WICK SQUARES OF THE GAUSSIAN FREE FIELD AND RIEMANNIAN RIGIDITY.

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ABSTRACT. In this short note, we show that on a compact Riemannian manifold (M, g) of dimension $(d = 2, 3)$ whose metric has negative curvature, the partition function $Z_g(\lambda)$ of a massive Gaussian Free Field or the fluctuations of the integral of the Wick square $\int_M : \phi^2 : dv$ determine the lenght spectrum of (M, g) and imposes some strong geometric constraints on the Riemannian structure of (M, g). In any finite dimensional family of Riemannian metrics of negative sectional curvature, there is only a **finite number of isometry classes** of metrics with given partition function $Z_g(\lambda)$ or such that the random variable $\int_M : \phi^2 : dv$ has given law.

1. INTRODUCTION.

Let (M, g) be a smooth, closed, compact Riemannian manifold. Consider the Laplace–Beltrami operator Δ which admits a discrete spectral resolution [3, Lemma 1.6.3 p. 51] which means there is an increasing sequence of eigenvalues :

$$\sigma(\Delta) = \{0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \to +\infty\}$$

and corresponding L^2-basis of eigenfunctions $(e_\lambda)_\lambda$ so that $\Delta e_\lambda = \lambda e_\lambda$.

1.0.1. *Gaussian Free Fields.* We next briefly recall the definition of the Gaussian free field (GFF) associated to Δ. Our definition is probabilistic and represents the Gaussian Free Field ϕ, also called bosonic Euclidean quantum field in the physics litterature, as a random distribution on M [4, Corollary 3.8 p. 21] [5, eq (1.7) p. 3] [6] (see also [7, section 4.2] for a related definition in a planar domain D).

Definition 1.1 (Gaussian Free Field)**.** *The Gaussian free field ϕ associated to (M, g) is defined as follows : denote by $(e_\lambda)_{\lambda\in\sigma(\Delta)}$ the spectral resolution of Δ. Consider a sequence $(c_\lambda)_{\lambda\in\sigma(\Delta)}, c_\lambda \in \mathcal{N}(0, 1)$ of independent, identically distributed, centered Gaussian random variables. Then we define the Gaussian Free Field ϕ as the random series*

$$\phi = \sum_{\lambda\in\sigma(\Delta)\setminus\{0\}} \frac{c_\lambda}{\sqrt{\lambda}} e_\lambda \tag{1.1}$$

where the sum runs over the positive eigenvalues of Δ and the series converges almost surely as distribution in $\mathcal{D}'(M)$.

The covariance of the Gaussian free field defined above is the Green function :

$$\boxed{\mathbf{G}(x, y) = \sum_{\lambda\in\sigma(\Delta)\setminus\{0\}} \frac{1}{\lambda} e_\lambda(x)e_\lambda(y)}$$

where the above series converges in $\mathcal{D}'(M \times M)$.

We next recall the definition of polygon Feynman amplitudes.

Definition 1.2 (Feynman amplitudes)**.** *Let (M, g) be a closed compact Riemannian manifold and* $\mathbf{G}$ *the Green function of the Laplace–Beltrami operator Δ.*

Define the formal product of Green function :

$$t_n = \mathbf{G}(x_1, x_2) \ldots \mathbf{G}(x_n, x_1) \tag{1.2}$$

as an element in $C^\infty(M^n \setminus \text{diagonals})$.

1.0.2. *From TQFT to Riemann invariants.* In topological field theories of Chern–Simons [8, 9, 10] and of BF type [11, 12, 13] [14, 3.4], one has a correspondance :

$$\boxed{\text{closed manifolds} \longrightarrow \text{partition function } Z\left(M\right)=\sum_{n=0}^{\infty}h^nF_n(M)} \tag{1.3}$$

where the $F_n\left(M\right)$ are invariants of the C^∞-structure and do not depend on the choice of metrics needed to define the propagator of the theory. For non topological theories, it was proved by Belkale–Brosnan [15] and Bogner–Weinzierl [16] that Feynman amplitudes are special numbers called periods. As a consequence of the quantum field theory formalism of Segal [19], Stolz–Teichner [20, 21, 22], a QFT should give a correspondance from closed manifolds endowed with extra Riemannian or complex structure to complex numbers. On Riemannian manifolds, numbers of QFT might become sensitive to variations of the metric g and have no reasons to be periods anymore. The goal of the present paper is to study the dependence of the partition function Z in the Riemannian metric g.

1.0.3. *Motivations of our results.* In the present paper, we shall study the renormalized partition function

$$Z_g\left(\lambda\right)=\mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M :\phi^2(x): dv(x)\right)\right)\in\mathbb{C}[[\lambda]] \tag{1.4}$$

of a free bosonic theory when the dimension of (M,g) equals $2\leqslant d\leqslant 4$ where we need some extra renormalization when $d=4$, see Proposition 1.4. The partition function $Z_g(\lambda)\in\mathbb{C}[[\lambda]]$ depends only on the isometry class of (M,g) and a natural question would be what informations on (M,g) can be extracted from $Z_g\left(\lambda\right)$ as a formal power series.

We can also formulate a related question as follows : let $\varphi : M\mapsto M$ be a diffeomorphism and let $\mathbf{G}$ be the Green function of Δ_g. (M,φ^*g) is **isometric** to (M,g) and induces a diffeomorphism $\Phi : M\times M\mapsto M\times M$ such that the pull–back $\Phi^*\mathbf{G}\in\mathcal{D}'(M\times M)$ is the Green function of the Laplace–Beltrami operator Δ_{Φ^*g}. It follows that integrals of non divergent Feynman amplitudes are **isometry invariant** numbers and depend only on the Riemannian structure. What informations on the Riemannian structure (M,g) can be recovered from integrals of Feynman amplitudes over configuration space ? In what follows, we introduce some preliminary definitions before we state our two main results on Riemannian rigidity from quantum fields.

1.0.4. *The moduli space of metrics.* The set of C^∞ Riemannian metrics on M with the usual Fréchet topology on smooth 2-tensors is denoted by $\mathbf{Met}(M)$ [1]. We have the natural action of $\mathbf{Diff}(M)$, the set of diffeomorphisms of M acting by pull–back on $\mathbf{Met}(M)$, then we define the **moduli space of Riemannian metrics** as a quotient space :

$$\boxed{\mathcal{R}(M)=\mathbf{Met}(M)/\mathbf{Diff}(M)} \tag{1.5}$$

where a sequence of isometry classes $[g_n]\underset{n\rightarrow+\infty}{\rightarrow}[g]$ if there is a sequence of representatives g_n of $[g_n]$ which converges to g in the C^∞–topology [23, p. 602] [24, p. 233] (see also [25, p. 175]).

1.0.5. *Fluctuations of the integrated Wick square.* In quantum field theory on curved space times, one is interested in the behaviour of the stress–energy tensor and its fluctuations under quantization of the fields the metric stays classical. For instance, many works of Moretti [26, 27, 28, 29, 30] deal with the renormalization of various quantum field theoretic quantities, for instance the stress–energy tensor, using zeta regularization and local point splitting methods. In the present paper, we study fluctuations of the integral of the Wick square $\int_M :\phi^2(x): dv(x)$ on the manifold M which is a simpler observable and is the integral of the *field fluctuations* in Moretti's work. In aQFT, it also appears in the work of Sanders [31] and is interpreted as a local temperature.

In probability, the Wick square is also related to loop measures associated to some random walks on graphs [32, 33] and it is assumed that the continuous Wick square should be related to some loop measures.

