion of Theorem 6.4 (estimates included).

PROOF OF LEMMA 6.8. We note that, thanks to the open mapping theorem, it suffices to prove only the existence part. Following the strategy in [**2**, Section 7] we start with the case of a cube $Q := (-1,1)^{d-1}\times(0,1)$ and its lower boundary $\partial_d Q := (-1,1)^{d-1}\times\{0\}$. More precisely, we consider systems of the form

$$\begin{cases} du = 0, & \text{in } Q \\ u = \gamma, & \text{on } \partial_d Q \end{cases}. \tag{6.6}$$

In most of the cases we identify $\partial_d Q$ with $(-1,1)^{d-1}$. We will also use the notation $x = \left(x', x_d\right) \in Q' := (-1,1)^{d-1}\times(-1,1)$ where $x' \in \partial_d Q$ and $x_d \in (-1,1)$.

PROOF OF SOLVABILITY OF (6.6). *Step 1.* Consider an l-form $\gamma \in C(\partial_d Q)\cap B_p^{\alpha-1/p,p}(\partial_d Q)$, of the form

$$\gamma(x') = \sum_{\substack{|I|=l \\ d\notin I}} \gamma_I(x')dx_I,$$

with $dx_d \wedge d\gamma = 0$ in the sense of distributions on $\partial_d Q$. Note that in this case, thanks to the special form of γ, the condition $dx_d\wedge d\gamma = 0$ reads $d_{\mathbb{R}^{d-1}}\gamma = 0$ (here, we indentify γ with a genuine form on $(-1,1)^{d-1}$ and "$d_{\mathbb{R}^{d-1}}$" is considered only in the variables $x_1,\dots,x_{d-1}$.) In order to apply the results in Section 2 we assume that $supp\ \gamma \subset \partial_d Q$. (However, this hypothesis is not necessary for the final result and can be easily removed.)

It can be seen immediately that there exist an l-form $u' \in C(\overline{Q})\cap F_q^{\alpha,p}(Q)$ and an l-form $\omega \in F_q^{\alpha-1,p}(Q)$ such that

$$\begin{cases} du' = dx_d\wedge\omega, & \text{in } Q \\ \mathfrak{T}u' = \gamma, & \text{on } \partial_d Q \end{cases}. \tag{6.7}$$

Indeed, consider a function $\rho \in C_c^\infty(\partial_d Q)$ with

$$\int_{\mathbb{R}^{d-1}} \rho = 1$$

and define an l-form by

$$u' := \gamma_I * \rho_{x_d} = \sum_{\substack{|I|=l \\ d\notin I}} \gamma_I * \rho_{x_d} dx_I,$$

where $\rho_t(x') = t^{1-d}\rho(x'/t)$ for $x' \in \partial_d Q$ and $t > 0$. According to (6.4) and (6.5) we have that $u' \in C(\overline{Q}) \cap F_q^{\alpha,p}(Q)$ and $\mathfrak{T}u' = \gamma$ on $\partial_d Q$ (see also [**2**, Section 7]). We now compute du'. By using the fact that $d\gamma = 0$, we obtain:

$$\begin{aligned} du' &= \sum_{\substack{|I|=l \\ d\notin I}} \sum_{1\le i\le d-1} \left(\partial_i \gamma_I\right) * \rho_{x_d} dx_i \wedge dx_I + \sum_{\substack{|I|=l \\ d\notin I}} \partial_d \left(\gamma_I * \rho_{x_d}\right) dx_d \wedge dx_I \\ &= \rho_{x_d} * (d\gamma) + dx_d \wedge \omega = dx_d \wedge \omega, \end{aligned} \tag{6.8}$$

where ω is the l-form

$$\omega := \sum_{\substack{|I|=l \\ d\notin I}} \partial_d u'_I dx_I \in F_q^{\alpha-1,p}(Q). \tag{6.9}$$

Step 2. In order to eliminate ω from (6.8), we rely on the following lemma.

LEMMA 6.9. *If $\omega \in F_q^{\alpha-1,p}(Q)$ is an l-form such that $\mathfrak{T}\omega$ is closed in Q, then there exists an l-form $w \in C(\overline{Q}) \cap F_q^{\alpha,p}(Q)$, satisfying*

$$\begin{cases} dw = dx_d \wedge \omega, & \text{in } Q \\ \mathfrak{T}w = 0, & \text{on } \partial_d Q \end{cases}.$$

PROOF OF LEMMA 6.9. We assume for simplicity that $\alpha > 1$. In what follows, $\mathfrak{R}$ is the reflection operator defined in Section 2 for $\sigma := \alpha - 1 > 0$. (When $\alpha \le 1$, the argument has to be modified as was done in [**2**, Section 7, Proof of Theorem 1.3, case i)].)

Since, $\mathfrak{T}\omega$ is closed in Q, one can easily check that $dx_d \wedge \mathfrak{R}\omega$ is closed in Q'. By standard regularity theory, we can find an l-form $\phi \in F_q^{\alpha,p}(Q')$ such that $d\phi = dx_d \wedge \mathfrak{R}\omega$ in Q'. Hence, by using Theorem 6.7, we obtain an l-form $\zeta \in C(\overline{Q'}) \cap F_q^{\alpha,p}(Q')$, such that $d\zeta = d\phi = dx_d \wedge \mathfrak{R}\omega$, in Q'. Let us observe that, by decomposing ζ as

$$\zeta = \sum_{|I|=l} \zeta_I dx_I = \sum_{\substack{|I|=l \\ d\notin I}} \zeta_I dx_I + \sum_{\substack{|I|=l \\ d\in I}} \zeta_I dx_I,$$

we get

$$d\zeta = \sum_{\substack{|I|=l \\ d\notin I}} \partial_d \zeta_I dx_d \wedge dx_I + \sum_{\substack{|I|=l \\ d\in I}} \sum_{1\le i\le d-1} \partial_i \zeta_I dx_i \wedge dx_I + \lambda,$$

where λ is a form whose terms do not contain dx_d as a factor. Since we have $d\zeta = dx_d \wedge \mathfrak{R}\omega$, the form λ must be identically zero. Hence,

$$dx_d \wedge \mathfrak{R}\omega = d\zeta = \sum_{\substack{|I|=l \\ d\notin I}} \partial_d \zeta_I dx_d \wedge dx_I + \sum_{\substack{|I|=l \\ d\in I}} \sum_{1\le i\le d-1} \partial_i \zeta_I dx_i \wedge dx_I. \tag{6.10}$$

We now construct a new l-form w as follows. First, one can check (see [**2**, Section 7]) that there exist real numbers $\beta_2, \beta_3, \ldots, \beta_{k+2}$ such that

$$\beta_1 + \beta_{k+2} a_1 = 1 \text{ and } \beta_j + \beta_{k+2} a_j = 0, \text{ for any } 2 \le j \le k+1, \tag{6.11}$$

and

$$\sum_{j=1}^{k+1} j\beta_j - \beta_{k+2} = 0. \tag{6.12}$$

(See Section 2 for the definition of the a_j's.)

We now define w by

$$w(x',x_d) := \sum_{\substack{|I|=l\\ d\notin I}} \left(\sum_{j=1}^{k+1} j\beta_j \zeta_I(x',\frac{x_d}{j}) - \beta_{k+2}\zeta_I(x',-x_d) \right) dx_I$$
$$+ \sum_{\substack{|I|=l\\ d\in I}} \left(\sum_{j=1}^{k+1} \beta_j \zeta_I(x',\frac{x_d}{j}) + \beta_{k+2}\zeta_I(x',-x_d) \right) dx_I.$$

Clearly, $w \in C(\overline{Q}) \cap F_q^{\alpha,p}(Q)$ and thanks to (6.12) we have $\mathfrak{T}w = 0$ on $\partial_d Q$. Let us see now that $dw = dx_d \wedge \omega$, in Q. Indeed, we have,

$$dw(x',x_d) = \sum_{\substack{|I|=l\\ d\notin I}} \sum_{1\le i\le d-1} \left(\sum_{j=1}^{k+1} j\beta_j \partial_i\zeta_I(x',\frac{x_d}{j}) - \beta_{k+2}\partial_i\zeta_I(x',-x_d) \right) dx_i \wedge dx_I$$
$$+ \sum_{\substack{|I|=l\\ d\notin I}} \left(\sum_{j=1}^{k+1} \beta_j \partial_d\zeta_I(x',\frac{x_d}{j}) + \beta_{k+2}\partial_d\zeta_I(x',-x_d) \right) dx_d \wedge dx_I$$
$$+ \sum_{\substack{|I|=l\\ d\in I}} \sum_{1\le i\le d-1} \left(\sum_{j=1}^{k+1} \beta_j \partial_i\zeta_I(x',\frac{x_d}{j}) + \beta_{k+2}\partial_i\zeta_I(x',-x_d) \right) dx_i \wedge dx_I.$$

Changing the order of summation, we get

$$dw(x',x_d) = \sum_{j=1}^{k+1} j\beta_j \lambda(x',\frac{x_d}{j}) - \beta_{k+2}\lambda(x',-x_d)$$
$$+ \sum_{j=1}^{k+1} \beta_j \left(\sum_{\substack{|I|=l\\ d\notin I}} \partial_d\zeta_I(x',\frac{x_d}{j}) dx_d \wedge dx_I + \sum_{\substack{|I|=l\\ d\in I}} \sum_{1\le i\le d-1} \partial_i\zeta_I(x',\frac{x_d}{j}) dx_i \wedge dx_I \right)$$
$$+ \beta_{k+2} \left(\sum_{\substack{|I|=l\\ d\notin I}} \partial_d\zeta_I(x',-x_d) dx_d \wedge dx_I + \sum_{\substack{|I|=l\\ d\in I}} \sum_{1\le i\le d-1} \partial_i\zeta_I(x',-x_d) dx_i \wedge dx_I \right).$$

Using (6.10) and (6.11) we get that, in Q we have

$$\begin{aligned} dw(x',x_d) =& dx_d \wedge \left(\sum_{j=1}^{k+1} \beta_j \omega(x',\frac{x_d}{j}) + \beta_{k+2}\mathfrak{R}\omega(x',-x_d) \right) \\ =& dx_d \wedge \left(\sum_{j=1}^{k+1} \beta_j \omega(x',\frac{x_d}{j}) + \beta_{k+2} \sum_{j=1}^{k+1} a_j \omega(x',\frac{x_d}{j}) \right) \\ =& dx_d \wedge \sum_{j=1}^{k+1} (\beta_j + \beta_{k+2} a_j) \omega(x',\frac{x_d}{j}) \\ =& dx_d \wedge \omega(x',x_d), \end{aligned}$$

which concludes the proof of the lemma. □

Step 3. In order to obtain u satisfying the boundary condition on $\partial_d Q$, we rely on the following result.

LEMMA 6.10. *Consider an l-form $\gamma \in C(\partial_d Q) \cap B_p^{\alpha-1/p,p}(\partial_d Q)$ such that*

$$\gamma(x') = \sum_{\substack{|I|=l\\ d\in I}} \gamma_I(x') dx_I.$$

Then, there exists $u \in C(\overline{Q}) \cap F_q^{\alpha,p}(Q)$ *such that*

$$\begin{cases} du = 0, & in\ Q \\ u = \gamma, & on\ \partial_d Q \end{cases}.$$

PROOF OF LEMMA 6.10. One can assume that $supp\ \gamma \subset \partial_d Q$. Considering the function ρ_t as before, we define on Q the form

$$u := (-1)^{l-1} \sum_{\substack{|I|=l-1 \\ d \notin I}} \sum_{1 \le i \le d} \partial_i \left(x_d \left(\gamma_{I \cup \{d\}} * \rho_{x_d} \right) \right) dx_i \wedge dx_I = (-1)^{l-1} d \left(x_d \left(\gamma * \rho_{x_d} \right) \right).$$

Note that, thanks to (6.4) and (6.5), we have $u \in C(\overline{Q}) \cap F_q^{\alpha,p}(Q)$ and also we can check that tr $\partial_i \left(x_d \left(\gamma_{I \cup \{d\}} * \rho_{x_d} \right) \right) = 0$ on $\partial_d Q$ whenever $1 \le i \le d-1$ and $tr\ \partial_d \left(x_d \left(\gamma_{I \cup \{d\}} * \rho_{x_d} \right) \right) = \gamma_{I \cup \{d\}}$ on $\partial_d Q$ (see also [**2**, Lemma 7.2]). As a consequence we get that the trace of u on $\partial_d Q$ equals γ:

$$tr\ u = (-1)^{l-1} \sum_{\substack{|I|=l-1 \\ d \notin I}} \gamma_{I \cup \{d\}} dx_d \wedge dx_I = \gamma.$$

Since the form u is exact, we must have $du = 0$ in Q. This completes the proof of Lemma 6.10. □

Step 4. Thanks to the above results, we can now handle the case where the trace has both tangential and nontangential components.

LEMMA 6.11. *Consider an l-form* $\gamma \in C(\partial_d Q) \cap B_p^{\alpha-1/p,p}(\partial_d Q)$ *such that* $\nu \wedge d\gamma = 0$ *on* $\partial_d Q$. *Then, there exists an l-form* $u \in C(\overline{Q}) \cap F_q^{\alpha,p}(Q)$ *satisfying*

$$\begin{cases} du = 0, & in\ Q \\ u = \gamma, & on\ \partial_d Q \end{cases}. \tag{6.13}$$

PROOF OF LEMMA 6.11. We decompose γ as $\gamma = \mathfrak{T}\gamma + \mathfrak{N}\gamma$. The condition $dx_d \wedge d\gamma = 0$ reads $d_{\mathbb{R}^{d-1}} \mathfrak{T}\gamma = 0$. Thanks to *Step1* (see (6.7)), there exist an l-form $u' \in C(\overline{Q}) \cap F_q^{\alpha,p}(Q)$ and an l-form $\omega \in F_q^{\alpha,p}(Q)$ such that

$$\begin{cases} du' = dx_d \wedge \omega, & \text{in } Q \\ \mathfrak{T}u' = \mathfrak{T}\gamma, & \text{on } \partial_d Q \end{cases}.$$

This ω automatically satisfies $\mathfrak{T}\omega = 0$ in Q. According to Lemma 6.9, we can find an l-form $w \in C(\overline{Q}) \cap F_q^{\alpha,p}(Q)$ satisfying

$$\begin{cases} dw = dx_d \wedge \omega, & \text{in } Q \\ \mathfrak{T}w = 0, & \text{on } \partial_d Q \end{cases}.$$

It follows that

$$\begin{cases} d\left(u' - w\right) = 0, & \text{in } Q \\ \mathfrak{T}\left(u' - w\right) = \mathfrak{T}\gamma, & \text{on } \partial_d Q \end{cases}.$$

We can now apply Lemma 6.10 to the trace form $-tr\ \mathfrak{N}\left(u' - w\right) + \mathfrak{N}\gamma$ in order to obtain the existence of an l-form $u_1 \in C(\overline{Q}) \cap F_q^{\alpha,p}(Q)$ such that

$$\begin{cases} du_1 = 0, & \text{in } Q \\ u_1 = -\mathfrak{N}\left(u' - w\right) + \mathfrak{N}\gamma, & \text{on } \partial_d Q \end{cases}.$$

To conclude the proof of Lemma 6.11 it suffices to set $u := u' - w + u_1$. □

PROOF OF LEMMA 6.8 ON A SMOOTH EPIGRAPH. Consider a function $\psi \in C_c^\infty((-1,1)^{d-1})$ and the corresponding domain $\widetilde{Q}$ together with its "lower boundary" $\partial_d \widetilde{Q}$ defined by

$$\widetilde{Q} := \left\{(x', x_d) \in (-1,1)^{d-1} \times \mathbb{R} |\ \psi(x') < x_d < 1 + \psi(x')\right\},$$
$$\partial_d \widetilde{Q} := \left\{(x', x_d) \in (-1,1)^{d-1} \times \mathbb{R} |\ \psi(x') = x_d\right\}.$$

(By an abuse of terminology we will call a domain like $\widetilde{Q}$ above an epigraph and we will say that $\partial_d \widetilde{Q}$ is its graph.)

The function $F : \widetilde{Q} \to Q$ with $F(x', x_d) := (x', x_d - \psi(x'))$ is a diffeomorphism. We observe that F naturally extends to an diffeomorphism (also denoted by F) from the closure of $\widetilde{Q}$ to the closure of Q. Also, the restriction of F to $\partial_d \widetilde{Q}$ (again denoted by F) is a diffeomorphism from $\partial_d \widetilde{Q}$ to $\partial_d Q$.

Now suppose $\gamma \in C(\partial_d \widetilde{Q}) \cap B_p^{\alpha-1/p,p}(\partial_d \widetilde{Q})$ is an l-form with the property that $\nu \wedge d\gamma = 0$ on $\partial_d \widetilde{Q}$. If F^* is the pullback of F, then $dx_d \wedge dF^*\gamma = 0$ on $\partial_d Q$ (equivalently, $\mathfrak{T} F^*\gamma$ is closed on $\partial_d Q$). We also have (see [**6**, Proposition 6, p. 16]) that $F^*\gamma \in C(\partial_d Q) \cap B_p^{\alpha-1/p,p}(\partial_d Q)$.

This enables us to use the case of the cube to find an l-form $u \in C(\overline{Q}) \cap F_q^{\alpha,p}(Q)$ that solves the system

$$\begin{cases} du = 0, & \text{in } Q \\ u = F^*\gamma, & \text{on } \partial_d Q \end{cases}.$$

Now, choosing $\widetilde{u} := \left(F^{-1}\right)^* u$ we get $d\widetilde{u} = \left(F^{-1}\right)^* du = 0$ in $\widetilde{Q}$ and $\operatorname{tr}\widetilde{u} = \left(F^{-1}\right)^* F^*\gamma = \gamma$ on $\partial_d \widetilde{Q}$. Also (see [**6**, Proposition 6, p. 16]) we get $\widetilde{u} \in C(\overline{\widetilde{Q}}) \cap F_q^{\alpha,p}(\widetilde{Q})$. Hence, we have obtained a solution of the system

$$\begin{cases} d\widetilde{u} = 0, & \text{in } \widetilde{Q} \\ \widetilde{u} = \gamma, & \text{on } \partial_d \widetilde{Q} \end{cases}. \tag{6.14}$$

Gluing the pieces.

Now we find a global solution. Suppose that Ω is a bounded smooth domain. It is easy to see that there exist some open sets $V_1, \ldots, V_n, V_1', \ldots, V_n'$ such that $\overline{V_j} \subset V_j'$, $\partial\Omega \subset V_1 \cup \ldots \cup V_n$, and each $V_j' \cap \Omega$ being isometric with a smooth epigraph (whose corresponding graph is $V_j' \cap \partial\Omega$). We choose a family of functions $\phi_1, \ldots, \phi_n \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\phi_j \subset V_j$ and $\phi_1 + \ldots + \phi_n = 1$ on Ω. As in (6.14), we can find for each $j \in \{1, \ldots, n\}$ an l-form $u_j \in C(\overline{V_j' \cap \Omega}) \cap F_q^{\alpha,p}(V_j' \cap \Omega)$ such that $du_j = 0$ in $V_j' \cap \Omega$ and $u_j = \gamma$ on $V_j' \cap \partial\Omega$. We extend each u_j by 0 outside $\overline{V_j' \cap \Omega}$ and we define the l-form $u := \phi_1 u_1 + \ldots + \phi_n u_n$ on $\mathbb{R}^d$. Clearly, $u \in C(\overline{\Omega}) \cap F_q^{\alpha,p}(\Omega)$ and

$$\begin{cases} du = L, & \text{in } \Omega \\ u = \gamma, & \text{on } \partial\Omega \end{cases}, \tag{6.15}$$

where $L = L(u_1, \ldots, u_n) \in F_q^{\alpha,p}(\Omega)$ is an $(l+1)$-form. Note that the differential regularity of the source term L is $\alpha > \alpha - 1$. This will be used in what follows.

