A smooth introduction to the wavefront set



2014 J. Phys. A: Math. Theor. 47 443001

(http://iopscience.iop.org/1751-8121/47/44/443001)

**IOP** Publishing | Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor **47** (2014) 443001 (30pp) doi:10.1088/1751-8113/47/44/443001

**Topical Review**

# A smooth introduction to the wavefront set

**Christian Brouder[1], Nguyen Viet Dang[2] and Frédéric Hélein[2]**

[1] Institut de Minéralogie, de Physique des Matériaux et de Cosmochimie, Sorbonne Universités, UMR CNRS 7590, UPMC Université Paris 06, Muséum National d'Histoire Naturelle, IRD UMR 206, 4 place Jussieu, F-75005 Paris, France
[2] Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université Paris, Diderot-Paris 7, CNRS UMR7586, Case 7012, Bât. Sophie Germain, F-75205 Paris cedex 13, France

E-mail: christian.brouder@upmc.fr

Received 3 April 2014, revised 29 July 2014
Accepted for publication 13 August 2014
Published 14 October 2014

**Abstract**
The wavefront set of a distribution describes not only the points where the distribution is singular, but also the 'directions' of the singularities. Because of its ability to control the product of distributions, the wavefront set was a key element of recent progress in renormalized quantum field theory in curved spacetime, quantum gravity, the discussion of time machines or quantum energy inequalitites. However, the wavefront set is a somewhat subtle concept whose standard definition is not easy to grasp. This paper is a step-by-step introduction to the wavefront set, with examples and motivation. Many different definitions and new interpretations of the wavefront set are presented. Some of them involve a Radon transform.

Keywords: wave front set, quantum field theory, microlocal analysis

(Some figures may appear in colour only in the online journal)

## 1. Introduction

Feynman propagators are distributions, and Stueckelberg realized very early that renormalization was essentially the problem of defining a product of distributions [1–3]. This point of view was clarified by Bogoliubov, Shirkov, Epstein and Glaser [4–6] but was later almost forgotten.

In a ground-breaking paper [7], Radzikowski showed that the *wavefront set* of a distribution was a crucial concept to define quantum fields in curved spacetime. This idea was fully developed into a renormalized scalar field theory in curved spacetimes by Brunetti, Fredenhagen [8], Hollands and Wald [9]. This approach was rapidly extended to the case of Dirac fields [10–16], to gauge fields [17–19] and even to the quantization of gravitation [20].

This tremendous progress was made possible by a complete reformulation of quantum field theory, where the wavefront set of distributions plays a central role, for example to determine the algebra of microcausal functions and to define a spectral condition for time-ordered products and quantum states [21–24]. The wavefront set was also a decisive tool to discuss the existence of time-machine spacetimes [25], quantum energy inequalities [26] and cosmological models [27].

Until the early 1990s, the wavefront set was rarely used to solve physical problems. We only know of a few works in crystal optics [28, 29] and quantum field theory on curved spacetimes [30, 31]. This is probably due to the fact that this concept is not familiar to most physicists and not easy to grasp. But now, the wavefront set is here to stay and we think that a smooth and physically motivated introduction to it is worthwhile. This is the purpose of the present paper.

There are textbook descriptions of the wavefront set [32–41], but they do not give any clue on its physical meaning and advanced textbooks are notoriously laconic (the outstanding exception being the book by Gregory Eskin [40]).

The main use of the wavefront set in quantum field theory is to provide a condition for the product of distributions. Indeed, the Feynman propagator is a distribution and the products of propagators present in a Feynman diagrams are not well defined. The wavefront set gives a precise description of the region of spacetime where the product is well defined and the value of the Feynman diagram on the whole spacetime is then obtained by an extension procedure [8].

After this introduction, we discuss in simple terms the problem of the multiplication of one-dimensional distributions. This elementary example reveals a natural condition for two distributions to be multiplied and this condition leads to the definition of the wavefront set. After giving elementary examples of wavefront sets, we discuss in detail the wavefront set of the characteristic function of a domain $\Omega$ in the plane (i.e. a function which is equal to 1 on $\Omega$ and to 0 outside it). To bring a physical feel of the concept, we give two new characterizations of the wavefront set of such functions: the first one uses a Radon transform, the second one counts the number of intersections of straight lines with the boundary of $\Omega$. These two characterizations do not employ any Fourier transform. The next section explores the wavefront set of a distribution defined by an oscillatory integral. This technique is crucial to calculate the wavefront set of the Wightman and Feynman propagators in quantum field theory. The main properties of the wavefront set are listed without proof. The last section enumerates other definitions of the wavefront set.

## 2. Multiplication of distributions

We shall introduce the wavefront set as a condition required to multiply distributions. We first recall that a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is a continuous linear map from the set of smooth compactly supported functions $\mathcal{D}(\mathbb{R}^n)$ to the complex numbers, and we denote $u(f)$ by $\langle u, f \rangle$. For example, if $\delta$ is the Dirac delta distribution, then $\langle \delta, f \rangle = f(0)$. If $g$ is a locally integrable function, then we can consider it as a distribution by associating to $g$ the distribution $\langle u_g, f \rangle = \int g(x) f(x) \mathrm{d}x$ (for a nice introduction to distributions see for example [36]).

It is well known that distributions can generally not be multiplied [42]. The first reason is the very definition of distributions as objects which generalize the functions but for which the 'value at some point' has no sense in general. But, motivated by questions in theoretical physics (e.g. quantum field theory), we may ask under which circumstances it is possible to *extend* the product of ordinary functions to distributions. In most cases this is just impossible. For instance we cannot make sense of the square of $\delta$: a simple way to convince yourself of that is to study the family of functions $\chi_\varepsilon$: $\mathbb{R} \to \mathbb{R}$ for $\varepsilon > 0$ defined by $\chi_\varepsilon(x) = 1/\varepsilon$ if $|x| \leqslant \varepsilon/2$ and $\chi_\varepsilon(x) = 0$ otherwise. For any $f \in \mathcal{D}(\mathbb{R})$ we have $\int_{\mathbb{R}} \chi_\varepsilon(x) f(x) \mathrm{d}x = \varepsilon^{-1} \int_{-\varepsilon/2}^{\varepsilon/2} f(x) \mathrm{d}x = \varepsilon^{-1}(\varepsilon f(0) + O(\varepsilon^3))$ and $\lim_{\varepsilon \to 0} \chi_\varepsilon = \delta$. However, the square of $\chi_\varepsilon$ does not converge to a distribution: $\int_{\mathbb{R}} \chi_\varepsilon^2(x) f(x) \mathrm{d}x = \varepsilon^{-2} \int_{-\varepsilon/2}^{\varepsilon/2} f(x) \mathrm{d}x = \varepsilon^{-2}(\varepsilon f(0) + O(\varepsilon^3))$ diverges for $\varepsilon \to 0$.

In some other cases it is possible to define a product, but we loose some good properties. Consider the example of the Heaviside step function $H$, which is defined by $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geqslant 0$. Its associated distribution, denoted by $\theta$, is

$$\langle \theta, f \rangle = \int_{-\infty}^{\infty} H(x) f(x) \mathrm{d}x = \int_0^{\infty} f(x) \mathrm{d}x.$$

The *function* $H$ can obviously be multiplied with itself and $H^n = H$ for any integer $n > 0$. As we shall see, it is possible to define a product of distributions such that $\theta^n = \theta$ as a distribution. But then, we loose the compatibility of the product with the Leibniz rule because, by taking the derivative of both sides we would obtain $n\theta^{n-1}\theta' = \theta'$. The identity $\theta' = \delta$ and $\theta^{n-1} = \theta$ would give us $n\theta\delta = \delta$ for all integers $n > 1$. Since the left-hand side depends linearly on $n$ and the right-hand side does not and is not equal to zero, we reach a contradiction.

The Leibniz rule is essential for applications in mathematical physics and we shall define a product of distributions obeying the Leibniz rule. We first enumerate some conditions under which distributions can be safely multiplied.

### 2.1. In which cases can we multiply distributions?

*2.1.1. A distribution times a smooth function.* The product of distributions is well defined when one of the two distributions is a smooth function. Indeed, consider a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ and a smooth function $\phi \in C^\infty(\mathbb{R}^n)$. Then, for all test function $f \in \mathcal{D}(\mathbb{R}^n)$ we can define the product of $u$ and $\phi$ by $\langle u\phi, f \rangle = \langle u, \phi f \rangle$.

*2.1.2. Distributions with disjoints singular supports.* We can also define the product of two distributions when the singularities of the distributions are disjoint. To make this more precise, we recall that the *support of a function f*, denoted by supp $f$, is the closure of the set of points where the function is not zero [32, p 14]. For example, the support of the Heaviside function is supp $H = [0, +\infty]$. Note that although a function is zero outside its support, it can also vanish at isolated points of its support, because of the closure condition of the definition. For example the support of the sine function is $\mathbb{R}$ although $\sin(n\pi) = 0$.

However, the *support of a distribution* cannot be defined as the support of a function because the value of a distribution at a point is generally not defined. Hence we define the support by duality: we say that the point $x$ does not belong to the support of the distribution $u$ if and only if there is an open neighborhood $U$ of $x$ such that $u$ is zero on $U$, in other words if $\langle u, f \rangle = 0$ for all test functions $f$ whose support is contained in $U$ [36, p 12]. For example supp $\delta = \{0\}$ and supp $\theta = [0, +\infty]$. Similarly, we can define the singular support of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$, denoted by sing supp $u$, by saying that $x \notin$ sing supp $u$ if and only if there is a neighborhood $U$ of $x$ such that the restriction of $u$ to $U$ is a smooth function, in other words if there is a smooth function $\phi \in C^\infty(U)$ such that $\langle u, f \rangle = \langle \phi, f \rangle = \int \phi(x) f(x) \mathrm{d}x$ for all test functions $f$ supported on $U$ [36, p 108]. For example sing supp $\delta = \{0\}$, sing supp $\theta = \{0\}$.

A more elaborate example is the distribution $u \in \mathcal{D}'(\mathbb{R})$, defined by: $u(x) = (x + i0^+)^{-1}$, i.e. $u$ is the limit in $\mathcal{D}'(\mathbb{R})$ of $u_\varepsilon(x) := (x + \mathrm{i}\epsilon)^{-1}$, when $\varepsilon > 0$ and $\varepsilon \to 0$, this means that [43,

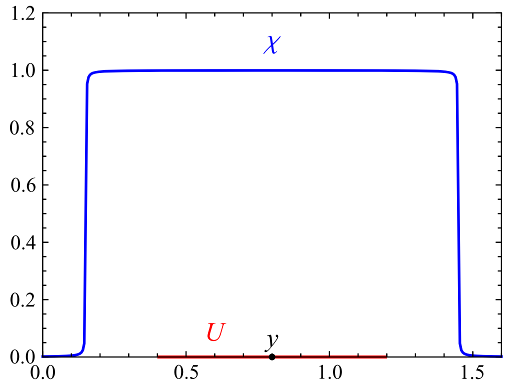


**Figure 1.** In this figure we take $y = 0.8$, the open set is $U = (0.4, 1.2)$ and the smooth function $\chi$ is supported on $(0.1, 1.5)$.

section 2]

$$\langle u, f\rangle = \lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x+\mathrm{i}\epsilon} = \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\infty}\frac{f(x)-f(-x)}{x}\mathrm{d}x - \mathrm{i}\pi f(0).$$

If $y \neq 0$, consider the open set $U = (y - |y|/2, y + |y|/2)$. Take a smooth function $\chi$ such that $\chi(x) = 1$ for $|x - y| < 3|y|/4$ and $\chi(x) = 0$ for $|x - y| > 7|y|/8$. Then, for any $f$ supported on $U$ we have $f(0) = 0$ and $f = f\chi$. Thus,

$$\begin{aligned}\langle u, f\rangle = \ &\langle u, \chi f\rangle = \int_{|y|/8}^{\infty}\frac{\chi(x)f(x) - \chi(-x)f(-x)}{x}\mathrm{d}x\\ = \ &\int_{-\infty}^{\infty}\frac{\chi(x)f(x)}{x}\mathrm{d}x = \langle\phi, f\rangle,\end{aligned}$$

where $\phi(x) = \chi(x)/x$ is smooth because $\chi(x) = 0$ for $|x| < |y|/8$ (see figure 1). As a consequence, every $y \neq 0$ is not in the singular spectrum of $u$ and sing supp $u = \{0\}$ because the imaginary part of $u$ is proportional to a Dirac $\delta$ distribution.

We can now state an important theorem [32, p 55].

**Theorem 1.** *If $u$ and $v$ are two distributions in $\mathcal{D}'(\mathbb{R}^n)$ such that* sing supp $u \cap$ sing supp $v = \emptyset$, *then the product $uv$ is well defined.*

**Proof.** We first notice that, if $f \in \mathcal{D}(\mathbb{R}^n)$ is supported outside the singular support of $v$, then $vf$ is smooth and we can define the product by $\langle uv, f\rangle = \langle u, vf\rangle$. Similary, $\langle uv, f\rangle = \langle v, uf\rangle$ if $f$ is supported outside the singular support of $u$. This definition of $uv$ extends to all test functions $f$ by using a smooth function $\chi$ which is equal to zero on a neighborhood of the singular support of $u$ and equal to one on a neighborhood of the singular support of $v$. Then $\langle uv, f\rangle = \langle v, u\chi f\rangle + \langle u, v(1 - \chi)f\rangle$. This product is associative and commutative [32, p 55]. □

*2.1.3. The singular oscillations of the distributions are transversal.* Consider the two distributions $u = \delta\otimes 1$ and $v = 1\otimes\delta$ in $\mathcal{D}'(\mathbb{R}^2)$, i.e., $\forall\ \varphi \in \mathcal{D}(\mathbb{R}^2)$, $\langle u, \varphi\rangle = \int_{\mathbb{R}}\varphi(0, y)\mathrm{d}y$ and $\langle v, \varphi\rangle = \int_{\mathbb{R}}\varphi(x, 0)\mathrm{d}x$. Then we can define their product by $uv = (\delta\otimes 1)(1\otimes\delta) := \delta\otimes\delta = \delta^{(2)}$, i.e. $\langle uv, \varphi\rangle = \varphi(0, 0)$, since $\langle uv, \varphi\rangle = \int\int u(x)v(y)\varphi(x, y)\mathrm{d}x\mathrm{d}y =$

$\int u(x)\Big(\int v(y)\varphi(x, y)\mathrm{d}y\Big)\mathrm{d}x = \int u(x)\varphi(x, 0)\mathrm{d}x = \varphi(0, 0)$ by the Fubini theorem for distributions. Here $u$ and $v$ are singular on the lines $\{x = 0\}$ and $\{y = 0\}$ respectively, which have a non empty intersection $\{(0, 0)\}$. However the oscillations of both distributions are orthogonal at that point, so that this definition makes sense. But actually the orthogonality is not essential and, as we will see, the important point is the transversality.

Indeed we can extend this example to measures which are supported by non orthogonal lines: let $\alpha: \mathbb{R}^2 \to \mathbb{R}^2$ be a *linear invertible* map and set $\alpha = (\alpha^1, \alpha^2)$ and $u_\alpha := \alpha^* u = u \circ \alpha$ and $v_\alpha := \alpha^* v = v \circ \alpha$, where $\forall\, w \in \mathcal{D}'(\mathbb{R}^2)$, $\forall\, \varphi \in \mathcal{D}(\mathbb{R}^2)$, $\langle \alpha^* w, \varphi\rangle := (\det\alpha)^{-1}\langle w, \varphi \circ \alpha^{-1}\rangle$. These distributions are well-defined and they are singular on the line of equation $\alpha^1 = 0$ and $\alpha^2 = 0$ respectively. Moreover we can define $u_\alpha v_\alpha$ by setting $u_\alpha v_\alpha := \alpha^*(uv) = \alpha^*(\delta^{(2)})$. Hence here $u_\alpha v_\alpha = (\det\alpha)^{-1}\delta^{(2)}$ and we see that the product makes sense as long as $\det\alpha \neq 0$, which means that the singular supports of $u_\alpha$ and $v_\alpha$ are *transversal*.