[1] it is an open convex cone of the space of 2–tensors

On Riemannian manifolds of negative curvature, there is a strong analogy between Brownian motion on the base manifold M, the continuous version of random walks, and the geodesic flow on the unitary cosphere bundle S^*M over M. Our main results, Proposition 1.4 and Theorem 1, give an explicit relation between fluctuations of the Wick squares, the partition functions $Z_g(\lambda)$, periodic geodesics and rigidity on manifolds with negative curvature.

1.0.6. *Periods of the geodesic flow.* We recall the definition of the periods of the geodesic flow [25, section 10.5].

Definition 1.3 (Periods)**.** *Let us consider the moduli space of Riemannian metrics $\mathcal{R}(M)$ on some smooth, closed, compact manifold M. For every element of $\mathcal{R}(M)$, choose a representative (M,g). We denote by $(\Phi^t)_t : S^*M \mapsto S^*M$ the geodesic flow acting on the unitary cosphere bundle S^*M. Then for every class $[(M,g)] \in \mathcal{R}(M)$, we define the **periods** $\mathcal{P}(g)$ as the set :*

$$\boxed{\mathcal{P}(g) = \{T > 0 \text{ s.t. } \Phi^T(x;\xi) = (x;\xi) \text{ for } (x;\xi) \in S^*M\} \subset \mathbb{R}_{>0}.} \tag{1.6}$$

The set $\mathcal{P}(g)$ is called the **lenght spectrum** of (M,g).

1.1. **Main results.** Recall we defined the formal product t_n of Green functions in definition 1.2. For a compact operator A, we will denote by $\sigma(A)$ the set of singular values of A. Our first result reads :

Proposition 1.4. *Given a closed compact Riemannian manifold (M,g) of dimension $2 \leqslant d \leqslant 4$, a function $V \in C^\infty(M)$, define the sequence of numbers $c_n(g) = \int_{M^n} t_n(x_1,\dots,x_n)V(x_1)\dots V(x_n)dv_n$, where dv_n is the Riemannian density in $|\Lambda^{top}|M^n$. Let $\phi_\varepsilon = e^{-\varepsilon\Delta}\phi$ be the heat regularized GFF, $:\phi_\varepsilon^2(x) := \phi_\varepsilon^2(x) - \mathbb{E}\left(\phi_\varepsilon^2(x)\right)$ and define the renormalized partition functions :*

$$\begin{aligned} Z_g(\lambda) &= \lim_{\varepsilon\to 0^+} \mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x) :\phi_\varepsilon^2(x): dv\right)\right), \text{ when } d = (2,3), \\ Z_g(\lambda) &= \lim_{\varepsilon\to 0^+} \mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x) :\phi_\varepsilon^2(x): dv - \frac{\lambda^2\int_M V^2(x)dv}{128\pi^2}|\log(\varepsilon)|\right)\right), \text{ when } d = 4. \end{aligned}$$

Then the sequence $c_n(g)$ is well–defined for $n > \frac{d}{2}$ and the partition functions Z_g satisfies the following identity for small $|\lambda|$:

$$Z_g(\lambda) = \exp\left(P(\lambda) + \sum_{n>\frac{d}{2}} \frac{(-1)^n c_n(g)\lambda^n}{2n}\right) \tag{1.7}$$

*where P is a polynomial of degree 2 in λ, $P = 0$ when $d < 4$ and Z_g^{-2} extends as an **entire function** on the complex plane $\mathbb{C}$ whose zeroes lie in $-\sigma(V\Delta^{-1})$.*

Note that $V\Delta^{-1} \in \Psi^{-2}(M)$ is a pseudodifferential operator of negative degree hence is a compact operator and $\sigma(\Delta^{-1}V)$ is well–defined. At this point, it was pointed out to the author by Claudio Dappiaggi that there should be some explicit relation between the renormalization done here and the methods from the papers [17, 34, 18] on Euclidean algebraic Quantum Field Theory which uses an Euclidean version of Epstein–Glaser renormalization. From the above, we deduce the following corollaries when $V = 1 \in C^\infty(M)$:

Corollary 1.5. *Let $(M_1,g_1),(M_2,g_2)$ be a pair of compact Riemannian manifolds without boundary of dimension $2 \leqslant d \leqslant 4$, then the following claims are equivalent :*

1. *$c_n(g_1) = c_n(g_2)$ for all $n > \frac{d}{2}$,*
2. *the partition functions coincide $Z_{g_1} = Z_{g_2}$,*
3. *$(M_1,g_1),(M_2,g_2)$ are **isospectral**.*

In particular the Einstein–Hilbert action, hence $\chi(M)$ when M is a surface, is determined by Z_g by the formula :

$$\boxed{S_{EH}(g) = Res|_{s=\frac{d}{2}-1} \sum_{\lambda, Z_g(\lambda)^{-2}=0} \lambda^{s-1}.}$$

and if (g_1, g_2) are metrics with negative sectional curvatures, then $\mathcal{P}(g_1) = \mathcal{P}(g_2)$ where the lenght spectrum is the singular support of the distribution :

$$\boxed{t \mapsto \sum_{\lambda, Z_g(\lambda)^{-2}=0} e^{it\sqrt{\lambda}} \in \mathcal{D}'(\mathbb{R}_{>0}).} \tag{1.8}$$

We next give the main Theorem of our note which deals with the rigidity of the Riemannian structure in negative curvature where the fluctuations of the Wick square are encoded by the probability law of the random variable $\int_M :\phi^2(x): dv$.

Theorem 1. *Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $d = (2, 3)$, ϕ is the Gaussian free field with covariance $\mathbf{G}$ with corresponding measure μ. Denote by $\phi_\varepsilon = e^{-\varepsilon\Delta}\phi$ to be the heat regularized GFF.*

Then the limit $\int_M :\phi^2(x): dv(x) = \lim_{\varepsilon\to 0^+} \int_M \phi_\varepsilon^2(x)dv(x) - \mathbb{E}\left(\int_M \phi_\varepsilon^2(x)dv(x)\right)$ converges as a random variable in $L^p(\mathcal{D}'(M), \mu), p \geqslant 2$ with the following properties :

1. *Let N be a* ***non necessarily compact, finite dimensional*** *submanifold of* **Met** (M)*. Then the set of metrics $g \in N \cap \mathcal{R}(M)_{<0}$ such that the random variable $\int_M :\phi^2(x): dv$* ***has given law is finite****.*
2. *When $d = 3$, for a sequence $(M_i, g_i)_{i\in\mathbb{N}}$ of Riemannian 3–manifolds of negative curvature such that the random variable $\int_M :\phi^2(x): dv$* ***has a fixed given law****, one can extract a subsequence such that M_i has* ***fixed diffeomorphism type*** *and $g_i \to g$ to some metric g in the C^∞ topology.*

The above two results hold true for manifolds (M, g) with given partition function $Z_g(\lambda)$.

Our result gives an example of metric dependent (non topological) Quantum Field Theory where the knowledge of the partition function gives both some **topological and metrical** constraints on the Riemannian manifold (M, g).

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2. Proof of Proposition 1.4.

The results of Proposition 1.4 are particular cases of the main results from [35]. However, for the sake of clarity, we shall give a self–contained proof which is simpler in our case since we work with scalar fields in low dimension $d \leqslant 4$.