In order to complete the proof of Lemma 6.8 we need to apply the following version of Theorem 6.2 adapted to the scale of the Triebel-Lizorkin spaces.

THEOREM 6.12. *Let Ω be a bounded smooth domain in $\mathbb{R}^d$. Consider the parameters $1 < p, q < \infty$, $s > 1$ and an integer $1 \le l \le d-2$. Suppose $v \in F_q^{s,p}(\Omega)$ is an l-form satisfying $\nu \wedge dv = 0$ on $\partial\Omega$. Then, there exists an l-form $u \in F_q^{s,p}(\Omega)$ satisfying*

$$\begin{cases} du = dv, & \text{on } \Omega \\ \operatorname{tr} u = 0, & \text{on } \partial\Omega \end{cases}.$$

Moreover, u can be chosen such that

$$\|u\|_{F^{s,p}_q(\Omega)} \lesssim \|v\|_{F^{s,p}_q(\Omega)}.$$

SKETCH OF PROOF OF THEOREM 6.12. Notice that, by the same method as above we can obtain an analogue of (6.15) for the "noncritical" case. Namely, for a given l-form $g \in B^{s-1/p,p}_p(\partial\Omega)$ satisfying $\nu \wedge dg = 0$ on $\partial\Omega$, we have

$$\begin{cases} d\phi = L_1, & \text{in } \Omega \\ \operatorname{tr}\phi = g, & \text{on } \partial\Omega \end{cases}. \tag{6.16}$$

for some l-forms $\phi \in F^{s,p}_q(\mathbb{R}^d)$ and $L_1 \in F^{s,p}_q(\Omega)$.

Fix $1 < p < \infty$. At this moment we know that, thanks to Theorem 6.2, the above the statement of Theorem 6.12 is true for any integer $s \geq 2$ and $q = 2$. For any pair (s,q) of parameters, consider now the following assertion

$A(s,q)$: "Theorem 6.12 is true for s and q."

We show that (6.16) and $A(s,2)$ for all the integers $s \geq 2$, are sufficient in order to conclude $A(s,q)$ for all the real numbers $s > 1$ and all $1 < q < \infty$.

Let $1 < \sigma < s < \sigma + 1$. We show that

$$A(\sigma+1,2) \text{ implies } A(s,q) \tag{6.17}$$

for any $1 < q < \infty$. Indeed, since $\sigma < s$ we have $F^{s,p}_q(\Omega) \hookrightarrow F^{\sigma,p}_2(\Omega)$. Since, $L_1 \in F^{\sigma,p}_2(\Omega)$ there exists a compactly supported $(l+1)$-form $\widetilde{L}_1 \in F^{\sigma,p}_2(\mathbb{R}^d)$ such that $\widetilde{L}_1 = L_1$ on Ω. Let $\phi_1 := d^*\triangle^{-1}\widetilde{L}_1 \in F^{\sigma+1,p}_2(\Omega)$. We observe that

$$\nu \wedge d\phi_1 = \nu \wedge L_1 = \nu \wedge d\phi = \nu \wedge dg = 0 \text{ on } \partial\Omega.$$

By applying $A(\sigma+1,2)$ one can find an l-form $\phi_2 \in F^{\sigma+1,p}_2(\Omega) \hookrightarrow F^{s,p}_q(\Omega)$ such that

$$d\phi_2 = d\phi_1 = L_1 = d\phi \text{ in } \Omega,$$

and $tr\ \phi_2 = 0$ on $\partial\Omega$. If we define $\phi_3 := \phi - \phi_2 \in F^{s,p}_q(\Omega)$, then, using (6.16), we get

$$\begin{cases} d\phi_3 = 0, & \text{in } \Omega \\ \operatorname{tr}\phi_3 = g, & \text{on } \partial\Omega \end{cases}. \tag{6.18}$$

Now, suppose an l-form $v \in F^{s,p}_q(\Omega)$ is given such that $\nu \wedge dv = 0$ on $\partial\Omega$. Applying (6.18) to $g := tr\ v$, we obtain an l-form $\varphi \in F^{s,p}_q(\Omega)$ such that

$$\begin{cases} d\varphi = 0, & \text{in } \Omega \\ \operatorname{tr}\varphi = \operatorname{tr} v, & \text{on } \partial\Omega \end{cases}.$$

It suffices now to set $u := v - \varphi$ and we obtain $A(s,q)$ and hence (6.17) is proved.

As we mentioned above, thanks to Theorem 6.2 we have $A(s,2)$ for any integer $s \geq 2$. This, together with (6.17) give us that $A(s,q)$ for any real $s > 1$ which is not an integer and any $1 < q < \infty$. By applying once again (6.17) together with this last result, we obtain the full statement of Theorem 6.12. □

PROOF OF LEMMA 6.8 COMPLETED. We extend the $(l+1)$-form L from (6.15) to a compactly supported $(l+1)$-form $\widetilde{L} \in F^{\alpha,p}_q(\mathbb{R}^d)$ and we write $\widetilde{L} = d\phi$ for some l-form $\phi \in F^{\alpha+1,p}_q(\mathbb{R}^d)$. In particular, we have $d\phi = du$ in Ω and thanks to Proposition 6.6, $\nu \wedge d\phi = \nu \wedge du = \nu \wedge d\gamma = 0$ on $\partial\Omega$. Hence, by Theorem 6.12, there exists an l-form $\phi_1 \in F^{\alpha+1,p}_q(\Omega) \hookrightarrow C(\overline{\Omega}) \cap F^{\alpha,p}_q(\Omega)$ with tr $\phi_1 = 0$ on $\partial\Omega$ and such that $d\phi_1 = d\phi = du$ in Ω. Notice that, it suffices now to redefine u as $u - \phi_1 \in C(\overline{\Omega}) \cap F^{\alpha,p}_q(\Omega)$. □

The proof of Theorem 6.4 is complete. □

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Part 3

Miscellaneous

CHAPTER 7

Minimal BV-liftings of $W^{1,1}(\Omega,\mathbb{S}^1)$ maps in 2D are "often" unique

Let Ω be a smooth, bounded and simply connected domain in $\mathbb{R}^2$ and k a positive integer. We prove that the set of vectors $a=(a_1,...,a_k)\in\Omega^k$ for which each $u\in W^{1,1}(\Omega,\mathbb{S}^1)\cap C(\Omega\setminus\{a_1,...,a_k\})$ admits a unique (mod 2π) minimal BV-lifting is of full measure in Ω^k. (Here, $\mathbb{S}^1$ is the unit circle.)

In particular, this implies that the set of those $u\in W^{1,1}(\Omega,\mathbb{S}^1)$ that admit a unique (mod 2π) minimal BV-lifting is dense in $W^{1,1}(\Omega,\mathbb{S}^1)$. This answers a question of Brezis and Mironescu.

1. Introduction

Suppose Ω is a smooth, bounded and simply connected domain in $\mathbb{R}^2$. It is known (see [**4**, Section 6.2], [**3**], [**6**] and [**2**, Theorem 2.4]) that for each $u\in W^{1,1}(\Omega,\mathbb{S}^1)$ there exists a BV-lifting of u on Ω, i.e., there exists $\varphi\in BV(\Omega,\mathbb{R})$ such that $u=e^{i\varphi}$ on Ω. Clearly, φ is not unique; if φ is a BV-lifting, then so is $\varphi+2k\pi$, $k\in\mathbb{Z}$. We say that φ is a minimal BV-lifting of u if

$$|\varphi|_{BV}=\inf_{u=e^{i\phi}}|\phi|_{BV},$$

where

$$|\phi|_{BV}:=\|D\phi\|_{\mathcal{M}(\Omega,\mathbb{R}^2)}.$$

Clearly, the above infimum is attained. In general, the minimal lifting is not unique, even (mod 2π). For example, the following functions have more than one minimal BV-lifting (mod 2π):

a) $u(z):=z/|z|$, on $\Omega=B(0,1)$ (the unit disc);
b) $u(z):=(2z-1)^{-1}|2z-1|(2z+1)|2z+1|^{-1}$, on $\Omega=(-1,1)^2$.

(See Remark 7.10 below.)

In order to simplify the presentation, in what follows, uniqueness of liftings is meant (mod 2π). We do not specify this anymore.

We are going to answer the following question raised in [**2**]: is the set of functions $u\in W^{1,1}(\Omega,\mathbb{S}^1)$ which admit a unique minimal BV-lifting, residual in $W^{1,1}(\Omega,\mathbb{S}^1)$?

The answer is positive. More specifically, we have the following result.

THEOREM 7.1. *Suppose Ω is a smooth, bounded and simply connected domain in $\mathbb{R}^2$. Consider the set*

$$U:=\{u\in W^{1,1}(\Omega,\mathbb{S}^1)\mid u \text{ has a unique minimal } BV\text{-lifting}\}.$$

Then, U is a G_δ dense subset of $W^{1,1}(\Omega,\mathbb{S}^1)$.

This will be proved by using the geometrical description of the minimal liftings given in [**2**] combined with some "generic" geometric properties of k-tuples in Ω^k, where k is a positive integer. In fact, our proof will give a somewhat more precise result. Consider $u\in W^{1,1}(\Omega,\mathbb{S}^1)\cap C(\Omega\setminus\{a_1,...,a_k\})$, where $a_1,...,a_k$ are distinct points in Ω. It is easy to see (see Remark 7.9 below) that whether or not u admits a unique minimal BV-lifting, depends only on the vector of singularities $a=(a_1,...,a_k)\in\Omega^k$ and the vector of degrees $d=(d_1,...,d_k)$, with $d_j:=\deg(u,a_j)$ (the degree of u on a small circle around a_j). We have that in "almost all cases" the minimal BV-lifting of u is unique:

THEOREM 7.2. *Suppose Ω is a smooth, bounded and simply connected domain in $\mathbb{R}^2$. Let k be a positive integer. The set of vectors $a=(a_1,...,a_k)\in\Omega^k$ for which each $u\in W^{1,1}(\Omega,\mathbb{S}^1)\cap C(\Omega\setminus\{a_1,...,a_k\})$ admits a unique minimal BV-lifting (regardless the choice of $d_1,...,d_k\in\mathbb{Z}$) is of full measure in Ω^k.*

It is easy to see, using the results in [**1**], that Theorem 7.2 implies Theorem 7.1 (see Lemma 7.13 below).

2. "Generic" properties of k-tuples in Ω^k

In this part, Ω is an open subset of $\mathbb{R}^2$ such that $\Omega\neq\emptyset,\mathbb{R}^2$.

We start by fixing some notation. Given a point $x\in\Omega$, we will denote by P_x its set of projections on the boundary of Ω, i.e.,

$$P_x:=\{y\in\partial\Omega\mid dist(x,\partial\Omega)=|x-y|\}.$$

We say that $x\in\Omega$ has a unique projection on $\partial\Omega$ if P_x contains only one point. Also, given a set $A\subset\mathbb{R}^2$ we denote by $\operatorname{diam}A$ its diameter.

For the convenience of the reader we mention some elementary geometric facts.

Fact 1. *Consider $r>0$. Suppose P is a point in the open ball $B(O,r)\subset\mathbb{R}^2$, which is not its center. Consider $\alpha\in[0,2\pi]$ and let $Q_\alpha\in\partial B(O,r)$ be such that the angle $\measuredangle POQ_\alpha$ equals α. Then, the distance $|PQ_\alpha|$ is a strictly increasing function of α, for $\alpha\in[0,\pi]$.*

Fact 2. *Suppose P is a point in the open ball $B(O,r)\subset\mathbb{R}^2$, which is not its center. Consider $\alpha<\beta$ two angles in $[0,\pi]$. Suppose Q_α is as above and $Q'_\beta\in\mathbb{R}^2\setminus B(O,r)$ is a point such that that the angle $\measuredangle POQ'_\beta$ equals β. Then, $|PQ_\alpha|<\left|PQ'_\beta\right|$.*

Fact 1 is a direct consequence of the cosine formula. Fact 2 is a direct consequence of Fact 1 and the cosine formula. Indeed, with the above notation, we have from Fact 1 that $|PQ_\alpha|<\left|PQ_\beta\right|$. Now, since the function $x\to x^2-2x|OP|\cos\beta$ is increasing on $(|OP|,\infty)$ and $\left|OQ'_\beta\right|\geq\left|OQ_\beta\right|=r>|OP|$, we have

$$\begin{aligned}\left|PQ'_\beta\right|^2&=\left|OQ'_\beta\right|^2-2\left|OQ'_\beta\right||OP|\cos\beta+|OP|^2\\&>\left|OQ_\beta\right|^2-2\left|OQ_\beta\right||OP|\cos\beta+|OP|^2\\&=\left|PQ_\beta\right|^2>|PQ_\alpha|^2.\end{aligned}$$

Using these facts we prove the following geometric lemma.

LEMMA 7.3. *Let Ω be an open subset of $\mathbb{R}^2$ such that $\Omega\neq\emptyset,\mathbb{R}^2$. Suppose that $B(x_0,r)\subset\Omega$. Then, for any $\varepsilon>0$, there exist two numbers α, $\delta>0$ depending only on ε, and a cone C_α of angle α, with vertex x_0, such that for any $x\in C_\alpha\cap B(x_0,\delta r)$ we have* $\operatorname{diam}P_x<\varepsilon$.

PROOF. Choose $x_1\in P_{x_0}$. We can suppose without loss of generality that $r=|x_1-x_0|$. For each $0<\beta<2\pi$ we consider the open cone C_β of angle β with vertex x_0 and axis determined by the vector x_1-x_0. Let $0<\alpha<\pi/4$ be an angle that will be chosen later. Fact 2 implies that

$$\overline{B}(x,|x-x_1|)\setminus B(x_0,|x_0-x_1|)\subset\overline{C}_{2\alpha}\tag{7.1}$$

for any $x\in C_\alpha$. Indeed, suppose by contradiction that there exists $y\in\overline{B}(x,|x-x_1|)\setminus B(x_0,|x_0-x_1|)$ such that $y\notin\overline{C}_{2\alpha}$. In particular, we have $y\in\mathbb{R}^2\setminus B(x_0,r)$ and

$$|\measuredangle(y-x_0,x-x_0)|>\alpha/2>|\measuredangle(x_1-x_0,x-x_0)|.$$

Fact 2 gives now that $|y-x|>|x_1-x|$, which contradicts the fact that $y\in\overline{B}(x,|x-x_1|)$.

Now, for any $\varepsilon' > 0$ there exists a $\delta > 0$ depending only on ε', such that, if $|x - x_0| < \delta r$, then

$$\overline{B}(x, |x - x_1|) \subset \overline{B}\left(x_0, \left(1 + \varepsilon'\right)|x_0 - x_1|\right). \tag{7.2}$$

Fix $\varepsilon' > 0$ and choose $\delta > 0$ as above. From (7.1) and (7.2) we get that, for any $x \in C_\alpha$ with $|x - x_0| < \delta r$, we have the inclusion

$$\overline{B}(x, |x - x_1|) \backslash B(x_0, |x_0 - x_1|) \subset A_{\alpha,\varepsilon'}, \tag{7.3}$$

where

$$A_{\alpha,\varepsilon'} := \left(\overline{C}_{2\alpha} \cap \overline{B}\left(x_0, \left(1 + \varepsilon'\right)|x_0 - x_1|\right)\right) \backslash B(x_0, |x_0 - x_1|).$$

If $x' \in P_x$, then $\left|x - x'\right| \le |x - x_1|$, and hence $P_x \subseteq \overline{B}(x, |x - x_1|)$. Also, we have $P_x \subseteq \partial\Omega$, and since $B(x_0, |x_0 - x_1|)$ contains no point from $\partial\Omega$, it follows that $P_x \subseteq \overline{B}(x, |x - x_1|) \backslash B(x_0, |x_0 - x_1|)$. Hence, thanks to (7.3), we get $P_x \subset A_{\alpha,\varepsilon'}$.

It remains to observe that, if α and ε' are sufficiently small, then $\operatorname{diam} A_{\alpha,\varepsilon'} < \varepsilon$. This implies

$$\operatorname{diam} P_x \le \operatorname{diam} A_{\alpha,\varepsilon'} < \varepsilon$$

for any $x \in C_\alpha \cap B(x_0, \delta r)$. □

Using the above lemma we are able to prove the following (possibly known) proposition concerning the smallness of the set of points with nonunique projections on the boundary.

PROPOSITION 7.4. *Let $\Omega \subset \mathbb{R}^2$ be an open set such that $\Omega \neq \emptyset, \mathbb{R}^2$. If M is the set of the points of Ω which have unique projection on $\partial\Omega$, then $M^c := \Omega \backslash M$ is a Lebesgue null set.*

PROOF. First we note that

$$M = \bigcap_{n=1}^{\infty} M_n \text{ where } M_n := \{x \in \Omega \mid \operatorname{diam} P_x < 1/n\}.$$

We will show that each M_n contains a Lebesgue measurable set of full measure and hence the exterior measure of each M_n^c is 0. This will show in particular that each M_n is measurable, M is measurable and

$$m\left(M^c\right) \le \sum_{n=1}^{\infty} m\left(M_n^c\right) = 0.$$

Fix $n \ge 1$. With $\varepsilon = 1/n$, let α and δ be as in Lemma 7.3. If $B(x_0, r) \subset \Omega$ and Q is a square centred at x_0 and such that $Q \subset B(x_0, \delta r)$, by applying Lemma 7.3, we can find a cone C of angle α with vertex x_0 such that $C \cap Q \subset M_n$. Note that

$$\frac{m(C \cap Q)}{m(Q)} \ge \eta, \tag{7.4}$$

where $0 < \eta < 1$ only depends on α and hence it only depends on n.

Consider a nonempty open set $V \subset \Omega$. We claim that we may write

$$V = \bigcup_{j=1}^{\infty} Q_j,$$

with Q_j essentially disjoint squares such that, for each j, there exists some ball $B(x_j, r_j) \subset \Omega$ (where x_j is the center of Q_j) with $Q_j \subset B(x_j, \delta r_j)$. Indeed, it suffices to consider first the Whitney decomposition

$$V = \bigcup_{k=1}^{\infty} \widetilde{Q}_k$$

of V, then cut each $\widetilde{Q}_k$ into a finite number of squares of size $< \delta r_0$, where r_0 is the distance from $\widetilde{Q}_k$ to $\partial\Omega$.

Applying (7.4), we get a collection of cones C^1, C^2,... such that $C^j \cap Q_j$ are essentially disjoint and $m(C^j \cap Q_j) \geq \eta m(Q_j)$ for all $j \geq 1$. Now, for $\mathscr{A} := \cup_{j\geq 1}(C^j \cap Q_j)$ we have

$$m(\mathscr{A}) = \sum_{j=1}^{\infty} m(C^j \cap Q_j) \geq \eta \sum_{j=1}^{\infty} m(Q_j) = \eta m(V).$$

Note that, since each $C^j \cap Q_j$ is included in M_n, we have $\mathscr{A} \subset M_n$. This implies that, for any nonempty open set $V \subset \Omega$ (of finite measure) and any $\theta > 0$, there exists a closed set $A \subset V \cap M_n$ such that

$$\frac{m(A)}{m(V)} \geq \eta - \theta. \tag{7.5}$$

We now introduce the following quantity

$$R := \inf_{V\subset\Omega} \sup_{A\subset M_n\cap V} \frac{m(A)}{m(V)},$$

where inf is taken over all nonempty open sets $V \subset \Omega$ and sup is taken over all closed sets $A \subset V \cap M_n$. By (7.5), we have $\eta \leq R \leq 1$. We show that $R = 1$.