*2.1.4. The singularities of the distributions are transversal in the complex world.* This last case looks as the most mysterious at first glance and concerns complex valued distributions. Consider the distribution $u(x) = 1/(x + \mathrm{i}0^+)$ defined previously, i.e. the limit of $u_\varepsilon(x) = 1/(x + \mathrm{i}\varepsilon) = \frac{x}{x^2 + \varepsilon^2} - \frac{\mathrm{i}\varepsilon}{x^2 + \varepsilon^2}$ when $\varepsilon > 0$ and $\varepsilon \to 0$ (hence $u = \mathrm{pv}(\frac{1}{x}) - \mathrm{i}\pi\delta_0$.) Observe that $(u_\varepsilon)' = -(u_\varepsilon)^2$, $\forall\, \varepsilon > 0$. Thus since $(u_\varepsilon)'$ converges to $u'$ in $\mathcal{D}'(\mathbb{R})$, we can set $u^2 := -u'$. Moreover since any polynomial relation in $u_\varepsilon$ and its derivatives which follows from Leibniz rule is satisfied ($u_\varepsilon$ being a smooth function), the same holds for $u$. One can define similarly the square of $\overline{u}(x) = 1/(x - \mathrm{i}0^+)$. However this recipe fails for defining the product of $u$ by $\overline{u}$.

A similar mechanism works for making sense of the square of the *Wightman function* (see section 6). One way to understand what happens is to remark that we multiply distributions which are boundary values of holomorphic functions on the same domain.

In order to really understand all these examples and go beyond, we need to revisit them by using refined tools such as: the Radon transform and the Fourier transform. This will lead us to Hörmander's definition of wavefront sets.

### *2.2. The product of distributions by using Fourier transform*

We remark that the Fourier transform of a product of distributions (when it is defined) is the convolution of the Fourier transforms of these distributions [36, p 102]: $\widehat{uv} = \hat{u} \star \hat{v}$, if it exists. Therefore, we can define the product of two distributions $u$ and $v$ as the inverse Fourier transform of $\hat{u} \star \hat{v}$. However, this definition, which requires the Fourier transforms of $u$ and $v$ to be defined and their convolution product to make sense, can be improved. Indeed it does not take into account the fact that the product of two distributions is *local*, i.e. that its definition on the neighborhood of a point depends only on the restriction of the distributions on that neighborhood. Therefore, we can localize the distributions by multiplying them with a test function: if $u \in \mathcal{D}'(U)$ and $f \in \mathcal{D}(U)$, then $fu$ is a distribution with compact support in $U$ and we can extend it to a distribution defined on $\mathbb{R}^n$ by setting it to equal to zero outside $U$. Let us still denote by $fu$ this compactly supported distribution on $\mathbb{R}^n$. It has a Fourier transform $\widehat{fu}(k)$ which is an entire analytic function of $k$ by the Paley–Wiener–Schwartz theorem.

Following the physicist's convention [44], [45, p 32], we define the Fourier transform of $u$ by

$$\mathcal{F}(u)(k) = \hat{u}(k) = \int_{\mathbb{R}^n} \mathrm{d}x \mathrm{e}^{\mathrm{i}k\cdot x} u(x),$$

where $k \cdot x = \sum_{\mathrm{i}} k_{\mathrm{i}} x^{\mathrm{i}}$ (we could interpret this quantity as an Euclidean scalar product between two vectors in $\mathbb{R}^n$; however as we will see in section 6 it is better to understand $k$ as a *covector* and the product $k \cdot x$ as a *duality* product, this is the reason for the lower indices used for the coordinates of $k$ and the upper indices used for the coordinates of $x$). More rigorously, the above definition applies to functions $f$ of rapid decrease and, for a tempered distribution $u$, the Fourier transform is defined by $\langle \hat{u}, f \rangle = \langle u, \hat{f} \rangle$. The inverse Fourier transform is

$$u(x) = \int \frac{\mathrm{d}k}{(2\pi)^n} \mathrm{e}^{-\mathrm{i}k\cdot x} \hat{u}(k),$$

where $n$ is the dimension of spacetime. The same convention was used, for example, by Franco and Acebal [46]. Note the relation between this Fourier transform and the one used in other references: $\hat{u}(k) = \mathcal{F}_H(u)(-k)$ [32, 47], or $\hat{u}(k) = (2\pi)^{n/2} \mathcal{F}_{RS}(u)(-k)$ [35, 39, 41].

We can now give a definition of the product of two distributions. Note that there are alternative definitions, under different hypotheses (and we will meet another one later on.) For a general overwiew about the existing options, see [48, 49].

**Definition 2.** Let $u$ and $v$ in $\mathcal{D}'(\mathbb{R}^n)$. We say that $w \in \mathcal{D}'(\mathbb{R}^n)$ is the product of $u$ and $v$ if and only if, for each $x \in \mathbb{R}^n$, there exists some $f \in \mathcal{D}(\mathbb{R}^n)$, with $f = 1$ near $x$, so that for each $k \in \mathbb{R}^n$ the integral

$$\widehat{f^2 w}(k) = \left(\widehat{fu} \star \widehat{fv}\right)(k) = \frac{1}{(2\pi)^n} \int \widehat{fu}(q) \widehat{fv}(k - q) \mathrm{d}q, \tag{1}$$

*is absolutely convergent.*

When it exists, this product has many desirable properties: it is unique, commutative, associative (when all intermediate products are defined) and it coincides with the product of theorem 1 when the singular supports of $u$ and $v$ are disjoint [35, p 90].

Let us consider some examples.

**Example 3.** If $u = v = \delta$, the product is not defined.

**Proof.** For any test function $f$ satisfying the hypothesis of the definition, $f\delta(x) = f(0)\delta(x) = \delta(x)$ and $\widehat{f\delta}(k) = 1$, so that $\int \widehat{f\delta}(q) \widehat{f\delta}(k - q) \mathrm{d}q = \int \mathrm{d}q$, which is not absolutely convergent. $\square$

**Example 4.** If $u = v = \theta$, the product is well defined.

**Proof.** For any $f \in \mathcal{D}(\mathbb{R})$, $\widehat{f\theta}(k) = \int_0^\infty \mathrm{e}^{\mathrm{i}kx} f(x) \mathrm{d}x$ satisfies the uniform bound $|\widehat{f\theta}(k)| \leqslant \| f \|_{L^1} := \int_{\mathbb{R}} |f(x)| \mathrm{d}x$. Moreover an integration by part gives us also $\widehat{f\theta}(k) = \frac{\mathrm{i}}{k} [f(0) + g(k)]$ with $g(k) := \int_0^\infty \mathrm{e}^{\mathrm{i}kx} f'(x) \mathrm{d}x$ and we thus have the uniform bound $|\widehat{f\theta}(k)| \leqslant \frac{1}{|k|} (|f(0)| + \| f' \|_{L^1})$. Hence, for any $k \in \mathbb{R}$, $|\widehat{f\theta}(k)| \leqslant C(1 + |k|)^{-1}$ for $C = \| f' \|_{L^1} + \| f \|_{L^1} + |f(0)|$ and the integral defining $(\widehat{fu} \star \widehat{fv})(k)$ is absolutely convergent because

$$\int_{\mathbb{R}} \left|\widehat{f\theta}(q)\widehat{f\theta}(k-q)\right| \mathrm{d}q \leqslant \int_{\mathbb{R}} \frac{C^2 \mathrm{d}q}{(|k-q|+1)(|q|+1)} \leqslant \tilde{C} \int_{\mathbb{R}} \frac{\mathrm{d}q}{(|q|+1)^2},$$

where $\tilde{C} = C^2 \sup_q \frac{(|q|+1)}{(|k-q|+1)}$ is finite. □

**Example 5.** If $u(x) = v(x) = 1/(x + \mathrm{i}0^+)$, the product exists.

**Proof.** By contour integration, $\hat{u}(k) = -2\mathrm{i}\pi\theta(-k)$. Thus,

$$\widehat{fu}(k) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}q \hat{f}(q)\hat{u}(k-q) = -\mathrm{i}\int_k^{\infty} \hat{f}(q)\mathrm{d}q,$$

tends to $-2\pi\mathrm{i}f(0) = -2\pi\mathrm{i}$ for $k \to -\infty$.

To show that the integral in equation (1) is absolutely convergent, we define the smooth function $F(k) = \int_k^{+\infty} \hat{f}(q)\mathrm{d}q$. The Fourier transform of a test function $f$ is fast decreasing: for any integer $N$, there is a constant $C_N$ for which $|\hat{f}(q)| \leqslant C_N(1+|q|)^{-N}$ [32, p 252]. Thus, for $k \geqslant 0$

$$|F(k)| \leqslant C_N \int_k^{\infty} (1+q)^{-N}\mathrm{d}q = \frac{C_N}{N-1}(1+k)^{1-N},$$

is fast decreasing and for any $k \in \mathbb{R}$

$$|F(k)| \leqslant C_N \int_{-\infty}^{\infty} (1+|q|)^{-N}\mathrm{d}q = \frac{2C_N}{N-1}.$$

Therefore, the right-hand side of equation (1) can be written $-(2\pi)^{-1}\left(\int_{-\infty}^k + \int_k^0 + \int_0^{+\infty}\right)F(q)F(k-q)\mathrm{d}q$. The first integral is absolutely convergent because $|F(q)F(k-q)| \leqslant 2C_N^2(N-1)^{-2}(1+|k-q|)^{1-N}$, the second because the integrand is smooth and the domain is finite and the third integral because $|F(q)F(k-q)| \leqslant 2C_N^2(N-1)^{-2}(1+q)^{1-N}$.

To compute the product $w = u^2$ we take $f = 1$ and we calculate directly

$$\widehat{u^2}(k) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}(q)\hat{u}(k-q)\mathrm{d}q = -2\pi\int_{\mathbb{R}} \theta(-q)\theta(q-k) = 2\pi k\theta(-k).$$

□

Note that the Fourier transform of the derivative of a distribution $v$ is given by $\widehat{v'}(k) = -\mathrm{i}k\hat{v}(k)$. Thus we recover the relation $\widehat{u^2} = -\widehat{u'}$ i.e. $u(x)^2 = (x+\mathrm{i}0^+)^{-2} = -\frac{\mathrm{d}}{\mathrm{d}x}(x+\mathrm{i}0^+)^{-1}$.

**Example 6.** If $u(x) = 1/(x+\mathrm{i}0^+)$ and $v(x) = 1/(x-\mathrm{i}0^+)$, the product does not exist.

**Proof.** We have $\hat{u}(k) = -2\mathrm{i}\pi\theta(-k)$ and $\hat{v}(k) = 2\mathrm{i}\pi\theta(k)$. Thus,

$$\widehat{fv}(k) = \frac{1}{2\pi}\int \mathrm{d}q \hat{f}(q)\hat{v}(k-q) = \mathrm{i}\int_{-\infty}^{k} \hat{f}(q)\mathrm{d}q,$$

which decreases fast for $k \to -\infty$ and tends to $2\pi\mathrm{i}f(0) = 2\pi\mathrm{i}$ for $k \to +\infty$. We define $G(k) = \int_{-\infty}^{k} \hat{f}(q)\mathrm{d}q$ and recall that $F(k) = \int_k^{+\infty} \hat{f}(q)\mathrm{d}q$ so that $F + G = 2\pi$. The right-hand side of equation (1) can be written as the limit for $M \to \infty$ of $(2\pi)^{-1}I_M(k)$ with

$$\begin{aligned} I_M(k) = & \int_{-M}^{\infty} F(q) G(k-q) \mathrm{d}q \\ = & \ 2\pi \int_{-M}^{\infty} F(q) \mathrm{d}q - \int_{-M}^{\infty} F(q) F(k-q) \mathrm{d}q. \end{aligned}$$

We saw in the previous example that the second term is absolutely convergent and for the first term we use $F = 2\pi - G$ to write

$$\int_{-M}^{\infty} F(q) \mathrm{d}q = \int_{0}^{\infty} F(q) \mathrm{d}q + 2\pi \int_{-M}^{0} \mathrm{d}q - \int_{-M}^{0} G(q) \mathrm{d}q.$$

The decay properties of $F$ and $G$ imply that the first and third terms are absolutely convergent, but the second term is $2\pi M$ which diverges for $M \to \infty$. Thus, there is no test function $f$ with $f(0) = 1$ such that $I_M(k)$ converges: the product of distributions does not exist. □

In example 5, the distribution $u^2$ was calculated without using the localizing test function $f$. In general this is not possible. For example, consider

**Example 7.**

$$u(x) = \frac{1}{x + \mathrm{i}0^+} + \frac{1}{x + a - \mathrm{i}0^+},$$

*with* $a \neq 0$. *Then,* $u^2$ *exists.*

Indeed, denote by $u_1$ and $u_2$ the two terms on the right-hand side. We showed that $u_1^2$ exists and the same reasoning implies that $u_2^2$ exists. The cross term $u_1 u_2$ exists because the singular support of $u_1$, which is $\{0\}$, is disjoint from the singular support of $u_2$, which is $\{-a\}$. Thus, $u^2$ exists although the Fourier transform of $u$ (i.e. $\hat{u}(k) = -2\mathrm{i}\pi\theta(-k) + 2\mathrm{i}\pi \mathrm{e}^{-\mathrm{i}ka}\theta(k)$) is slowly decreasing in both directions. Therefore, the role of the localizing test function $f$ is not only to make the Fourier transform of $fu$ exist (even when the Fourier transform of $u$ does not), but also to isolate the singularities of $u$. In example 7, the two singular points of $u$ are $x = 0$ and $x = -a$. To localize the distribution around $x = 0$, we multiply $u$ by a smooth function $f$ such that $f(0) = 1$ and $f(x) = 0$ for $|x| > |a|/2$, so that $\widehat{fu}(k) = -\mathrm{i}\int_k^{\infty} \hat{f}(q) \mathrm{d}q$ is fast decreasing in the direction of $k > 0$ because the contribution of $1/(x + a - \mathrm{i}0^+)$ is eliminated. Conversely, if we multiply the distribution by a smooth function $g$ such that $g(-a) = 1$ and $g(x) = 0$ for $|x + a| > |a|/2$, then $\widehat{gu}(k) = \mathrm{i}\int_{-\infty}^{k} \hat{g}(q) \mathrm{d}q$, which is fast decreasing in the direction $k < 0$.

*2.2.1. Discussion.* In the previous examples, we saw that the calculation of the product of two distributions by using the Fourier transform looks rather tricky. In particular, it seems that we have to know the Fourier transform of the product of each distribution with an arbitrary function.

Moreover even when we are able to define it, the product of distribution does not always satisfy the Leibniz rule $\partial(uv) = (\partial u)v + u(\partial v)$. For instance the product of $\theta$ makes sense (example 4) but does not respect the Leibniz rule (see section 2). On the other hand the square of $1/(x + \mathrm{i}0^+)$ can be defined (see example 5) and this definition agrees with Leibniz rule.

Fortunately, Hörmander devised a powerful condition on a pair of distributions to: (1) guarantee the existence of their product without computing it; (2) ensure that this product satisfies the Leibniz rule.

