FRIED CONJECTURE IN SMALL DIMENSIONS

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Abstract. We study the twisted Ruelle zeta function $\zeta_X(s)$ for smooth Anosov vector fields X acting on flat vector bundles over smooth compact manifolds. In dimension 3, we prove Fried conjecture, relating Reidemeister torsion and $\zeta_X(0)$. In higher dimensions, we show more generally that $\zeta_X(0)$ is locally constant with respect to the vector field X under a spectral condition. As a consequence, we also show Fried conjecture for Anosov flows near the geodesic flow on the unit tangent bundle of hyperbolic 3-manifolds. This gives the first examples of non-analytic Anosov flows and geodesic flows in variable negative curvature where Fried conjecture holds true.

1. Introduction

Let $\mathcal{M}$ be a smooth ($\mathcal{C}^\infty$), compact, connected and oriented manifold of dimension n and $E \to \mathcal{M}$ a smooth Hermitian vector bundle with fibers $\mathbb{C}^r$ equipped with a flat connection ∇. Parallel transport via ∇ induces a conjugacy class of representation $\rho : \pi_1(\mathcal{M}) \to \mathrm{GL}(\mathbb{C}^r)$, which is unitary as soon as ∇ preserves $\langle \cdot, \cdot \rangle_E$. One can then define a twisted de Rham complex on the space $\Omega(\mathcal{M}; E)$ of smooth twisted forms with twisted exterior derivative d^∇, and we denote by $H^k(\mathcal{M}; \rho)$ its cohomology of degree k. We say that the complex (or ρ) is acyclic if $H^k(\mathcal{M}; \rho) = 0$ for each k. If ρ is acyclic and unitary, Ray and Singer introduced a secondary invariant which is defined by the value at 0 of the derivative of the spectral zeta function of the Laplacian [RaSi71]. They showed that this quantity $\tau_\rho(\mathcal{M})$ is in fact independent of the choice of the metric used to define the Laplacian, thus an invariant of the flat bundle. This is the so-called *analytic torsion* and it was conjectured by Ray and Singer to be equal to the *Reidemeister torsion* [Re, Fra, DR]. This conjecture was proved independently by Cheeger [Ch] and Müller [Mu1] and it was extended to unimodular flat vector bundles by Müller [Mu2] and to arbitrary flat vector bundles by Bismut and Zhang [BiZh]. For an introduction to the different notions of torsion, we refer the reader to [Mn].

In the context of hyperbolic dynamical systems, Fried conjectured and proved in certain cases that the analytic torsion can in fact be related to the value at 0 of a certain dynamical zeta function [Fr4] that we will now define. Given a (primitive) closed hyperbolic orbit γ of a smooth vector field X, one can define its orientation index ε_γ to be equal to 1 when its unstable bundle $E_u(\gamma)$ is orientable and to -1 otherwise. If now X is a smooth Anosov vector field on $\mathcal{M}$, we can define the *Ruelle zeta function twisted* by the representation ρ as :

$$\zeta_{X,\rho}(\lambda) := \prod_{\gamma \in \mathcal{P}} \det(1 - \varepsilon_\gamma \rho([\gamma]) e^{-\lambda \ell(\gamma)}), \quad \mathrm{Re}(\lambda) > C \tag{1.1}$$

where $\mathcal{P}$ denotes the set of primitive closed orbits of X and $\ell(\gamma)$ the corresponding periods. Here $C > 0$ is some large enough constant depending on X and ρ. If ρ is unitary and acyclic and if X is the geodesic vector field on the unit tangent bundle $\mathcal{M} = SM$ of a hyperbolic manifold M, Fried showed that $\zeta_{X,\rho}(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$ using Selberg trace formula [Fr3] and the work or Ruelle [Rue]. Then he proved [Fr2] the remarkable formula (with $\dim(\mathcal{M}) = 2n_0 + 1$) :