Let V be as above. Choose $0 < \theta < R$. We can find a closed set $A_0 \subset V \cap M_n$ such that $m(A_0)/m(V) > R - \theta$. The set $V\setminus A_0$ is nonempy and open. Hence, by (7.5) we can find $A_1 \subset (V\setminus A_0) \cap M_n$ such that $m(A_1)/m(V\setminus A_0) > R - \theta$. We have that $A_0 \cup A_1 \subset V \cap M_n$ and

$$\begin{aligned}\frac{m(A_0 \cup A_1)}{m(V)} &= \frac{m(A_0)}{m(V)} + \frac{m(A_1)}{m(V)} \geq \frac{m(A_0)}{m(V)} + (R-\theta)\frac{m(V\setminus A_0)}{m(V)} \\ &= \frac{m(A_0)}{m(V)} + (R-\theta)\left(1 - \frac{m(A_0)}{m(V)}\right) = (1-R+\theta)\frac{m(A_0)}{m(V)} + R - \theta \\ &\geq (1-R+\theta)(R-\theta) + R - \theta.\end{aligned}$$

Since θ can be chosen arbitrarily small, we get

$$R \geq (1-R)R + R.$$

Hence, we have $R = 0$ or $R = 1$. Since $R \geq \eta > 0$, we get $R = 1$.

This shows that M_n has full measure in Ω, concluding the proof of the Proposition 7.4. □

A shorter proof of this proposition can be given by using Rademacher's differentiation theorem. The following proof was suggested to the author by P. Bousquet.

ANOTHER PROOF OF PROPOSITION 7.4. Consider the function $\varphi : \Omega \to \mathbb{R}$ defined by $\varphi(x) := (dist(x,\partial\Omega))^2$. Choose $x \in \Omega$ such that φ is differentiable in x. Fix $v \in \mathbb{R}^2$. If $x' \in P_x$, then

$$\begin{aligned}\varphi(x+tv) &\leq \left|x + tv - x'\right|^2 = \left|x - x'\right|^2 + 2t\left\langle v, x - x'\right\rangle + t^2|v|^2 \\ &= \varphi(x) + 2t\left\langle v, x-x'\right\rangle + t^2|v|^2,\end{aligned}$$

for any $t \in \mathbb{R}$ with $x + tv \in \Omega$. Hence, if $t > 0$ is as above, we get

$$\frac{\varphi(x+tv) - \varphi(x)}{t} \leq 2\left\langle v, x - x'\right\rangle + t|v|^2,$$

and letting $t \to 0$, we obtain $\left\langle \nabla\varphi(x), v\right\rangle \leq 2\left\langle v, x-x'\right\rangle$. By a similar argument (considering $t < 0$) we get $\left\langle \nabla\varphi(x), v\right\rangle \geq 2\left\langle v, x-x'\right\rangle$. Since v is arbitrary, we get $\nabla\varphi(x) = 2\left(x - x'\right)$. In particular, we obtain that $P_x = \left\{x'\right\}$ (x has unique projection on $\partial\Omega$). (This argument is taken from [**5**, p. 14].)

Since φ is locally Lipschitz, the set of points $x \in \Omega$ such that φ is differentiable in x is of full measure in Ω. By the above observation we get Proposition 7.4. □

LEMMA 7.5. *Suppose* $d_1, d_2 \in \mathbb{N}^*$ *and* $K \subset (0,1)^{d_1} \times (0,1)^{d_2}$ *is a closed set with* $m(K) > 0$. *For any* $y \in (0,1)^{d_2}$ *define*

$$K_y := \left\{x \in (0,1)^{d_1} \mid (x,y) \in K\right\}.$$

Then, there exists a measurable set $A \subset (0,1)^{d_2} \times (0,1)^{d_2}$, *with* $m(A) > 0$, *such that for all the pairs* $(y_1, y_2) \in A$, $m\left(K_{y_1} \cap K_{y_2}\right) > 0$. *In particular, there exists a point* $P = (y_1, y_2) \in A$ *such that all of its* $2d_2$ *coordinates are pairwise distinct and* $m\left(K_{y_1} \cap K_{y_2}\right) > 0$.

PROOF. For $(y_1, y_2) \in (0,1)^{d_2} \times (0,1)^{d_2}$ we write

$$m\left(K_{y_1} \cap K_{y_2}\right) = \int_{(0,1)^{d_1}} \mathbb{1}_{K_{y_1}}(x) \mathbb{1}_{K_{y_2}}(x)\,dx = \int_{(0,1)^{d_1}} \mathbb{1}_K(x, y_1) \mathbb{1}_K(x, y_2)\,dx.$$

Integrating on $(0,1)^{d_2} \times (0,1)^{d_2}$, and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\int_{(0,1)^{d_2}} \int_{(0,1)^{d_2}} m\left(K_{y_1} \cap K_{y_2}\right) dy_1 dy_2 &= \int_{(0,1)^{d_1}} \left(\int_{(0,1)^{d_2}} \mathbb{1}_K(x, y_1)\,dy_1\right) \left(\int_{(0,1)^{d_2}} \mathbb{1}_K(x, y_2)\,dy_2\right) dx \\
&= \int_{(0,1)^{d_1}} \left(\int_{(0,1)^{d_2}} \mathbb{1}_K(x, y)\,dy\right)^2 dx \\
&\geq \left(\int_{(0,1)^{d_1}} \int_{(0,1)^{d_2}} \mathbb{1}_K(x, y)\,dy dx\right)^2 \\
&= (m(K))^2 > 0,
\end{aligned}$$

whence the first claim.

To get the second claim we observe that the set of the points in $(0,1)^{d_2} \times (0,1)^{d_2}$ for which at least two of the $2d_2$ real coordinates coincide, is contained in a finite union of hyperplanes, and hence is a Lebesgue null set. Hence, its complement is of full measure and intersects A. ☐

Now we use the above lemma to prove the following.

LEMMA 7.6. *Let* $\Omega \subset \mathbb{R}^2$ *be an open set such that* $\Omega \neq \emptyset, \mathbb{R}^2$, *and* $k \in \mathbb{N}^*$. *Consider some real numbers* a_i, $1 \leq i \leq k$, α_{ij}, $1 \leq i < j \leq k$ *not all zero and* $c \in \mathbb{R}$. *Almost everywhere on* Ω^k *we have*

$$\sum_{1 \leq i \leq k} a_i dist(x_i, \partial\Omega) + \sum_{1 \leq i < j \leq k} \alpha_{ij} \left|x_i - x_j\right| \neq c.$$

PROOF. Consider the function $f : \Omega^k \mapsto \mathbb{R}$ defined by

$$f(X) = \sum_{1 \leq i \leq k} a_i dist(x_i, \partial\Omega) + \sum_{1 \leq i < j \leq k} \alpha_{ij} \left|x_i - x_j\right|,$$

where $X = (x_1, ..., x_k) \in \Omega^k$.

Suppose by contradiction that the set $M := \left\{x \in \Omega^k \mid f(X) = c\right\}$ has nonzero measure. Since f is continuous, M is closed and by applying Lemma 7.5, we can find some $Y_1 = (x_2^1, ..., x_k^1), Y_2 = (x_2^2, ..., x_k^2) \in \Omega^{k-1}$ such that the elements $x_2^1, ..., x_k^1, x_2^2, ..., x_k^2$ are pairwise distinct and $m\left(M_{Y_1} \cap M_{Y_2}\right) > 0$. We have that, for any $x \in M' := M_{Y_1} \cap M_{Y_2} \subset \Omega$,

$$a_1 dist(x, \partial\Omega) + \sum_{j=2}^{k} \alpha_{1j} \left|x - x_j^1\right| = c_1, \tag{7.6}$$

$$a_1 dist(x, \partial\Omega) + \sum_{j=2}^{k} \alpha_{1j} \left|x - x_j^2\right| = c_2, \tag{7.7}$$

where c_1 and c_2 are some constants. By subtracting the above equalities, we get for any $x \in M'$,

$$\sum_{j=2}^{k} \alpha_{1j} \left|x - x_j^1\right| - \sum_{j=2}^{k} \alpha_{1j} \left|x - x_j^2\right| = c_3$$

for some constant c_3. The function $g : \Omega \setminus \{x_2^1, ..., x_d^1, x_2^2, ..., x_d^2\} \to \mathbb{R}$ defined by

$$g(x) = \sum_{j=2}^{k} \alpha_{1j} \left|x - x_j^1\right| - \sum_{j=2}^{k} \alpha_{1j} \left|x - x_j^2\right|$$

(which is real analytic) is constant on M'. Since $m(M') > 0$, it follows that $g \equiv c_3$ on Ω.

Suppose now that $\alpha_{1j_0} \neq 0$ for some $j_0 \geq 2$. We can write

$$\alpha_{1j_0} \left|x - x_{j_0}^1\right| = -\sum_{\substack{j=2\\ j\neq j_0}}^{k} \alpha_{1j} \left|x - x_j^1\right| + \sum_{j=2}^{k} \alpha_{1j} \left|x - x_j^2\right| + c_3 \tag{7.8}$$

on Ω. However, in a neighborhood of $x_{j_0}^1$, the right hand side of (7.8) is a C^1 function, while the left hand side is not. Hence, we must have $\alpha_{1j} = 0$ for all $j \geq 2$.

By a similar argument we get that all the coefficients α_{ij} are zero.

The relation (7.6) reads now as $a_1 dist(x, \partial\Omega) = c_1$ on M'. Suppose $a_1 \neq 0$ and consider the set

$$\mathscr{S} := \{x \in \Omega \mid dist(x, \partial\Omega) = c_1/a_1\}.$$

Since $M' \subset \mathscr{S}$, the set $\mathscr{S}$ has positive measure. Hence, there exists a Lebesgue point x_0 in $\mathscr{S}$, i.e., some $x_0 \in \mathscr{S}$ satisfying

$$\lim_{r\to 0} \frac{m(\mathscr{S} \cap B(x_0, r))}{m(B(x_0, r))} = 1. \tag{7.9}$$

Let $x_1 \in \partial\Omega$ such that $|x_0 - x_1| = dist(x_0, \partial\Omega)$. Using the notation from the proof of Lemma 7.3, we have that $C_{2\pi/3} \cap \mathscr{S} \cap B(x_0, |x_0 - x_1|) = \emptyset$. Indeed, if $x \in C_{2\pi/3} \cap B(x_0, |x_0 - x_1|)$, then $dist(x, \partial\Omega) < c_1/a_1$. Hence,

$$\lim_{r\to 0} \frac{m(\mathscr{S} \cap B(x_0, r))}{m(B(x_0, r))} = \lim_{r\to 0} \frac{m((\mathscr{S} \cap B(x_0, r)) \setminus C_{2\pi/3})}{m(B(x_0, r))} \leq \frac{2\pi - 2\pi/3}{2\pi} = \frac{2}{3},$$

which contradicts (7.9).

Hence $a_1 = c_1 = 0$. By a similar argument we get also that all the coefficients a_i are zero, obtaining a contradiction. □

With this results we can easily prove the following

PROPOSITION 7.7. *Let $\Omega \subset \mathbb{R}^2$ be an open set such that $\Omega \neq \emptyset, \mathbb{R}^2$, and $k \in \mathbb{N}^*$. For almost all $X = (x_1, ..., x_k) \in \Omega^k$ we have that the numbers $dist(x_i, \partial\Omega)$, $1 \leq i \leq k$, $\left|x_i - x_j\right|$, $1 \leq i < j \leq k$ are linearly independent over $\mathbb{Z}$ and each x_i has a unique projection on $\partial\Omega$.*

(We will say that a point X as above has the property (P).)

PROOF. Let $v_1, v_2, ...,$ be an enumeration of the set $\mathbb{Z}^N \setminus \{0\}$, where $N := k + \binom{k}{2}$, and for each $X = (x_1, ..., x_k) \in \Omega^k$ consider the vector

$$\Delta(X) := \Big((dist(x_i, \partial\Omega))_{1\leq i\leq k}, \left(\left|x_i - x_j\right|\right)_{1\leq i<j\leq k} \Big) \in \mathbb{R}^N.$$

Let $\Lambda_n := \left\{X \in \Omega^k \mid \langle v_n, \Delta(X)\rangle = 0\right\}$ for $n \geq 1$. By Lemma 7.6 we have that $m(\Lambda_n) = 0$ for all $n \geq 1$. Hence, the set

$$\Lambda := \left\{X \in \Omega^k \mid \text{there exists } v \in \mathbb{Z}^N \setminus \{0\} \text{ with } \langle v, \Delta(X)\rangle = 0\right\} = \bigcup_{n=1}^{\infty} \Lambda_n$$

is Lebesgue null.

This fact combined with Proposition 7.4 gives the result. □

REMARK 7.8. It is easy to see that Lemma 7.3, Proposition 7.4, Lemma 7.6 and Proposition 7.7 remain true in $\mathbb{R}^d$ for $d \geq 3$. The adaptations of the above proofs are obvious.

3. Geometric properties of liftings in 2D

From now on we suppose that Ω is a smooth, bounded and simply connected domain in $\mathbb{R}^2$. We are going to apply the Proposition 5.37 in order to obtain the prevalence of the set of those $u \in W^{1,1}(\Omega, \mathbb{S}^1)$, with a finite number of singularities, that admit a unique minimal BV-lifting. We will use the conventions and several facts from [**2**, Chapter 3] to describe the minimal liftings (and the minimal configurations) of a given $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ with a finite number of singularities. We quickly recall these conventions and facts.

Consider a function $u \in W^{1,1}(\Omega, \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, ..., a_k\})$ where $a_1, \ldots, a_k \in \Omega$ are pairwise distinct points. To the vector of singularities $a = (a_1, a_2, ..., a_k)$ we associate the vector of degrees $d = (d_1, d_2, ..., d_k)$ where $d_j := \deg(u, a_j)$ is the degree of u computed on a small circle around a_j. We consider a fictitious point $a_{k+1} \in \partial\Omega$, of degree

$$d_{k+1} = -\sum_{j=1}^{k} d_j.$$

We split the family of points $a_1, a_2, ..., a_k, a_{k+1}$ in two disjoint parts: the family of "positive points" whose degree is positive and the family of "negative points" whose degree is negative. We omit the points of zero degree. The points from the first family will be denoted P_l and those from the second family N_l. With these points we create a list $\{P_l, N_l\}_{1 \leq l \leq m}$ by repeating $|d_j|$ times each point of degree d_j. It is easy to see that there are as many positive and negative points, and therefore these points can be matched in pairs.

We introduce the following pseudometric on $\overline{\Omega}$:

$$dist_s(A_1, A_2) := \min\{|A_1 - A_2|, dist(A_1, \partial\Omega) + dist(A_2, \partial\Omega)\},$$

for $A_1, A_2 \in \overline{\Omega}$.

With this we define the quantity:

$$L(a,d) := \min_{\sigma \in S_m} \sum_{l=1}^{m} dist_s\left(P_l, N_{\sigma(l)}\right). \tag{7.10}$$

We recall that ([**2**, Chapter 3, Lemma 3.4]) we can further add points from $\partial\Omega$ to the collection $\{P_l, N_l\}_{1 \leq l \leq m}$, to obtain a possibly larger collection $\{P_l, N_l\}_{1 \leq l \leq n}$ satisfying the properties:

$$\sum_{l=1}^{n} \delta_{P_l} = \sum_{\substack{j=1 \\ d_j > 0}}^{k} d_j \delta_{a_j}, \quad \sum_{l=1}^{n} \delta_{N_l} = \sum_{\substack{j=1 \\ d_j < 0}}^{k} d_j \delta_{a_j} \ \text{ in } \mathscr{D}'(\Omega), \tag{7.11}$$

and

$$L(a,d) = \sum_{l=1}^{n} |P_l - N_l|. \tag{7.12}$$

We will say that a collection of oriented segments $(P_l, N_l)_{1 \leq l \leq n}$ (counted with multiplicities) formed with points satisfying (7.11) and (7.12) is a *minimal configuration* associated with (a,d). Note that, in general there is more than one minimal configurations for a given u.

A *connection* associated with (a,d) is an $\mathbb{R}^2$-valued measure μ on Ω of the form

$$\mu = \sum_{i=1}^{\infty} \nu_i \mathscr{H}^1 \lfloor (S_i \cap \Omega),$$

where S_i are Borel subsets of C^1 oriented curves in $\mathbb{R}^2$ of normal vectors ν_i, with

$$\sum_{i=1}^{\infty} \mathscr{H}^1 \lfloor (S_i \cap \Omega) < \infty,$$

and satisfying

$$\operatorname{curl}\mu = \sum_{j=1}^{k} d_j \delta_{a_j}.$$

A *minimal connection* (associated with (a,d)) is a connection μ (associated with (a,d)) such that $\|\mu\|_{\mathcal{M}} = L(a,d)$.

It is known (see [**2**, Chapter 3]) that there is a one-to-one correspondence between the minimal connections and the minimal liftings of a given $u \in W^{1,1}(\Omega,\mathbb{S}^1) \cap C(\Omega \setminus \{a_1,...,a_k\})$. (Recall that, by our convention, two minimal liftings are equal if they differ by an integer multiple of 2π.)

REMARK 7.9. The above one-to-one correspondence between the minimal liftings and the minimal connections gives us that the property that $u \in W^{1,1}(\Omega,\mathbb{S}^1) \cap C(\Omega \setminus \{a_1,...,a_k\})$ admits a unique minimal BV-lifting depends only on the vector of singularities $a = (a_1,...,a_k) \in \Omega^k$ and the vector of degrees $d = (d_1,...,d_k)$.

REMARK 7.10. Let us discuss the examples, presented in the introduction, of maps having several minimal BV-liftings.

a) In the case of $u(z) := z/|z|$, on $\Omega = B(0,1)$, we have one singularity at the origin, of degree $+1$. The minimal configurations are given by the pairs (P_1,N_1) where $P_1 = 0$ and N_1 is any point on $\partial D(0,1)$ (considered with the degree -1). Hence, there are infinitely many minimal configurations. Each one of these configurations corresponds to a minimal connection, hence we have an infinite number of minimal BV-liftings for this u.

b) In the case of $u(z) := (2z-1)^{-1}|2z-1|(2z+1)|2z+1|^{-1}$, on $\Omega = (-1,1)^2$, we have two singularities, $a_1 = -1/2$, respectively $a_2 = 1/2$, of degrees $d_1 = +1$, respectively $d_2 = -1$. We have in this case exactly two minimal configurations. One configuration is given by the collection of oriented segments (P_1,N_1), (P_2,N_2), where $P_1 := -1/2$ (of degree $+1$), $N_1 := -1$ (of degree -1), $N_2 := 1/2$ (of degree $+1$), $P_2 := 1$ (of degree $+1$). Another minimal configuration is given by the oriented segment (P_1,N_2) (the same notation). Each one of these configurations corresponds to a minimal connection, hence we have two minimal BV-liftings for this u.

REMARK 7.11. In order to prove Theorem 7.2, we will use a property weaker than the bijective correspondence between the minimal connections and the minimal liftings. More specifically, we rely on the fact that there is a surjective correspondence between the minimal configurations and the minimal liftings of a given $u \in W^{1,1}(\Omega,\mathbb{S}^1) \cap C(\Omega \setminus \{a_1,...,a_k\})$. In particular, if there exists only one minimal configuration for u as above, then, there exists only one minimal lifting of u. (See [**2**, Chapter 3, Remark 3.8].)