As a preparation for this condition, we can analyze why the product exists in example 5 and not in example 6. In example 5, the support of $\hat{u}$ is $(-\infty, 0)$ and, because of the convolution formula $\hat{u}(q)\hat{u}(k-q)$, the support of $\hat{u}(q)\hat{u}(k-q)$ as a function of $q$ is the finite interval $[k, 0]$ if $k \leqslant 0$ and is empty if $k > 0$. Thus, the integral over $q$ is absolutely convergent. On the other hand, in example 6 the support of $\hat{u}(q)\hat{v}(k-q)$ is $(-\infty, \min(k, 0))$, which is infinite.

In general, for the convolution integral to be well defined, we just need that the product $\widehat{fu}(q)\widehat{fv}(k-q)$ decreases fast enough for large $q$ for the integral over $q$ to be absolutely convergent. Note also that, for any distribution $u$ and for any smooth function $f$ with compact support, since $fu$ is a distribution with compact support, its Fourier transform $\widehat{fu}$ grows at most polynomially at infinity, i.e. there exists some $p \in \mathbb{N}$ and some constant $C > 0$ such that $|\widehat{fu}(k)| \leqslant C(1+|k|)^p$ everywhere. Hence it is enough that one of the two factors in the product $\widehat{fu}(q)\widehat{fv}(k-q)$ is fast decreasing at infinity to ensure that the product is fast decreasing. In example 5, $\widehat{fu}(q)$ decreases very fast for $q \to +\infty$ but does not decrease for $q \to -\infty$. If $\widehat{fu}(q)$ decreases slowly in some directions $q$, this must be compensated by a fast decrease of $\widehat{fv}(k-q)$ in the same direction $q$. This is exactly what happens in example 5 and not in example 6.

Lastly example 7 confirms that a general condition for the existence of a product of distributions should use the Fourier transform of distributions localized around singular points.

It is now time to introduce the key notion for defining Hörmander's product of distributions: the *wavefront set*.

## 3. The wavefront set

We want to find a sufficient condition by which the product of distributions defined in equation (1) is absolutely convergent. In this integral, the distribution $fv$ is compactly supported because $f \in \mathcal{D}(\mathbb{R}^n)$. Thus, there is constant $C$ and an integer $m$ such that $|\widehat{fv}(k-q)| \leqslant C(1+|k-q|)^m$. The smallest $m$ for which this inequality is satisfied is called the *order* of the distribution $fv$. The integral (1) would be absolutely convergent if we had $|\widehat{fu}(q)| \leqslant C'(1+|q|)^{-m-n-1}$. However, since we also wish the product of distributions to be compatible with derivatives through the Leibniz rule $\partial(uv) = (\partial u)v + u(\partial v)$ and since a derivative of order $n$ increases the order of $u$ by $n$, what we really need is that $|\widehat{fu}(q)|$ decreases faster than any inverse power of $1+|q|$. We give now a precise definition of the property of fast decrease.

### 3.1. Outside the wavefront set: the regular directed points

We start by defining some basic tools: the conical neighborhoods and the fast decreasing functions.

**Definition 8.** A *conical neighborhood* of a point $k \in \mathbb{R}^n/\{0\}$ is a set $V \subset \mathbb{R}^n$ such that $V$ contains the ball $B(k, \epsilon) = \{q \in \mathbb{R}^n;\ |q-k| < \epsilon\}$ for some $\epsilon > 0$ and, for any $p$ in $V$ and any $\alpha > 0$, $\alpha p$ belongs to $V$.

An example of conical neighborhood of $k_0$ is given in figure 2.

**Definition 9.** A smooth function $g$ is said to be *fast decreasing* on a conical neighborhood $V$ if, for any integer $N$, there is a constant $C_N$ such that $|g(q)| \leqslant C_N(1+|q|)^{-N}$ for all $q \in V$.

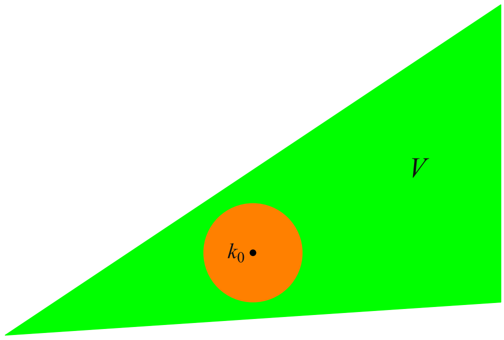


**Figure 2.** Example of a conical neighborhood of $k_0$.

For example, the function $e^{-q^2}$ is fast decreasing on $\mathbb{R}^n$. We need functions to be fast decreasing in a conical neighborhood and not only along a specific direction (which would be the case if $C_N$ were a function of $q$), because a single direction has zero measure and we would not be able to control the integral 1. According to the discussion of the previous section we see that the integral 1 converges absolutely if the directions where $\widehat{fv}(k-q)$ decrease slowly correspond to regions where $\widehat{fu}(q)$ is fast decreasing.

We define now the 'nice points' around which $\widehat{fu}$ is fast decreasing. They are called *regular directed points* [35, p 92].

**Definition 10.** For a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$, a point $(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n/\{0\})$ is called a *regular directed point* of $u$ if and only if there exist: (i) a function $f \in \mathcal{D}(\mathbb{R}^n)$ with $f(x) = 1$ and (ii) a closed conical neighborhood $V \subset \mathbb{R}^n$ of $k$, such that $\widehat{fu}$ is fast decreasing on $V$.

The relevance of the concept of regular directed point also stems from the following theorem [32, p 252]

**Theorem 11.** *A compactly supported distribution $u \in \mathcal{E}'(U)$ is a smooth function if and only if $\hat{u}(q)$ is fast decreasing on $\mathbb{R}^n$.*

This theorem is physically reasonable because, if $f$ is a smooth function, then $f(x)e^{ik\cdot x}$ oscillates widely when $k$ is large, so that the average of this expression (i.e. $\widehat{f}(k)$) is very small. Theorem 11 implies that any singularity of a distribution can be detected by an absence of fast decrease in some direction: a point $x$ is in the singular support if and only if there is a direction $k$, where the Fourier transform is not fast decreasing. However, if $x \in \text{sing supp } u$, there can be directions $k$ such that $(x, k)$ is regular directed. In example 5, we saw that $\widehat{fu}(k)$ is rapidly decreasing for $k > 0$ but not for $k < 0$. This brings us finally to the definition of the wavefront set.

### *3.2. The definition of the wavefront set and the product theorem*

**Definition 12.** The *wavefront set* of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is the set, denoted by $\mathrm{WF}(u)$, of points $(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n/\{0\})$ which are not regular directed for $u$.

In other words, for each point of the singular support of $u$, the wavefront set of $u$ is composed of the directions where the Fourier transform of $fu$ is not fast decreasing, for $f$ a sufficiently small support. The name 'wavefront set' comes from the fact that the singularities of the solutions of the wave equation move within it [32, p 274], so that the wavefront set describes the evolution of the wavefront. The wavefront set is a refinement of the singular support, in the sense that the singular support of $u$ is the set of points $x \in \mathbb{R}^n$, such that $(x, k) \in \mathrm{WF}(u)$ for some nonzero $k \in \mathbb{R}^n$.

Now we see how this definition can be used to determine the product of two distributions $u$ and $v$. Broadly speaking, if a point $x$ belongs to the singular support of $u$ and $v$, then the product of $u$ and $v$ exists at $x$ if, for all directions $q$, either $\widehat{fu}(q)$ or $\widehat{fv}(k - q)$ is rapidly decreasing. In particular, if $(x, q)$ belongs to $\mathrm{WF}(u)$, then $(x, -q)$ must not belong to $\mathrm{WF}(v)$. This is called *Hörmander's condition* and the precise theorem is [32, p 267].

**Theorem 13.** **(Product theorem)**. *Let $u$ and $v$ be distributions in $\mathcal{D}'(U)$. Assume that there is no point $(x, k)$ in $\mathrm{WF}(u)$ such that $(x, -k)$ belongs to $\mathrm{WF}(v)$, then the product $uv$ can be defined. Moreover, if so, then*

$$\mathrm{WF}(uv) \subset S_+ \bigcup S_u \bigcup S_v, \tag{2}$$

*where $S_+ = \{(x, k + q) | (x; k) \in \mathrm{WF}(u)$ and $(x; q) \in \mathrm{WF}(v)\}$, $S_u = \{(x; k) | (x; k) \in \mathrm{WF}(u)$ and $x \in \mathrm{supp}\,(v)\}$ and $S_v = \{(x; k) | (x; k) \in \mathrm{WF}(v)$ and $x \in \mathrm{supp}\,(u)\}$.*

**Remarks**.

(1) This theorem is absolutely fundamental for the theory of renormalization in curved spacetimes. With this simple criterion, we can prove that a product of distributions exists even if we cannot calculate their Fourier transforms and even if we do not know the explicit form of the distributions.

(2) The condition involving the support of $u$ in $S_v$ and the support of $v$ in $S_u$ in $\mathrm{WF}(uv)$ is given in [38, p 84] but is usually not stated explicitly [32, p 267] [33, p 21] [35, p 95], [34, p 527], [36, p 153], [37, p 193], [40, p 61]. This support condition is imperative to calculate the wavefront set of example 19 or of the Feynman propagator in section 6.2.

(3) When Hörmander's condition holds, then the product of distributions satisfies the Leibniz rule for derivatives, because derivatives do not extend the wavefront set [32, p 256]).

(4) Note that if $u$ and $v$ satisfy Hörmander's condition, then their product exists in the sense of definition 2. The converse is not true in general. However, if the product of distributions is extended beyond Hörmander's condition, then it is generally not compatible with the Leibniz rule, as shown by the example of the Heaviside distribution at the beginning of section 2.

(5) Hörmander's condition of the Product theorem can be rephrased by saying that $S_+$ does not meet the zero section (of the cotangent bundle over $U$), i.e. that $S_+ \cap (U \times \{0\}) = \emptyset$.

(6) For any pair $A$ and $B$ of subsets of $U \times \mathbb{R}^n$, we can define $A \oplus B := \{(x, k + q) | (x, k) \in A, (x, q) \in B\}$. We then observe that $S_+ = \mathrm{WF}(u) \oplus \mathrm{WF}(v)$ and hence Hörmander's condition amounts to saying that $\mathrm{WF}\,(u) \oplus \mathrm{WF}(v)$ does not intersect the zero section. On the other hand if we set $\underline{\mathrm{WF}}(u) := \mathrm{WF}(u) \cup (\mathrm{supp}\, u \times \{0\})$, etc., we then always have $\underline{\mathrm{WF}}(u) \oplus \underline{\mathrm{WF}}(v) = S_+ \cup S_u \cup S_v \cup (\mathrm{supp}(uv) \times \{0\})$. Moreover if Hörmander's condition holds then $\mathrm{supp}(uv) \times \{0\}$ is disjoint from $S_+ \cap S_u \cap S_v$ and thus conclusion (2) is equivalent to the inclusion $\underline{\mathrm{WF}}(uv) \subset \underline{\mathrm{WF}}(u) \oplus \underline{\mathrm{WF}}(v)$.

### *3.3. Simple examples and applications of the product theorem*

We give a few very simple examples.

**Example 14.** The simplest example is $\delta(x)$ in $\mathcal{D}'(\mathbb{R}^n)$, for which $\mathrm{WF}(\delta) = \{(0; k)|k \in \mathbb{R}^n, k \neq 0\}$. Thus, the powers of $\delta$ cannot be defined.

**Proof.** The singular support of $\delta(x)$ is $\{0\}$ and $\widehat{f\delta}(k) = f(0)$ is not fast decreasing if $f(0) \neq 0$. This proves that $\mathrm{WF}(\delta) = \{(0; k)|k \in \mathbb{R}^n, k \neq 0\}$. To show that the product is not allowed, consider any point $(0; k)$ of $\mathrm{WF}(\delta)$, then $(0; -k)$ is also a point of $\mathrm{WF}(\delta)$ and the Hörmander condition is not satisfied. □

**Example 15.** The wavefront set of the Heaviside distribution $\theta$ is $\mathrm{WF}(\theta) = \{(0; k);\ k \neq 0\}$. There is a constant $C$ such that $|\widehat{\theta f}(k)| \leqslant C/(1 + |k|)$ for all $k$.

**Proof.** The Heaviside distribution is smooth for $x < 0$ and $x > 0$ because it is constant there. Thus, the only possible singular point is $x = 0$. Consider a smooth compactly supported function $f$ such that $f(0) = 1$. We have for $k \neq 0$

$$\begin{aligned}\widehat{\theta f}(k) &= \int_0^\infty \mathrm{e}^{\mathrm{i}kx} f(x)\mathrm{d}x = \frac{(-\mathrm{i})}{k}\int_0^\infty \left(\mathrm{e}^{\mathrm{i}kx}\right)' f(x)\mathrm{d}x \\ &= \frac{\mathrm{i}f(0)}{k} + \frac{\mathrm{i}}{k}\int_0^\infty \mathrm{e}^{\mathrm{i}kx} f'(x)\mathrm{d}x,\end{aligned} \tag{3}$$

where the prime denotes a derivative with respect to $x$ and we integrated by parts. A further integration by part gives us

$$\widehat{\theta f}(k) = \frac{\mathrm{i}f(0)}{k} - \frac{f'(0)}{k^2} - \frac{1}{k^2}\int_0^\infty \mathrm{e}^{\mathrm{i}kx} f''(x)\mathrm{d}x. \tag{4}$$

Let $L$ be the length of $\mathrm{supp} f$ and, for $n = 0, 1, 2$, let $M_n$ be a constant such that $|f^{(n)}(x)| \leqslant M_n$ for all $x$. Using $f(0) = 1$, identity (4) implies that $|\widehat{\theta f}(k) - \frac{\mathrm{i}}{k}| \leqslant \frac{M_1 + LM_2}{k^2}$. Hence $(0, k) \in WF(\theta)$, $\forall\ k \neq 0$. On the other hand (3) implies both $|\widehat{\theta f}(k)| \leqslant LM_0$ and $|\widehat{\theta f}(k)| \leqslant \frac{1 + LM_1}{|k|}$. We hence deduce that $|\widehat{\theta f}(k)| \leqslant \frac{C}{1 + |k|}$, for some constant $C$. □

The wavefront set of $\theta$ is the same as the wavefront set of $\delta$. This explains why the powers of $\theta$ are not allowed in the sense of Hörmander.

**Example 16.** $u(x) = 1/(x + \mathrm{i}0^+)$, then $\mathrm{WF}(u) = \{(0; k), k < 0\}$. Thus, $u^2$ exists and $\mathrm{WF}(u^2) = \mathrm{WF}(u)$.

**Proof.** The proof is obvious from example 5 (see also [35, p 94], where one must recall that the sign is opposite because of the different convention for the Fourier transform). □

**Example 17.** $v(x) = 1/(x - \mathrm{i}0^+)$, then $\mathrm{WF}(v) = \{(0; k), k > 0\}$. Thus, $v^2$ exists and $\mathrm{WF}(v^2) = \mathrm{WF}(v)$, but we cannot conclude that $uv$ exists (it does not, as we saw in example 6).

**Example 18.** We consider again example 7

$$u(x) = \frac{1}{x + \mathrm{i}0^+} + \frac{1}{x + a - \mathrm{i}0^+},$$

with $a \neq 0$. Then, $\mathrm{WF}(u) = \{(0; k), k < 0\} \cup \{(-a; k), k > 0\}$ and $u^2$ exists, with $\mathrm{WF}(u^2) = \mathrm{WF}(u)$.