We first discuss the case of dimension $d = (2, 3)$. Start from the relation [36, Prop 9.3.1 p. 211]

$$\mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x) :\phi_\varepsilon^2(x): dv(x)\right)\right) = \exp\left(-\frac{1}{2}Tr_{L^2}\left(\log\left(I + \lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right) - \lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right)\right)$$

which holds true for small $|\lambda|$ since $e^{-2\varepsilon\Delta}\Delta^{-1}V \in \Psi^{-\infty}(M)$ is smoothing. Then by Lemma 5.1, there is an explicit relation connecting Fredholm determinants $\det_F$, Gohberg–Krein's determinants $\det_2$ and functional traces (see also [36, p. 212]), this relation reads :

$$\begin{aligned} \exp\left(-\frac{1}{2}Tr_{L^2}\left(\log\left(I + \lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right) - \lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right)\right) &= \det_F\left(I + \lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^{-\frac{1}{2}} \exp\left(\frac{\lambda}{2}Tr_{L^2}(e^{-2\varepsilon\Delta}\Delta^{-1}V)\right) \\ &= \det_2\left(I + \lambda e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^{-\frac{1}{2}} \end{aligned}$$

which follows immediately from the properties of Gohberg–Krein's determinants $\det_2$.

Observe that the function $p_t : \xi \in \mathbb{R} \mapsto e^{-t|\xi|^2}$ defines a family $(p_t)_{t\in[0,+\infty)}$ of symbols in $S^0_{1,0}(\mathbb{R})$ such that $p_t \underset{t\to 0}{\to} 1$ in $S^{+0}_{1,0}(\mathbb{R})$ where $p \in S^0_{1,0}(\mathbb{R})$ iff $|\partial^j_\xi p(\xi)| \leqslant C_j\,(1+|\xi|)^{-j}$ [37, Lemm 1.2 p. 295]. Indeed, for all $k\in\mathbb{N}$, $(t\xi)^k e^{-t\xi^2}$ reaches its maximum when $\frac{d}{dt}\left((t\xi)^k e^{-t\xi^2}\right) = (k-t\xi^2)\xi(t\xi)^{k-1}e^{-t\xi^2} = 0$ hence for $t=\frac{k}{\xi^2}$, $\sup_t (t\xi)^k e^{-t\xi^2} = (\frac{k}{\xi})^k e^{-k}$. Then by a simple induction on the number of derivatives, we can show that $(1+|\xi|)^j|\partial^j_\xi e^{-t\xi^2}| \leqslant C_j$ uniformly on t, hence $p_t\in S^0_{1,0}$ uniformly on t. We also have for all $\delta>0$, $t\leqslant \frac{\delta^{1+2\varepsilon}}{200}$ implies that $\sup_\xi |(1+|\xi|)^{-\varepsilon}(e^{-t\xi^2}-1)| \leqslant \delta$ which means that $\sup_\xi |(1+|\xi|)^{-\varepsilon}(e^{-t\xi^2}-1)| \to 0$ when $t\to 0^+$ which implies the convergence $p_t\to 1$ in $S^{+0}_{1,0}$. By a result of Strichartz [37, Thm 1.3 p. 296],

$$p_t(\sqrt{\Delta}) = e^{-t\Delta} \underset{t\to 0^+}{\to} Id \text{ in } \Psi^{+0}_{1,0}(M). \tag{2.1}$$

By composition of pseudodifferential operators, we find that $e^{-2\varepsilon\Delta}\Delta^{-1}V \underset{\varepsilon\to 0}{\to} \Delta^{-1}V$ in the space $\Psi^{-2+0}(M)$ of pseudodifferential operators of order $-2+\varepsilon, \forall\varepsilon>0$ which implies that the convergence occurs in the ideal $\mathcal{I}_2$ of Hilbert–Schmidt operators by [38, Prop B 20]. By continuity of $H\in\mathcal{I}_2 \mapsto \det_2(I+H)$, we find that

$$Z_g(\lambda) = \lim_{\varepsilon\to 0^+} \mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x) : \phi^2_\varepsilon(x) : dv\right)\right) = \det_2\left(I+\lambda\Delta^{-1}V\right)^{-\frac{1}{2}}. \tag{2.2}$$

From the properties of $\det_2\left(I+\lambda\Delta^{-1}V\right)$ recalled in Lemma 5.1, we find that the divisor of Z_g^{-2} coincides with the subset $\{z \text{ s.t. } z\lambda=-1, \lambda\in\sigma(\Delta^{-1}V)\}\subset\mathbb{C}$ hence when $V=1$, the partition function Z_g determines the spectrum $\sigma(\Delta)$ of the Laplace–Beltrami operator Δ.

2.1. **Explicit counterterms in dimension** $d\leqslant 4$**.** When $d=(2,3)$, Δ^{-1} is only Hilbert–Schmidt but not trace class and we only need the Wick renormalization to renormalize the partition function which is exactly what is done by considering Gohberg–Krein's renormalized determinant $\det_2$. When $d=4$, for small $|\lambda|$, we have the series expansion :

$$\log\mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x) : \phi^2_\varepsilon(x) : dv(x)\right)\right) = \frac{-1}{2}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\lambda^k}{k}Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^k\right) - \lambda Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)\right)$$

$$= \frac{-1}{2}\left(\frac{-\lambda^2}{2}Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^2\right) + \sum_{k=3}^{\infty}\frac{(-1)^{k+1}\lambda^k}{k}Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^k\right)\right)$$

where we need to renormalize both terms $Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)$ and $Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^2\right)$ since for all $k\geqslant 3$, equation 2.1 implies $\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^k \underset{\varepsilon\to 0^+}{\to} \left(\Delta^{-1}V\right)^k \in \Psi^{-2k}(M)$ which are trace class. The first term $Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)$ is subtracted by Wick renormalization but we have the term $Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^2\right)$ left. We shall use pseudodifferential calculus to extract the singular parts of this term. The second term can be arranged

$$Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^2\right) = Tr_{L^2}\left(e^{-4\varepsilon\Delta}\Delta^{-2}V^2\right) + Tr_{L^2}\left(e^{-2\varepsilon\Delta}\Delta^{-1}[V, e^{-2\varepsilon\Delta}\Delta^{-1}]V\right),$$

where the family of heat operators $(e^{-\varepsilon\Delta})_{\varepsilon\in[0,1]}$ is bounded in $\Psi^0(M)$ by equation 2.1, the commutator term $[V, e^{-2\varepsilon\Delta}\Delta^{-1}]$ is therefore bounded in $\Psi^{-3}(M)$ **uniformly** in the parameter $\varepsilon\in[0,1]$. By composition in the pseudodifferential calculus, we thus find that $e^{-2\varepsilon\Delta}\Delta^{-1}[V, e^{-2\varepsilon\Delta}\Delta^{-1}]V \in \Psi^{-5}(M)$, uniformly in $\varepsilon\in[0,1]$ and is therefore of trace class by Proposition [38, Prop B 20] and since we are in dimension $d=4$, uniformly in the small parameter $\varepsilon\in[0,1]$.

Finally, we found that $Tr_{L^2}\left(\left(\Delta^{-1}e^{-\varepsilon\Delta}Ve^{-\varepsilon\Delta}\right)^2\right) = Tr_{L^2}\left(e^{-4\varepsilon\Delta}\Delta^{-2}V^2\right) + \mathcal{O}(1)$. Then the singular part of $Tr_{L^2}\left(e^{-4\varepsilon\Delta}\Delta^{-2}V^2\right)$ is easily extracted using the heat kernel asymptotic expansion [39] as :

$$\begin{aligned} Tr_{L^2}\left(e^{-4\varepsilon\Delta}\Delta^{-2}V^2\right) &= \frac{1}{2}\int_0^1 Tr_{L^2}\left(e^{-(4\varepsilon+t)\Delta}V^2\right)t^{2-1}dt + \mathcal{O}(1) = \frac{1}{2(4\pi)^2}\int_0^1 \frac{t}{(4\varepsilon+t)^2}dt\int_M V^2(x)dv + \mathcal{O}(1) \\ &= \frac{\int_M V^2(x)dv}{32\pi^2}\int_{4\varepsilon}^{1+4\varepsilon}(u^{-1}-4\varepsilon u^{-2})du + \mathcal{O}(1) = \frac{\int_M V^2(x)dv(x)}{32\pi^2}|\log(\varepsilon)| + \mathcal{O}(1). \end{aligned}$$