$$|\zeta_{X,\rho}(0)^{(-1)^{n_0}}| = \tau_\rho(\mathcal{M}), \tag{1.2}$$

where ρ is the lift to $\pi_1(\mathcal{M})$ of an acyclic and unitary representation $\rho_0 : \pi_1(M) \to U(\mathbb{C}^r)$. Fried interpreted this formula as an analogue of the Lefschetz fixed point formula answering his own question in the case of geodesic flows [Fr1, p. 441] : *is there a general connection between the analytic torsion of Ray and Singer and closed orbits of some flow (e.g. geodesic flow) ?* He then extended this formula [Fr4, Fr5] to various families of flows such as Morse-Smale flows and formula (1.2) was also generalized to non-positively curved locally symmetric spaces by Moscovici-Stanton [MoSt] and Shen [Sh]. To generalize the above results, Fried makes the following conjecture in [Fr4, p. 66]: *it is even conceivable that* (φ_t, E) *is Lefschetz for any acyclic* E *with a flat density and any* C^ω *contact flow* φ_t. For geodesic flows, he also conjectured in [Fr5, p. 181] : *One may hope to generalize these results to variable negative curvature ...* Yet, as stated by Zworski [Zw2, p. 5] : *in the case of smooth manifolds of variable negative curvature, (1.2) remains completely open.*

For analytic Anosov flows, generalizing earlier works of Ruelle [Rue], Rugh showed in [Ru] that $\zeta_{X,\rho}$ has meromorphic continuation to the whole complex plane when $\dim(\mathcal{M}) = 3$. This was later extended to higher dimensions by Fried [Fr5]. Then, Sanchez-Morgado [Sa1, Sa2] proved that (1.2) holds for transitive analytic Anosov flows in dimension 3 if there exists a closed orbit γ such that, for each $j \in \{0, 1\}$, $\ker(\rho([\gamma]) - \varepsilon_\gamma^j \mathrm{Id}) = 0$ – see also [Fr4] for related assumptions in the case of Morse-Smale flows. More recently, the meromorphic continuation of Ruelle zeta functions was proved in the case of hyperbolic dynamical systems with less regularity (say $\mathcal{C}^\infty$). The case of Anosov diffeomorphisms was handled by Liverani [Liv2] while the case of Axiom A diffeomorphisms was treated by Kitaev [Ki] and Baladi-Tsujii [BaTs2]. Afterwards, Giulietti, Liverani and Pollicott proved that the meromorphic continuation of $\zeta_{X,\rho}$ holds for *smooth* Anosov flows [GLP]. An alternative proof of this latter fact was given by Dyatlov-Zworski [DyZw1] via microlocal techniques, and extended by Dyatlov-Guillarmou [DyGu1, DyGu2] to Axiom A cases. In the case of smooth *contact* Anosov vector fields in dimension 3 and of the trivial representation $1 : [\gamma] \in \pi_1(\mathcal{M}) \to 1 \in \mathbb{C}^*$, Dyatlov-Zworski [DyZw2] subsequently proved that the vanishing order of $\zeta_{X,1}(\lambda)$ at 0 is $\lambda^{b_1(\mathcal{M})-2}$ [DyZw2] where $b_1(\mathcal{M})$ is the first Betti number of $\mathcal{M}$ – see also [Ha2] in the case with boundary. Recent account about these progresses can be found in [Go, Zw2]. We also refer to the book of Baladi [Ba] for a complete introduction to the spectral analysis of zeta functions in the case of diffeomorphisms. Building on these recent results in the smooth case, the purpose of this work is to bring new insights on Fried's questions regarding the links between Ruelle zeta functions and analytic torsion.

2. Statement of the main results

Our first result answers Fried's question in dimension 3 for smooth Anosov flows.

Theorem 1. *Suppose that* $\dim(\mathcal{M}) = 3$ *and let* E *be a smooth Hermitian vector bundle with a flat connection* ∇ *inducing a unitary and acyclic representation* $\rho : \pi_1(\mathcal{M}) \to U(\mathbb{C}^r)$*. Let* X_0 *be a smooth Anosov vector field preserving a smooth volume form. Then, there is a nonempty neighborhood* $\mathcal{U}(X_0) \subset C^\infty(\mathcal{M}; T\mathcal{M})$ *of* X_0 *so that*

$$\forall X \in \mathcal{U}(X_0), \quad \zeta_{X,\rho}(0) = \zeta_{X_0,\rho}(0) \neq 0.$$