**Example 19.** (See [35, p 97]). Let $\delta_1$ and $\delta_2$ be the distributions in $\mathcal{D}'(\mathbb{R}^2)$ defined by $\langle \delta_1, f\rangle = \int \mathrm{d}y f(0, y)$ and $\langle \delta_2, f\rangle = \int \mathrm{d}x f(x, 0)$. Then, $\mathrm{WF}(\delta_1) = \{(0, y; \lambda, 0) | y \in \mathbb{R}, \lambda \neq 0\}$ and $\mathrm{WF}(\delta_2) = \{(x, 0; 0, \mu) | x \in \mathbb{R}, \mu \neq 0\}$. Thus, $\delta_1\delta_2$ exists and $\mathrm{WF}(\delta_1\delta_2) \subset \{(0, 0; \lambda, \mu), \lambda \neq 0, \mu \neq 0\} \cup \{(0, 0; \lambda, 0), \lambda \neq 0\} \cup \{(0, 0; 0, \mu), \mu \neq 0\}$, where we used $\mathrm{supp}\,(\delta_2) = \{(x, 0) | x \in \mathbb{R}\}$ and $\mathrm{supp}\,(\delta_1) = \{(0, y) | y \in \mathbb{R}\}$. Note that the estimate of the wavefront set of $\delta_1\delta_2$ would be much worse if the support of $\delta_2$ and $\delta_1$ had not been taken into account in $S_{\delta_1}$ and $S_{\delta_2}$ of the Product Theorem. In that case the inclusion is in fact an equality because $\mathrm{WF}(\delta_1\delta_2) = \{(0, 0; \lambda, \mu), (\lambda, \mu) \neq (0, 0)\}$.

**Proof.** Let $y \in \mathbb{R}$, we want to calculate $\mathrm{WF}\big(\delta_1\big)$ at $(0, y)$. Take a test function $f(x_1, x_2)$ which is equal to one around $(0, y)$. Then,

$$\widehat{f\delta_1}(k) = \int \mathrm{d}x_1 \mathrm{d}x_2 f(x_1, x_2)\delta(x_1)\mathrm{e}^{\mathrm{i}k_1x_1 + \mathrm{i}k_2x_2} = \int \mathrm{d}x_2 f\big(0, x_2\big)\mathrm{e}^{\mathrm{i}k_2x_2}.$$

Take $k = (k_1, k_2)$ and observe the decay of $\widehat{f\delta_1}(\lambda k)$. If $k_2 \neq 0$ this is a fast decreasing function of $\lambda$ because $f(0, x_2)$ is a smooth compactly supported function of $x_2$. If $k_2 = 0$, then we have $\widehat{f\delta_1}(k_1, 0) = \int \mathrm{d}x_2 f(0, x_2)$, which is independent of $k_1$, so that $\widehat{f\delta_1}(\lambda k_1, 0)$ is not fast decreasing. This proves that $\mathrm{WF}(\delta_1)$ has the given form. A similar proof yields $\mathrm{WF}(\delta_2)$. The rest follows from the fact that $\delta_1\delta_2$ is the two-dimensional delta function. □

## 4. The wavefront set of a characteristic function

Now that we know the definition of the wavefront set, we shall get the feel of it by studying in detail the characteristic distribution $u$ of a region $\Omega$ of $\mathbb{R}^n$, defined by $\langle u, f\rangle = \int_\Omega f(x)\mathrm{d}x$. We shall also revisit it in section 5.2.

### *4.1. The upper half-plane*

For concreteness we start from the characteristic distribution of the upper half-plane

$$\langle u, f\rangle = \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_0^{\infty} \mathrm{d}x_2 f(x_1, x_2).$$

This is the distribution corresponding to the function equal to one on the upper half-plane (i.e. if $x_2 \geqslant 0$) and to zero on the lower half-plane (i.e. if $x_2 < 0$). It is intuitively clear that the singular support of $u$ is the line $(x_1, 0)$. Now take a point $(x_1, 0)$ of the singular support and a test function $f$ which is non-zero on $(x_1, 0)$. What are the directions of slow decrease of $\widehat{fu}$? It seems clear that $\widehat{fu}(k)$ decreases fast when $k$ is along $(1, 0)$, because we do not feel the step of $u$ if we walk along it and do not cross it. But what about the other directions? Does the wavefront set contain all the directions that cross the step or just the direction $(0, 1)$ which is perpendicular to it?

The wavefront set of $u$ can be obtained by noticing that $u$ is the (tensor) product of the constant function 1 for the variable $x_1$ by the Heaviside distribution $\theta(x_2)$. Then, a standard theorem [32, p 267] gives us $\mathrm{WF}(u) = \{(x_1, 0; 0, \lambda), \lambda \neq 0\}$. In other words, the wavefront set detects the direction perpendicular to the step. It is instructive to make an explicit calculation to understand why it is so.

We use an idea of Strichartz [37, p 194] and consider test functions $f(x_1, x_2) = f_1(x_1)f_2(x_2)$. This is not really a limitation because any test function can be approximated by a finite sum of such products. Then $\widehat{uf}(k) = \widehat{f_1}(k_1)\widehat{\theta f_2}(k_2)$. We want to show that, if $k_1 \neq 0$, for every integer $N$ there is a constant $C_N$ such that $|\widehat{uf}(\tau k)| \leqslant C_N(1 + \tau|k|)^{-N}$ for every $\kappa > 0$. We already know that there is a constant $D_N$ such that $|\widehat{f_1}(\tau k_1)| \leqslant D_N(1 + \tau|k_1|)^{-N}$ because $f_1$ is smooth and a constant $C$ such that $|\widehat{f_2}(\tau k_2)| \leqslant C(1 + \tau|k_2|)^{-1}$ (see example 15). We are going to show that, if the component $k_1$ of $k$ is not zero, the fast decrease of $\widehat{f_1}(\tau k_1)$ induces the fast decrease of $\widehat{uf}(\tau k)$. If $k_1 \neq 0$, we have $|k| \leqslant \alpha|k_1|$ where $\alpha = |k|/|k_1|$. Note that $\alpha \geqslant 1$ because $|k_1| \leqslant |k|$. Hence $(1 + \tau|k|) \leqslant \alpha(1 + \tau|k_1|)$ and

$$\left|\widehat{uf}(\tau k)\right| \leqslant CD_N\left(1 + \tau\left|k_1\right|\right)^{-N}\left(1 + \tau\left|k_2\right|\right)^{-1} \leqslant CD_N\alpha^N(1 + \tau|k|)^{-N},$$

where we bounded $(1 + \tau|k_2|)^{-1}$ by 1. Finally, if $k_1 \neq 0$, then $|\widehat{uf}(\tau k)| \leqslant C_N(1 + \tau|k|)^{-N}$ for all $\kappa > 0$ with $C_N = \alpha^N CD_N$. This result was obtained for a single vector $k$, but it can be extended to a cone around $k$ by increasing the value of $\alpha$.

### *4.2. Characteristic function of general domains*

More generally, we can consider the characteristic function of any domain $\Omega$ in $\mathbb{R}^n$ limited by a smooth surface $S$. The characteristic function of $\Omega$ is the function $\chi_\Omega$ such that $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ if $x \notin \Omega$. The characteristic function $\chi_\Omega$ corresponds to a distribution $u_\Omega$ defined by $\langle u_\Omega, f\rangle = \int_\Omega f(x)\mathrm{d}x$. The wavefront set of $u_\Omega$ is given by [50, p 129]

**Proposition 20.** *Let $\Omega \subset \mathbb{R}^n$ be a region with smooth boundary $S$ and let $u_\Omega = \chi_\Omega$ be the characteristic distribution of $\Omega$. Then* $\mathrm{WF}(u_\Omega) = \{(x, k); x \in S, \text{ and } k \text{ normal to } S\}$.

Notice that the vectors $k$ are perpendicular to the boundary $S$ of $\Omega$ (see figure 3 for the example of a disk). This can be understood by a hand-waving argument. Since the boundary $S$ is smooth, by using a test function with very small support around $x \in S$, the boundary looks flat around $x$ and we can apply the argument of the upper-half plane (generalized to $\mathbb{R}^n$) previously discussed. The set of vectors $k$ which are perpendicular to all tangent vectors to $S$ at $x$ is called the *conormal* of $S$ at $x$ and is denoted by $C_x$ (see figure 3 for the example where $n = 2$ and $\Omega$ is the unit disk). The set $C = \{(x, k);\ k \in C_x\}$ is called the *conormal bundle* of $S$. The previous proposition says that the wavefront set of $u_\Omega$ is the conormal bundle of $S$.

The wavefront set of a characteristic distribution has many applications. Its ability to give an accurate description of the boundary of shapes makes it particularly efficient for image analysis [51] and tomography [52].

### *4.3. Counting intersections*

We close this section by stating that the wavefront set of the characteristic distribution of a bounded smooth domain $\Omega$ in the plane can be determined by the following striking procedure. For each straight line $L_{k,a}$ in the plane, denote by $n_{k,a}$ the number of times the straight line intersects the boundary (see figure 4). For generic domains, the wavefront set of $u_\Omega$ can

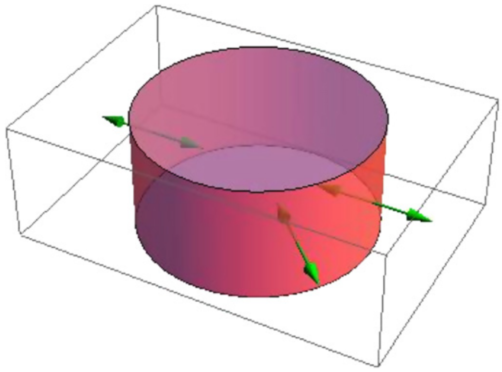

**Figure 3.** The characteristic function of the unit disk (pink) is equal to 1 for $x^2 + y^2 \leqslant 1$ to zero for $x^2 + y^2 > 1$. Some vectors of the wavefront set are indicated as green arrows. For a given point $(x, y)$ of the boundary $x^2 + y^2 = 1$, the points $(x, y; k_x, k_y)$ of the wavefront set are such that $(k_x, k_y)$ is perpendicular to the boundary, thus $(k_x, k_y) = (\lambda x, \lambda y)$ for all $\lambda \neq 0$. In this figure we represent the characteristic function, the tangent bundle and the cotangent bundle in the same coordinates.

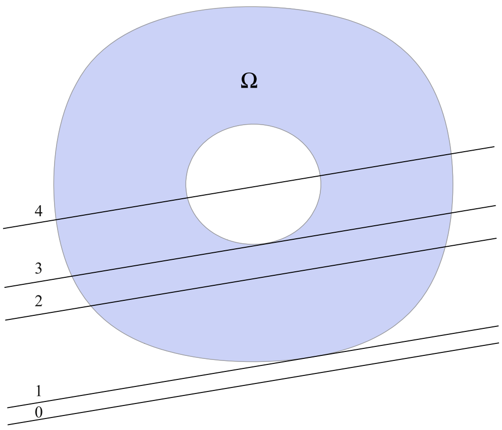


**Figure 4.** Counting the number of times a straight line crosses the boundary of $\Omega$. From the bottom to the top, this number is 0, 1, 2, 3 and 4. It is possible to reconstruct $\Omega$ from the set of straight lines and their numbers of crossings.

be recovered from the set of integers $n_{k,a}$ [53]. In particular, this information is sufficient to recover the shape of $\Omega$. This remark can have applications in image analysis.

In some exceptional cases, this result holds only up to localization or the replacement of the number of intersections by the number of connected parts of the intersection [53]. This characterization of the wavefront set can be extended to surfaces in $\mathbb{R}^3$ if we replace the number of intersections $n_{k,a}$ by the Euler characteristic of the intersection of a given surface with all possible half-spaces [53].

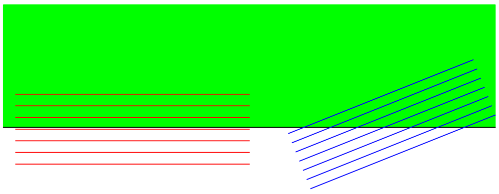

**Figure 5.** The upper half-plane is green. An integration over the blue lines (which are not parallel to the edge) gives a smooth function of the distance from the first line. An integration over the red lines (parallel to the edge) jumps when the line reaches the edge.

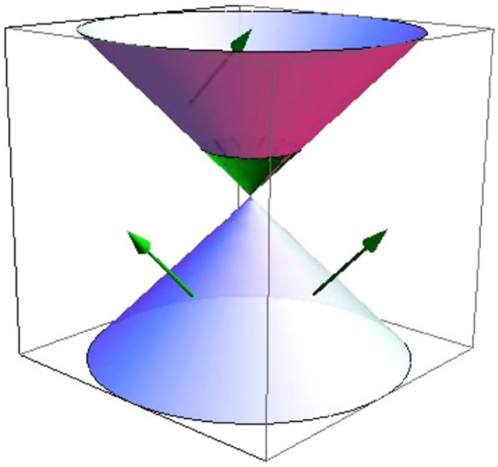

**Figure 6.** Wavefront set of $\Delta_+$: the wavefront set at the origin is an upper cone. Note that, in this figure, three different spaces are identified: the configuration space $\mathbb{R}^3$, the tangent space and the cotangent space over each point of the configuration point. The tangent and cotangent spaces are identified through the Euclidian metric. This implies that the covectors in$\mathbf{WF}$(u) are perpendicular to the tangent planes.

## 5. Use of the Radon transform

### *5.1. The wavefront set of a measure supported by a hypersurface*

In an attempt to better understand the wavefront set, we came up with the following idea. As seen in example (c) in section 2.1, a distribution may be singular and may enjoy partial regularity properties simultaneously. Consider for instance a smooth submanifold $\Gamma \subset \mathbb{R}^n$ and the distribution which is the measure $\mu$ supported by $\Gamma$ with the Euclidean density. The singular character of $\mu$ shows up by restricting $\mu$ to a smooth path which crosses transversally $\Gamma$: this gives us a Dirac mass type singularity. However if we probe $\mu$ by moving in a parallel to $\Gamma$ we may be tempted to say that heuristically the distribution varies smoothly. Such a test cannot be performed by following a path which lies inside $\Gamma$, because the restriction of $\mu$ to such a path would not make sense! However we may replace such a path by a dual wave. In the most naive approach, this consists in a family of hypersurfaces $(H_t)_t$ which cross transversally (e.g. orthogonally) our path and which forms locally a foliation of an open subset of

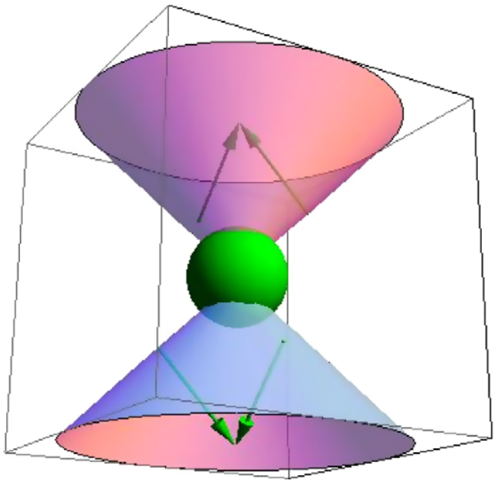

**Figure 7.** Wavefront set of $\Delta_F$, the wavefront set at the origin is a ball.

$\mathbb{R}^n$. Each $H_t$ can be thought as a wavefront in this Huygens type picture. This is another indication that we must interpret $p$ as a covector.