We need to introduce some new notation. Let $u \in W^{1,1}(\Omega,\mathbb{S}^1) \cap C(\Omega \setminus \{a_1,...,a_k\})$ and (a,d) be given as above, and suppose the vector $a = (a_1,a_2,...,a_k) \in \Omega^k$ has the property (P) described in Proposition 5.37, namely, the numbers $dist(a_i,\partial\Omega)$, $1 \le i \le k$, $|a_i - a_j|$, $1 \le i < j \le k$ are linearly independent over $\mathbb{Z}$ and each a_i has unique projection on $\partial\Omega$. Let P be a positive point and N a negative point as above. We observe that one and only one of the following may happen:

(i) $dist_s(P,N) = |P-N|$;

(ii) $dist_s(P,N) = |P-N'| + |P'-N|$ for some $P',N' \in \partial\Omega$ with $dist(P,\partial\Omega) = |P-N'|$ and $dist(N,\partial\Omega) = |P'-N|$. Thanks to property (P), the points N' and P' are unique.

Indeed, the definition of d_s ensures that the pair (P,N) is in at least one of the above cases. Also, thanks to the fact that $|P-N|$, $|P-N'|$, $|P'-N|$ are linearly independent over $\mathbb{Z}$, we have that only one of the above situations is possible.

Consider the set of oriented segments

$$M := \{(P_i, N_j) \mid 1 \le i, j \le m,\ (P_i, N_j) \text{ is in case (i)}\}$$
$$\cup \left\{\left(P_i, N_j'\right) \mid 1 \le i, j \le m,\ (P_i, N_j) \text{ is in case (ii)}\right\}$$
$$\cup \{(P_i', N_j) \mid 1 \le i, j \le m,\ (P_i, N_j) \text{ is in case (ii)}\},$$

respectivelly the set of numbers

$$M_d := \{|P_i - N_j| \mid 1 \le i, j \le m,\ (P_i, N_j) \text{ is in case (i)}\}$$
$$\cup \left\{\left|P_i - N_j'\right| \mid 1 \le i, j \le m,\ (P_i, N_j) \text{ is in case (ii)}\right\}$$
$$\cup \{|P_i' - N_j| \mid 1 \le i, j \le m,\ (P_i, N_j) \text{ is in case (ii)}\}.$$

Clearly, the function $\delta : M \mapsto M_d$, defined by $\delta(P, N) := |P - N|$, is a bijection.

Fix $\sigma \in S_m$. Consider the sum

$$L_\sigma := \sum_{l=1}^{m} dist_s\left(P_l, N_{\sigma(l)}\right). \tag{7.13}$$

Note that, thanks to the definition of $dist_s$, this is a sum with elements from M_d. Proposition 5.37 allows us to define the set

$$\mathfrak{C}_\sigma := \left\{\left(\delta^{-1}(r), n\right) \mid (r, n) \in M_d \times \mathbb{N},\ r \text{ appears exactly } n \text{ times in (7.13)}\right\}.$$

If

$$\mathfrak{C}_\sigma = \left\{\left(\delta^{-1}(r_1), n_1\right), \ldots, \left(\delta^{-1}\left(r_p\right), n_p\right)\right\},$$

let C_σ be the collection

$$\underbrace{\delta^{-1}(r_1), \ldots, \delta^{-1}(r_1)}_{n_1 \text{ times}}, \ldots, \underbrace{\delta^{-1}\left(r_p\right), \ldots, \delta^{-1}\left(r_p\right)}_{n_p \text{ times}}.$$

Thanks to Proposition 5.37, we immediately see that if $\sigma_1, \sigma_2 \in S_m$ are such that $L_{\sigma_1} = L_{\sigma_2}$, then $C_{\sigma_1} = C_{\sigma_2}$. If σ is minimal, i.e, $L_\sigma = L(a, d)$, then C_σ is a minimal configuration. In particular, it follows that there is only one minimal configuration. Hence, we get Theorem 7.2 (see Remark 7.11). □

REMARK 7.12. Consider a connection μ associated with (a, d) as above. We can associate with μ a unique 1-rectifiable current given by

$$\mathscr{C} := \sum_{i=1}^{\infty} \tau_i \mathscr{H}^1 \lfloor (S_i \cap \Omega),$$

where τ_i is obtained from ν_i by a rotation of $-\pi/2$ (hence τ_i is tangent to the C^1 curve that supports S_i). We have

$$\partial \mathscr{C} = \sum_{j=1}^{k} d_j \delta_{a_j}. \tag{7.14}$$

Also to each 1-rectifiable current satisfying (7.14) we can associate a unique connection μ. In case where μ is a minimal connection, $\mathscr{C}$ is a mass minimizing 1-rectifiable current.

In the language of geometric measure theory, Remark 7.9 and Remark 7.11 give the following: if there exists only one minimal configuration for (a, d) as above, then, there exists only one mass minimizing 1-rectifiable current (i.e., "least length curve") with (measure geometric) boundary $\sum_{j=1}^{k} d_j \delta_{a_j}$. (See [**2**, Chapter 3, Section 3.9.4] for details.) Thus the proof of Theorem 7.2 implies the following: for a.e. $(a_1, \ldots, a_k) \in \Omega^k$, and for every $(d_1, \ldots, d_k) \in \mathbb{Z}^k$, there exists exactly one least length curve with boundary $\sum_{j=1}^{k} d_j \delta_{a_j}$.

Now we show how Theorem 7.2 implies Theorem 7.1. From now on, we consider domains Ω which are bounded, simply connected and smooth.

Fix $k \in \mathbb{N}$. Let $d = (d_1,...,d_k) \in (\mathbb{Z}\setminus\{0\})^k$ and consider the set W_d of those $u \in W^{1,1}(\Omega,\mathbb{S}^1)$ for which there exist some distinct $a_1,...,a_k \in \Omega$ such that $u \in W^{1,1}(\Omega,\mathbb{S}^1) \cap C(\Omega\setminus\{a_1,...,a_k\})$ and $\deg(u,a_j) = d_j$ for all $1 \le j \le k$. The set W_d is a metric space with the norm induced by $W^{1,1}(\Omega,\mathbb{S}^1)$.

It is easy to see that each $u \in W_d$ can be written as $u = u_a e^{i\psi}$ with $a = (a_1,...,a_k)$ as above, with u_a given by the formula

$$u_a(z) := \prod_{j=1}^{k} \left(\frac{z-a_j}{|z-a_j|} \right)^{d_j}, \ z \in \Omega,$$

and $\psi \in W^{1,1}(\Omega,\mathbb{R})$.

This can be proved as follows. From [**2**, Chapter 3], we have

$$J(u) = J(u_a) = \pi \sum_{j=1}^{k} d_j \delta_{a_j}$$

where $J(u) := \operatorname{curl}(u \wedge \nabla u)/2$. Hence, if $v := u_a^{-1}u = \overline{u}_a u$, then $v \in W^{1,1}(\Omega,\mathbb{S}^1)$ and

$$J(v) = J(\overline{u}_a) + J(u) = -J(u_a) + J(u) = 0. \tag{7.15}$$

Combining (7.15) with and from [**2**, Chapter 2, Lemma 2.8], we find that there exists some $\psi \in W^{1,1}(\Omega,\mathbb{R})$ such that $v = e^{i\psi}$.

We have the following.

LEMMA 7.13. *Fix $k \in \mathbb{N}$. For each $d \in (\mathbb{Z}\setminus\{0\})^k$, the set $U_d := U \cap W_d$ is dense in W_d.*

PROOF. Let $\varepsilon > 0$ and $u \in W_d$. From the above observation, we can write $u = u_a e^{i\psi}$ for some $a = (a_1,...,a_k) \in \Omega^k$, and $\psi \in W^{1,1}(\Omega,\mathbb{R})$. If $a' \in \Omega^k$ and the distance $|a-a'|$ is sufficiently small, then $\|\nabla(u_a - u_{a'})\|_{L^1} < \varepsilon/2$ and

$$\int_\Omega |u_a - u_{a'}| \, |\nabla\psi| \, dx < \varepsilon/\left(2 + 2\|\nabla\psi\|_{L^1}\right).$$

For $u' := u_{a'} e^{i\psi}$ we have

$$\|\nabla(u-u')\|_{L^1} \le \|\nabla(u_a - u_{a'})\|_{L^1} + \int_\Omega |u_a - u_{a'}| \, |\nabla\psi| \, dx < \varepsilon.$$

Note that Theorem 7.2 allows us to choose $a' \in \Omega^k$ as above and such that $u' \in U_d$ admits a unique minimal BV-lifting. ☐

Note that, since the set of those $u \in W^{1,1}(\Omega,\mathbb{S}^1)$ with a finite number of singularities is dense in $W^{1,1}(\Omega,\mathbb{S}^1)$ (see [**1**]), Lemma 7.13 immediately implies that U is dense in $W^{1,1}(\Omega,\mathbb{S}^1)$. This gives the density part in Theorem 7.1.

To complete the proof of Theorem 7.1, we show that U is a G_δ set in $W^{1,1}(\Omega,\mathbb{S}^1)$. We present below the argument.

For each $u \in W^{1,1}(\Omega,\mathbb{S}^1)$, we consider the set $\mathcal{L}(u)$ of all minimal BV-liftings ϕ of u satisfying

$$\left| \frac{1}{m(\Omega)} \int_\Omega \phi(x)\, dx \right| \le \pi. \tag{7.16}$$

We endow $\mathcal{L}(u)$ with the L^1 metric and we consider $\rho : \mathcal{L}(u) \times \mathcal{L}(u) \mapsto [0,\infty)$ defined by

$$\rho(\varphi_1,\varphi_2) := \inf_{k\in\mathbb{Z}} \|\varphi_1 - \varphi_2 + 2\pi k\|_{L^1}, \quad (\varphi_1,\varphi_2) \in \mathcal{L}(u) \times \mathcal{L}(u).$$

Define

$$\operatorname{diam}_\rho \mathscr{L}(u) := \sup_{\phi_1,\phi_2 \in \mathscr{L}(u)} \rho\left(\phi_1,\phi_2\right), \tag{7.17}$$

and consider the sets

$$D_n := \left\{u \in W^{1,1}\left(\Omega,\mathbb{S}^1\right) \mid \operatorname{diam}_\rho \mathscr{L}(u) < 1/n\right\},\ n \geq 1.$$

We easily check that $U = \cap_{n\geq 1} D_n$ and hence it suffices to prove that each D_n is open in $W^{1,1}\left(\Omega,\mathbb{S}^1\right)$.

For this purpose we start by establishing some useful properties.

First, let $(u_m)_{m\geq 1}$ be a sequence in $W^{1,1}\left(\Omega,\mathbb{S}^1\right)$ converging to some $u \in W^{1,1}\left(\Omega,\mathbb{S}^1\right)$, and let $\left(\varphi^m\right)_{m\geq 1}$ be a sequence in $BV(\Omega,\mathbb{R})$ such that φ^m is a minimal lifting of u_m for each $m \geq 1$. If φ^m converges to some $\varphi \in BV(\Omega,\mathbb{R})$ in the L^1 norm, then φ is a minimal lifting of u.

Indeed,

$$\left\|u_m - e^{i\varphi}\right\|_{L^1(\Omega)} = \left\|e^{i\varphi^m} - e^{i\varphi}\right\|_{L^1(\Omega)} \leq \left\|\varphi^m - \varphi\right\|_{L^1(\Omega)} \to 0,$$

when $m \to \infty$. It follows that $u_m \to e^{i\varphi} \in BV\left(\Omega,\mathbb{S}^1\right)$ in the sense of distributions and hence $u = e^{i\varphi}$, i.e., φ is a BV-lifting of u.

Using [**2**, Corollary 2.4], we have that

$$\Sigma(u_m) = \left\|u_m \wedge \nabla u_m - D\varphi^m\right\|_{\mathscr{M}}, \tag{7.18}$$

where $\Sigma(v)$, for $v \in W^{1,1}\left(\Omega,\mathbb{S}^1\right)$, is defined as being the quantity

$$\Sigma(v) := \inf_{\phi \in BV(\Omega,\mathbb{R})} \left\|v \wedge \nabla v - D\phi\right\|_{\mathscr{M}}.$$

In order to show the minimality of φ, it suffices to show that

$$\left\|u \wedge \nabla u - D\varphi\right\|_{\mathscr{M}} \leq \left\|u \wedge \nabla u - D\psi\right\|_{\mathscr{M}},$$

for any $\psi \in BV(\Omega,\mathbb{R})$ (see [**2**, Corollary 2.4]).

Fix $\psi \in BV(\Omega,\mathbb{R})$ as above. By (7.18) we have, for all $m \geq 1$,

$$\left\|u_m \wedge \nabla u_m - D\varphi^m\right\|_{\mathscr{M}} \leq \left\|u_m \wedge \nabla u_m - D\psi\right\|_{\mathscr{M}}. \tag{7.19}$$

Since $u_m \wedge \nabla u_m \to u \wedge \nabla u$ in L^1, we immediately see that

$$\left\|u_m \wedge \nabla u_m - D\psi\right\|_{\mathscr{M}} \to \left\|u \wedge \nabla u - D\psi\right\|_{\mathscr{M}}.$$

Also, $D\varphi^m \to D\varphi$ in the sense of distributions and hence, from (7.19) we get

$$\left\|u \wedge \nabla u - D\varphi\right\|_{\mathscr{M}} \leq \varliminf_{m\to\infty} \left\|u_m \wedge \nabla u_m - D\varphi^m\right\|_{\mathscr{M}} \leq \left\|u \wedge \nabla u - D\psi\right\|_{\mathscr{M}}.$$

A second observation is that the supremum in (7.17) is attained. Indeed, by the above observation, $\mathscr{L}(u)$ is compact in $L^1(\Omega)$. Since $\mathscr{L}(u) \times \mathscr{L}(u)$ is compact in $L^1(\Omega) \times L^1(\Omega)$ and ρ is continuous, there exist $\varphi_1, \varphi_2 \in \mathscr{L}(u)$ such that

$$\rho\left(\varphi_1,\varphi_2\right) = \operatorname{diam}_\rho \mathscr{L}(u). \tag{7.20}$$

Going back to the proof Theorem 7.1, it remains to prove that D_n^c is a closed set. We have that:

$$D_n^c = \left\{u \in W^{1,1}\left(\Omega,\mathbb{S}^1\right) \mid \operatorname{diam}_\rho \mathscr{L}(u) \geq 1/n\right\}.$$

Suppose that $(u_m)_{m\geq 1}$ is a sequence in D_n^c converging to some $u \in W^{1,1}\left(\Omega,\mathbb{S}^1\right)$. From (7.20), there exist two sequences $\left(\varphi_1^m\right)_{m\geq 1}$, $\left(\varphi_2^m\right)_{m\geq 1}$ with $\varphi_1^m, \varphi_2^m \in \mathscr{L}(u_m)$ for all $m \geq 1$, such that $u_m = e^{i\varphi_1^m} = e^{i\varphi_2^m}$ and

$$\rho\left(\varphi_1^m,\varphi_2^m\right) = \operatorname{diam}_\rho \mathscr{L}(u_m) \geq 1/n. \tag{7.21}$$

Since $(u_m)_{m\geq 1}$ is bounded in $W^{1,1}\left(\Omega,\mathbb{S}^1\right)$, we get $\left\|\varphi_1^m\right\|_{BV}, \left\|\varphi_2^m\right\|_{BV} \lesssim \|u_m\|_{W^{1,1}} \lesssim 1$. Hence, there exist $\varphi_1, \varphi_2 \in BV(\Omega,\mathbb{R})$ such that $\varphi_1^m \to \varphi_1$, $\varphi_2^m \to \varphi_2$ in L^1, possibly up to a subsequence. According to our observation, φ_1 and φ_2 are minimal liftings of u. We have from (7.21) and the continuity of ρ that $\rho\left(\varphi_1,\varphi_2\right)\geq 1/n$. Also, φ_1, φ_2 satisfy (7.16). We get that $u\in D_n^c$. The proof of Theorem 7.1 is complete. □

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CHAPTER 8

On the continuity of Fourier multipliers on $\dot{W}^{l,1}(\mathbb{R}^d)$ and $\dot{W}^{l,\infty}(\mathbb{R}^d)$

Suppose $d \geq 2$. Kazaniecki and Wojciechowski proved in 2013 ([**3**]) that every Fourier multiplier on $\dot{W}^{1,1}(\mathbb{R}^d)$ is a bounded continuous function on $\mathbb{R}^d$. This is a generalization of an old result of Bonami and Poornima concerning homogeneous multipliers of degree zero. We further generalize the result of Kazaniecki and Wojciechowski. We prove that, given an integer $l \geq 1$, every multiplier on $\dot{W}^{l,1}(\mathbb{R}^d)$ or on $\dot{W}^{l,\infty}(\mathbb{R}^d)$ is a bounded continuous function on $\mathbb{R}^d$. We obtain these results via a substantial simplification of the Riesz products technique used in [**3**]. Another feature of our approach is that it does not rely on transference theorems.

1. Introduction

In this chapter, we study the continuity properties of functions which are Fourier multipliers on the homogeneous Sobolev spaces $\dot{W}^{l,1}(\mathbb{R}^d)$ and $\dot{W}^{l,\infty}(\mathbb{R}^d)$, where $l \geq 1$ is an arbitrary integer.

Given a nonnegative integer l and a parameter $1 \leq p \leq \infty$, the space $\dot{W}^{l,p}(\mathbb{R}^d)$ consists of those distributions f on $\mathbb{R}^d$ for which $\nabla^l f \in L^p(\mathbb{R}^d)$. This space is endowed with the seminorm given by

$$\|f\|_{\dot{W}^{l,p}(\mathbb{R}^d)} = \left\|\nabla^l f\right\|_{L^p(\mathbb{R}^d)} = \max_{|\alpha|=l} \|\nabla^\alpha f\|_{L^p(\mathbb{R}^d)}.$$

Given a function $m \in L^1_{loc}(\mathbb{R}^d)$, we say that m is a Fourier multiplier on $\dot{W}^{l,p}(\mathbb{R}^d)$ if, for each Schwartz function $f \in \mathscr{S}(\mathbb{R}^d)$ the distribution $m\widehat{f}$ is temperate and if, in addition, the following estimate holds:

$$\|T_m f\|_{\dot{W}^{l,p}} \leq C\|f\|_{\dot{W}^{l,p}}, \ \forall f \in \mathscr{S}(\mathbb{R}^d), \tag{8.1}$$

for some constant $C < \infty$, where T_m is defined by the relation

$$\widehat{T_m f} = m\widehat{f}, \ \forall f \in \mathscr{S}(\mathbb{R}^d).$$

The least constant C in the above inequality will be called the norm of m and will be denoted by $\|T_m\|$ (which is a quantity depending on p and l).

The Fourier transform that we work with is given by the following formula

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} f(x)dx, \ \forall \mathscr{S}(\mathbb{R}^d).$$

Some classical examples of multipliers on $\dot{W}^{l,p}(\mathbb{R}^d)$ in the case $1 < p < \infty$ (for any l), are the functions $m_j(\xi) := \xi_j/|\xi|$, defined for $\xi \in \mathbb{R}^d \setminus \{0\}$ and any $j = 1,2,...,d$. In this case we have $T_{m_j} = R_j$, where $R_1,...,R_d$ are the Riesz transforms on $\mathbb{R}^d$. Let us observe that the functions m_j are homogeneous of degree zero, i.e., $m_j(\lambda\xi) = m_j(\xi)$, for all $\xi \in \mathbb{R}^d \setminus \{0\}$ and any $\lambda > 0$. Also, m_j are not continuous at zero.