We conclude by the observation that for small $|\lambda|$:

$$\begin{aligned} Z_g(\lambda) &= \lim_{\varepsilon\to 0^+}\mathbb{E}\left(\exp\left(-\frac{\lambda}{2}\int_M V(x):\phi_\varepsilon^2(x):dv - \frac{\lambda^2\int_M V^2(x)dv}{128\pi^2}|\log(\varepsilon)|\right)\right) \\ &= \lim_{\varepsilon\to 0^+}\exp\left(\underbrace{\frac{\lambda^2}{4}Tr_{L^2}\left(\left(\Delta^{-1}e^{-2\varepsilon\Delta}V\right)^2\right) - \frac{\lambda^2\int_M V^2(x)dv}{128\pi^2}|\log(\varepsilon)|}_{\mathcal{O}(1)} + \sum_{k=3}^{\infty}\frac{(-1)^k\lambda^k}{2k}Tr_{L^2}\left(\left(e^{-2\varepsilon\Delta}\Delta^{-1}V\right)^k\right)\right) \\ &= \lim_{\varepsilon\to 0^+}\exp\left(\frac{\lambda^2}{4}Tr_{L^2}\left(\left(\Delta^{-1}e^{-2\varepsilon\Delta}V\right)^2\right) - \frac{\lambda^2\int_M V^2(x)dv}{128\pi^2}|\log(\varepsilon)|\right)\det{}_3\left(I+\lambda\Delta^{-1}e^{-2\varepsilon\Delta}V\right)^{-\frac{1}{2}} \\ &= e^{P(\lambda)}\det{}_3\left(I+\lambda\Delta^{-1}V\right)^{-\frac{1}{2}} \end{aligned}$$

where we recognized Gohberg–Krein's renormalized determinant $\det_3$ which converges since $\Delta^{-1}e^{-2\varepsilon\Delta}V \underset{\varepsilon\to 0^+}{\to} \Delta^{-1}V \in \Psi^{-2}(M)$ hence in the Schatten ideal $\mathcal{I}_3$ and P is a polynomial of degree 2.

Now we conclude similarly as for dimension $d=(2,3)$, $\det_3\left(I+\lambda\Delta^{-1}V\right)$ vanishes with multiplicity on the set $-\sigma(\Delta^{-1}V)$ which implies that Z_g determines $\sigma(\Delta)$ when $V=1$.

2.1.1. *Conclusion of the proof.* The proof of identity

$$Z_g(\lambda) = \exp\left(P(\lambda) + \sum_{n>\frac{d}{2}}\frac{(-1)^n c_n(g)\lambda^n}{2n}\right) \tag{2.3}$$

follows immediately from the fact that for $n > \frac{d}{2}$, composition in the pseudodifferential calculus implies that $(\Delta^{-1}V)^n \in \Psi^{-2n}(M)$ is trace class hence the integrals $c_n(g) = \int_{M^n} t_n(x_1,\dots,x_n)V(x_1)\dots V(x_n)dv_n$ are convergent and equal to $Tr_{L^2}\left((\Delta^{-1}V)^n\right)$. The conclusion follows from the relation of Gohberg–Krein's determinants $\det_p$ with functional traces summarized in Lemma 5.1.

3. Proof of Corollary 1.5.

Corollary 1.5 is an immediate consequence of Proposition 1.4 and of the deep Theorem of Colin de Verdière [1, 2], Duistermaat–Guillemin [40, Thm 4.5 p. 60] relating the spectrum of the Laplacian and the lenght spectrum. We recall, in the particular case of metrics with negative curvature,

Theorem 2 (Trace formula)**.** *Let (M,g) be a smooth compact Riemannian manifold with negative sectional curvatures and Δ_g the Laplace-Beltrami operator. Then the spectrum $\sigma(\Delta_g)$ determines the non marked lenght spectrum by the* ***trace formula*** *:*

$$2Re\left(\sum_{\lambda\in\sigma(\Delta_g)} e^{i\sqrt{\lambda}t}\right) = \sum_\gamma \frac{\ell_\gamma}{m_\gamma|\det(I-P_\gamma)|^{\frac{1}{2}}}\delta(t-\ell_\gamma) + L^1_{loc}, \tag{3.1}$$

[2] *where* ℓ_γ, m_γ *are the period and multiplicity of the orbit* γ *and* P_γ *is the Poincaré return map. Furthermore, the singularities of the wave trace equals the lenght spectrum :*

$$ss\left(2Re\left(\sum_{\lambda\in\sigma(\Delta_g)} e^{i\sqrt{\lambda}t}\right)\right) = \{\ell_\gamma | [\gamma]\in\pi_1(M)\} \tag{3.2}$$

which implies the Laplace spectrum $\sigma(\Delta_g)$ *determines the lenght spectrum of* (M,g).

For geodesic flows in negative curvature, the set of periods forms a discrete subset of $\mathbb{R}_{>0}$ hence each period is isolated and the corresponding periodic orbits are isolated and in finite number. In the notation of [40], each Z_j has dimension 1 and $d_j = 1$. In that case, Duistermaat–Guillemin (see also [41, Thm 3 p. 195]) give a leading term for the real part of the distributional flat trace $2Re\left(Tr^\flat(U(.))\right)\in\mathcal{D}'(\mathbb{R}_{>0})$ of the wave propagator $U(t)=e^{it\sqrt{\Delta_g}}$:

$$2Re\left(Tr^\flat(U(.))\right) = \sum_{[\gamma]\in\pi_1(M)} \frac{i^{-\sigma_\gamma}\ell_\gamma e^{-iT.\gamma}}{m_\gamma|\det(I-P_\gamma)|^{\frac{1}{2}}}\delta(t-\ell_\gamma) + L^1_{\text{loc}}.$$

In case the metric has negative curvature, each closed geodesic make a non-zero contribution to the singular support of U since the Maslov index $\sigma_\gamma = 0$ for all γ as noted in [42, Coro 1.1 p. 73] and the subprincipal symbol of $\sqrt{\Delta_g}$ vanishes hence the term $e^{-iT.\gamma}=1$ which gives the desired result.

The identity

$$S_{EH}(g) = Res|_{s=\frac{d}{2}-1}\sum_{\lambda, Z_g(\lambda)^{-2}=0}\lambda^{s-1}$$

follows immediately from the spectral interpretation of the integral of the scalar curvature (Einstein–Hilbert action) [43, Thm 6.1 p. 26]. Let us briefly recall the principle of this derivation. The first heat invariant of the scalar Laplacian is directly related to the scalar curvature, for $Re(s)>\frac{d}{2}$, the sum $\sum_{\lambda\in\sigma(\Delta),\lambda>0}\lambda^{-s}$ converges by Weyl's law and coincide with $Tr_{L^2}\left(\Delta^{-s}\right)$. By the heat kernel expansion, the trace $Tr_{L^2}\left(\Delta^{-s}\right)$ admits an analytic continuation as a meromorphic function whose poles at $s=\frac{d}{2}-1$ are related to the first heat invariant.

4. Proof of Theorem 1.

4.0.2. *Existence of Wick square as random variable.* We use a very nice result on Gaussian measures that can be found in Glimm–Jaffe [36, Prop 9.3.1 p. 211]. In dimension $d=(2,3)$, the operator $\Delta^{-\frac{1}{2}}V\Delta^{-\frac{1}{2}}\in\Psi^{-2}(M)$ is Hilbert–Schmidt and therefore the Wick renormalized functional $\int_M : V\phi^2(x) : dv$ is a **well–defined random variable** in all $L^p(\mathcal{D}'(M),\mu), p\in[2,+\infty)$ where μ is the Gaussian Free Field measure on $\mathcal{D}'(M)$ with covariance Δ^{-1}.