Let's explore this idea in the simple case where $\Gamma$ is a smooth curve. Choose a point $x_0 \in \Gamma$ and a covector $p \in \mathbb{R}^n$, and define the linear form $\alpha$: $\mathbb{R}^n \to \mathbb{R}$ by $\alpha(x) := p \cdot x$ and assume that $\alpha|_{T_{x_0}\Gamma} \neq 0$. We will test $\mu$ locally around $x_0$ by using a plane wave whose wavefronts are the hyperplanes $H_{\alpha,a}$ of equation $\alpha(x) = a$, for $a \in \mathbb{R}$ close to $\alpha(x_0)$. Choose an open neighborhood $U \subset \mathbb{R}^n$ of $x_0$ such that there exists a parametrization $\gamma$: $I \to U$ of $\Gamma \cap U$. Then for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with support contained in $U$, we have

$$\langle \mu, \varphi \rangle = \int_I \varphi(\gamma(t)) \left| \dot{\gamma}(t) \right| \, \mathrm{d}t.$$

Moreover we may choose $U$ such that $\alpha|_{T_x\Gamma} \neq 0$, $\forall\, x \in \Gamma \cap U$. We remark then that $\alpha \circ \gamma$ is a diffeomorphism into its image.

Let $\omega$ be an open subset of $U$ such that $\overline{\omega} \subset U$ and let $\chi \in \mathcal{D}(\mathbb{R}^n)$ with support contained in $U$ and such that $\chi = 1$ on $\overline{\omega}$. Let $f \in \mathcal{D}(\mathbb{R})$ with support in $\alpha(\omega \cap \Gamma)$. Set $\varphi := \chi(f \circ \alpha)$ and observe that $f \circ \alpha = \varphi$ on $U \cap \Gamma$. Hence we can define $\langle \mu, f \circ \alpha \rangle$ by setting

$$\langle \mu, f \circ \alpha \rangle := \langle \mu, \varphi \rangle = \int_I f \circ \alpha \circ \gamma(t) \left| \dot{\gamma}(t) \right| \, \mathrm{d}t.$$

By performing the change of variable $a = \alpha \circ \gamma(t)$, $\mathrm{d}a = |\alpha(\dot{\gamma}(t))| \mathrm{d}t$, $A = \alpha \circ \gamma(I)$, we obtain

$$\langle \mu, f \circ \alpha \rangle = \int_A f(a) \frac{\mathrm{d}a}{|\alpha(\tau(a))|},$$

where $\tau(a)$ is the tangent vector to $\Gamma$: $\tau(\alpha \circ \gamma(t)) = \dot{\gamma}(t)/|\dot{\gamma}(t)|$. We see that we can extend this definition by replacing $f$ by a Dirac mass $\delta_{\mathrm{a}}$ at some value $a \in A$. We then get $\langle \mu, \alpha^* \delta_a \rangle = 1/|\alpha(\tau(a))| = 1/|p \cdot \tau(a)|$, a smooth function of $a$. However it appears clearly that this quantity becomes singular when $\alpha(\tau(a)) = p \cdot \tau(a) = 0$: this corresponds to points of $\Gamma$ such that $T_x\Gamma$ is contained in the kernel of $\alpha$.

Note that we may replace $\alpha$ by $\widetilde{\alpha}(x) = \widetilde{p} \cdot x$, for $\widetilde{p} \in \mathbb{R}^n$ close to $p$: by choosing $U$ suitably we can show that the previous computation remains valid for $(\widetilde{\alpha}, a)$ close to

$(\alpha, \alpha(x_0))$. Geometrically $\langle\mu, \tilde{\alpha}^*\delta_a\rangle$ corresponds to the integral of $\mu$ on the hyperplane $H_{\tilde{\alpha},a}$ (more precisely a neighborhood in $H_{\tilde{\alpha},a}$ of $x_0$), i.e. the value of the local Radon transform at this hyperplane.

The Radon transform provides a determination of the wavefront set (up to sign) without Fourier transformation. Thus, it can give a way to guess the wavefront set of a distribution, as shall be illustrated in the next section.

### *5.2. The Radon transform of the characteristic function of the half-plane*

We go back to the distribution $u$ introduced in section 4.1, i.e. the characteristic function of the upper half-plane $\Omega$ in $\mathbb{R}^2$. Any half-line $\{(x, \lambda k)|\lambda \in (0, +\infty)\}$ in the wavefront set of $u$ is characterized by a point $x$ and a unit direction $k$. Consider a straight line perpendicular to $k$ and move it along $k$. Then, something should happen to the restriction of $u$ to the line when the line crosses the point $x$. To be more precise, consider a straight line $L_{k,a}$ defined by the equation $k\cdot x = a$ (the line perpendicular to $k$ that goes through the point $(a/k_1, 0)$ if $k_1 \neq 0$). By changing the value of $a$ we move the line along $k$. The integral of $fu$ over the line $L_{k,a}$ is $\mathcal{R}(fu)(k, a) = \int_{L_{k,a}\cap\Omega} f(x)\mathrm{d}\ell$, the value of the Radon transform of $fu$ at $(k, a)$. Let us check in this case the result which will be proved in section 5.3, i.e. that the wavefront set of $u$ can be obtained by looking at the points where the Radon transform $\mathcal{R}(fu)(k, a) = \int_{L_{k,a}\cap\Omega} f(x)\mathrm{d}\ell$ is not a smooth function of $a$. This means here that if a line $L_{k,a}$ is not parallel to the step, then a small variation of $a$ is smooth, while it will jump at the step if $L_{k,a}$ is the $x_1$ axis (see figure 5). To prove this, let $k$ be a unit vector and $v$ a unit vector perpendicular to $k$. Then the points of the straight line $L_{k,a}$ are $x = ak + tv$ and

$$\mathcal{R}(fu)(k, a) = \int_{-\infty}^{\infty} f(ak + tv)\theta\big(ae_2\cdot k + te_2\cdot v\big)\mathrm{d}t,$$

where $e_2$ is the unit vector along the $x_2$ axis. If we choose an angle $-\pi/2 < \phi \leqslant \pi/2$ such that $k = e_1 \sin\phi + e_2\cos\phi$ and $v = e_1\cos\phi - e_2\sin\phi$ (where $e_1$ is the unit vector along the $x_1$ axis) we obtain $\mathcal{R}(fu)(k, a) = \int_{-\infty}^{\infty} f(ak + tv)\theta(a\cos\phi - t\sin\phi)\mathrm{d}t$. We must consider three cases.

$$\begin{aligned}
\mathcal{R}(fu)(k, a) &= \int_{a\,\mathrm{cotg}\phi}^{\infty} f(ak + tv)\mathrm{d}t \text{ if } \phi > 0,\\
\mathcal{R}(fu)(k, a) &= \theta(a)\int_{-\infty}^{\infty} f(ae_2 + te_1)\mathrm{d}t \text{ if } \phi = 0 \text{ and } a \neq 0,\\
\mathcal{R}(fu)(k, a) &= \int_{-\infty}^{a\,\mathrm{cotg}\phi} f(ak + tv)\mathrm{d}t \text{ if } \phi < 0.
\end{aligned}$$

We indeed see that $\mathcal{R}(fu)(e_2, a)$ jumps from 0 for $a = 0^-$ to $\int_{-\infty}^{\infty} f(te_1)\mathrm{d}t$ for $a = 0^+$.

### *5.3. The wavefront set up to sign and the Radon transform*

Let us start with the following definition.

**Definition 21.** For any distribution $u$ the *wavefront set up to a sign* of $u$ is the set

$$\mathrm{WF}^{\pm}(u) := \{(x, p)|(x, p) \in \mathrm{WF}(u) \text{or} (x, -p) \in \mathrm{WF}(u)\}.$$

This notion is slightly coarser than the wavefront set. However it gives interesting information about its geometry. Note that $T^*M\backslash\mathrm{WF}^{\pm}(u)$ is the set of *absolutely regular directed points*. These are the points $(x, p)$ such that there exists $f \in \mathcal{D}(\mathbb{R}^n)$ satisfying $f(x) = 1$ and a closed conic neighborhood $V \subset \mathbb{R}^n$ of $p$ such that $\widehat{fu}$ is fast decreasing on $V \cup (-V)$. The set $\mathrm{WF}^{\pm}(u)$ or equivalently its complementary set can be characterized by using the Radon transform.

The Radon transform is defined by averaging functions over affine subspaces. Here we use affine hyperplanes of $\mathbb{R}^n$. First consider the case of a continuous function with compact support $u \in C_c^0(\mathbb{R}^n)$. For any $(\nu, s) \in S^{n-1} \times \mathbb{R}$, let $H_{\nu,s}$ be the hyperplane of equation $\nu \cdot x = s$ and set

$$\mathcal{R}(u)(\nu, s) := \int_{H_{\nu,s}} u(x) \mathrm{d}\sigma(x),$$

where $\sigma$ is the Lebesgue measure on $H_{\nu,s}$. This defines a function $\mathcal{R}(u)$ on $S^{n-1} \times \mathbb{R}$, the *Radon transform* of $u$. This function is linked to the Fourier transform of $u$ by

$$\begin{aligned} \hat{u}(p) = & \int_{\mathbb{R}^n} \mathrm{d}x \ \mathrm{e}^{\mathrm{i}p\cdot x} u(x) = \int_{\mathbb{R}} \mathrm{d}s \ \mathrm{e}^{\mathrm{i}s|p|} \mathcal{R}(u)(p/|p|, s) \\ = & \ \mathcal{F}\left( \mathcal{R}(u)\left( \frac{p}{|p|}, \cdot \right)\right)(|p|), \end{aligned}$$

hence conversely

$$\mathcal{R}(u)(\nu, s) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}k \ \mathrm{e}^{-\mathrm{i}ks} \hat{u}(k\nu).$$

Now consider a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$, let $(x, p)$ be an absolutely regular directed point of $u$. Let $f \in \mathcal{D}(\mathbb{R}^n)$ such that $f(x) = 1$ and a closed conic neighborhood $V \subset \mathbb{R}^n$ of $p$ such that $\widehat{fu}$ is fast decreasing on $V$. For any $\nu \in V \cap S^{n-1}$, $k \to \widehat{fu}(k\nu)$ is a smooth fast decreasing function of $k \in \mathbb{R}$. We can thus define its inverse Fourier transform and set

$$\mathcal{R}(fu)(\nu, s) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}k \ \mathrm{e}^{-\mathrm{i}ks} \widehat{fu}(k\nu).$$

Note, for any fixed $\nu$, $s \to \mathcal{R}(fu)(\nu, s)$ has a compact support because $f$ has a compact support. Since $\forall N \in \mathbb{N}, \exists C_N > 0$ such that $\forall q \in V, |\widehat{fu}(q)| \leqslant C_N(1 + |q|)^N$, it implies that, $\forall m \leqslant N - 2$, $\forall \nu \in V \cap S^{n-1}$, $|\frac{\mathrm{d}^m}{(\mathrm{d}s)^m} \mathcal{R}(fu)(\nu, s)| \leqslant C' C_N$, for $C' = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\mathrm{d}k}{(1 + |k|)^2}$. Hence $\mathcal{R}(fu)$ is uniformly smooth in $s$ on $(V \cap S^{n-1}) \times \mathbb{R}$.

Conversely let $u$ be a distribution and assume that, for some $(x, \nu) \in \mathbb{R}^n \times S^{n-1}$, there exists $f \in \mathcal{D}(\mathbb{R}^n)$ and a closed neighborhood $V \cap S^{n-1}$ of $\nu$ in $S^{n-1}$ such that we can make sense of the Radon transform $\mathcal{R}(fu)$ of $fu$ on $(V \cap S^{n-1}) \times \mathbb{R}$ (e.g. by proving that there exists a sequence $(fu)_\varepsilon$ of smooth functions with compact support which converges to $fu$ in $\mathcal{D}'(\mathbb{R}^n)$ and that the sequence $\mathcal{R}((fu)_\varepsilon)$ converges also in $\mathcal{D}'(S^{n-1} \times \mathbb{R})$ to a distribution which we call $\mathcal{R}(fu)$). Conversely let $u$ be a distribution and observe that, for any $f \in \mathcal{D}(\mathbb{R}^n)$ and any closed neighborhood $V \cap S^{n-1}$ of $\nu$ in $S^{n-1}$, we can make sense of the Radon transform $\mathcal{R}(fu)$ of $fu$ on $(V \cap S^{n-1}) \times \mathbb{R}$, e.g. by noticing that $fu$ is a compactly supported distribution, thus $\widehat{fu}$ is real analytic with polynomial growth by Paley–Wiener–Schwartz, therefore the restriction $\widehat{fu}(k\nu)$ is analytic with polynomial growth in $k \in \mathbb{R}$ uniformly in $\nu \in V \cap \mathbb{S}^{n-1}$, hence a tempered distribution in $\mathcal{S}'(\mathbb{R})$. Its inverse Fourier transform $\mathcal{R}(fu)(\nu, s)$ is thus a tempered distribution in $s$.) Assume moreover, $\forall N \in \mathbb{N}$, $\exists \Gamma_N > 0$ such that $\forall \eta \in V$, $|\frac{\mathrm{d}^m}{(\mathrm{d}s)^m} \mathcal{R}(fu)(\nu, s)| \leqslant \Gamma_N$. Then, since $\forall \eta \in V \cap S^{n-1}$, $s \longmapsto \mathcal{R}(fu)(\eta, s)$ is compactly supported we can define its Fourier transform in $s$ and set $\widehat{fu}(p) = \int_{\mathbb{R}} \mathrm{d}s \ \mathrm{e}^{\mathrm{i}|p|s} \mathcal{R}(fu)(p/|p|, s)$,

$\forall\, p \in V$. It follows then that $|\widehat{fu}\,(p)| \leqslant \Gamma_N |\mathrm{supp}\mathcal{R}\,(fu)(p/|p|,\,\cdot\,)||p|^{-N}$ and hence $\widehat{fu}$ is fast decreasing in $V$. As a conclusion:

**Theorem 22.** *Let $u \in \mathcal{D}'(U)$ be a distribution and $(x, k) \in U \times (\mathbb{R}^n/\{0\})$. Then $(x, k)$ does not belong to $WF^{\pm}(u)$ iff there exists $f \in \mathcal{D}(U)$ such that $\mathcal{R}\,(fu)$ is smooth on a neigborhood of $(k/|k|,\, k \cdot x/|k|)$ in $U \times (V \cap S^{n-1})$.*

## 6. Oscillatory integrals

In proposition 20, the singular support of the characteristic distribution is the submanifold $S$. Hörmander gives another example of a distribution where the singular support is a submanifold [32, p 261]. This example is important because it exhibits a distribution defined by an oscillatory integral (as the Wightman propagator).