If $p = 1$, the situation is different. When $l = 0$, the m_j's fail to be multipliers on $\dot{W}^{0,1}(\mathbb{R}^d) = L^1(\mathbb{R}^d)$, since the R_j's are not bounded on $L^1(\mathbb{R}^d)$. In fact, if m is a multiplier on $L^1(\mathbb{R}^d)$, then it is easy to see that m is the Fourier transform of a finite measure and hence $m \in C_b(\mathbb{R}^d)$. The case of the multipliers on $L^\infty(\mathbb{R}^d)$ is similar.

Suppose $d \geq 2$. Then there exist Fourier multipliers on $\dot{W}^{1,1}(\mathbb{R}^d)$ which are not Fourier transforms of finite measures (see [**7**, Proposition 2.2]). In fact, the proof in [**7**] concerning $\dot{W}^{l,1}(\mathbb{R}^d)$ applies to all the spaces $\dot{W}^{l,p}(\mathbb{R}^d)$, with $l \geq 1$ and $1 \leq p \leq \infty$.

Let us illustrate this when $d = 2$ and $l = 1$, via a simple example from [**7**]. As a consequence of Ornstein's L^1 non-inequality (see [**6**]), there exists a distribution u, supported in the unit ball, such that $\partial_1^2 u$, $\partial_1^2 u$ are L^1 functions on $\mathbb{R}^2$ and $\partial_1\partial_2 u$ is not a finite measure. We define $m := \widehat{\partial_1\partial_2 u}$. Clearly, m is not the Fourier transform of a finite measure, however m is a multiplier on $\dot{W}^{1,p}\left(\mathbb{R}^2\right)$. Indeed,

$$T_m f = \partial_1\partial_2 u * f,$$

for any Schwartz function f.

Hence,

$$\nabla T_m f = \left(\partial_1^2 u * \partial_2 f, \partial_2^2 u * \partial_1 f\right)$$

and thus, by Young's inequality,

$$\|\nabla T_m f\|_{L^p} \le \left\|\partial_1^2 u\right\|_{L^1} \|\partial_2 f\|_{L^p} + \left\|\partial_2^2 u\right\|_{L^1} \|\partial_1 f\|_{L^p} = \left(\left\|\partial_1^2 u\right\|_{L^1} + \left\|\partial_2^2 u\right\|_{L^1}\right) \|\nabla f\|_{L^p}.$$

Using Ornstein's L^1 non-inequality, Bonami and Poornima proved in 1982 that the only Fourier multipliers on $\dot{W}^{1,1}\left(\mathbb{R}^d\right)$ which are homogeneous functions of degree zero and continuous outside the origin are the constant functions. More precisely, they proved the following (see [**1**, Theorem 2]).

THEOREM 8.1. *Suppose $d \ge 2$, $l \ge 1$ are two integers and let Ω be a continuous function on $\mathbb{R}^d \setminus \{0\}$, homogeneous of degree zero. If Ω is a Fourier multiplier on $\dot{W}^{l,1}\left(\mathbb{R}^d\right)$, then Ω is a constant.*

When $l = 1$, this result was generalized by Kazaniecki and Wojciechowski in 2013 as follows (see [**3**, Theorem 1.1]).

THEOREM 8.2. *Suppose $d \ge 2$. If m is a Fourier multiplier on $\dot{W}^{1,1}\left(\mathbb{R}^d\right)$, then $m \in C_b\left(\mathbb{R}^d\right)$.*

Since any function homogeneous of degree zero that is continuous on $\mathbb{R}^d$ has to be constant, we see that Theorem 8.2 implies Theorem 8.1 when $l = 1$. In order to prove Theorem 8.2, Kazaniecki and Wojciechowski used Theorem 8.1 and some Riesz product technique reminiscent of [**10**]. Also, for technical reasons, some classical results concerning multipliers, as for example de Leeuw's transference theorems, were involved in the argument. The central role is played by the Riesz products technique, a key tool being a relatively difficult lemma of Wojciechowski (see [**9**, Lemma 1], [**10**, Lemma 1]) concerning the L^1-norm of some trigonometric polynomials.

We follow the ideas in [**3**] in order to prove a generalisation of Theorem 8.2 for the case of $\dot{W}^{l,1}\left(\mathbb{R}^d\right)$, where $l \ge 1$. The proof is also based on the Riesz products technique and the constructions we use are very similar to those in [**3**]. However, rather than using Wojciechowski's lemma we rely on much easier facts instead (see Lemmas 8.14 and 8.16 below). The other ingredient is Theorem 8.1. We do not use transference theorems and, apart from Bonami and Poornima's result, the proof is quite elementary. Another advantage of our approach is that it also applies to multipliers on $\dot{W}^{l,\infty}\left(\mathbb{R}^d\right)$.

Our results are the following.

THEOREM 8.3. *Suppose $d \ge 2$ and $l \ge 1$ are integers. If m is a Fourier multiplier on $\dot{W}^{l,\infty}\left(\mathbb{R}^d\right)$, then $m \in C_b\left(\mathbb{R}^d\right)$.*

THEOREM 8.4. *Suppose $d \ge 2$ and $l \ge 1$ are integers. If m is a Fourier multiplier on $\dot{W}^{l,1}\left(\mathbb{R}^d\right)$, then $m \in C_b\left(\mathbb{R}^d\right)$.*

REMARK 8.5. In fact, what we prove in these theorems is that m is a.e. equal to some bounded continuous function.

The proofs go as follows. First, following the idea in the proof of Lemma 3.1 in [**3**] we show, by simple arguments, that, whenever m is a multiplier on $\dot{W}^{l,1}\left(\mathbb{R}^d\right)$ or on $\dot{W}^{l,\infty}\left(\mathbb{R}^d\right)$, we need to have $m \in C_b\left(\mathbb{R}^d \setminus \{0\}\right)$. (More specifically, as a preliminary step in our analysis, we define a function which equals m a.e. on $\mathbb{R}^d$ and is continuous and bounded on $\mathbb{R}^d \setminus \{0\}$.) Next, using this conclusion, we prove that, if m is a multiplier on $\dot{W}^{l,1}\left(\mathbb{R}^d\right)$ or on $\dot{W}^{l,\infty}\left(\mathbb{R}^d\right)$, then m has to be continuous at the origin. For this part of the proof, we use constructions based on Riesz products.

While the proofs of these facts are similar, we start by studying the multipliers on $\dot{W}^{l,\infty}(\mathbb{R}^d)$, since the proof is simpler in this setting. For this purpose, we adapt and simplify the ideas in [**3**] to the case of $\dot{W}^{l,\infty}(\mathbb{R}^d)$, with the help of Lemma 8.14 below. Next, we study multipliers on $\dot{W}^{l,1}(\mathbb{R}^d)$. We show by a duality method that the boundedness of T_m implies the existence of bounded solutions for some underdetermined differential equation. We conclude that such solutions do not exist if m is not continuous. Here, the technique is similar to the one in [**2**], which was in turn inspired by the one in [**10**].

2. Continuity outside the origin

Suppose that m is a multiplier of $\dot{W}^{l,p}(\mathbb{R}^d)$ for some $1 \le p \le \infty$ and $l \ge 0$. Let us notice that the norm of T_m is invariant by dilations and isometries. More precisely, if $\lambda \in \mathbb{R}\setminus\{0\}$, respectively $R \in \mathscr{O}(d)$ and we set $m_\lambda(\xi) := m(\lambda\xi)$, respectively $m^R(\xi) := m(R\xi)$, $\forall\, \xi \in \mathbb{R}^d$, then

$$\left\|T_{m_\lambda}\right\| = \|T_m\| \text{ and } \left\|T_{m^R}\right\| \sim_{l,d} \|T_m\|, \tag{8.2}$$

where the $\|T_m\|$ is the norm of $T_m : \dot{W}^{l,p}(\mathbb{R}^d) \to \dot{W}^{l,p}(\mathbb{R}^d)$.

Let us justify this when $m \in L^1(\mathbb{R}^d)$; the general case is obtained by approximation. For any $d \times d$ real invertible matrix A and any Schwartz function f on $\mathbb{R}^d$, we have the following identity:

$$\widehat{f^A}(\xi) = \frac{1}{|\det A|}\widehat{f}\left(A^{-1}\xi\right),$$

where $f^A(x) := f(Ax)$ for any $x \in \mathbb{R}^d$. Via a change of variables, we find that

$$\begin{aligned} T_{m^A}f(x) =& (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} m(A\xi)\widehat{f}(\xi)\,d\xi = \frac{(2\pi)^{-d}}{|\det A|}\int_{\mathbb{R}^d} e^{i\langle x,A^{-1}\xi\rangle} m(\xi)\widehat{f}\left(A^{-1}\xi\right)d\xi \\ =& (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i\left\langle (A^{-1})^t x,\xi\right\rangle} m(\xi)\widehat{f^A}(\xi)\,d\xi = T_m f^A\left(\left(A^{-1}\right)^t x\right) \text{ for a.e. } x \in \mathbb{R}^d. \end{aligned}$$

When, $A = \lambda I$, we get that $T_{m_\lambda}f = T_m f_\lambda(\cdot/\lambda)$. We obtain

$$\begin{aligned} \left\|\nabla^l T_{m_\lambda} f\right\|_{L^p} =& \lambda^{-l}\left\|\left(\nabla^l T_m f_\lambda\right)(\cdot/\lambda)\right\|_{L^p} = \lambda^{-l}\lambda^{d/p}\left\|\nabla^l T_m f_\lambda\right\|_{L^p} \\ \le& \|T_m\|\,\lambda^{-l}\lambda^{d/p}\left\|\nabla^l f_\lambda\right\|_{L^p} = \|T_m\|\left\|\nabla^l f\right\|_{L^p}. \end{aligned}$$

Hence, $\left\|T_{m_\lambda}\right\| \le \|T_m\|$ for any $\lambda \ne 0$. This gives the first equivalence in (8.2).

When, $A = R$, where R is orthogonal, we get $T_{m^R}f = T_m f^R(R\cdot)$. Since the absolute value of each entry of R is bounded by 1, we obtain

$$\begin{aligned} \left\|\nabla^l T_{m^R} f\right\|_{L^p} \lesssim_{l,d}& \left\|\left(\nabla^l T_m f^R\right)(Rx)\right\|_{L^p} = \left\|\nabla^l T_m f^R\right\|_{L^p} \\ \le& \|T_m\|\left\|\nabla^l f^R\right\|_{L^p} \lesssim_{l,d} \|T_m\|\left\|\nabla^l f\right\|_{L^p}. \end{aligned}$$

Hence, $\left\|T_{m^R}\right\| \lesssim_{l,d} \|T_m\|$ for any orthogonal $d \times d$ matrix R. This gives the second equivalence in (8.2).

Let us first observe that the multipliers of $\dot{W}^{l,1}(\mathbb{R}^d)$ and the multipliers of $\dot{W}^{l,\infty}(\mathbb{R}^d)$ are bounded and continuous on $\mathbb{R}^d\setminus\{0\}$.

LEMMA 8.6. *If m is a multiplier on $\dot{W}^{l,1}(\mathbb{R}^d)$, then $m \in C_b\left(\mathbb{R}^d\setminus\{0\}\right)$.*

REMARK 8.7. Actually, the statement means that m has a bounded continuous representative on $\mathbb{R}^d \setminus 0$.

PROOF. We recall that $l \ge 1$. We essentially follow the argument in [**3**, Lemma 3.1]. We first prove that m has a continuous representative on $\mathbb{R}^d \setminus \{0\}$. Indeed, let ρ be any Schwartz function such that $\widehat{\rho}(\xi) \ne 0$, $\forall\, \xi$ (e.g. a standard Gaussian). Set $\alpha^j := (\delta^j_k\, l)_{1\le k\le d}$, where δ^j_k is the Kronecker delta. Since $\partial^{\alpha^j}(T_m\rho) \in L^1$, we find that $m_j := (i\xi)^{\alpha^j}\widehat{T_m\rho}$ is a continuous bounded function, and

$$(i\xi_j)^l\, m(\xi)\widehat{\rho}(\xi) = m_j(\xi) \text{ for a.e. } \xi \in \mathbb{R}^d. \tag{8.3}$$

Set, for $\xi \neq 0$,

$$\widetilde{m}(\xi) := \frac{m_j(\xi)}{(i\xi_j)^l\, \widehat{\rho}(\xi)}, \text{ if } \xi_j \neq 0.$$

It is not clear if the above definition is correct, since the result may depend not only on ξ, but also on the choice of the coordinate ξ_j. However, (8.3) implies first that this definition is correct for a.e. ξ, next, using the continuity of m_j, the definition is correct *for every* ξ and that, in addition $\widetilde{m}$ is continuous. Clearly (from (8.3)), we have note that $m = \widetilde{m}$ a.e. and

$$(i\xi_j)^l\, \widetilde{m}(\xi)\,\widehat{\rho}(\xi) = m_j(\xi) \text{ for every } \xi \in \mathbb{R}^d \setminus \{0\}. \tag{8.4}$$

Using the fact that each m_j is bounded, we find from (8.4) that $\widetilde{m}$ is bounded on the unit sphere $\mathbb{S}^{d-1}$. More specifically, we have

$$\max_{|\xi|=1} |\widetilde{m}(\xi)| \lesssim \|T_m\|. \tag{8.5}$$

Combining (8.5) and (8.2), we find that $\widetilde{m}$ is bounded. □

LEMMA 8.8. *If m is a multiplier on $\dot{W}^{l,\infty}(\mathbb{R}^d)$, then $m \in C_b(\mathbb{R}^d \setminus \{0\})$.*

PROOF. As in the proof of Lemma 8.6, we first prove that m has a representative which is continuous on $\mathbb{R}^d \setminus \{0\}$. Set m' defined by $m'(\xi) := m(-\xi)$. Clearly, m' is also a multiplier on $\dot{W}^{l,\infty}(\mathbb{R}^d)$, with the same norm as m.

It follows that

$$\left\|\partial_1^l T_{m'}\varphi\right\|_{L^\infty(\mathbb{R}^d)} \leq \|T_m\| \left\|\nabla^l \varphi\right\|_{L^\infty(\mathbb{R}^d)}, \tag{8.6}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^2)$.

Consider now the normed subspace

$$V := \left\{\nabla^l \varphi \mid \varphi \in C_c^\infty\left(\mathbb{R}^d\right)\right\} \subset \left(C_0\left(\mathbb{R}^d\right)\right)^\beta,$$

endowed with the norm induced by $\left(C_0\left(\mathbb{R}^d\right)\right)^\beta$, where $\beta := \#\left\{\alpha \in \mathbb{N}^d \mid |\alpha| = l\right\}$. Let ρ be as in the proof of Lemma 8.6. We consider the linear functional $L_\rho : V \to \mathbb{R}$ defined by

$$L_\rho\left(\nabla^l \varphi\right) := \left\langle \rho, \partial_1^l T_{m'}\varphi\right\rangle, \ \forall \varphi \in C_c^\infty\left(\mathbb{R}^d\right).$$

Thanks to (8.6), L_ρ is well-defined and bounded on V and $\left\|L_\rho\right\| \leq \|T_m\| \left\|\rho\right\|_{L^1(\mathbb{R}^d)}$.

Using the Hahn-Banach theorem, we obtain a bounded extension $\widetilde{L}_\rho$ of L_ρ to $\left(C_0\left(\mathbb{R}^d\right)\right)^\beta$. Moreover, we can choose $\widetilde{L}_\rho \in \left(\left(C_0\left(\mathbb{R}^d\right)\right)^\beta\right)^* = \left(M\left(\mathbb{R}^d\right)\right)^\beta$ such that its norm equals $\left\|L_\rho\right\|$. Let $\left(\mu_\alpha\right)_{|\alpha|=l} \in \left(M\left(\mathbb{R}^d\right)\right)^\beta$ be an element representing $\widetilde{L}_\rho$. We have that

$$\left\|\mu_\alpha\right\|_{M(\mathbb{R}^d)} \leq \|T_m\| \left\|\rho\right\|_{L^1(\mathbb{R}^d)}, \tag{8.7}$$

for any multiindex α, with $|\alpha| = l$. Also, we have

$$\begin{aligned}\left\langle \partial_1^l T_m \rho, \varphi\right\rangle &= (-1)^l \left\langle \rho, \partial_1^l T_{m'}\varphi\right\rangle = (-1)^l L_\rho\left(\nabla^l \varphi\right) = (-1)^l \widetilde{L}_\rho\left(\nabla^l \varphi\right) \\ &= (-1)^l \sum_{|\alpha|=l} \left\langle \mu_\alpha, \nabla^\alpha \varphi\right\rangle = \sum_{|\alpha|=l} \left\langle \nabla^\alpha \mu_\alpha, \varphi\right\rangle,\end{aligned}$$

i.e.,

$$\partial_1^l T_m \rho = \sum_{|\alpha|=l} \nabla^\alpha \mu_\alpha, \tag{8.8}$$

in the sense of tempered distributions on $\mathbb{R}^d$.

Taking the Fourier transform in (8.8), we obtain

$$(i\xi_1)^l\, m(\xi)\,\widehat{\rho}(\xi) = \sum_{|\alpha|=l} (i\xi)^\alpha\, \widehat{\mu_\alpha}(\xi) := m_1(\xi) \text{ a.e. on } \mathbb{R}^d. \tag{8.9}$$

Similar identities hold for the partial derivatives $\partial_j^l T_m f$, $j = 2,\dots,d$. Noting that each $\widehat{\mu_\alpha}$ is a continuous function (since μ_α is a finite measure), we continue as in the proof of Lemma 8.6 and find some $\widetilde{m} \in C(\mathbb{R}^d \setminus \{0\})$ such that $m = \widetilde{m}$ a.e.

The boundedness of m is obtained exactly as in Lemma 8.6 □

REMARK 8.9. In the case where $1 < p < \infty$, it is not true that if m is a multiplier on $\dot{W}^{l,p}\left(\mathbb{R}^d\right)$, then $m \in C_b\left(\mathbb{R}^d \setminus \{0\}\right)$. For example if $m(\xi) := \operatorname{sgn}(\xi_1)$, then T_m is the Hilbert transform on the first coordinate and hence m is a multiplier of any space $\dot{W}^{l,p}\left(\mathbb{R}^d\right)$, with $1 < p < \infty$. However, m is singular on the whole hyperplane $\{\xi_1 = 0\}$.

Also, if $p = 2$, any bounded measurable function is a multiplier. Hence, in this case, the multiplier may be even less regular.

It remains to study, in $\dot{W}^{l,1}$ and $\dot{W}^{l,\infty}$, the continuity of the multipliers at the origin.

3. Almost radial limits

Following [**3**, Section 2], we will say that a function $f : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ has *almost radial limits* at the origin if the following condition is satisfied.

$$\begin{aligned}&\text{If } (v_n)_{n\geq 1}, (w_n)_{n\geq 1} \subset \mathbb{R}^d \setminus \{0\} \text{ are two sequences converging to } 0\\ &\text{and } \lim_{n\to\infty} f(v_n) \neq \lim_{n\to\infty} f(w_n), \text{ then } \liminf_{n\to\infty}\left|\frac{v_n}{|v_n|} - \frac{w_n}{|w_n|}\right| > 0.\end{aligned} \tag{I}$$

Note that, if (I) does not hold for $f = m$, which is bounded, then there exists a sequence $(v_n)_{n\geq 1} \subset \mathbb{R}^d \setminus \{0\}$, converging to 0 and such that

$$\frac{v_n}{|v_n|} \to \nu \in \mathbb{S}^{d-1},\ m(v_{2n}) \to b_1,\ m(v_{2n+1}) \to b_2, \text{ with } b_1, b_2 \in \mathbb{C},\ b_1 \neq b_2.$$

By considering the possible limits (up to subsequences) of $(m(-v_{2n}))$ and $(m(-v_{2n+1}))$, we obtain the following. If $m : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ is a bounded function which does not have almost radial limits, then there exists a sequence $(v_n)_{n\geq 1} \subset \mathbb{R}^d \setminus \{0\}$, converging to 0, and such that (at least) one of the two happens:

$$\begin{aligned}\frac{v_n}{|v_n|} \to \nu \in \mathbb{S}^{d-1},\ m(v_{2n}) \to b_1,\ m(-v_{2n}) \to b_1,\ m(v_{2n+1}) \to b_2,\\ m(-v_{2n+1}) \to b_2, \text{ with } b_1, b_2 \in \mathbb{C},\ b_1 \neq b_2,\end{aligned} \tag{IIs}$$

or

$$\frac{v_n}{|v_n|} \to \nu \in \mathbb{S}^{d-1},\ m(v_n) \to b_1,\ m(-v_n) \to b_2, \text{ with } b_1, b_2 \in \mathbb{C},\ b_1 \neq b_2. \tag{IIa}$$

We will refer to the first case as the *symmetric case*, and to the second as the *asymmetric case*.