4.0.3. *Spectrum of* Δ *and the law of the Wick square* $\int_M : V\phi^2(x) : dv$. Furthermore, the law of the random variable $\int_M : V\phi^2(x) : dv$, more precisely its moments are related to the partition function $Z_g(\lambda)$ by the observation that the series

$$Z_g(\lambda) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\mathbb{E}\left(\left(\int_M : V\phi^2(x) : dv\right)^n\right)$$

converges absolutely for $\|\lambda\Delta^{-\frac{1}{2}}V\Delta^{-\frac{1}{2}}\|_{\mathcal{I}_2} < 1$ where $\|.\|_{\mathcal{I}_2}$ is the Hilbert–Schmidt norm. Therefore by Proposition 1.4, the law of $\int_M : V\phi^2(x) : dv$ determines Z_g and its zeroes hence the spectrum of Δ when $V=1$. Therefore the second claim from Theorem 1 follows from the rigidity results of Anderson [44] for isospectral manifolds of dimension 3 s.t. the lenght of its shortest geodesic is bounded from below, see Theorem 5.

[2] which is an equality in the sense of distributions in $\mathcal{D}'(\mathbb{R}_{>0})$

4.1. **Rigidity in negative curvature.** To conclude the proof of Theorem 1, it remains to show that :

Proposition 4.1. *Let M be a smooth closed compact manifold and $\mathcal{N}$ be some finite dimensional submanifold in $\textbf{Met}(M)$ which is not necessarily compact. Set $\varepsilon > 0$ and consider the set $\mathcal{R}(M)_{\leqslant -\varepsilon}$ of isometry classes of metrics with negative curvature $\mathfrak{K}_g \leqslant -\varepsilon < 0$. Set $\tilde{\mathcal{N}}$ to be the image of $\mathcal{N}$ under the quotient map $\textbf{Met}(M) \mapsto \mathcal{R}(M)$. Then the set of isospectral metrics in $\tilde{\mathcal{N}} \cap \mathcal{R}(M)_{\leqslant -\varepsilon}$ is finite.*

We prove Proposition 4.1 by giving a simple adaptation of a result due to Sarnak [23] in dimension 2 and Sharafutdinov [45] for hyperbolic metrics that in dimensions $d = (2, 3)$, for a finite dimensional manifold of metrics of negative curvature, there are only a **finite number of isospectral metrics**. The next paragraph is devoted to the first ingredients of our proof which are results from Croke–Sharafutdinov on the decomposition of metrics as sum of a solenoidal and potential part.

4.1.1. *Space of metrics.* We work on a smooth closed compact manifold M of dimension $d = 2, 3$. The convergence of isometry classes $[g_n] \to [g]$ means that there is a sequence of representatives $g_n \to g$ in the C^∞ topology for 2-tensors. It was proved by Croke–Sharafutdinov [46, Thm 2.2 p. 1269] [47, Thm 3.8 p. 26] that

Theorem 3. *Let (M, g) be a compact Riemannian manifold s.t. the geodesic flow on S^*M is Anosov. Then every symmetric 2–tensor $T \in C^\infty\left(S^2T^*M\right)$ admits the following **unique decomposition***

$$T = T^s + D\theta \tag{4.1}$$

where T^s is the g–solenoidal part of the tensor [3] *and $D\theta = \sigma\nabla\theta$ is the* potential part *where $\theta \in C^\infty(T^*M)$ is a 1–form, ∇ is the covariant derivative w.r.t. g and σ is the symmetrization operator* [46, p. 1267-1268].

For any purely *potential* 2-tensor $f = D\theta$, it is proved [47, Prop 3.10 p. 28] that $\pi_2^*(D\theta) = X(\pi_1^*\theta)$ where X is the Lie derivative along the generator of the geodesic flow on SM and (π_1^*, π_2^*) are the natural operators acting on 1 and 2-tensors which lift them to functions on the sphere bundle SM. We have the following [48, Thm 2.1] :

Lemma 4.2. *Let M be a smooth closed compact manifold. For any metric g_0 of strictly negative curvature* [4]*, there exists a neighborhood $\mathcal{U}$ of g_0 in $C^\infty(S^2T^*M)$ such that for any $g \in \mathcal{U}$, there is a metric $g' = \Phi^*g$ isometric to g such that $g - g_0$ is **solenoidal** w.r.t g_0.*

Let us explain the idea behind the Lemma as explained to me by Thibault Lefeuvre. One tries to solve the equation $D_{g_0}\Phi^*g' = 0$ with the implicit function Theorem. Intuitively, the picture one should have in mind is that in the space $\textbf{Met}(M)$ of metrics (viewed as an open cone of the space of 2–tensors hence as a Fréchet manifold), the tangent space $T_{g_0}\textbf{Met}(M)$ to g_0 admits the decomposition

$$T_{g_0}\textbf{Met}(M) = \text{solenoidal tensors for } g_0 \oplus \text{potential tensors for } g_0.$$

The space of potential tensors for g_0 is precisely the tangent part to the orbit through g_0 of the action of the group of diffeomorphisms which is $T_{g_0}(\textbf{Diff}(M).g_0)$. Hence starting from g_0 and adding a small solenoidal part exactly means moving in the transversal direction to the orbits of $\textbf{Diff}(M)$ which means after projection that we are moving in the quotient space $\mathcal{R}(M) = \textbf{Met}(M)/\textbf{Diff}(M)$.

4.1.2. *The compactness results.* The second ingredient of our proof of Proposition 4.1 uses compactness results on the space of isospectral metrics. Note that two isospectral Riemannian surfaces (M_1, g_1) and (M_2, g_2) have the same genus since the second heat invariant $a_1 = \frac{1}{6}\int_M \mathfrak{K}_g$, which is also a spectral invariant, is proportional to the integral of the scalar curvature $\mathfrak{K}_g$ on M hence it determines the Euler characteristic thus the genus of M by Gauss–Bonnet. This is no longer true in dimension $d = 3$. We start by the compactness result of Osgood–Philips–Sarnak [49] which deals with isospectral families surfaces.

Theorem 4 (Compactness for $d = 2$)**.** *An isospectral set of isometry classes of metrics on a closed surface is sequentially compact in the C^∞–topology.*

[3] also called divergence free part

[4] It is claimed in [48] that the Theorem holds true for metrics whose geodesic flow has one dense orbit

For $d=3$, we shall use the celebrated result of Brooks-Petersen–Perry [42] and Anderson [44].

Theorem 5 (Compactness for $d=3$)**.** *The space of smooth compact isospectral* 3*-manifolds* (M,g) *for which the lenght of the shortest closed geodesic is bounded from below*

$$\ell_M \geqslant \ell > 0 \tag{4.2}$$

is compact in the C^∞ *topology. In particular, there are only finitely many diffeomorphism types of isospectral* 3*-manifolds which satisfy 4.2.*

Let us explain the meaning of the above statement in practice. Let $(M_i, g_i)_i$ denotes a sequence of isospectral smooth compact 3–manifolds without boundary. Then there is a finite number of manifolds $(M'_1, \dots, M'_k)$ and on each M'_j a compact family of metrics $\mathcal{M}'_j$ such that each of the manifolds M_j is diffeomorphic to one of the M'_i and isometric to an element of $\mathcal{M}'_i$.