**Example 23.** Let $M$ be a smooth submanifold of $\mathbb{R}^n$ defined near a point $x_0 \in M$ by $\phi_1(x) = \ldots = \phi_k(x) = 0$, where $\mathrm{d}\phi_1, \ldots, \mathrm{d}\phi_k$ are linearly independent at $x_0$. If the function $a \in \mathcal{D}(\mathbb{R}^n)$ has support near $x_0$, we define the distribution $\langle u, f\rangle = (2\pi)^k \int \mathrm{d}x a(x)\delta(\phi_1, \ldots, \phi_k) f(x)$, where $\delta$ is the delta function in $\mathbb{R}^k$. This can be rewritten

$$\langle u, f\rangle = \int_{\mathbb{R}^n} \mathrm{d}x f(x) \int_{\mathbb{R}^k} \mathrm{d}\xi\; a(x)\mathrm{e}^{\mathrm{i}\phi(x,\xi)},$$

where $\phi(x, \xi) = \sum_{i=1}^{k} \phi_i(x)\xi_i$ and $\xi_i \in \mathbb{R}$. Then $WF(u) = \{(x, -\mathrm{d}_x\phi(x, \xi));\ \phi_1(x) = \ldots = \phi_k(x) = 0,\ x \in \mathrm{supp}\, a\}$, where

$$\mathrm{d}_x\phi(x, \xi) = \frac{\partial\phi(x, \xi)}{\partial x_1}\mathrm{d}x_1 + \ldots + \frac{\partial\phi(x, \xi)}{\partial x_n}\mathrm{d}x_n.$$

We can use this result to recover the wavefront set of example 20 when $n = 2$, $\Omega$ is the unit disk and $S$ is the unit circle. We have a single function $\phi_1(x_1, x_2) = x_1^2 + x_2^2 - 1$, so that $\phi(x, \xi) = (x_1^2 + x_2^2 - 1)\xi$, the critical set is given by $\mathrm{d}_\xi\phi(x, \xi) = \phi_1(x_1, x_2) = 0$ and $\mathrm{d}_x\phi(x, \xi) = (2x_1\mathrm{d}x_1 + 2x_2\mathrm{d}x_2)\xi$. If we switch to polar coordinates, we obtain $\mathrm{d}_x\phi(x, \xi) = 2\rho\xi\mathrm{d}\rho$, which is a direction perpendicular to the unit circle at $x$. Note that $\xi$ can have both signs, thus both $\mathrm{d}\rho$ and $-\mathrm{d}\rho$ belong to the wavefront set. This example confirms an important characteristics of the wavefront set. The direction $k$ are not vectors but covectors. Indeed, $\mathrm{d}_x\phi(x, \xi)$ can be expanded over the (covector) basis $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ of $T_x^*M$ and not over the vector basis $\partial_{x_1}, \ldots, \partial_{x_n}$ of $T_xM$. To determine the nature of the directions $k$ in the wavefront set, we can also look at the way the wavefront set transforms under a smooth mapping $\mathbb{R}^n \to \mathbb{R}^n$. The detailed calculation [37, p 195] confirms that $k$ are covectors because they transform covariantly. This point is important for distributions on manifolds.

The previous result can be extended to more generaly oscillatory integrals (in the following we always assume that the *phase function* $\phi$ is homogeneous of degree 1, i.e. $\phi(x, \lambda\xi) = \lambda\phi(x, \xi)$, $\forall\, \lambda > 0$, see [32, p 260] for details).

**Theorem 24.** *If a distribution $u$ is defined by an oscillatory integral*

$$u(f) = \int_{\mathbb{R}^n} \mathrm{d}x f(x) \int_{\mathbb{R}^s} \mathrm{d}\xi\; a(x, \xi)\mathrm{e}^{\mathrm{i}\phi(x,\xi)}\mathrm{d}\xi,$$

*where $\phi$ is a phase function and $a$ an asymptotic symbol, then $WF(u) \subset \{(x; -\mathrm{d}_x\phi(x, \xi))|\ \mathrm{d}_\xi\phi(x, \xi) = 0\}$.*

We refer to the literature for a precise definition of phase functions and asymptotic symbols [32, p 236] [35, p 99]. We can give a hand-waving argument to understand the origin of this wavefront set. The Fourier transform of $u$ is given by $\int \mathrm{d}x \int \mathrm{d}\xi a(x, \xi)\mathrm{e}^{\mathrm{i}\phi(x,\xi)+\mathrm{i}k\cdot x}$. By using the stationary-phase method, we see that the directions of slow decrease are the directions where the phase $\phi(x, \xi) + k \cdot x$ is critical with respect to $(x, \xi)$. They are determined by the equations $k + \mathrm{d}_x\phi(x, \xi) = 0$ and $\mathrm{d}_\xi\phi(x, \xi) = 0$.

### 6.1. The Wightman propagator

With the help of theorem 24 we can calculate the wavefront set of a fundamental distribution of quantum field theory: the Wightman propagator in Minkowski spacetime [35, p 66]

$$
\begin{aligned}
W_2(f, g) &= \int_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{d}x\mathrm{d}y \langle 0 \,|\varphi(x)\varphi(y)|\, 0\rangle f(x)g(y) \\
&= -\mathrm{i}\int_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{d}x\mathrm{d}y\Delta_+(x-y)f(x)g(y),
\end{aligned}
$$

where [35, p 70]

$$
\Delta_+(x) = \frac{\mathrm{i}}{2(2\pi)^3}\int_{\mathbb{R}^3}\frac{\mathrm{d}\mathbf{k}}{k_0}\mathrm{e}^{-\mathrm{i}k\cdot x},
$$

with $\mathbf{k} = (k_1, k_2, k_3)$, $k_0 = \sqrt{|\mathbf{k}|^2 + m^2}$, $k = (k_\mu) = (k_0, \mathbf{k})$ and $k\cdot x = \sum_{\mu=0}^{3} k_\mu x^\mu$.

The analytic form of $\Delta_+(x)$ is given by Scharf [54, p 90]. We can write $\Delta_+(f)$ in the oscillatory integral form of theorem 24 by setting, for $\xi \in \mathbb{R}^3$, [35, p 100]

$$
\phi(x, \xi) = -x^0\,|\xi| - x^j\xi_j, \quad a(x, \xi) = \frac{\mathrm{e}^{-\mathrm{i}x^0\left(\omega_\xi - |\xi|\right)}}{\omega_\xi},
$$

where $|\xi| = \sqrt{(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2}$, $\omega_\xi = \sqrt{|\xi|^2 + m^2}$ and $x^j\xi_j = \sum_{j=1}^{3} x^j\xi_j$. To prove this, just write

$$
-ik\cdot x = -i\left(x^jk_j + x^0k_0\right) = i\left(-x^0|\mathbf{k}| - x^jk_j\right) - ix^0\left(k_0 - |\mathbf{k}|\right),
$$

and replace $\mathbf{k}$ by $\xi$. The modification of the phase is necessary to make $a(x, \xi)$ an asymptotic symbol. We can now calculate the wavefront set of the Wightman propagator [35, p 106]

**Proposition 25.** *The wavefront set of* $\Delta_+$ *is* $\mathrm{WF}(\Delta_+) = S_0 \cup S_+ \cup S_-$, *where*

$$
\begin{aligned}
S_0 &= \left\{(0; |\mathbf{k}|, \mathbf{k}) \,\middle|\, \mathbf{k} \in \left(\mathbb{R}^3/\left\{0\right\}\right)\right\}, \\
S_\pm &= \left\{(\pm|\mathbf{x}|, \mathbf{x}; \lambda\,|\mathbf{x}|, \mp\lambda\mathbf{x}) \,\middle|\, \mathbf{x} \in \left(\mathbb{R}^3/\left\{0\right\}\right), \lambda > 0\right\}.
\end{aligned}
$$

*More compactly [39, p 118],* $\mathrm{WF}(\Delta_+) = \{(x; k);\ k_0 = |\mathbf{k}|, x^0 = \lambda k_0, x^i = -\lambda k_i, \lambda \in \mathbb{R}\}$ *(see figure 6).*

The advantage of the physical convention for the Fourier transform is that positive energies correspond to $k_0 > 0$. The wavefront set of $\Delta_+$ for curved (globally hyperbolic) spacetime is given by Strohmaier [39].

**Proof.** According to theorem 24, we first calculate the set of critical points $\{\mathrm{d}_\xi\phi = 0\}$ for $\phi(x, \xi) = -x^0|\xi| - x^i\xi_i$. We find $-x^0\left(\sum_{i=1}^{3}\frac{\xi_i d\xi_i}{|\xi|}\right) - \sum_{i=1}^{3} x^i\mathrm{d}\xi_i = 0$, which implies

$x^i = -x^0\frac{\xi_i}{|\xi|}$ and thus $x^0 = \lambda|\xi|$ and $x^i = -\lambda\xi_i$ for $\lambda = \frac{x^0}{|\xi|}$. Conversely, if we plug $x^0 = \lambda|\xi|$, $x^i = -\lambda\xi_i$ in $\mathrm{d}_\xi\phi$ for any $\lambda \in \mathbb{R}$, we find $\mathrm{d}_\xi\phi = \sum_{i=1}^3(\lambda\xi_i - \lambda\xi_i)\mathrm{d}\xi_i = 0$. Then theorem 24 claims that $WF(\Delta_+)$ is a subset of $\{(x;\, \mathrm{d}_x\phi);\ \mathrm{d}_\xi\phi(x;\,\xi) = 0\}$:

$$\begin{aligned}\mathrm{WF}(\Delta_+) &\subset \left\{\left(x^0,\, \mathbf{x};\, \left|\xi\right|,\, \xi_i\right)\middle| x^0 = \lambda|\xi|,\, x^i = -\lambda\xi_i,\, \lambda \in \mathbb{R}\right\}\\ &\subset \left\{\left(x^0,\, \mathbf{x};\, k_0,\, k_i\right)\middle| k_0 = |\mathbf{k}|,\, x^0 = \lambda k_0,\, x^i = -\lambda k_i,\, \lambda \in \mathbb{R}\right\}.\end{aligned}$$

We leave to the reader the proof of the decomposition $\{(x^0, \mathbf{x};\, k_0,\, k_i)|k_0 = |\mathbf{k}|,\, x^0 = \lambda k_0,\, x^i = -\lambda k_i,\, \lambda \in \mathbb{R}\} = S_0 \cup S_+ \cup S_-$.

Note that theorem 24 states only that $\mathrm{WF}(\Delta_+) \subset S_0 \cup S_+ \cup S_-$. We refer to the literature to show that $\subset$ can be replaced by $=$ [35, p 107]. The singular support of $\Delta_+$ is the light cone $x^0 = \pm|\mathbf{x}|$, the cotangent vectors $k$ are light-like, have positive energy $k_0$ and are perpendicular to $x$. □

### 6.2. The Feynman propagator

**Proposition 26.** *The Feynman propagator* $\Delta_F(x) = \theta(x^0)\Delta_+(x) + \theta(-x^0)\Delta_+(-x)$ *exists and its wavefront set is* $\mathrm{WF}(\Delta_F) = D^* \cup C_F$, *where* $D^* = \{(0;\, k)|\, k \neq 0\}$ *is the wavefront set of the Dirac delta function and* $C_F = \{(x;\, k)|\,(x^0)^2 - |\mathbf{x}|^2 = 0,\, x^0 \neq 0,\, k_0 = \lambda x^0,\, k_i = -\lambda x^i,\, \lambda > 0\}$ *(see figure 7).*

**Proof.** $\theta(x^0)\Delta_+(x)$ is a product of distributions, we must first show that it exists. As a distribution in $\mathbb{R}^4$, $\theta(x^0)$ is defined by $\theta(x^0)(f) = \int_{x^0 \geq 0} f(x)\mathrm{d}x$. Therefore, it is the tensor product of the Heaviside distribution in the variable $x^0$ by the unit distribution in the variables $x^1,\ldots,x^3$: $\theta(x^0) = \theta \otimes 1$. The distribution 1 is smooth and its wavefront set is empty. Thus, by property (i) of section 7, we have $\mathrm{WF}(\theta(x^0)) \subset \{(0,\, \mathbf{x};\, \pm\lambda,\, 0)|\, \mathbf{x} \in \mathbb{R}^3,\, \lambda > 0\}$. In fact, the inclusion can be replaced by an equal sign [35, p 108]. By theorem 13, we see that the product $\theta(x^0)\Delta_+(x)$ exists. Indeed, sing supp $\theta(x^0) \cap$ sing supp $\Delta_+ = \{0\}$ and, at $x = 0$, the allowed cotangent vectors are $k = (\pm\lambda,\, 0)$ with $\lambda > 0$ for $\theta(x^0)$ and $q = (|\mathbf{k}|,\, \mathbf{k})$ with $\mathbf{k} \neq 0$ for $\Delta_+$. Thus, $q + k \neq 0$ and the product exists. A similar calculation for $\theta(-x^0)\Delta_F(-x)$ shows that $\Delta_F$ is a well defined distribution on $\mathbb{R}^4$. However, the estimate of the wavefront set given by the product theorem is not precise enough because of the contribution of $\mathrm{WF}(\theta)$. To calculate the wavefront set of $\Delta_F$, we use the causality method of Bogoliubov. Let $x = (x^0,\, \mathbf{x}) \in \mathbb{R}^4/\{0\}$. If $x^0 > 0$ then there is a neighborhood $U$ of $x$ such that $\forall\, y \in U,\, y^0 > 0$. Therefore, $\Delta_F|_U = \theta(x^0)\Delta_+(x)|_U = \Delta_+(x)|_U$ thus $\mathrm{WF}(\Delta_F|_U) = \mathrm{WF}(\Delta_+|_U) = S_+|_U$ and by definition of $S_+$:

$$\mathrm{WF}\left(\Delta_F\middle|_U\right) = \left\{(x;\, k);\ \ x^0 = |\mathbf{x}|,\, x^0 = \lambda k_0,\, x^i = -\lambda k_i,\, \lambda > 0,\, x \in U\right\}.$$

If $x^0 < 0$ then there is a neighborhood $U$ of $x$ such that $\Delta_F|_U = \theta(-x^0)\Delta_+(-x)|_U = \Delta_+(-x)|_U$. Thus

$$\begin{aligned}\mathrm{WF}\left(\Delta_F\middle|_U\right) &= \left\{(x;\, k);\ (-x;\, -k) \in S_+,\, x \in U\right\}\\ &= \left\{(x;\, k);\ x^0 = -|\mathbf{x}|,\, x^0 = \lambda k_0,\, x^i = -\lambda k_i,\, \lambda > 0,\, x \in U\right\}.\end{aligned}$$

If $x^0 = 0$, then $x$ is space-like because $x \neq 0$. Thus, there exists some orthochronous Lorentz transformation $R \in SO^\uparrow(1,\, 3)$ such that $(Rx)^0 > 0$. From the definition of $\Delta_F$ we deduce that $\Delta_F(Rx) = \theta((Rx)^0)\ \Delta_+(Rx) + \theta(-(Rx)^0)\Delta_+(-Rx)$. Since $\Delta_F$ and $\Delta_+$ are invariant by orthochronous Lorentz transformations, this implies $\Delta_F(x) = \theta((Rx)^0)\Delta_+(x) + \theta(-(Rx)^0)\Delta_+(-x)$. Hence we recover the case $x^0 > 0$ and $\Delta_F$ is smooth on a neighborhood of $x$ because $x$ is not light-like. This gives us $\mathrm{WF}(\Delta_F)|_{x\neq 0} = C_F$. To complete the proof of the proposition, recall that $(\Box + m^2)\Delta_F = -\mathrm{i}\delta$ [44, p 124]. Thus property (h) of section 7 implies $\mathrm{WF}(\delta) = D^* \subset \mathrm{WF}(\Delta_F)$. Since no wavefront set at $x = 0$ can be larger than $D^*$, we obtain $\mathrm{WF}(\Delta_F)|_{x=0} = D^*$ and the proposition is proved. $\Box$

The calculation of $\mathrm{WF}(\Delta_F)$ was first made by Duistermaat and Hörmander [55] after discussion with Wightman. The analytic expression for the Feynman propagator in position space is given by Zhang *et al* [56]. The wavefront set of the advanced and retarded solutions to the wave equation is calculated in [55] and [39, p 115].