The plan of the proofs of Theorems 8.3 and 8.4 consists of establishing the desired results separately in cases (I), (IIs) and (IIa). In case (I), the proof relies on Theorem 8.1 or on its $\dot{W}^{l,\infty}$ variant, Theorem 8.10 below.

4. Proof of Theorems 8.3 and 8.4 in case (I)

The case of $\dot{W}^{l,1}\left(\mathbb{R}^d\right)$. First, as in [**3**], we observe that a bounded function having almost radial limits at 0 also has (genuine) radial limits at 0, and therefore we may define the function

$$\Omega(\xi) := \lim_{n\to\infty} m(\xi/n). \tag{8.10}$$

Clearly, the function Ω is homogeneous of degree 0. Now, one can easily see that Ω is a multiplier on $\dot{W}^{l,1}\left(\mathbb{R}^d\right)$. Indeed, let f be a Schwartz function and let ψ be an arbitrary Schwartz function with $\|\psi\|_{L^\infty} \leq 1$. Thanks to (8.2) we have, for any multiindex α with $|\alpha| = l$ and any $n \geq 1$,

$$\left|\left\langle \nabla^\alpha T_{m_{1/n}} f, \psi\right\rangle\right| \leq \|T_m\| \, \|f\|_{\dot{W}^{l,1}}.$$

On the other hand, the dominated convergence theorem gives, with $c = c_{\alpha,d} := \imath^{|\alpha|}(2\pi)^{-d}$,

$$\left\langle \nabla^\alpha T_{m_{1/n}} f, \psi \right\rangle = c \int_{\mathbb{R}^d} \xi^\alpha m(\xi/n) \widehat{f}(\xi) \widehat{\psi}(\xi) d\xi \to c \int_{\mathbb{R}^d} \xi^\alpha \Omega(\xi) \widehat{f}(\xi) \widehat{\psi}(\xi) d\xi = \left\langle \nabla^\alpha T_\Omega f, \psi \right\rangle,$$

and hence,

$$\left| \left\langle \nabla^\alpha T_\Omega f, \psi \right\rangle \right| \leq \|T_m\| \, \|f\|_{\dot{W}^{l,1}}.$$

By Lemma 8.6, we have that $\Omega \in C\left(\mathbb{R}^d \setminus \{0\}\right)$. We are now in position to apply Theorem 8.1 and obtain that Ω is constant. From this and condition (I), we deduce that m is continuous at the origin. □

The case of $\dot{W}^{l,\infty}\left(\mathbb{R}^d\right)$. As above, we conclude that the function Ω defined by (8.10) is a multiplier on $\dot{W}^{l,\infty}\left(\mathbb{R}^d\right)$. In particular, by Lemma 8.8, we have that $\Omega \in C\left(\mathbb{R}^d \setminus \{0\}\right)$. In order to complete (as above) the proof in this case, it suffices to establish the following analogue of Theorem 8.1. □

THEOREM 8.10. *Let $d \geq 2$ and $l \geq 1$ be some integers and let $\Omega \in C\left(\mathbb{R}^d \setminus \{0\}; \mathbb{C}\right)$ be homogeneous of degree zero. If Ω is a multiplier on $\dot{W}^{l,\infty}\left(\mathbb{R}^d\right)$, then Ω is a constant.*

PROOF OF THEOREM 8.10. We adapt the arguments from [**5**]. As in the proof of Lemma 8.8, for any Schwartz function ρ whose integral is 1, one can find some finite measures μ_α such that

$$\partial_1^l T_{\Omega'} \rho = \sum_{|\alpha|=l} \nabla^\alpha \mu_\alpha, \tag{8.11}$$

where $\Omega'(\xi) := \Omega(-\xi)$.

Now, if $\varphi \in \mathscr{S}\left(\mathbb{R}^d\right)$ and $\varphi_\varepsilon(x) := \varphi(\varepsilon x)$ for some $\varepsilon > 0$, then

$$\left(\partial_1^l T_\Omega \varphi_\varepsilon\right)(x) = \varepsilon^l \left(\partial_1^l T_\Omega \varphi\right)(\varepsilon x), \tag{8.12}$$

since Ω is homogeneous of degree zero.

Combining (8.11) and (8.12), we find that

$$\varepsilon^l \int_{\mathbb{R}^d} \rho(x) \left(\partial_1^l T_\Omega \varphi\right)(\varepsilon x)\, dx = \varepsilon^l \sum_{|\alpha|=l} \int_{\mathbb{R}^d} \left(\nabla^\alpha \varphi\right)(\varepsilon x)\, d\mu_\alpha(x). \tag{8.13}$$

Since Ω is bounded, $\partial_1^l T_{\Omega'} \varphi$ is the inverse Fourier transform of an L^1 function and hence, $\partial_1^l T_{\Omega'} \varphi$ is continuous and bounded. Dividing both sides in (8.13) by ε^l and taking $\varepsilon \to 0$, we get by the dominated convergence theorem,

$$\left(\partial_1^l T_\Omega \varphi\right)(0) = \sum_{|\alpha|=l} \mu_\alpha(\mathbb{R}^d) \left(\nabla^\alpha \varphi\right)(0),$$

for any $\varphi \in \mathscr{S}\left(\mathbb{R}^d\right)$. This implies that

$$\left(\partial_1^l T_\Omega \varphi\right)(x) = \sum_{|\alpha|=l} \mu_\alpha(\mathbb{R}^d) \left(\nabla^\alpha \varphi\right)(x),$$

for any $\varphi \in \mathscr{S}\left(\mathbb{R}^d\right)$ and any $x \in \mathbb{R}^d$. Hence, by taking the Fourier transform, we get

$$\xi_1^l \Omega(\xi) \widehat{\varphi}(\xi) = \sum_{|\alpha|=l} \mu_\alpha(\mathbb{R}^d) \xi^\alpha \widehat{\varphi}(\xi),$$

and we have

$$\xi_1^l \Omega(\xi) = \sum_{|\alpha|=l} \mu_\alpha(\mathbb{R}^d) \xi^\alpha =: p_1(\xi).$$

We can write

$$\Omega(\xi) = \frac{p_1(\xi)}{\xi_1^l}, \tag{8.14}$$

as an equality of two continuous functions in the domain where $\xi_1 \neq 0$. Similarly, there exists a homogeneous polynomial p_d of degree l such that

$$\Omega(\xi) = \frac{p_d(\xi)}{\xi_d^l}, \tag{8.15}$$

as an equality of two continuous functions in the domain where $\xi_d \neq 0$. From (8.14) and (8.15), we get

$$\xi_d^l p_1(\xi) = \xi_1^l p_d(\xi) \text{ everywhere in } \mathbb{R}^d. \tag{8.16}$$

By identifying the coefficients in (8.16), we see that p_1 must be a multiple of ξ_1^l, thus a constant multiple of ξ_1^l (since p_1 is of degree l). Going back to (8.14), we find that Ω is constant in the region $\{\xi_1 \neq 0\}$. Similarly, Ω is constant in the region $\{\xi_d \neq 0\}$, and thus constant. □

From now on, we investigate cases (IIs) and (IIa), which are more involved.

5. Proof of Theorem 8.3

We argue by contradiction. We assume that m is not continuous in 0 and we show that (8.1) does not hold. The following easy lemma will enable us to replace some estimates involving Schwartz functions with similar estimates involving instead functions which are linear combinations of some exponentials. This last type of functions will be used to explicitly construct a sequence of functions violating (8.1).

LEMMA 8.11. *Let m be a multiplier on $\dot{W}^{l,\infty}(\mathbb{R}^d)$ for some integer $l \geq 0$. Consider the set of functions*

$$P_m := \left\{ \sum_{j=1}^{n} c_j e^{i\langle \cdot, q_j\rangle} \;\middle|\; n \in \mathbb{N}^*,\ q_1, \dots, q_n \in \mathbb{R}^d \setminus \{0\} \text{ and } c_1, \dots, c_n \in \mathbb{C} \right\}.$$

Let $T'_m : P_m \to P_m$ be defined by

$$T'_m\left(\sum_{j=1}^{n} c_j e^{i\langle \cdot, q_j\rangle}\right) := \sum_{j=1}^{n} c_j m(q_j) e^{i\langle \cdot, q_j\rangle},$$

for any $n \in \mathbb{N}^*$, $q_1, \dots, q_n \in \mathbb{R}^d \setminus \{0\}$ and $c_1, \dots, c_n \in \mathbb{C}$.
We have that

$$\left\| T'_m f \right\|_{\dot{W}^{l,\infty}(\mathbb{R}^d)} \leq \| T_m \| \, \| f \|_{\dot{W}^{l,\infty}(\mathbb{R}^d)}, \tag{8.17}$$

for any function $f \in P_m$.

REMARK 8.12. Note that, since the exponentials $e^{i\langle \cdot, q_j\rangle}$ are linearly independent and P_m is formed only with (finite) linear combinations of these exponentials, the definition of T'_m is correct.

PROOF. We note that at this point we know that m is continuous and bounded on $\mathbb{R}^d \setminus \{0\}$. This will be used in the proof below.

Consider a function $\eta \in C_c^\infty(\mathbb{R}^d)$ whose integral is 1. Fix $q \in \mathbb{R}^d \setminus \{0\}$. For any small $\varepsilon > 0$, we set $\varphi_q^\varepsilon(t) := e^{i\langle t, q\rangle} \widehat{\eta}(\varepsilon t)$ on $\mathbb{R}^d$. Since φ_q^ε is a Schwartz function, we have

$$\widehat{T_m \varphi_q^\varepsilon}(\xi) = m(\xi) \widehat{\varphi_q^\varepsilon}(\xi),$$

in the sense of tempered distributions on $\mathbb{R}^d$. A direct computation gives

$$\begin{aligned}
\widehat{\varphi_q^\varepsilon}(\xi) &= \int_{\mathbb{R}^d} e^{-i\langle t, \xi - q\rangle} \widehat{\eta}(\varepsilon t)\, dt = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} e^{-i\left\langle t, \frac{\xi - q}{\varepsilon}\right\rangle} \widehat{\eta}(t)\, dt \\
&= \frac{1}{\varepsilon^d} \widehat{\widehat{\eta}}\left(\frac{\xi - q}{\varepsilon}\right) = \frac{(2\pi)^d}{\varepsilon^d} \eta\left(\frac{q - \xi}{\varepsilon}\right).
\end{aligned}$$

Hence,

$$\widehat{T_m\varphi_q^{\varepsilon}}(\xi)=m(\xi)\frac{(2\pi)^d}{\varepsilon^d}\eta\left(\frac{q-\xi}{\varepsilon}\right), \tag{8.18}$$

Note that, since $m\in L^\infty$, the right hand side of (8.18) is L^1; we obtain using the Fourier inversion formula,

$$T_m\left(\varphi_q^{\varepsilon}\right)(t)=\int_{\mathbb{R}^d}e^{i\langle t,\xi\rangle}m(\xi)\frac{1}{\varepsilon^d}\eta\left(\frac{q-\xi}{\varepsilon}\right)d\xi=\int_{\mathbb{R}^d}e^{i\langle t,q-\varepsilon\xi\rangle}m(q-\varepsilon\xi)\eta(\xi)d\xi, \tag{8.19}$$

in the sense of tempered distributions. We naturally identify $T_m\left(\varphi_q^{\varepsilon}\right)$ with the right hand side of (8.19).

Now we can prove (8.17). Let $f\in P_m$, with

$$f(t)=\sum_{j=1}^n c_je^{i\langle t,q_j\rangle}.$$

Using (8.19) we have

$$\begin{aligned}\left\|\nabla^l\sum_{j=1}^n c_j\int_{\mathbb{R}^d}m(q_j-\varepsilon\xi)e^{i\langle t,q_j-\varepsilon\xi\rangle}\eta(\xi)d\xi\right\|_{L_t^\infty}&=\left\|\nabla^lT_m\left(\sum_{j=1}^n c_j\varphi_{q_j}^{\varepsilon}\right)(t)\right\|_{L_t^\infty}\\&\le\|T_m\|\left\|\nabla^l(f(t)\widehat{\eta}(\varepsilon t))\right\|_{L_t^\infty}.\end{aligned}$$

In other words, for every multiindex $\alpha\in\mathbb{N}^d$ with $|\alpha|=l$,

$$\left\|\sum_{j=1}^n c_j\int_{\mathbb{R}^d}(q_j-\varepsilon\xi)^\alpha m(q_j-\varepsilon\xi)e^{i\langle t,q_j-\varepsilon\xi\rangle}\eta(\xi)d\xi\right\|_{L_t^\infty}\le\|T_m\|\left\|\left(\nabla^lf(t)\right)\widehat{\eta}(\varepsilon t)\right\|_{L_t^\infty}+\varepsilon\|T_m\|C_{f,\eta},$$

where $C_{f,\eta}$ is a finite constant depending only on f and η. Letting $\varepsilon\to0$ we find that:

$$\left\|\nabla^\alpha\sum_{j=1}^n c_jm(q_j)e^{i\langle t,q_j\rangle}\right\|_{L_t^\infty}=\left\|\sum_{j=1}^n c_jq_j^\alpha m(q_j)e^{i\langle t,q_j\rangle}\right\|_{L_t^\infty}\le\|T_m\|\left\|\nabla^lf\right\|_{L^\infty}.$$

Here, we use the fact that $\widehat{\eta}(0)=1$ and the obvious fact that

$$\left\|\left(\nabla^lf(t)\right)\widehat{\eta}(0)\right\|_{L_t^\infty}\le\liminf_{\varepsilon\to0}\left\|\left(\nabla^lf(t)\right)\widehat{\eta}(\varepsilon t)\right\|_{L_t^\infty}.$$

The proof of Lemma 8.11 is complete. □

REMARK 8.13. For simplicity, from now on, we will denote both operators T_m and T'_m by T_m. We keep this convention even in the case where $p=1$. As we will see this will turn out to be convenient in some computations.

The symmetric case, (IIs). In what follows we suppose for simplicity that $d=2$. Also, we suppose without loss of generality that $b_1=1$, $b_2=0$ and $v=(1,0)$. This is possible thanks to the rotation and dilation invariance we have discussed.

We will need the following simple lemma (see the Appendix).

LEMMA 8.14. *Fix $N\in\mathbb{N}^*$. There exists a finite sequence $(\sigma_k)_{1\le k\le N}$ in $\{0,1\}$ such that*

$$\left|\sum_{k=1}^N\frac{\sigma_k}{k}\prod_{j=1}^{k-1}\left(1+\frac{i}{j}\right)\right|\ge\frac{1}{\pi}\ln N. \tag{8.20}$$

Suppose $N\in\mathbb{N}^*$ is fixed and $\sigma_1,\ldots,\sigma_N\in\{0,1\}$ are some fixed numbers such that inequality (8.20) holds. We construct, by backward induction on k, a sequence $(a_k)_{1\le k\le N}$ in $\mathbb{R}^2$ satisfying the following properties:

(P1) for each $k \in \{1,...,N\}$ we have

$$\left| m\left(\varepsilon_k a_k + \sum_{1\le j\le k-1} \varepsilon_j a_j\right) - \sigma_k \right| < \frac{1}{4^N},$$

for all $\varepsilon_1,...,\varepsilon_k \in \{-1,0,1\}$ with $\varepsilon_k \neq 0$;

(P2) for each $k \in \{1,...,N-1\}$ we have

$$4\,|a_k(1)| < |a_{k+1}(1)| \quad \text{and} \quad 4\,|a_k(2)| < |a_{k+1}(2)|;$$

(Here, $a_k(1)$ and $a_k(2)$ are the two coordinates of a_k.)

(P3) for each $k \in \{1,...,N\}$ we have

$$0 < \left| a_k(1) + \sum_{1\le j\le k-1} \varepsilon_j a_j(1) \right|, \left| a_k(2) + \sum_{1\le j\le k-1} \varepsilon_j a_j(2) \right| < 1,$$

for all $\varepsilon_1,...,\varepsilon_{k-1} \in \{-1,0,1\}$;

(P4) for each $k \in \{1,...,N\}$ we have

$$\frac{\left|a_k(2) + \sum_{1\le j\le k-1} \varepsilon_j a_j(2)\right|}{\left|a_k(1) + \sum_{1\le j\le k-1} \varepsilon_j a_j(1)\right|} < \frac{1}{4^N},$$

for all $\varepsilon_1,...,\varepsilon_{k-1} \in \{-1,0,1\}$.

(A similar construction appears in [**3**, Subsection 2.2] and in [**10**].) The construction goes as follows. We first modify the sequence $(v_n)_{n\ge1}$ in (IIs) such that $v_n(2) \neq 0$ for all $n \ge 1$. This is possible, since m is continuous on $\mathbb{R}^2 \setminus \{0\}$. At each step we choose a_k to be a term in the set $\{v_n \mid n \equiv 0 (\bmod 2)\}$ or $\{v_n \mid n \equiv 1 (\bmod 2)\}$ if $\sigma_k = 1$ or $\sigma_k = 0$ respectively. It remains to see that at each step the term a_k can be chosen sufficiently small in order to satisfy the above conditions. Since $v_n \to 0$, we can choose a vector a_k with both components nonzero and such that $|m(\varepsilon_k a_k) - \sigma_k| < (1/2)4^{-N}$, for any $\varepsilon_k \in \{-1,1\}$. Since m is continuous outside the origin, there exists $r_k > 0$ such that $|m(\xi) - \sigma_k| < 4^{-N}$ for any $\xi \in B(a_k, r_k) \cup B(-a_k, r_k)$. Hence, if $a_{k-1},...,a_1$ are sufficiently small, then (P1) is satisfied. We have that $\nu = (1,0)$, and hence if $a_k = v_n$, for n sufficiently large, and $a_{k-1},...,a_1$ are sufficiently small, then (P4) is satisfied. It is easy to see that the remaining conditions can be satisfied too.