4.1.3. *Proof of Proposition 4.1 by a contradiction argument.* We assume by contradiction that the set of isospectral metrics in $\tilde{\mathcal{N}} \cap \mathcal{R}(M)_{-\varepsilon}$ has an infinite number of classes. Therefore, we assume there exists an infinite sequence $(g_n)_n$ of smooth isospectral metrics on M whose isometry classes $([g_n])_n$ are 2 by 2 distinct. So if $(g_n)_n$ is a sequence of isospectral metrics of negative curvature $< -\varepsilon$, the above compactness Theorems tell us that we may extract a subsequence such that $g_n \to g$ in the C^∞–topology. In dimension 3, we can apply the compactness Theorem 5 since the spectrum determines the lenght of the shortest closed geodesic by Theorem 2.

By lemma 4.2, we may assume without generality that the sequence of representatives g_n is chosen in such a way that the difference $\varepsilon_n = g_n - g \in C^\infty(S^2T^*M)$ is **solenoidal** w.r.t. g. This will be very important in the sequel since we shall use the injectivity of the X-ray transform for solenoidal tensors w.r.t. g. We assume by contradiction that the sequence of metrics g_n is non stationary, which means that the sequence $\varepsilon_n = g_n - g_0$ never vanishes for every n and $\varepsilon_n \to 0$ in $C^\infty(S^2T^*M)$. For each free homotopy class $[\gamma]$ in $\pi_1(M)$, for every n, there is a unique closed geodesic $\gamma_n : [0, T_n] \mapsto M$ of g_n (resp γ of g) in the class $[\gamma]$. For each closed curve γ in SM, we define a Radon measure $\delta_\gamma \in \mathcal{D}'(SM)$ by equation (5.8) in subsection 5.3 in the appendix. By Proposition 5.3 proved in the appendix, we have the convergence $\delta_{\gamma_n} \to \delta_\gamma$ in the sense of Radon measures on SM. By the convergence of metrics $g_n \to g$, for every free homotopy class $[\gamma]$ in $\pi_1(M)$, for every n, we have the convergence $\ell_{g_n}(\gamma_n) \to \ell_g(\gamma)$ by [50, Lemma 4.1 p. 11].

4.1.4. *Inequalities satisfied by* ε_n. From the fact that the metrics are isospectral and the lenght spectrum is discrete, we deduce that $\ell_{g_n}(\gamma_n) = \ell(\gamma)$ for every $n \geqslant N_\gamma$ where the integer N_γ depends on $[\gamma] \in \pi_1(M)$. By equation (5.8) defining the Radon measures $\delta_\gamma \in \mathcal{D}'(SM)$ carried by closed curves γ, the lenght of the curve γ for the metric g is defined as $\ell(\gamma) = \delta_\gamma(g)$. The key observation is that since γ_n is minimizing for g_n in the class $[\gamma]$ implies the inequality $\delta_{\gamma_n}(g_n) \leqslant \delta_\gamma(g_n)$ but for $n \geqslant N_\gamma$, we find that $\delta_{\gamma_n}(g_n) = \delta_\gamma(g_0)$ from which we deduce the inequality $\delta_\gamma(g_0) \leqslant \delta_\gamma(g_n)$ which implies

$$\delta_\gamma\left(\frac{\varepsilon_n}{\|\varepsilon_n\|_\infty}\right) \geqslant 0. \tag{4.3}$$

Conversely since γ minimizes the lenght for g_0 we have a reverse inequality $\delta_{\gamma_n}(g_n) = \delta_\gamma(g_0) \leqslant \delta_{\gamma_n}(g_0)$ which implies the second inequality :

$$\delta_{\gamma_n}\left(\frac{\varepsilon_n}{\|\varepsilon_n\|_\infty}\right) \leqslant 0. \tag{4.4}$$

Now we would like to know if we can extract a subsequence from $\frac{\varepsilon_n}{\|\varepsilon_n\|_\infty} \in C^\infty(S^2T^*M)$ **with non trivial limit** so that we obtain inequalities on the X-ray transform which are independent of n. We assumed in Proposition 4.1 that the sequence of metrics $(g_n)_n$ belongs to some finite dimensional manifold $\mathcal{N}$ embedded in $\mathcal{R}(M)$. This means we assume there exists an abstract smooth manifold $\mathcal{N}$ (not necessarily compact) and a C^∞- map $\iota \ : \mathcal{N} \longmapsto \mathbf{Met}(M) \mapsto \mathcal{R}(M)$ where the last arrow is the projection induced by the quotient and the smoothness is understood w.r.t. the C^∞ structures. Put a Riemannian metric $\tilde{g}$ on $\mathcal{N}$ and denote by v_n a sequence of tangent vectors in $T_{g_0}\mathcal{N}$ such that $\iota\left(\exp_{g_0}(v_n)\right) = g_0 + \frac{\varepsilon_n}{\|\varepsilon_n\|_\infty}$ where exp is the

Riemannian exponential map induced by the metric $\tilde{g}$. Since the exponential map $v \in T_{g_0}\mathcal{N} \mapsto \exp_{g_0}(v)$ is a diffeomorphism near the origin whose differential at 0 is the identity, we may find that the norm of the sequence v_n is equivalent to the distance $\text{dist}\left(g_0 + \frac{\varepsilon_n}{\|\varepsilon_n\|_\infty}, g_0\right)$. Since $\mathcal{N}$ has finite dimension and the sequence $\frac{\varepsilon_n}{\|\varepsilon_n\|_\infty}$ has sup norm 1, the sequence of tangent vectors v_n is contained in some bounded subset of $T_{g_0}\mathcal{N}$ which avoids 0. Then by compactness of bounded subsets in **finite dimension**, we can extract a subsequence of $(v_n)_n$ s.t. $v_n \to v_\infty \neq 0 \in T_{g_0}\mathcal{N}$. Hence, up to extracting a subsequence, we may assume that $\frac{\varepsilon_n}{\|\varepsilon_n\|_\infty} \underset{n\to\infty}{\longrightarrow} u \in C^\infty(S^2T^*M)$ in the C^∞ topology where $u \neq 0$ and $\|u\|_\infty = 1$.

Passing to the limit in both inequalities 4.3 and 4.4 and using the fact that $\delta_{\gamma_n} \to \delta_\gamma$ in the sense of Radon measures, we find that the limit u satisfies $I_2(u)_\gamma = \delta_\gamma(u) \geqslant 0$ and $I_2(u)_\gamma = \delta_\gamma(u) \leqslant 0$ hence for any free homotopy class $[\gamma] \in \pi_1(M)$, $I_2(u)_\gamma = \delta_\gamma(u) = 0$. But since u is solenoidal w.r.t. g and $u \in \ker(I_2)$, we conclude that $u = 0$ by injectivity of the X-ray transform I_2 acting on solenoidal g tensors [46, Thm 1.3] which contradicts $u \neq 0$ in $C^\infty(S^2T^*M)$.

5. Appendix.

5.1. **Spectral identities related to Fredholm determinants.** We quickly recall some identities relating the Fredholm determinant $\det_F(I+B)$ for trace class operator $B : H \mapsto H$ acting on some separable Hilbert space H and functional traces of powers of B. The Fredholm determinant $\det_F(I+B)$ is defined in [51, equation (3.2) p. 32] as

$$\det_F(I+zB) = \sum_{k=0}^{\infty} z^k Tr(\Lambda^k B) \tag{5.1}$$

where $\Lambda^k B : \Lambda^k H \mapsto \Lambda^k H$ acting on the fermionic Fock space $\Lambda^k H$ is trace class. Using the bound $\|\Lambda^k B\|_1 \leqslant \frac{\|B\|_1}{k!}$ [51, Lemma 3.3 p. 33], it is immediate that $\det_F(I+zB)$ is an **entire** function in $z \in \mathbb{C}$ (see also [52, Thm 2.1 p. 26]).