Note that the wavefront set condition does not allow to multiply Feynman propagators. This implies that a renormalization procedure is required. By using the causal condition put forward by Stueckelberg [57] and Bogoliubov [58][3], Brunetti and Fredenhagen [8] showed that Feynman diagrams can be inductively built as well-defined distributions on $M^{\times k}$, where $M$ is the spacetime manifold and $k$ is the number of vertices of the Feynman diagram, except along the thin diagonal (i.e. where $(x_1 = \cdots = x_k)$). The distribution is then defined on the full $M^{\times k}$ by an extension whose ambiguity is organized by the renormalization group [59, 60].

## 7. Properties of the wavefront set

We now give without proof a number of properties of the wavefront set. Let $u$ and $v \in \mathcal{D}'(\mathbb{R}^n)$. Then

(a) $\mathrm{WF}(u)$ is a closed subset of $\mathbb{R}^n \times (\mathbb{R}^n/\{0\})$ [35, p 92].
(b) For each $x \in \mathbb{R}^n$, $\mathrm{WF}_x(u) = \{k;\ (x, k) \in \mathrm{WF}(u)\}$ is a cone, i.e. $k \in \mathrm{WF}_x(u)$ and $\lambda > 0$ implies $\lambda k \in \mathrm{WF}_x(u)$ [35, p 92].
(c) $\mathrm{WF}(u + v) \subset \mathrm{WF}(u) \cup \mathrm{WF}(v)$ [35, p 92].
(d) sing supp $u = \{x;\ WF_x(u) \neq \varnothing\}$ [35, p 93].
(e) If $u$ is a tempered distribution and $\hat{u}$ has support in a closed cone $C$, then for each $x$, $\mathrm{WF}_x(u) \subset C$ [35, p 93].
(f) Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be two open sets. For any smooth ($C^\infty$) map $f: U \to V$ we define

$$N_f := \left\{ (f(x), k) \in V \times \mathbb{R}^n;\ k \circ \mathrm{d}f_x = 0 \right\},$$

where $k \circ \mathrm{d}f_x := (k_i \mathrm{d}y^i) \circ \mathrm{d}f_x := k_i \mathrm{d}f_x^i$. Consider the *pull-back operator* $u \longmapsto f^* u := u \circ f$ defined on smooth maps $u$ on $V$. Then it is possible to extend this operator to the space of distributions $u \in \mathcal{D}'(V)$ which satisfy $N_f \cap WF(u) = \varnothing$ in an unique way (if we furthermore require some continuity assumptions, see [32, theorem 8.2.4]). Moreover the wavefront set of $f^* u$ is contained in the set

$$f^* WF(u) := \left\{ \left(x, k \circ \mathrm{d}f_x\right) \middle| (f(x), k) \in WF(u) \right\}.$$

[32, theorem 8.2.4] (beware that, in the definition of the inverse image of a distribution by a diffeomorphism in [35, p 93], the expression for the wavefront set of $f^* u$ is not correct.)

[3] German translation in [86], English translation in [87, pp 23–118]

(g) If $u \in \mathcal{D}'(U)$ and $f \in C^\infty(U)$, then $\mathrm{WF}(fu) \subset \{\mathrm{supp}\, f \times (\mathbb{R}^n/\{0\})\} \cap \mathrm{WF}(u)$ [41, p 344].
(h) If $u \in \mathcal{D}'(U)$ and $P$ is a partial differential operator with smooth coefficients, then $\mathrm{WF}(Pu) \subset \mathrm{WF}(u)$ [32, p 256].
(i) If $u \in \mathcal{D}'(U)$ and $v \in \mathcal{D}'(V)$, then $\mathrm{WF}(u \otimes v) \subset (\mathrm{WF}(u) \times \mathrm{WF}(v)) \cup ((\mathrm{supp}\, u \times \{0\}) \times \mathrm{WF}(v)) \cup (\mathrm{WF}(u) \times (\mathrm{supp}\, v \times \{0\}))$ [32, p 267].
(j) If $u \in \mathcal{D}'(U \times U)$ is such that (formally) $u(x, y) = v(x - y)$ for some $v \in \mathcal{D}'(V)$, then $\mathrm{WF}(u) = \{(x, y; k, -k);\ (x - y; k) \in \mathrm{WF}(v)\}$ [39, p 118] and [32, p 270].

As an application of the pull-back theorem, we calculate the wavefront set of $\Delta_F$ for a massless particle, whose analytic expression is [44, p 133]

$$\Delta_F(x) = \frac{1}{4\pi^2}\frac{1}{x^2 - i0},$$

where $x^2 = (x^0)^2 - |\mathbf{x}|^2$. We first prove that this distribution is well defined on $\mathbb{R}^4/\{0\}$. So $\Delta_F$ is just the pull–back of $(2\pi)^{-2}(t - \mathrm{i}0)^{-1}$ by the $C^\infty$ map.

$$f\colon x \in \mathbb{R}^4/\{0\} \to \left(x^0\right)^2 - |\mathbf{x}|^2 \in \mathbb{R}. \tag{5}$$

Indeed, this map is smooth and $N_f = \{(f(x), k);\ 2k(x^0\mathrm{d}x^0 - x^i\mathrm{d}x^i) = 0\}$. We know that $\mathrm{WF}(1/(t - \mathrm{i}0^+)) = \{(0; k);\ k > 0\}$. Thus, the condition $N_f \cap WF(u) = \varnothing$ implies $x \neq 0$ and $\Delta_F$ is therefore well defined in $\mathcal{D}'(\mathbb{R}^4/\{0\})$. Furthermore, by property (f) $WF(\Delta_F|_{x\neq 0})$ is included in the pull-back of $\mathrm{WF}(1/(t - \mathrm{i}0^+))$ by $f$. We obtain:

$$\mathrm{WF}\left(\Delta_F\Big|_{x\neq 0}\right) \subset \left\{(x;\ \lambda \circ \mathrm{d}f);\ (f(x); \lambda) \in \mathrm{WF}(t - \mathrm{i}0)^{-1}\right\}.$$

Therefore, $\mathrm{WF}(\Delta_F|_{x\neq 0}) = \{(x; k)|f(x) = 0, k = \lambda \mathrm{d}f(x) \subset \lambda(x^0, -\mathbf{x}), \lambda > 0, x \neq 0\}$. To conclude, observe that $\Delta_F$ is a homogeneous distribution, therefore by a theorem of Hörmander ([32, Theorem 3.2.4]), it admits an extension in $\mathcal{D}'(\mathbb{R}^4)$. The wavefront set of $\Delta_F$ at $x = 0$ is calculated as in the proof of proposition 26 by using $\Box\Delta_F = -\mathrm{i}\delta$ and we recover proposition 26 for $m = 0$.

## 8. The many faces of the wavefront set

In this section we give several definitions of the wavefront set. Each of them can be useful in specific contexts.

### *8.1. The frequency set*

It is possible to define the wavefront set in terms of the *frequency set* of distributions $u$, denoted by $\Sigma(u)$ [32, p 254], which is the projection of the wavefront set of $u$ on the momentum (i.e. cotangent) space.

**Definition 27.** Let $u \in \mathcal{E}'(\mathbb{R}^n)$, we define $\Sigma(u)$ to be the closed cone in $\mathbb{R}^n/\{0\}$ having no conic neighborhood $V$ such that, $|\hat{u}(k)| \leqslant C_N(1 + |k|)^{-N}$ for $k \in V$ and for all $N = 1, 2, \ldots$.

Friedlander and Joshi define the *frequency set* $\Sigma(u)$ by

**Definition 28.** Let $u \in \mathcal{E}'(\mathbb{R}^n)$, then the direction $k_0$ is not in $\Sigma(u) \subset \mathbb{R}^n/\{0\}$ iff there is a conic neighborhood $V$ of $k_0$ such that, for all $N$, there is a $C'_N$ such that $|\hat{u}(k)| \leqslant C'_N \langle k \rangle^{-N}$, for all $k$ in $V$, where $\langle k \rangle = (1 + |k|^2)^{1/2}$.

Duistermaat (implicitly) proposed a third definition

**Definition 29.** Let $u \in \mathcal{E}'(\mathbb{R}^n)$, then the direction $k_0$ is not in $\Sigma(u) \subset \mathbb{R}^n/\{0\}$ iff there is a neighborhood $W$ of $k_0$ such that, for all $N$, there is a constant $D_N$ such that $|\hat{u}(\tau k)| \leqslant D_N \tau^{-N}$ for $\tau \to \infty$ uniformly in $k \in W$.

The proof of the equivalence of these definitions is left to the reader.

### *8.2. Several definitions of the wavefront set*

The frequency set is used in several definitions of the wavefront set. According to Hörmander [32, p 254]

**Definition 30.** Let $U$ be an open set of $\mathbb{R}^n$, $u \in \mathcal{D}'(U)$ and $\Sigma_x(u) = \bigcap_\phi \Sigma(\phi u)$, where $\phi$ runs over all elements of $\mathcal{D}(U)$ such that $\phi(x) \neq 0$. The wavefront set of $u$ is the closed subset of $U \times (\mathbb{R}^n/\{0\})$ defined by

$$\mathrm{WF}(u) = \Big\{ (x; k) \in U \times \big(\mathbb{R}^n \backslash \{0\}\big);\ k \in \Sigma_x(u) \Big\}.$$

For Duistermaat [33, p 16] the wavefront set is:

**Definition 31.** If $u \in \mathcal{D}'(U)$, then WF($u$) is defined as the complement in $U \times (\mathbb{R}^n/\{0\})$ of the collection of all $(x_0, k_0) \in (\mathbb{R}^n/\{0\})$ such that for some neighborhood $U$ of $x_0$, $V$ of $k_0$ we have for each $\phi \in \mathcal{D}(U)$ and each $N$: $\widehat{\phi u}(\tau k) = O(\tau^{-N})$ for $\tau \to \infty$, uniformly in $k \in V$.

An equivalent definition was used by Chazarain and Piriou [34, p 501], who use the name *singular spectrum* but the notation WF$u$.

For Friedlander and Joshi [36, p 145] (after correction of a misprint) and Strichartz [37, p 191]

**Definition 32.** Let $Y$ be an open set of $\mathbb{R}^n$ and $u \in \mathcal{D}'(U)$, then we shall say that $(x_0, k_0) \in U \times (\mathbb{R}^n/\{0\})$ is not in WF($u$) iff there exists $\phi \in \mathcal{D}(U)$ such that $\varphi(x_0) \neq 0$ and $k_0 \notin \Sigma(\phi u)$.

For Eskin [40, p 58]

**Definition 33.** Let $U$ be an open set of $\mathbb{R}^n$ and $u \in \mathcal{D}'(U)$, then we shall say that $(x_0, k_0) \in U \times (\mathbb{R}^n/\{0\})$ is not in WF($u$) iff there exists $\phi \in \mathcal{D}(U)$ such that $\varphi(x_0) \neq 0$ and $|\widehat{\phi u}(k)| \leqslant C_N(1 + |k])^{-N}$ for all $N$ and all $k \neq 0$ satisfying $|\frac{k}{|k|} - \frac{k_0}{|k_0|}| < \delta$ for some $\delta > 0$.

The proof of the equivalence of these definitions is left to the reader.

### *8.3. More definitions of the wavefront set*

In this section we gather alternative definitions of the wavefront set, which show that the wavefront set is the single solution of many different problems.

*8.3.1. Coordinate invariant definition.* A coordinate invariant definition of the wavefront set was given by Duistermaat [33, p 16] following a first attempt by Gabor [61]. We consider a smooth $n$-dimensional manifold $M$, its cotangent bundle $T^*M$ and the zero section $Z$ of $T^*M$ (i.e. $Z = \{(x; k) \in T^*M;\ \ k = 0\}$). Then, the wavefront set of a distribution $u \in \mathcal{D}'(M)$ is a closed conic subset of $\dot{T}^*M = T^*M \backslash Z$.

**Definition 34.** If $M$ is a smooth $n$-dimensional manifold, $u \in \mathcal{D}'(M)$ and $(x_0; k_0) \in \dot{T}^*M$, then $(x_0; k_0) \notin \mathrm{WF}(u)$ iff, for any smooth function $\psi$: $M \times \mathbb{R}^p \to \mathbb{R}$, with $\mathrm{d}_x\psi(x_0, a_0) = k_0$, there are open neighborhoods $U$ of $x_0$ and $A$ of $a_0$ such that, for any $\phi \in \mathcal{D}(U)$ we have for all $N \geqslant 1$: $\langle u, \mathrm{e}^{\mathrm{i}\tau\psi(\cdot,a)}\phi\rangle = O(\tau^{-N})$ for $\tau \to \infty$, uniformly in $a \in A$.

This definition is surprisingly general because the phase function $\psi$ is only required to be smooth and to satisfy $\mathrm{d}_x\psi(x_0, a_0) = k_0$. The usual definition of the wavefront set is recovered by choosing $A = \mathbb{R}^n$, $a = k$ and $\psi(x, a) = k \cdot x$. In the coordinate invariant definition, the open set $A$ is used to parametrize the covectors $k_0$ of the wavefront set but its dimension $p$ is not necessarily equal to $n$. Still, this general definition is equivalent to the standard one (see [33, p 17], [34, p 542] and [62]).

*8.3.2. Pseudo-differential operators.* The original definition of the wavefront set was given by Hörmander in terms of pseudo-differential operators [63], [64, p 89]

$$\mathrm{WF}(u) = \bigcap\Big\{\mathrm{char}P;\ \ Pu \in C^\infty\big(\mathbb{R}^n\big)\Big\},$$

where $P$ runs over the pseudo-differential operators of all orders [64, p 85]. If $p_m(x, k)$ is the principal symbol of $P$, then $\mathrm{char}P = \{(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n/\{0\});\ \ p_m(x, k) = 0\}$ is the set of characteristic points of $P$ [64, p 87]. A proof of the equivalence with the other definitions can be found in [65, p 307] (see also [38, p 78]).

This definition can be extended to a refinement of the wavefront set for vector-valued distributions, called the *polarization set*, introduced by Dencker [66]. If $E \to M$ is a complex vector bundle of rank $r$ over the $n$-dimensional manifold $M$, then

$$\mathrm{WF}_{pol}(u) = \bigcap\Big\{N_P;\ \ Pu \in C^\infty(M)\Big\},$$

where, locally, $N_P = \{(x, k; w) \in \dot{T}^*M \times \mathbb{C}^r;\ \ w \in \mathrm{ker}p_m(x, k)\}$. Dencker used the polarization set to describe electromagnetic waves in uniaxial crystals [67].

*8.3.3. Wavelets and Co.* In the usual definitions of the wavefront set, the distribution $u$ is multiplied by a large family of test functions $f$ and the product is Fourier transformed. It is in fact possible to use a single function $f$ and to scale it. More precisely, let $f$ be an even Schwartz function that does not vanish at zero and, for any $\alpha$ with $0 < \alpha < 1$, form the family of Schwartz functions $f_t(y) = t^{\alpha n/2} f\big(t^\alpha(y - x)\big)$ for $t > 0$. Then, $(x, k)$ is not in the wavefront set of the tempered distribution $u$ iff there exists an open subset $U$ of $\mathbb{R}^n$ such that $\widehat{uf_t}(tq)$ is fast decreasing in the variable $t > 0$ uniformly in $q \in U$ [68, p 159]. This definition was first proposed by Córdoba and Fefferman for $\alpha = 1/2$ and $f$ a Gaussian function [69]. It is then similar to the FBI-transform (see [70] for a nice presentation and [71] for a geometric version).

Although wavelets cannot be used to measure the wavefront set because they are isotropic, some variants of them, known as curvelets [51], shearlets [72] or conical wavelets [73] provide an interesting resolution of the wavefront set.