In addition, if $b_1(\mathcal{M}) \neq 0$ *or if there exists a closed orbit* γ *of* X_0 *such that, for each* $j \in \{0, 1\}$, $\ker(\rho([\gamma]) - \varepsilon_\gamma^j \mathrm{Id}) = 0$*, then* $|\zeta_{X,\rho}(0)|^{-1} = \tau_\rho(\mathcal{M})$ *is the Reidemeister torsion for each* $X \in \mathcal{U}(X_0)$*.*

The second part of the Theorem is based on the approximation of smooth volume preserving Anosov flows by analytic transitive Anosov flows and the result of Sanchez-Morgado [Sa2], while the first part follows from a variation formula for $\zeta_{X,\rho}(0)$ with respect to X which shows that $X \mapsto \zeta_{X,\rho}(0)$ is locally constant for unitary and acyclic representations in dimension 3. Observe that a vector field in $\mathcal{U}(X_0)$ may not preserve a smooth volume form even if X_0 does. This variation property of the Ruelle zeta function at 0 is in fact our main result and it holds more generally for smooth Anosov vector fields in any dimension under a certain non-resonance at $\lambda = 0$ assumption. In order to state it, we need to recall the notion of *Pollicott-Ruelle resonances.*

Given a vector field X_0 and connection ∇, one can define the Lie derivative $\mathbf{X}_0 := d^\nabla \iota_{X_0} + \iota_{X_0} d^\nabla$ acting on smooth differential forms $\Omega(\mathcal{M}; E)$. Then, one can find some $C > 0$ depending on X_0 and ρ such that

$$R_{\mathbf{X}_0}(\lambda) := \int_0^{+\infty} e^{-t\lambda} e^{-t\mathbf{X}_0} dt : \Omega(\mathcal{M}; E) \to \Omega'(\mathcal{M}, E)$$

is holomorphic for $\mathrm{Re}(\lambda) > C$ where $\Omega'(\mathcal{M}; E)$ is the space of currents with values in E. For smooth Anosov flows, it was first proved by Butterley and Liverani that $R_{\mathbf{X}_0}(\lambda)$ has a meromorphic extension to the whole complex plane [BuLi]. The poles of this meromorphic extension are called Pollicott-Ruelle resonances and this result was based on the construction of appropriate functional spaces for the differential operator $\mathbf{X}_0$ – see also [BKL, GoLi] in the case of diffeomorphisms and [Liv1, GLP] for flows. Building on earlier works for diffeomorphisms [BaTs1, FRS], Faure and Sjöstrand introduced microlocal methods to analyse the spectrum of Anosov flows and, among other things, they gave another proof of this result – see also [Ts, DyZw1, FaTs]. Using this meromorphic extension, our main result reads as

Theorem 2. *Let* E *be a smooth vector bundle with a flat connection* ∇*. Then the set of smooth Anosov vector fields* X *such that* 0 *is not a pole of the meromorphic extension of* $R_{\mathbf{X}}(\lambda) : \Omega(\mathcal{M}; E) \to \Omega'(\mathcal{M}; E)$ *forms an open subset* $\mathcal{U} \subset C^\infty(\mathcal{M}, T\mathcal{M})$*, and the map* $X \in \mathcal{U} \longmapsto \zeta_{X,\rho}(0)$ *is locally constant and nonzero.*

This result is valid in any dimension and without any assumption on the fact that ρ is unitary or that X preserves some smooth volume form. Note from [DaRi, Th. 2.1] that our condition on the poles of $R_{\mathbf{X}}(\lambda)$ implies that ρ is acyclic. If we suppose in addition that $\mathcal{M}$ is 3-dimensional, that ρ is unitary and that X preserves a smooth volume form, then we will show that the converse is true and thus deduce the first part of Theorem 1. This spectral assumption also implies that $\zeta_{X,\rho}(0) \neq 0$ as a consequence of [GLP, DyZw1] – see e.g. [DyZw2, § 3.1]. In the case of nonsingular Morse-Smale flows [Fr4, Th. 3.1], Fried proved that $\zeta_{X,\rho}(0)$ is equal to the Reidemeister torsion under certain assumptions on the eigenvalues of $\rho([\gamma])$ for every closed orbits. This geometric condition was in fact shown to be equivalent to the spectral condition we have here [DaRi, § 2.6].