For any compact operator B, we will denote by $(\lambda_k(B))_k$ its eigenvalues counted with multiplicity. For any $B \in \mathcal{I}_1$, by Lidskii Theorem [51, Equation 3.2 p. 46], B^m is trace class for any integer m, $Tr(B^m) = \sum_k \lambda_k(B^m)$ where $\lambda_k(B^m)$ are the singular values of B^m. By the spectral mapping Theorem, $\sum_k \lambda_k(B^m) = \sum_k \lambda_k(B)^m$. It follows that $Tr(B^m) = \left(\sum_k \lambda_k(B)^m\right)$ and :

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}z^m}{m} Tr(B^m) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}z^m}{m}\left(\sum_k \lambda_k(B)^m\right)$$

where both sides converge for $|z| < \|B\|_1^{-1}$ by the inequality for the Schatten 1-norm $\sum_k |\lambda_k(B)^m| \leqslant \|B^m\|_1 \leqslant \|B\|_1^m$ which follow from [52, eq (4.5) p. 57 and (5.4) p. 60] as in [51, proof Thm 5.4 p. 69]. We can interchange the sums which yields :

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}z^m}{m} \sum_k \lambda_k(B)^m = \sum_k \sum_{m=1}^{\infty} \frac{(-1)^{m+1}z^m}{m} \lambda_k(B)^m = \sum_k \log\left(1 + z\lambda_k(B)\right).$$

Therefore a consequence of Lidskii's Theorem reads when $|z|\|B\|_1 < 1$:

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}z^m}{m} Tr(B^m) = \sum_k \log\left(1 + z\lambda_k(B)\right). \tag{5.2}$$

By [51, Theorem 3.7], the Fredholm determinant can be identified with a Hadamard product :

$$\prod_k \left(1 + z\lambda_k(B)\right) = \det_F\left(1 + zB\right) \tag{5.3}$$

from which it follows that the Fredholm determinant is related to the functional traces by the following sequence of identities :

$$\det_F(1+zB) = \prod_k \exp(\log(1+z\lambda_k(B))) \tag{5.4}$$

$$= \exp\left(\sum_k \log(1+z\lambda_k(B))\right) = \underbrace{\exp\left(\sum_{m=1}^{\infty}(-1)^{m+1}z^m Tr(B^m)\right)} \tag{5.5}$$

where the term underbraced involving traces is well–defined only when $|z|\|B\|_1 < 1$ and the first equality comes from the infinite product representation and the last equality follows from equation 5.2. Note the important fact that $\exp\left(\sum_{m=1}^{\infty}(-1)^{m+1}z^m TR_{L^2}(B^m)\right)$ which is defined on the disc $\mathbb{D}=\{|z|\|B\|_1<1\}$ has **analytic continuation** as an **entire function** of $z\in\mathbb{C}$.

5.2. **Gohberg–Krein's determinants.** Set $p=[\frac{d}{2}]+1$ and let A belong to the Schatten ideal $\mathcal{I}_p$. Following [51, chapter 9], we consider the operator

$$R_p(A) = [(I+A)\exp(\sum_{n=1}^{p-1}\frac{(-1)^n}{n}A^n) - I] \in \mathcal{I}_1$$

which is trace class by [51, Lemma 9.1 p. 75] since $A\in\mathcal{I}_p$. Then following [51, p. 75], we define the regularized determinant as $\det_p(I+zA)=\det_F(1+R_p(zA))$ where $\det_F$ is the Fredholm determinant defined above. The quantity $\det_p$ is well defined since $B=R_p(A)$ is trace class. The function $R_p(z)$ is an entire function hence by the **spectral mapping Theorem** for entire functions, we find that $\lambda_k(R_p(A))=R_p(\lambda_k(A))$ where $R_p(z)=zh(z)$ for h an entire function (the domain is $\mathbb{C}$). Therefore for every k, note that $\lambda_k(B)=R_p(\lambda_k(A))$ hence for every fixed k we have the identity $\log(1+z\lambda_k(B))=\log(1+z\lambda_k(R_p(A)))=\log(1+z(R_p(\lambda_k(A)))$. Then plugging these identities in the formula for $\det_F(I+R_p(A))$ yields :

$$\begin{aligned}\det_F(1+R_p(A)) &= \prod_k(1+z(R_p(\lambda_k(A)))=\prod_k\exp(\log(1+z(R_p(\lambda_k(A))))\\ &= \exp\left(\sum_k\log(1+z(R_p(\lambda_k(A)))\right).\end{aligned}$$

For fixed k, we also have the identity :

$$\begin{aligned}\log(1+z(R_p(\lambda_k(A)))&=\log((1+z\lambda_k(A))\exp\left(\sum_{n=1}^{p-1}(-1)^n n^{-1}\lambda_k(A)^n\right))\\ &=\log(1+z\lambda_k(A)+\sum_{n=1}^{p-1}(-1)^n n^{-1}\lambda_k(A)^n=\sum_{n=p}^{\infty}\frac{(-1)^{n+1}z^n}{n}\lambda_k^n(A).\end{aligned}$$

From which we deduce the important identity :

$$\det_F(1+R_p(A))=\exp\left(\sum_k\sum_{n=p}^{\infty}\frac{(-1)^{n+1}z^n}{n}\lambda_k^n(A)\right).$$

We want to intepret the above formula in terms of functional traces of powers of A. Hence the next idea is to invert summations in k and in n for z small enough because of the following estimates :

$$\begin{aligned}\sum_k z^n|\lambda_k(A)|^n &= \|z^nA^n\|_1=\|z^nA^jA^{p[\frac{n}{p}]}\|_1\leqslant|z|^n\|A^j\|_\infty\|A^{p[\frac{n}{p}]}\|_1\\ &\text{by Hölder's inequality for trace class operators}\\ &\leqslant|z|^n\|A^j\|_\infty\|A^p\|_1^{[\frac{n}{p}]}\leqslant|z|^n\|A^j\|_\infty\|A\|_p^{[\frac{n}{p}]p}\leqslant C|z|^n\|A\|_p^n\end{aligned}$$

where $C=\sup_{j\in\{1,\dots,p\}}\|A^j\|_\infty$. Therefore sum inversion yields :

$$\begin{aligned}\sum_k \log\left(1+z(R_p(\lambda_k(A))\right) &= \sum_k\sum_{n=p}^{\infty}\frac{(-1)^{n+1}z^n}{n}\lambda_k^n(A)=\sum_k\left(\log(1+z\lambda_k(A))-\sum_{n=1}^{p}\frac{(-1)^{n+1}z^n}{n}\lambda_k^n(A)\right)\\ &= \sum_{n=p}^{\infty}\frac{(-1)^{n+1}z^n}{n}\sum_k\lambda_k^n(A)=\sum_{n=p}^{\infty}\frac{(-1)^{n+1}z^n}{n}Tr(A^n)\end{aligned}$$

where the last equality again follows from Lidskii's Theorem. Therefore, we have the following :

Lemma 5.1 (Gohberg–Krein's determinants and functional traces)**.** *For all $A\in\mathcal{I}_p$, the Gohberg–Krein determinant $\det_p(1+zA)$ is an **entire function** in $z\in\mathbb{C}$ and is related to traces $Tr(A^n)$ for $n>\frac{d}{2}$ by the following formulas :*

$$\boxed{det_p(1+zA)=\exp\left(\sum_{n=p}^{\infty}\frac{(-1)^{n+1}z^n}{n}Tr(A^n)\right)=\prod_k\left[(1+z\lambda_k(A))\exp\left(\sum_{n=1}^{p-1}(-1)^n n^{-1}\lambda_k(A)^n\right)\right]}$$

where the infinite product vanishes exactly when $z\lambda_k(A)=-1$ with multiplicity.