Consider the set

$$\Lambda_N := \left\{ \sum_{k=1}^{N} \varepsilon_k a_k \;\middle|\; \varepsilon_1,...,\varepsilon_N \in \{-1,0,1\}, \text{ not all } 0 \right\}. \tag{8.21}$$

Thanks to (P2), for each $q \in \Lambda_N$ the representation

$$q = \sum_{k=1}^{N} \varepsilon_k a_k, \text{ for some } \varepsilon_1,...,\varepsilon_N \in \{-1,0,1\},$$

is unique. Let us also observe that, for each $q \in \Lambda_N$ we have (from (P3) and (P4)),

$$0 < |q(1)|, |q(2)| < 1 \tag{8.22}$$

and

$$\frac{|q(2)|}{|q(1)|} < \frac{1}{4^N}. \tag{8.23}$$

Define the function

$$R_N(t) := -1 + \prod_{k=1}^{N} \left(1 + \frac{i}{k} \cos\langle t, a_k\rangle\right), \; t \in \mathbb{R}^2. \tag{8.24}$$

By (8.57) (see the Appendix),

$$R_N(t) = \sum_{k=1}^{N} \sum_{\substack{\varepsilon_1,\dots,\varepsilon_k\in\{-1,0,1\}\\ \varepsilon_k\neq 0}} \left(\prod_{\varepsilon_j\neq 0} \frac{i}{2j}\right) e^{i\langle t,\varepsilon_1 a_1+\dots\varepsilon_k a_k\rangle} = \sum_{q\in\Lambda_N} c_q e^{i\langle t,q\rangle}, \tag{8.25}$$

for some coefficients c_q with $\left|c_q\right| \le 1$. Thanks to (8.22) we have that $q(1)\neq 0$, for any $q\in\Lambda_N$. This allows us to define the function

$$h_N(t) := \sum_{q\in\Lambda_N} \frac{c_q}{(q(1))^l} e^{i\langle t,q\rangle}, \text{ on } \mathbb{R}^2. \tag{8.26}$$

We claim that

$$\left\|\nabla^l h_N\right\|_{L^\infty(\mathbb{R}^2)} \le 4. \tag{8.27}$$

Indeed, we have

$$\begin{aligned}\left\|\partial_1^l h_N\right\|_{L^\infty(\mathbb{R}^2)} = \|R_N\|_{L^\infty(\mathbb{R}^2)} &\le 1 + \prod_{k=1}^{N}\left(1+\frac{1}{k^2}\right)^{1/2}\\ &\le 1 + \prod_{k=1}^{N} e^{1/2k^2} \le 1 + e^{\pi^2/12} \le 4.\end{aligned} \tag{8.28}$$

On the other hand, if l_1, l_2 are nonnegative integers with $l_1 + l_2 = l$ and $l_1 < l$, we have (using (8.22), (8.23)),

$$\begin{aligned}\left\|\partial_1^{l_1}\partial_2^{l_2} h_N\right\|_{L^\infty(\mathbb{R}^2)} &= \left\|\sum_{q\in\Lambda_N} \frac{(q(2))^{l_2}}{(q(1))^{l-l_1}} c_q e^{i\langle t,q\rangle}\right\|_{L^\infty(\mathbb{R}^2)} \le \sum_{q\in\Lambda_N} \frac{|q(2)|^{l_2}}{|q(1)|^{l-l_1}}\\ &= \sum_{q\in\Lambda_N}\left(\frac{|q(2)|}{|q(1)|}\right)^{l_2} \le \sum_{q\in\Lambda_N} 4^{-Nl_2} \le |\Lambda_N| 4^{-N} \le 3^N 4^{-N} \le 1.\end{aligned}$$

We are now going to estimate $\|T_m h_N\|_{\dot W^{l,\infty}(\mathbb{R}^2)}$. Since by (8.26), $h_N\in P_m$, with P_m as in Lemma 8.11, we may define $T_m h_N$ via Lemma 8.11 (see Remark 8.13). More specifically, we will prove that

$$\|T_m h_N\|_{\dot W^{l,\infty}(\mathbb{R}^2)} \ge \frac{1}{\pi}\ln N - 1. \tag{8.29}$$

In order to see this, it suffices to prove that

$$\left\|\partial_1^l T_m h_N\right\|_{L^\infty(\mathbb{R}^2)} \ge \frac{1}{\pi}\ln N - 1. \tag{8.30}$$

We have

$$\left\|\partial_1^l T_m h_N\right\|_{L^\infty(\mathbb{R}^2)} = \left\|T_m \partial_1^l h_N\right\|_{L^\infty(\mathbb{R}^2)} = \|T_m R_N\|_{L^\infty(\mathbb{R}^2)}.$$

Using (8.25) and (8.57), we obtain

$$T_m R_N(t) = \sum_{k=1}^{N} \sum_{\substack{\varepsilon_1,\dots,\varepsilon_k\in\{-1,0,1\}\\ \varepsilon_k\neq 0}} m(\varepsilon_1 a_1 + \dots\varepsilon_k a_k)\left(\prod_{\varepsilon_j\neq 0}\frac{i}{2j}\right) e^{i\langle t,\varepsilon_1 a_1+\dots\varepsilon_k a_k\rangle}. \tag{8.31}$$

Introducing the function

$$Z(t) := \sum_{k=1}^{N} \sum_{\substack{\varepsilon_1,\dots,\varepsilon_k\in\{-1,0,1\}\\ \varepsilon_k\neq 0}} \sigma_k \left(\prod_{\varepsilon_j\neq 0}\frac{i}{2j}\right) e^{i\langle t,\varepsilon_1 a_1+\dots\varepsilon_k a_k\rangle}, \tag{8.32}$$

we have (by the identities (8.57) and (8.59) in the Appendix),

$$Z(t) = \sum_{k=1}^{N} \frac{i\sigma_k}{k}\cos\langle t,a_k\rangle \prod_{j=1}^{k-1}\left(1+\frac{i}{j}\cos\langle t,a_j\rangle\right).$$

Lemma 8.14 yields

$$\|Z\|_{L^\infty(\mathbb{R}^2)} \geq |Z(0)| \geq \frac{1}{\pi}\ln N. \tag{8.33}$$

Also, using the property (P1), together with (8.31) and (8.32), we get

$$\|T_m R_N - Z\|_{L^\infty(\mathbb{R}^2)} \leq |\Lambda_N| 4^{-N} \leq 3^N 4^{-N} \leq 1. \tag{8.34}$$

Using (8.33), (8.34) and the triangle inequality, we arrive at

$$\|T_m R_N\|_{L^\infty(\mathbb{R}^2)} \geq \|Z\|_{L^\infty(\mathbb{R}^2)} - \|T_m R_N - Z\|_{L^\infty(\mathbb{R}^2)} \geq \frac{1}{\pi}\ln N - 1,$$

concluding the proof of (8.30).

By taking $N \to \infty$, (8.27) and (8.29) give us that m is not a multiplier on $\dot{W}^{l,\infty}(\mathbb{R}^2)$. □

REMARK 8.15. To deal with the case $d > 2$ we may suppose that $\nu = (1,0,\ldots,0)$; we consider constructions like $R_N \otimes 1$, where R_N is defined as above on $\mathbb{R}^2$ and the constant function 1 is defined on $\mathbb{R}^{d-2}$.

The asymmetric case, (IIa). This case is very similar to the previous one. We again suppose without loss of generality that $b_1 = 1$, $b_2 = 0$ and $\nu = (1,0)$. In a similar way we construct a sequence $(a_k)_{1\leq k\leq N}$ satisfying the above properties (P2)- -(P4) and (P1') below:

(P1') for each $k \in \{1,\ldots,N\}$ we have

$$\left| m\left(\varepsilon_k a_k + \sum_{1\leq j\leq k-1} \varepsilon_j a_j\right) - \frac{1+\varepsilon_k}{2}\sigma_k \right| < \frac{1}{4^N},$$

for all $\varepsilon_1,\ldots,\varepsilon_k \in \{-1,0,1\}$ with $\varepsilon_k \neq 0$.

With this new sequence $(a_k)_{1\leq k\leq N}$ we define Λ_N as in (8.21). We again have (8.22), (8.23). We also define R_N and h_N as in (8.24) and (8.26) respectively. The inequality (8.27) holds in this case too and it remains to show that

$$\left\|\partial_1^l T_m h_N\right\|_{L^\infty(\mathbb{R}^2)} = \|T_m R_N\|_{L^\infty(\mathbb{R}^2)} \geq \frac{1}{\pi}\ln N - 1. \tag{8.35}$$

Using (8.57), we have

$$T_m R_N(t) = \sum_{k=1}^{N} \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_k\in\{-1,0,1\}\\ \varepsilon_k\neq 0}} m(\varepsilon_1 a_1 + \ldots \varepsilon_k a_k)\left(\prod_{\varepsilon_j\neq 0}\frac{i}{2j}\right) e^{i\langle t,\varepsilon_1 a_1+\ldots\varepsilon_k a_k\rangle}. \tag{8.36}$$

Introducing the function

$$Z(t) := \sum_{k=1}^{N} \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_k\in\{-1,0,1\}\\ \varepsilon_k\neq 0}} \frac{1+\varepsilon_k}{2}\sigma_k \left(\prod_{\varepsilon_j\neq 0}\frac{i}{2j}\right) e^{i\langle t,\varepsilon_1 a_1+\ldots\varepsilon_k a_k\rangle}, \tag{8.37}$$

we observe that, by (8.57) and (8.59), we have

$$Z(t) = \sum_{k=1}^{N} \frac{i\sigma_k}{k} e^{i\langle t,a_k\rangle} \prod_{j=1}^{k-1}\left(1 + \frac{i}{j}\cos\langle t,a_j\rangle\right).$$

Lemma 8.14 gives us that

$$\|Z\|_{L^\infty(\mathbb{R}^2)} \geq |Z(0)| \geq \frac{1}{\pi}\ln N. \tag{8.38}$$

The property (P1'), together with (8.36) and (8.37), give

$$\|T_m R_N - Z\|_{L^\infty(\mathbb{R}^2)} \leq |\Lambda_N| 4^{-N} \leq 3^N 4^{-N} \leq 1. \tag{8.39}$$

Using (8.38), (8.39) and the triangle inequality, we get

$$\|T_m R_N\|_{L^\infty(\mathbb{R}^2)} \geq \|Z\|_{L^\infty(\mathbb{R}^2)} - \|T_m R_N - Z\|_{L^\infty(\mathbb{R}^2)} \geq \frac{1}{\pi}\ln N - 1,$$

concluding the proof of (8.35). (For the case $d > 2$, see Remark 8.15.) □

6. Proof of Theorem 8.4

We prove now Theorem 8.4. Suppose m is not continuous in 0. As in the preceding section, we may assume that m is in one of the cases (IIs) or (IIa). Again we work under the hypothesis $d = 2$, $b_1 = 1$, $b_2 = 0$ and $\nu = (1,0)$.

The symmetric case, (IIs). We need the following analogue of Lemma 8.14 (see the Appendix):

LEMMA 8.16. *Fix $N \in \mathbb{N}^*$. There exists a finite sequence $(\sigma_k)_{1\leq k\leq N}$ in $\{0,1\}$ such that*

$$\left|\sum_{k=1}^{N} \frac{\sigma_k}{2k} \prod_{j=1}^{k-1}\left(1+\frac{i}{2j}\right)\right| \geq \frac{1}{2\pi}\ln N.$$

In what follows $(a_k)_{1\leq k\leq N}$ is a sequence in $\mathbb{Q}^2$ satisfying the properties (P1)- -(P4) for the sequence $(\sigma_k)_{1\leq k\leq N}$ from Lemma 8.16 above. It is easy to see that such a sequence exists. Using this sequence we construct the function R_N as in (8.24).

Suppose that m is a multiplier on $\dot{W}^{l,1}(\mathbb{R}^2)$. Then m' defined by $m'(\xi) := m(-\xi)$ is also a multiplier on $\dot{W}^{l,1}(\mathbb{R}^2)$ with the same norm as m. It follows that

$$\left\|\partial_1^l T_{m'}\varphi\right\|_{L^1(\mathbb{R}^2)} \leq \|T_m\| \left\|\nabla^l \varphi\right\|_{L^1(\mathbb{R}^2)}, \tag{8.40}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^2)$.

Consider now the normed subspace

$$V := \left\{\nabla^l\varphi \mid \varphi \in C_c^\infty(\mathbb{R}^2)\right\} \subset \left(L^1(\mathbb{R}^2)\right)^{2^l},$$

endowed with the norm induced by $\left(L^1(\mathbb{R}^2)\right)^{2^l}$. We consider the linear functional $L_N : V \to \mathbb{R}$ defined by

$$L_N\left(\nabla^l\varphi\right) := \left\langle R_N, \partial_1^l T_{m'}\varphi\right\rangle = \int_{\mathbb{R}^2} R_N(t)\partial_1^l T_{m'}\varphi(t)dt, \ \ \varphi \in C_c^\infty(\mathbb{R}^2).$$

Thanks to (8.40), L_N is bounded on V. Using the Hahn-Banach theorem, we get that there exists a bounded extension $\widetilde{L}_N$ of L_N, on $\left(L^1(\mathbb{R}^2)\right)^{2^l}$. Moreover, we can choose $\widetilde{L}_N \in \left(\left(L^1(\mathbb{R}^2)\right)^{2^l}\right)^* = \left(L^\infty(\mathbb{R}^2)\right)^{2^l}$ such that its norm equals $\|L_N\|$. Note that, by (8.28),

$$\|L_N\| \leq \|T_m\| \, \|R_N\|_{L^\infty(\mathbb{R}^2)} \leq 4\|T_m\|.$$

Let $(u_\alpha)_{|\alpha|=l} \in \left(L^\infty(\mathbb{R}^2)\right)^{2^l}$ be the element representing $\widetilde{L}_N$, where $\alpha \in \mathbb{N}^2$ are multiindexes. We have that

$$\|u_\alpha\|_{L^\infty(\mathbb{R}^2)} \leq 4\|T_m\|, \tag{8.41}$$

for any multiindex α, with $|\alpha| = l$. Also, we have (see Remark 8.13)

$$\begin{aligned}\left\langle \partial_1^l T_m R_N, \varphi\right\rangle &= (-1)^l \left\langle R_N, \partial_1^l T_{m'}\varphi\right\rangle = (-1)^l L_N\left(\nabla^l\varphi\right) = (-1)^l \widetilde{L}_N\left(\nabla^l\varphi\right) \\ &= (-1)^l \sum_{|\alpha|=l} \left\langle u_\alpha, \nabla^\alpha\varphi\right\rangle = \sum_{|\alpha|=l} \left\langle \nabla^\alpha u_\alpha, \varphi\right\rangle,\end{aligned}$$

i.e.,

$$\partial_1^l T_m R_N = \sum_{|\alpha|=l} \nabla^\alpha u_\alpha, \tag{8.42}$$

in the sense of distributions on $\mathbb{R}^2$.

As in (8.31) we have

$$T_m R_N(t) = \sum_{k=1}^{N} \sum_{\substack{\varepsilon_1,\dots,\varepsilon_k \in \{-1,0,1\} \\ \varepsilon_k \neq 0}} m(\varepsilon_1 a_1 + \dots \varepsilon_k a_k) \left(\prod_{\varepsilon_j \neq 0} \frac{i}{2j} \right) e^{i\langle t, \varepsilon_1 a_1 + \dots \varepsilon_k a_k \rangle}. \tag{8.43}$$

For each N we fix a positive integer $M = M(N)$ such that $Ma_k \in \mathbb{Z}^2$ for all $1 \le k \le N$. From (8.43) we get that $T_m R_N$ is a component-wise $2\pi M$-periodic function. Hence, $T_m R_N(Mt)$ is component-wise 2π-periodic.

We will show that u_α in (8.42) can be chosen to be component-wise $2\pi M$-periodic. In order to prove this we need the following easy lemma.

LEMMA 8.17. *Let $A > 0$ be a real number and suppose $u \in L^\infty(\mathbb{R}^2)$ is given. We consider the sequence of functions*

$$u_n(t) := \frac{1}{|B_n|} \sum_{\chi \in B_n} u\left(t + A\chi\right), \quad t \in \mathbb{R}^2,\ n \ge 1,$$

where $B_n := B(0,n) \cap \mathbb{Z}^2$. Then, there exists $g \in L^\infty(\mathbb{R}^2)$, component-wise A-periodic, with $\|g\|_{L^\infty(\mathbb{R}^2)} \le \|u\|_{L^\infty(\mathbb{R}^2)}$ and such that $u_n \to g$ up to a subsequence, in the sense of distributions.

PROOF OF LEMMA 8.17. Since $\|u_n\|_{L^\infty(\mathbb{R}^2)} \le \|u\|_{L^\infty(\mathbb{R}^2)}$ for any $n \ge 1$, by the sequential Banach-Alaoglu theorem, there exists $g \in L^\infty(\mathbb{R}^2)$ with $\|g\|_{L^\infty(\mathbb{R}^2)} \le \|u\|_{L^\infty(\mathbb{R}^2)}$ such that $u_n \to g$ in the w^*-topology of L^∞ up to a subsequence. (For simplicity we denote the subsequence also by $(u_n)_{n \ge 1}$.) In particular, $u_n \to g$ in the sense of distributions. Also, we easily get that g is component-wise A-periodic. Indeed, for $\varphi \in C_c^\infty(\mathbb{R}^2)$, and any $\chi_0 \in \mathbb{Z}^2$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} u_n\left(t + A\chi_0\right) \varphi(t)\, dt &= \int_{\mathbb{R}^2} u_n(t) \varphi\left(t - A\chi_0\right) dt \\ &\to \int_{\mathbb{R}^2} g(t) \varphi\left(t - A\chi_0\right) dt \\ &= \int_{\mathbb{R}^2} g\left(t + A\chi_0\right) \varphi(t)\, dt. \end{aligned} \tag{8.44}$$

Also,

$$\begin{aligned} \int_{\mathbb{R}^2} u_n\left(t + A\chi_0\right) \varphi(t)\, dt &= \frac{1}{|B_n|} \sum_{\chi \in B_n} \int_{\mathbb{R}^2} u\left(t + A\left(\chi + \chi_0\right)\right) \varphi(t)\, dt \\ &= \frac{1}{|B_n|} \sum_{\chi \in B_n + \chi_0} \int_{\mathbb{R}^2} u\left(t + A\chi\right) \varphi(t)\, dt \\ &= \frac{1}{|B_n|} \sum_{\chi \in B_n} \int_{\mathbb{R}^2} u\left(t + A\chi\right) \varphi(t)\, dt + r_n \\ &= \int_{\mathbb{R}^2} u_n(t) \varphi(t)\, dt + r_n, \end{aligned}$$

where

$$r_n := \frac{1}{|B_n|} \sum_{\chi \in (B_n + \chi_0) \setminus B_n} \int_{\mathbb{R}^2} u\left(t + A\chi\right) \varphi(t)\, dt - \frac{1}{|B_n|} \sum_{\chi \in B_n \setminus (B_n + \chi_0)} \int_{\mathbb{R}^2} u\left(t + A\chi\right) \varphi(t)\, dt.$$

Since $\left|(B_n + \chi_0) \setminus B_n\right|, \left|B_n \setminus (B_n + \chi_0)\right| \lesssim n$ and $|B_n| \sim n^2$, we have $r_n \to 0$. Hence,

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} u_n\left(t + A\chi_0\right) \varphi(t)\, dt = \lim_{n \to \infty} \int_{\mathbb{R}^2} u_n(t) \varphi(t)\, dt,$$

which together with (8.44) concludes the proof of Lemma 8.17. □

Now, since $\partial_1^l T_m R_N$ is component-wise $2\pi M$-periodic, we have $\left(\partial_1^l T_m R_N\right)_n = \partial_1^l T_m R_N$ for any $n \geq 1$. From (8.42) we get

$$\partial_1^l T_m R_N = \left(\partial_1^l T_m R_N\right)_n = \sum_{|\alpha|=l} \nabla^\alpha (u_\alpha)_n,$$

for any $n \geq 1$. Taking $n \to \infty$ and applying Lemma 8.17, with $A := 2\pi M$, we get

$$\partial_1^l T_m R_N = \sum_{|\alpha|=l} \nabla^\alpha g_\alpha, \tag{8.45}$$

for some component-wise $2\pi M$-periodic functions $g_\alpha \in L^\infty\left(\mathbb{R}^2\right)$ such that

$$\|g_\alpha\|_{L^\infty(\mathbb{R}^2)} \leq 4\,\|T_m\| \tag{8.46}$$

(from (8.41)).