Observe now that Theorem 2 says that the Ruelle zeta function evaluated at $\lambda = 0$ is locally constant under a certain spectral assumption. This result suggests that this value should be an invariant of the acyclic representation class $[\rho]$ but it does not say a priori that it should be equal to the Reidemeister torsion. In dimension 3, this is indeed the case under the extra assumptions that X_0 preserves a smooth volume form and that ρ is unitary as shown by Theorem 1. For contact Anosov flows and unitary representation ρ, we prove that it is enough (in order to apply Theorem 2) to verify that 0 is not a pole of the meromorphic extension of $R_{\mathbf{X}_0}(\lambda)$ restricted to $\Omega^{n_0}(\mathcal{M}, E)$ where $\dim(\mathcal{M}) = 2n_0 + 1$. For hyperbolic manifolds, using a factorisation of dynamical zeta functions associated to $\mathbf{X}$ in terms of infinite products of Selberg zeta functions associated to certain irreducible representations of $\mathrm{SO}(n_0)$, we can show that $\mathbf{X}$ has no 0 resonance in the acyclic case when $n = 5$ (see Proposition 7.7) and we deduce the following extension of Fried conjecture (1.2):

Theorem 3. *Suppose that $M = \Gamma\backslash\mathbb{H}^3$ is a compact oriented hyperbolic manifold of dimension 3 and denote by X_0 the geodesic vector field on $\mathcal{M} = SM$. Let E be a smooth Hermitian vector bundle with a flat connection ∇ on M inducing an acyclic and unitary representation $\rho : \pi_1(M) \to U(\mathbb{C}^r)$. Then, $\mathbf{X}_0$ has no resonance at 0 and there exists a nonempty neighborhood $\mathcal{U}(X_0) \subset C^\infty(\mathcal{M}; T\mathcal{M})$ of X_0 so that*[1]

$$\forall X \in \mathcal{U}(X_0), \quad \zeta_{X,\tilde{\rho}}(0) = \tau_\rho(M)^2,$$

where $\tilde{\rho}$ is the lift of ρ to $\mathcal{M}$.

In dimension $n_0 > 2$, the computations for the order of 0 as a resonance of $\mathbf{X}_0$ on $S(\Gamma\backslash\mathbb{H}^{n_0+1})$ are involved and do not always seem to be topological (cf Remark 5).

Organisation of the article. In section 3, we describe in detail the dynamical framework and construct the escape function needed to build appropriate functional spaces. In sections 4 and 5, we describe the variation of the Ruelle zeta function for $\mathrm{Re}(z)$ large. In section 6, we show the analytic continuation of our variation formula up to $z = 0$ relying on the microlocal methods of [FaSj, DyZw1]. In section 7, we use the variation formula and methods of [Sa2, DFG, DyZw2, DaRi] to discuss Fried conjecture. Finally, appendix A gives technical details on the escape function and appendix B discusses Selberg's trace on symmetric tensors.

[1] Recall from [Fr4] that $\tau_\rho(M)^2 = \tau_{\tilde{\rho}}(\mathcal{M})$.

Conventions. For a smooth compact manifold $\mathcal{M}$, we will always use the following terminology: $T_0^*\mathcal{M} := \{(x,\xi) \in T^*\mathcal{M}; \xi \neq 0\}$, $\mathcal{D}'(\mathcal{M}) = (C^\infty(\mathcal{M}))'$ is the space of distributions, $H^s(\mathcal{M}) := (1+\Delta)^{-s/2}L^2(\mathcal{M})$ if Δ is the Laplacian of some fixed Riemannian metric on $\mathcal{M}$. If B is a regularity space (such as $C^k, H^s, C^\infty, \mathcal{D}'$) and E a smooth vector bundle on $\mathcal{M}$, $B(\mathcal{M}; E)$ denotes the space of sections with regularity B. A set $\Gamma \subset T^*\mathcal{M}$ (or $\subset T_0^*M$) is called conic if $(x,\xi) \in \Gamma$ implies $(x, t\xi) \in \Gamma$ for all $t > 0$.