5.3. **The formalism of X-ray transform.** Periodic orbits γ of the vector field $X\in C^\infty(T(SM))$ which generates the geodesic flow of g on SM are defined as continuous maps :

$$\gamma: t\in[0,T_\gamma]\longmapsto(\gamma(t),\dot\gamma(t))\in SM \tag{5.6}$$

where γ is parametrized at unit speed. The closed geodesic γ defines a **distribution** in $\mathcal{D}'(SM)$, denoted by δ_γ, as follows :

$$\langle\delta_\gamma,f\rangle=\int_0^{T_\gamma}f\left(\gamma(t),\dot\gamma(t)\right)dt. \tag{5.7}$$

A symmetric m-tensor $f\in C^\infty\left(S^mT^*M\right)$ can be lifted as a function on the tangent bundle TM, polynomial of degree m in the fibers by the following $(x,v)\in TM\mapsto f(x,v)\in\mathbb{C}$. It follows that the distributions δ_γ act on symmetric m-tensor $C^\infty\left(S^mT^*M\right)$ as follows :

$$\langle\delta_\gamma,f\rangle=\int_0^{T_\gamma}f\left(\gamma(t),\dot\gamma(t)\right)dt. \tag{5.8}$$

By considering the collection of all maps $(\delta_\gamma)_{[\gamma]\in\pi_1(M)}$, for all periodic geodesics, we can define the X-ray transform.

Definition 5.2 (X-ray transform)**.** *The X-ray transform is defined as :*

$$I_2: f\in C^\infty\left(S^2T^*M\right)\longmapsto\left(\langle\delta_\gamma,f\rangle=\int_0^{T_\gamma}f\left(\gamma(t),\dot\gamma(t)\right)dt\right)_{[\gamma]\in\pi_1(M)} \tag{5.9}$$

which maps 2-tensors to sequences indexed by the free homotopy classes $\pi_1(M)$ of closed loops in M.

5.4. **Convergence of Radon measures corresponding to closed geodesics.** The goal of this paragraph is to show that if $g_n\mapsto g$ in the metrics of negative curvature, then for every free homotopy class $[\gamma]\in\pi_1(M)$, denote by γ_n (resp γ) the unique corresponding sequence of closed geodesic for g_n (resp g), the sequence δ_{γ_n} converges to δ_γ in the sense of Radon measures. We shall use the structural stability result of Anosov flows in the version of De La Llave–Marco–Moriyon [53, Thm A.2 p. 598].

Theorem 6 (Structural stability)**.** *Let (M,g) be a Riemannian manifold of negative curvature and set $\mathcal{M}=SM$ to be the sphere bundle of M. We denote by $X\in C^\infty(T\mathcal{M})$ the geodesic vector field of the metric g and by $C^0_X(\mathcal{M},\mathcal{M})$ the space of homeomorphisms from $\mathcal{M}$ to $\mathcal{M}$ which are C^1 along integral curves of X and $C^0(\mathcal{M})$ denotes continuous functions on $\mathcal{M}$. Then there exists a C^1 neighborhood $\mathcal{U}$ of X, a submanifold $\mathcal{N}\subset C^0_X(\mathcal{M},\mathcal{M})$ and a C^1 map :*

$$S:\mathcal{U}\longmapsto\mathcal{N}\times C^0(\mathcal{M}) \tag{5.10}$$

$$Y\longmapsto(\Phi_Y,h_Y) \tag{5.11}$$

satisfying the structure equation :

$$\boxed{(\Phi_Y^{-1*}h_Y)Y = \Phi_{Y*}X} \tag{5.12}$$

where $(\Phi_X, h_X) = (Id, 1) \in C^0_X(\mathcal{M}, \mathcal{M}) \times C^0(M)$.

The equation 5.12 follows from [53, equation (e) p. 592]

$$D\Phi_Y(x,v)(X(x,v)) = h_Y(x,v)\, Y(\Phi_Y(x,v)), \tag{5.13}$$

this implies that $D\Phi_Y\left(\Phi_Y^{-1}(x,v)\right)\left(X(\Phi_Y^{-1}(x,v))\right) = h_Y\left(\Phi_Y^{-1}(x,v)\right)Y(x,v)$ hence $\Phi_{Y*}X = \left(\Phi_Y^{-1*}h_Y\right)Y$. The above equation means that flows in a neighborhood $\mathcal{U}$ of X are conjugated to the flow generated by X up to reparametrization of time, more precisely let $\varphi_Y^t : \mathcal{M} \mapsto \mathcal{M}$ denotes the flow generated by $Y \in \mathcal{U} \subset C^1(T\mathcal{M})$, then there exists $\tau_Y \in C^0(\mathbb{R} \times \mathcal{M})$ s.t. :

$$\varphi_Y^t(x,v) = \Phi_Y \circ \varphi^{\tau_Y(t,x,v)} \circ \Phi_Y^{-1}(x,v) \tag{5.14}$$

where $\tau_Y(t,x,v) \to t$ in $C^0([0,T] \times \mathcal{M})$ for all $T > 0$ when $Y \to X$ in $C^1(T\mathcal{M})$.

A corollary of the above result

Proposition 5.3 (Convergence result for Radon measures.)**.** *Let $X \in C^\infty(T(SM))$ be a smooth Anosov vector field on SM. Let X_n be a sequence of vector fields which converges to X in $C^\infty(SM)$. Then for every free homotopy class $[\gamma] \in \pi_1(M)$, there exists $N_\gamma \in \mathbb{N}$ and a unique subsequence of periodic orbits $(\gamma_n)_{n \geqslant N_\gamma}$ of the vector field X_n which converges to a periodic orbit γ of X. The corresponding delta distributions $\delta_{\gamma_n}, n \geqslant N_\gamma$ will converge to the limit distribution δ_γ in the sense of Radon measures.*

Proof. Let $f \in C^0(SM)$ be a continuous test function. Denote by φ_n^t (resp φ^t) the flow generated by X_n (resp X) on SM. By definition $\delta_{\gamma_n}(f) = \int_0^{\ell(\gamma_n)} f \circ \varphi_n^t(x_n, v_n)dt$ for any $(x_n, v_n) \in \gamma_n$. The existence of the sequence $\gamma_n \to \gamma$ is a simple consequence of structural stability. Let $\Phi_n \in C^0_X(M,M)$ denotes the sequence of homeomorphisms conjugating the two flows whose existence comes from Theorem 6 :

$$\varphi_n^t(x,v) = \Phi_n \circ \varphi^{\tau_n(t,x,v)} \circ \Phi_n^{-1}(x,v)$$

where $\tau_n(t,x,v) \to t$ uniformly on $[0,T] \times SM$ for all $T > 0$ and $\Phi_n \to Id$ in $C^0(SM)$. Therefore for every (x,v) on the periodic orbit γ, the sequence $(x_n, v_n) = \Phi_n(x,v)$ lies in the periodic orbit γ_n by structural stability and converges to (x,v). It follows that when $n \to +\infty$,

$$\delta_{\gamma_n}(f) = \int_0^{\ell(\gamma_n)} f \circ \varphi_n^t(x_n, v_n)dt = \int_0^{\ell(\gamma_n)} f \circ \Phi_n \circ \varphi^{\tau_n(t,x_n,v_n)}(x,v)dt \underset{n\to+\infty}{\to} \int_0^{\ell(\gamma)} f \circ \varphi^t(x,v)dt$$

by dominated convergence and since the periods $\ell(\gamma_n) \underset{n\to+\infty}{\to} \ell(\gamma)$ converge [50, Lemma 4.1 p. 11] and $\frac{1}{2}\ell(\gamma) \leqslant \ell(\gamma_n) \leqslant 2\ell(\gamma)$ for all $n \geqslant N_\gamma$ [50, Remark 3]. It follows that the sequence of currents δ_{γ_n} will converge to the limit current δ_γ in the sense of Radon measures. □

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