## 9. Conclusion

We have presented a review of the various guises of the wavefront set. These different points of view should help grasp the meaning of this concept. We also proposed two new descriptions of the wavefront set of a characteristic distribution. Physically, we saw that the wavefront set is related to the fact that, in some directions, destructive interferences in Fourier space become weaker than for smooth functions. The wavefront set also describes the directions along which the singularities of the distribution propagate. We hope that we have convinced the reader that the wavefront set is a subtle but natural object. Its use is not limited to quantum field theory or many-body physics because, as stressed by Martinez, it is also related to the semi-classical limit [70, p 134].

In this introduction, we limited ourselves to the wavefront sets for the scalar field. For a Dirac or photon field, two definitions are possible: (i) the (standard) wavefront set of the propagator, which is the union of the wavefront sets of all its components in a given basis; (ii) the polarization set, described in section 8.3.2. The polarization set of the Dirac propagator was investigated in detail [10, 11]. The projection of the polarisation set on $\dot{T}^*M$ gives the (standard) wavefront set [66]. It is calculated for the Dirac field by Sanders [14]. Since the microlocal spectrum condition can be entirely defined in terms of the wavefront set (i), it turns out that the polarization set is not required in renormalized quantum field theory [74]. The polarization set for electromagnetic waves was discussed by Franco and Fagundes [73]. The wavefront set for gauge field propagators in the Feynman gauge is the same as that of the scalar field [17].

For pedagogical reasons, we focussed on the role of the wavefront set in the multiplication of distributions. However, the wavefront set also plays a crucial role to describe the propagation of singularites in quantum field theory [7] and the dispersion of waves [75], or to determine when a distribution can be pulled back by a smooth map [32]. Moreover, many proofs concerning distributions that are usually obtained by microlocal estimates can be derived more simply by considering the wavefront set [62].

It is ironic that, although the standard wavefront set is sufficient to build a quantum theory of gauge fields and gravitation, it is not enough to describe the optics of crystals (in particular the conical refraction). Higher order wavefront sets were proposed [76] to solve that problem.

Finally, note that we have restricted our discussion to the classical wavefront set. Many variations have been devised: analytic wavefront set (see [77] and [78] for a recent comparison of various definitions), homogeneous wavefront set [79], Gabor wavefront set [80], global wavefront set [81–83], discrete wavefont set [84], etc. Specific techniques can be applied to the (Sobolev) wavefront set of periodic distributions [85].

## Acknowledgments

We thank Etienne Balan for useful critical remarks and André Martinez for stressing the semi-classical aspect of the wavefront set.

## References

[1] Rivier D 1949 Une méthode d'élimination des infinités en théorie des champs quantifiés. Application au moment magnétique du neutron *Helv. Phys. Acta* **22** 265–318

[2] Stueckelberg E C G and Rivier D 1950 Causalité et structure de la matrice S *Helv. Phys. Acta* **23** 215–22
[3] Stueckelberg E C G and Green T A 1951 Elimination des constantes arbitraires dans la théorie des quanta *Helv. Phys. Acta* **24** 153–74
[4] Bogoljubow N N and Schirkow D W 1956 Probleme der Quantenfeldtheorie: II. Beseitigung der Divergenzen aus der Streumatrix *Fortschr. Phys.* **4** 438–517
[5] Bogoliubov N N and Shirkov D V 1959 *Introduction to the Theory of Quantized Fields* (New York: Interscience)
[6] Epstein H and Glaser V 1973 The role of locality in perturbation theory *Ann. Inst. Henri Poincaré* **19** 211–95
[7] Radzikowski M 1996 Micro-local approach to the Hadamard condition in quantum field theory on curved space-time *Commun. Math. Phys.* **179** 529–53
[8] Brunetti R and Fredenhagen K 2000 Microlocal analysis and interacting quantum field theories: renormalization on physical backgrounds *Commun. Math. Phys.* **208** 623–61
[9] Hollands S and Wald R M 2002 Existence of local covariant time ordered products of quantum fields in curved spacetime *Commun. Math. Phys.* **231** 309–45
[10] Kratzert K 2000 Singularity structure of the two point function of the free Dirac field on globally hyperbolic spacetime *Ann. Phys., Lpz.* **9** 475–98
[11] Hollands S 2001 The Hadamard condition for Dirac fields and adiabatic states on Robertson–Walker spacetimes *Commun. Math. Phys.* **216** 635–61
[12] D'Antoni C and Hollands S 2006 Nuclearity, local quasiequivalence and split property for Dirac quantum fields in curved spacetimes *Commun. Math. Phys* **261** 133–59
[13] Dappiaggi C, Hack T P and Pinamonti N 2009 The extended algebra of observables for Dirac fields and the trace anomaly of their stress-energy tensor *Rev. Math. Phys.* **21** 1241–312
[14] Sanders K 2010 The locally covariant Dirac field *Rev. Math. Phys.* **22** 381–430
[15] Rejzner K 2011 Fermionic fields in the functional approach to classical field theory *Rev. Math. Phys.* **23** 1009–33
[16] Zahn J 2014 The renormalized locally covariant Dirac field *Rev. Math. Phys.* **26** 1330012
[17] Hollands S 2008 Renormalized quantum Yang–Mills fields in curved spacetime *Rev. Math. Phys.* **20** 1033–172
[18] Fredenhagen K and Rejzner K 2012 Batalin–Vilkovisky formalism in the functional approach to classical field theory *Commun. Math. Phys.* **314** 93–127
[19] Fredenhagen K and Rejzner K 2013 Batalin–Vilkovisky formalism in perturbative algebraic quantum field theory *Commun. Math. Phys.* **317** 697–725
[20] Brunetti R, Fredenhagen K and Rejzner K 2013 Quantum gravity from the point of view of locally covariant quantum field theory, arXiv:1306.1058
[21] Brunetti R, Fredenhagen K and Köhler M 1996 The microlocal spectrum condition and Wick polynomials of free fields on curved spacetimes *Commun. Math. Phys.* **180** 633–52
[22] Verch R 1999 Wavefront sets in algebraic quantum field theory *Commun. Math. Phys.* **205** 337–67
[23] Hollands S and Ruan W 2003 The state space of perturbative quantum field theory in curved spacetimes *Ann. Inst. Henri Poincaré* **3** 635–57
[24] Sanders K 2010 Equivalence of the (generalised) Hadamard and microlocal spectrum condition for (generalised) free fields in curved spacetime *Commun. Math. Phys.* **295** 485–501
[25] Kay B S, Radzikowski M J and Wald R M 1997 Quantum field theory on spacetimes with compactly generated Cauchy horizon *Commun. Math. Phys.* **183** 533–56
[26] Fewster C J 2000 A general worldline quantum inequality *Class. Quantum Grav.* **17** 1897–911
[27] Pinamonti N 2011 On the initial conditions and solutions of the semiclassical Einstein equations in a cosmological scenario *Commun. Math. Phys.* **305** 563–604
[28] Ivrii V J 1977 Wave fronts of crystallo-optics system solutions *Sov. Math.—Dokl.* **18** 139–41
[29] Esser P 1987 Second analytic wave front set in crystal optics *Appl. Anal.* **24** 189–213
[30] Moreno C 1977 Spaces of positive and negative frequency solutions of field equations in curved space-times: I. The Klein–Gordon equation in stationary space-times *J. Math. Phys* **18** 2153–61
[31] Dimock J 1979 Scalar quantum field in an external gravitational field *J. Math. Phys.* **20** 2549–55
[32] Hörmander L 1990 The analysis of linear partial differential operators: I *Distribution Theory and Fourier Analysis* 2nd edn (Berlin: Springer)
[33] Duistermaat J J 1996 *Fourier Integral Operators* (Boston, MA: Birkhäuser)
[34] Chazarain J and Piriou A 1981 *Introduction à la théorie des équations aux dérivées partielles* (Paris: Gauthier-Villars)

[35] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics: II. Fourier Analysis, Self-adjointness* (New York: Academic)
[36] Friedlander F G and Joshi M 1998 *Introduction to the Theory of Distributions* 2nd edn (Cambridge: Cambridge University Press)
[37] Strichartz R S 2003 *A Guide to Distribution Theory and Fourier Transforms* (Singapore: World Scientific)
[38] Grigis A and Sjöstrand J 1994 *Microlocal Analysis for Differential Operators* (London: Cambridge University Press)
[39] Strohmaier A 2009 Microlocal analysis *Quantum Field Theory on Curved Spacetimes: Concepts and Mathematical Foundations* (*Lecture Notes in Physics* vol 786) ed C Bär and K Fredenhagen (Berlin: Springer) pp 113–24
[40] Eskin G 2011 *Graduate Studies in Mathematics* (*Lectures on Linear Partial Differential Equations* vol 123) (Providence, RI: American Mathematical Society)
[41] Wagschal C 2011 *Distributions, Analyse Microlocale, Equations aux dérivées partielles* (Paris: Hermann)
[42] Schwartz L 1954 Sur l'impossibilité de la multiplication des distributions *C. R. Acad. Sci. Paris* **239** 847–8
[43] Gel'fand I M and Shilov G E 1964 *Generalized Functions: vol I-Properties and Operations* (New York: Academic)
[44] Itzykson C and Zuber J B 1980 *Quantum Field Theory* (New York: McGraw-Hill)
[45] Peskin M E and Schroeder D V 1995 *An Introduction to Quantum Field Theory* (Reading, MA: Addison-Wesley)
[46] Franco D H T and Acebal J L 2007 Microlocal analysis and renormalization in finite temperature field theory *Int. J. Theor. Phys.* **46** 383–98
[47] Alinhac S and Gérard P 2007 Pseudo-differential operators and the Nash-Moser theorem (*Graduate Studies in Mathematics* vol 82) (Providence, RI: American Mathematical Society)
[48] Oberguggenberger M 1986 Products of distributions *J. Math.* **365** 1–11
[49] Oberguggenberger M 1992 *Multiplication of Distributions and Applications to Partial Differential Equations* (Harlow: Longman Scientific and Technical)
[50] Taylor M E 1981 *Pseudodifferential Operators* (Princeton, NJ: Princeton University Press)
[51] Candès E J and Donoho D L 2005 Continuous wavelet transform: I. Resolution of the wavefront set *Appl. Comput. Harmon. Anal.* **19** 162–97
[52] Xu Y, Wang L H V, Ambartsoumian G and Kuchment P 2009 Limited view thermoacoustic tomography *Photoacoustic Imaging and Spectroscopy* ed L H Wang (Boca Raton, FL: CRC Press) pp 61–73
[53] Dang N V 2014 The topological Radon transform, in preparation
[54] Scharf G 1995 *Finite Quantum Electrodynamics* 2nd edn (Berlin: Springer)
[55] Duistermaat J J and Hörmander L 1972 Fourier integral operators: II *Acta Math.* **128** 183–269
[56] Zhang H H, Feng K X, Qiu S W, Zhao A and Li X S 2010 On analytic formulas of Feynman propagators in position space *Chin. Phys.* C **34** 1576–82
[57] Stueckelberg E C G and Peterman A 1953 La normalisation des constantes dans la théorie des quanta *Helv. Phys. Acta* **26** 499–520
[58] Bogoliubov N N and Shirkov D W 1955 Problems of quantum field theory I (in Russian) *Usp. Fiz. Nauk* **55** 149–214
[59] Hollands S and Wald R M 2003 On the renormalisation group in curved spacetime *Commun. Math. Phys.* **237** 123–60
[60] Brunetti R, Dütsch M and Fredenhagen K 2009 Perturbative algebraic quantum field theory and the renormalization groups *Adv. Theor. Math. Phys.* **13** 1541–99
[61] Gabor A 1972 Remarks on the wave front of a distribution *Trans. Am. Math. Soc.* **170** 239–44
[62] Brouder C, Dang N V and Hélein F 2014 Boundedness and continuity of the fundamental operations on distributions having a specified wavefront set (with a counter-example by Semyon Alesker), in preparation
[63] Hörmander L 1971 Fourier integral operators. I *Acta Math.* **127** 79–183
[64] Hörmander L 1994 *The Analysis of Linear Partial Differential Operators: III. Pseudo-Differential Operators* (Berlin: Springer)
[65] Folland G B 1995 *Introduction to Partial Differential Equations* (Princeton, NJ: Princeton University Press)

[66] Dencker N 1982 On the propagation of polarization sets for systems of real principal type *J. Funct. Anal.* **46** 351–72
[67] Dencker N 1992 The propagation of polarization in double refraction *J. Funct. Anal.* **104** 414–68
[68] Folland G B 1989 *Harmonic Analysis in Phase Space* (Princeton, NJ: Princeton University Press)
[69] Córdoba A and Fefferman C 1978 Wave packets and Fourier integral operators *Commun. Partial Differ. Equ.* **3** 979–1005
[70] Martinez A 2002 *An Introduction to Semiclassical and Microlocal Analysis* (New York: Springer)
[71] Wunsch J and Zworski M 2000 The FBI transform on compact $C^\infty$ manifolds *Trans. Am. Math. Soc.* **353** 1151–67
[72] Grohs P 2012 Shearlets and microlocal analysis *Shearlets Multiscale Analysis for Multivariate Data* ed G Kutyniok and D Labate (New York: Springer) pp 39–67
[73] Franco D H T and Fagundes F N 2014 A note on directional wavelet transform and the propagation of polarization sets for solutions of Maxwell's equations *Rev. Math. Phys.* **26** 1430004
[74] Sahlmann H and Verch R 2001 Microlocal spectrum condition and Hadamard form for vector-valued quantum fields in curved spacetime *Rev. Math. Phys.* **13** 1203–46
[75] Ivanovici O 2014 Dispersion for the wave equation inside strictly convex domains I: the Friedlander model case *Ann. Math.* **180** 323–80
[76] Liess O 1993 *Conical Refraction and Higher Microlocalization* (*Lecture Notes in Mathematics* vol 1555) (Berlin: Springer)
[77] Strohmaier A, Verch R and Wollenberg M 2002 Microlocal analysis of quantum fields on curved spacetimes: Analytic wavefront sets and Reeh–Schlieder theorems *J. Math. Phys.* **43** 5514–30
[78] Harris B, He H Y and Ólafsson G 2014 Wave front sets of reductive Lie group representations, arXiv:1308.1863
[79] Nakamura S 2005 Propagation of the homogeneous wave front set for Schrödinger equations *Duke Math. J.* **126** 349–67
[80] Rodino L and Wahlberg P 2014 The Gabor wave front set *Mon. Math.* **173** 625–55
[81] Johansson K 2011 Association between temperate distributions and analytical functions in the context of wave-front sets *J. Pseudo-Differ. Operator Appl.* **2** 65–89
[82] Coriasco S, Johansson K and Toft J 2013 Global wave-front sets of Banach, Fréchet and modulation space types, and pseudo-differential operators *J. Differ. Equ.* **254** 3228–58
[83] Cappiello M and Schulz R 2014 Microlocal analysis of quasianalytic Gelfand-Shilov type ultradistributions, arXiv:1309.4236
[84] Johansson K, Pilipović S, Teofanov N and Toft J 2012 Gabor pairs, and a discrete approach to the wavefront set *Mon. Math.* **166** 181–99
[85] Maksimović S, Pilipović S, Sokoloski P and Vindas J 2014 Wave fronts via Fourier series coefficients, arXiv:1407.5155
[86] Bogoljubow N N and Schirkow D W 1955 Probleme der Quantentheorie der felder: I. Die Streumatrix *Fortschr. Phys.* **3** 439–95
[87] Bogoliubov N N 1995 *N. N. Bogoliubov Selected Works: IV. Quantum field theory* (Amsterdam: Gordon and Breach)