From now on, for each function ψ on $\mathbb{R}^2$, we write ψ^M for the function $\psi^M(t) := \psi(Mt)$.

Consider the function

$$G_N(t) := -1 + \prod_{k=1}^{N} (1 + \cos\langle t, a_k\rangle),\ t \in \mathbb{R}^2.$$

Notice that G_N^M is component-wise 2π-periodic. (We recall here that each Ma_k belongs to $\mathbb{Z}^2$.) Also, $(T_m R_N)^M$ and each g_α^M are component-wise 2π-periodic. From (8.45) we get

$$\partial_1^l (T_m R_N)^M = \sum_{|\alpha|=l} \nabla^\alpha g_\alpha^M,$$

in the sense of distributions on $\mathbb{R}^2$ and hence in the sense of distributions on $\mathbb{T}^2$. Taking convolution (on the torus $\mathbb{T}^2$) with G_N^M, we get

$$\partial_1^l \left((T_m R_N)^M * G_N^M\right) = \sum_{|\alpha|=l} \nabla^\alpha \left(g_\alpha^M * G_N^M\right). \tag{8.47}$$

It is easy to see that the spectrum of each $g_\alpha^M * G_N^M$ and the spectrum of $(T_m R_N)^M * G_N^M$, as functions on the torus $\mathbb{T}^2$, are included in $M\Lambda_N$ and therefore do not touch the set $\{0\} \times \mathbb{Z}$ (see (8.22)). Hence, we can apply the operator ∂_1^{-l} in (8.47) to obtain

$$(T_m R_N)^M * G_N^M = \sum_{|\alpha|=l} \nabla^\alpha \partial_1^{-l} \left(g_\alpha^M * G_N^M\right).$$

Hence,

$$\left\|(T_m R_N)^M * G_N^M\right\|_{L^\infty(\mathbb{T}^2)} \leq \sum_{|\alpha|=l} \left\|\nabla^\alpha \partial_1^{-l} \left(g_\alpha^M * G_N^M\right)\right\|_{L^\infty(\mathbb{T}^2)}. \tag{8.48}$$

We claim that

$$\left\|\nabla^\alpha \partial_1^{-l} \left(g_\alpha^M * G_N^M\right)\right\|_{L^\infty(\mathbb{T}^2)} \leq 8\,\|T_m\|, \tag{8.49}$$

for any multiindex α with $|\alpha| = l$. This estimate is similar to (8.27).

Indeed, for $\alpha = (l, 0)$ we have, using (8.46),

$$\begin{aligned}\left\|\partial_1^l \partial_1^{-l} \left(g_\alpha^M * G_N^M\right)\right\|_{L^\infty(\mathbb{T}^2)} = \left\|g_\alpha^M * G_N^M\right\|_{L^\infty(\mathbb{T}^2)} &\leq \left\|g_\alpha^M\right\|_{L^\infty(\mathbb{T}^2)} \left\|G_N^M\right\|_{L^1(\mathbb{T}^2)} \\ &\leq 2\,\|g_\alpha\|_{L^\infty(\mathbb{R}^2)} \leq 8\,\|T_m\|.\end{aligned} \tag{8.50}$$

Here, we have used the fact that $\left\|G_N^M\right\|_{L^1(\mathbb{T}^2)} \leq 2$. This can be justified as follows. We have

$$\prod_{k=1}^{N} (1 + \cos\langle t, Ma_k\rangle) \geq 0$$

and hence, thanks to (8.57) and (P2), we obtain

$$\left\|\prod_{k=1}^{N} (1 + \cos\langle \cdot, Ma_k\rangle)\right\|_{L^1(\mathbb{T}^2)} = 1.$$

We now turn to the proof of (8.49) for $\alpha \neq (l,0)$.

Writing

$$g_\alpha^M * G_N^M(t) = \sum_{q\in\Lambda_N} c'_q e^{i\langle t,Mq\rangle},$$

we get that (note that c'_q is a Fourier coefficient):

$$\left|c'_q\right| \leq \left\|g_\alpha^M * G_N^M\right\|_{L^\infty(\mathbb{T}^2)} \leq 8\,\|T_m\|,$$

for all $q \in \Lambda_N$.

Hence, if $\alpha = (l_1,l_2)$, with $l_1+l_2 = l$ and $l_1 < l$, we have (using (8.22), (8.23)),

$$\begin{aligned}\left\|\left(\partial_1^{l_1}\partial_2^{l_2}\right)\partial_1^{-l}\left(g_\alpha^M * G_N^M\right)\right\|_{L^\infty(\mathbb{T}^2)} &= \left\|\sum_{q\in\Lambda_N}\frac{(Mq(2))^{l_2}}{(Mq(1))^{l-l_1}}c'_q e^{i\langle t,Mq\rangle}\right\|_{L^\infty(\mathbb{T}^2)}\\ &\leq 8\,\|T_m\|\sum_{q\in\Lambda_N}\frac{|q(2)|^{l_2}}{|q(1)|^{l-l_1}} = 8\,\|T_m\|\sum_{q\in\Lambda_N}\left(\frac{|q(2)|}{|q(1)|}\right)^{l_2}\\ &\leq 8\,\|T_m\|\sum_{q\in\Lambda_N}4^{-Nl_2} \leq 8\,\|T_m\|\,|\Lambda_N|\,4^{-N} \leq 8\,\|T_m\|\,3^N 4^{-N}\\ &\leq 8\,\|T_m\|.\end{aligned} \tag{8.51}$$

We see that (8.50) and (8.51) imply (8.49).

We next obtain a contradiction. The starting point is the left-hand side of (8.48). We claim that

$$\left\|(T_mR_N)^M * G_N^M\right\|_{L^\infty(\mathbb{T}^2)} \geq \frac{1}{2\pi}\ln N - 1. \tag{8.52}$$

The method applied to obtain this estimate is similar to the one used to obtain (8.30).

By using (8.57) and (8.58) (see the Appendix) we have:

$$(T_mR_N)^M * G_N^M(t) = \sum_{k=1}^N \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_k\in\{-1,0,1\}\\ \varepsilon_k\neq 0}} m(\varepsilon_1 a_1 + \ldots \varepsilon_k a_k)\left(\prod_{\varepsilon_j\neq 0}\frac{i}{4j}\right)e^{i\langle t,\varepsilon_1 Ma_1+\ldots+\varepsilon_k Ma_k\rangle}. \tag{8.53}$$

Introducing the function

$$Z(t) := \sum_{k=1}^N \sum_{\substack{\varepsilon_1,\ldots,\varepsilon_k\in\{-1,0,1\}\\ \varepsilon_k\neq 0}} \sigma_k\left(\prod_{\varepsilon_j\neq 0}\frac{i}{4j}\right)e^{i\langle t,\varepsilon_1 Ma_1+\ldots+\varepsilon_k Ma_k\rangle},\ \text{on } \mathbb{T}^2, \tag{8.54}$$

we observe that, by (8.57) and (8.59), we have

$$Z(t) = \sum_{k=1}^N \frac{i\sigma_k}{2k}\cos\langle t,Ma_k\rangle\prod_{j=1}^{k-1}\left(1+\frac{i}{2j}\cos\left\langle t,Ma_j\right\rangle\right).$$

Lemma 8.16 gives us that

$$\|Z\|_{L^\infty(\mathbb{T}^2)} \geq |Z(0)| \geq \frac{1}{2\pi}\ln N. \tag{8.55}$$

Also, using the property (P1), together with (8.53) and (8.54), we get

$$\left\|(T_mR_N)^M * G_N^M - Z\right\|_{L^\infty(\mathbb{T}^2)} \leq |\Lambda_N|\,4^{-N} \leq 3^N 4^{-N} \leq 1. \tag{8.56}$$

Using (8.55), (8.56) and the triangle inequality, we obtain

$$\left\|(T_mR_N)^M * G_N^M\right\|_{L^\infty(\mathbb{T}^2)} \geq \|Z\|_{L^\infty(\mathbb{T}^2)} - \left\|(T_mR_N)^M * G_N^M - Z\right\|_{L^\infty(\mathbb{T}^2)} \geq \frac{1}{2\pi}\ln N - 1,$$

concluding the proof of (8.52).

Now, (8.48), (8.49) and (8.52) allow us to write

$$\frac{1}{2\pi}\ln N - 1 \le 8\,\|T_m\|\,2^l.$$

Since N is arbitrary, the last inequality implies that m is not a multiplier on $\dot{W}^{l,1}(\mathbb{R}^2)$. □

The asymmetric case, (IIa). This case is very similar to the previous one and we skip the proof. We can again suppose by contradiction that m is a multiplier on $\dot{W}^{l,1}(\mathbb{R}^2)$ and use this result to obtain a representation result similar to the one in (8.42). The only difference is that now we have to follow the "asymmetric case" as in the proof corresponding to multipliers on $\dot{W}^{l,\infty}(\mathbb{R}^2)$. The functions R_N and G_N will be constructed as above, starting, as in the case of $\dot{W}^{l,\infty}(\mathbb{R}^2)$, with a sequence $(a_k)_{1\le k\le N}$ in $\mathbb{Q}^2$ satisfying the conditions (P1'), (P2)–(P4). □

7. Appendix

Some useful identities. We quickly recall here some elementary facts and formulas concerning some trigonometric polynomials on the torus.

Fix a finite sequence $(a_k)_{1\le k\le N}$ in $\mathbb{Z}^d$. For each finite sequence $\alpha_1,\dots,\alpha_N$ of complex numbers we have the following expansion rule:

$$\prod_{k=1}^{N}(1+\alpha_k\cos\langle t,a_k\rangle) = 1+\sum_{k=1}^{N}\sum_{\substack{\varepsilon_1,\dots,\varepsilon_k\in\{-1,0,1\}\\ \varepsilon_k\neq 0}}\left(\prod_{\varepsilon_j\neq 0}\frac{\alpha_j}{2}\right)e^{i\langle t,\varepsilon_1a_1+\dots+\varepsilon_ka_k\rangle}. \tag{8.57}$$

A sequence $(a_k)_{k=1,N}$ in $\mathbb{Z}^d$ will be called *dissociated* if the only solution to the equation

$$\varepsilon_1a_1+\dots+\varepsilon_Na_N = \varepsilon'_1a_1+\dots+\varepsilon'_Na_N,$$

with $\varepsilon_1,\dots,\varepsilon_N,\varepsilon'_1,\dots,\varepsilon'_N\in\{-1,0,1\}$ is the trivial solution $\varepsilon_1=\varepsilon'_1,\dots,\varepsilon_N=\varepsilon'_N$. For example any sequence $(a_k)_{1\le k\le N}$ in $\mathbb{Z}^d$ which is lacunary on at least one component is dissociated. If $(a_k)_{1\le k\le N}$ is dissociated and $\alpha_1,\dots,\alpha_N$ and $\beta_1,\dots,\beta_N$ are complex numbers, by using (8.57) and the relation between convolution and the Fourier transform, we obtain that

$$\prod_{k=1}^{N}(1+\alpha_k\cos\langle\cdot,a_k\rangle) * \prod_{k=1}^{N}\left(1+\beta_k\cos\langle\cdot,a_k\rangle\right) = \prod_{k=1}^{N}\left(1+\frac{\alpha_k\beta_k}{2}\cos\langle\cdot,a_k\rangle\right), \tag{8.58}$$

as functions on the d-dimensional torus.

The following identity is also useful. We have

$$\prod_{k=1}^{N}(1+c_k) = 1+\sum_{k=1}^{N}c_k\prod_{j=1}^{k-1}\left(1+c_j\right) \tag{8.59}$$

for any complex numbers $c_1,\dots,c_N$.

A selection lemma. The following interesting fact is taken from [**8**] (Lemma 6.3, p. 118).

LEMMA 8.18. *Suppose $z_1,\dots,z_N$ are some complex numbers. Then, there exist $\sigma_1,\dots,\sigma_N\in\{0,1\}$ such that*

$$\left|\sum_{k=1}^{N}\sigma_kz_k\right| \ge \frac{1}{\pi}\sum_{k=1}^{N}|z_k|.$$

The proof is elementary and we skip it (see [**8**, Lemma 6.3]).

Let us define two sequences $\left(z_k^0\right)_{1\le k\le N}$ and $\left(z_k^1\right)_{1\le k\le N}$ by the expressions

$$z_k^{\beta} := \frac{1}{2^{\beta}k}\prod_{j=1}^{k-1}\left(1+\frac{i}{2^{\beta}j}\right) \text{ for } k=1,\dots,N,$$

where $\beta=0,1$ is an index. Here, the product over an empty set is by convention equal to 1.

It is easy to see that, using Lemma 8.18 applied to the sequence $\left(z_k^0\right)_{1\le k\le N}$ we get Lemma 8.14. Similarly, using Lemma 8.18 applied to the sequence $\left(z_k^1\right)_{1\le k\le N}$ we get Lemma 8.16.

Remarks on Wojciechowski's inequality. We discuss here some inequalities from the family of Lemma 8.14 and Lemma 8.16. Wojciechowski was the first one to use such inequalities in the proof of non-estimates. In particular, he obtained in [**9**] the following relatively difficult estimate (see [**9**, Lemma 1], [**10**, Lemma 1]):

LEMMA 8.19. *There exists a constant $C>0$ such that, for any integer $N\geq 2$ there exist $M=M(N)$ and a sequence $\sigma_1,\ldots,\sigma_N\in\{0,1\}$ such that*

$$\left\|\sum_{k=1}^{N}\sigma_k\cos\langle\cdot,a_k\rangle\prod_{j=1}^{k-1}\left(1+\cos\left\langle\cdot,a_j\right\rangle\right)\right\|_{L^1(\mathbb{T}^d)}\geq CN, \tag{8.60}$$

whenever the sequence $(a_k)_{1\leq k\leq N}$ in $\mathbb{Z}^d$ satisfies

$$|a_{k+1}|>M\,|a_k|,\ \textit{for}\ 1\leq k\leq N-1.$$

This lemma was already used in conjunction with the Riesz products technique in [**10**], [**3**], [**4**]. Lemma 8.19 was used in [**10**] to prove that there exists $g\in L^1\left(\mathbb{T}^2\right)$ such that there are no $f_0,f_1,f_2\in W^{1,1}\left(\mathbb{T}^2\right)$ with

$$g=f_0+\partial_1 f_1+\partial_2 f_2.$$

It was also used in [**4**] in order to prove some anisotropic Ornstein-type non-inequalities and in [**3**] to study the continuity of the multipliers on $\dot{W}^{1,1}\left(\mathbb{R}^d\right)$.

Here we want to point out that, in the above applications, a weaker form suffices: we only need to know that the lower bound in (8.60) goes to ∞ when $N\to\infty$. (In the case of the application of Lemma 8.19 given in [**10**], this was observed by Wojciechowski [**10**, Remark 1].) This weaker version can be achieved by much cheaper arguments than the ones used to obtain Lemma 8.19. In this direction we mention the following.

LEMMA 8.20. *For any integer $N\geq 2$ there exists a sequence $\sigma_1,\ldots,\sigma_N\in\{0,1\}$ such that*

$$\left\|\sum_{k=1}^{N}\sigma_k\cos\langle\cdot,a_k\rangle\prod_{j=1}^{k-1}\left(1+\cos\left\langle\cdot,a_j\right\rangle\right)\right\|_{L^1(\mathbb{T}^d)}\geq\frac{1}{2\pi}\sqrt{\frac{N}{e}},$$

for any dissociated sequence $(a_k)_{1\leq k\leq N}$ in $\mathbb{Z}^d$.

PROOF. The proof follows the ideas in [**2**]. By applying Lemma 8.18 to the sequence $(z_k)_{1\leq k\leq N}$, where

$$z_k:=\frac{1}{2\sqrt{N}}\left(1+\frac{i}{2\sqrt{N}}\right)^{k-1}\quad\text{for } k=1,\ldots,N, \tag{8.61}$$

we obtain a sequence $(\sigma_k)_{1\leq k\leq N}$ in $\{0,1\}$ such that

$$\left|\sum_{k=1}^{N}\frac{\sigma_k}{2\sqrt{N}}\left(1+\frac{i}{2\sqrt{N}}\right)^{k-1}\right|\geq\frac{1}{2\pi}\sum_{k=1}^{N}\frac{1}{\sqrt{N}}=\frac{\sqrt{N}}{2\pi}. \tag{8.62}$$

Suppose $(a_k)_{1\leq k\leq N}$ is a dissociated sequence in $\mathbb{Z}^d$. Consider the functions

$$g_N(t):=\prod_{k=1}^{N}\left(1+\frac{i}{\sqrt{N}}\cos\langle t,a_k\rangle\right)\quad\text{and}\quad G_N(t):=\prod_{k=1}^{N}\left(1+\cos\langle t,a_k\rangle\right),$$

defined on $\mathbb{T}^d$. Note that, by (8.58), we have

$$g_N*G_N(t)=\prod_{k=1}^{N}\left(1+\frac{i}{2\sqrt{N}}\cos\langle t,a_k\rangle\right). \tag{8.63}$$

Also, we consider the set

$$A:=\bigcup_{\substack{k=1\\ \sigma_k=1}}^{N}\{\varepsilon_1a_1+\ldots+\varepsilon_ka_k\,|\,\varepsilon_1,\ldots,\varepsilon_k\in\{-1,0,1\},\ \varepsilon_k\neq 0\}$$

and the projection P_A defined by $\widehat{P_A f}(n) = \widehat{f}(n)$ if $n \in A$ and $\widehat{P_A f}(n) = 0$ otherwise, for any trigonometric polynomial f on $\mathbb{T}^d$. Observe that, (8.59) and (8.63) give

$$P_A(g_N * G_N)(t) = \sum_{k=1}^{N} \frac{\sigma_k}{2\sqrt{N}} \cos\langle t, a_k\rangle \prod_{j=1}^{k-1} \left(1 + \frac{i}{2\sqrt{N}} \cos\langle t, a_j\rangle\right),$$

and thanks to (8.62),

$$|P_A(g_N * G_N)(0)| \geq \frac{\sqrt{N}}{2\pi}.$$

Since,

$$\|g_N\|_{L^\infty(\mathbb{T}^d)} = \left(1 + \frac{1}{N}\right)^{N/2} \leq \sqrt{e},$$

we obtain

$$\begin{aligned}\sqrt{e}\,\|P_A G_N\|_{L^1(\mathbb{T}^d)} \geq \|g_N\|_{L^\infty(\mathbb{T}^d)} \|P_A G_N\|_{L^1(\mathbb{T}^d)} &\geq |\langle g_N, P_A G_N\rangle| \\ &= |P_A(g_N * G_N)(0)| \geq \frac{\sqrt{N}}{2\pi}.\end{aligned}$$

It remains to observe that,

$$P_A G_N(t) = \sum_{k=1}^{N} \sigma_k \cos\langle t, a_k\rangle \prod_{j=1}^{k-1} \left(1 + \cos\langle t, a_j\rangle\right),$$

which concludes the proof. □

REMARK 8.21. In fact, it is possible to prove Lemma 8.20 without using Lemma 8.18. Indeed, the sequence $(z_k)_{1\leq k\leq N}$ defined in (8.61) has a quite simple form: the argument of z_k is $(k-1)\theta_N$ $(\mathrm{mod}\, 2\pi)$, where $\theta_N := \arctan\left(1/2\sqrt{N}\right)$. One can choose the sequence $(\sigma_k)_{1\leq k\leq N}$ explicitly: $\sigma_k = 1$, if $-\pi/4 \leq (k-1)\theta_N \pmod{2\pi} \leq \pi/4$, and $\sigma_k = 0$, otherwise.

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