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3. Dynamical and analytical preliminaries

Let X be a smooth vector field on a n-dimensional compact manifold $\mathcal{M}$, and denote by φ_t^X its flow on $\mathcal{M}$. Recall that a vector field is said to be *Anosov* if there exist some constants $C, \lambda > 0$ and a $d\varphi_t$-invariant continuous splitting

$$T\mathcal{M} = \mathbb{R}X \oplus E_u(X) \oplus E_s(X), \tag{3.1}$$

such that, for every $t \geq 0$,

$$\forall v \in E_s(X,x),\ \|d\varphi_t^X(x)v\| \leq Ce^{-\lambda t}\|v\|, \quad \forall v \in E_u(X,x),\ \|d\varphi_{-t}^X(x)v\| \leq Ce^{-\lambda t}\|v\|.$$

Here we have equipped $\mathcal{M}$ with a smooth Riemannian metric g that will be fixed all along the paper. The subset of Anosov vector fields

$$\mathcal{A} := \{X \in C^\infty(\mathcal{M}; T\mathcal{M}) : X \text{ is Anosov}\}$$

forms an open subset of $C^\infty(\mathcal{M}; T\mathcal{M})$ in the $\mathcal{C}^\infty$ topology. Next, we introduce the dual decomposition to (3.1):

$$T^*\mathcal{M} = E_0^*(X) \oplus E_u^*(X) \oplus E_s^*(X)$$

where $E_0^*(X)\,(E_u(X) \oplus E_s(X)) = \{0\}$, $E_{s/u}^*(X)\left(E_{s/u}(X) \oplus \mathbb{R}X\right) = \{0\}$. We have for every $t \geq 0$,

$$\begin{aligned} &\forall v \in E_s^*(X,x),\ \|(d\varphi_t^X(x)^T)^{-1}v\| \leq Ce^{-\lambda t}\|v\|, \\ &\forall v \in E_u^*(X,x),\ \|(d\varphi_{-t}^X(x)^T)^{-1}v\| \leq Ce^{-\lambda t}\|v\|. \end{aligned} \tag{3.2}$$

We define the *symplectic lift* of φ_t^X as follows:

$$\forall (x,\xi) \in T^*\mathcal{M},\ \Phi_t^X(x,\xi) := \left(\varphi_t^X(x), (d\varphi_t^X(x)^T)^{-1}\xi\right),$$

and the induced flow on $S^*\mathcal{M}$:

$$\tilde{\Phi}_t^X(x,\xi) := \left(\varphi_t^X(x), \frac{(d\varphi_t^X(x)^T)^{-1}\xi}{\left\|(d\varphi_t^X(x)^T)^{-1}\xi\right\|_{\varphi_t^X(x)}}\right).$$

The flow Φ_t^X is the Hamiltonian flow corresponding to the Hamiltonian $H(x,\xi) := \xi(X(x))$. The vector fields corresponding to these lifted flows will be denoted by X_H and $\tilde{X}_H$.

3.1. **Invariant neighborhoods.** Fix some $X_0 \in \mathcal{A}$. We will now recall how to construct cones adapted to the Anosov structure. For that purpose, we decompose any given $\xi \in T_x^*\mathcal{M}$ as

$$\xi = \xi_0 + \xi_u + \xi_s \in E_0^*(X_0, x) \oplus E_u^*(X_0, x) \oplus E_s^*(X_0, x),$$

and we define a new metric on $\mathcal{M}$

$$\|\xi\|_x' := \|\xi_0\|_x + \int_{-\infty}^0 e^{-\gamma t} \|(d\varphi_t^{X_0}(x)^T)^{-1}\xi_u\|_{\varphi_t^{X_0}(x)} dt + \int_0^{+\infty} e^{\gamma t} \|(d\varphi_t^{X_0}(x)^T)^{-1}\xi_s\|_{\varphi_t^{X_0}(x)} dt,$$

with $\gamma > 0$ small enough to ensure that the integrals converge. With these conventions, one has, for every $t_0 \geq 0$,

$$\forall \xi \in E_s^*(X_0, x), \quad \|(d\varphi_{t_0}^{X_0}(x)^T)^{-1}\xi\|' \leq e^{-\gamma t_0}\|\xi\|',$$
$$\forall \xi \in E_u^*(X_0, x), \quad \|(d\varphi_{-t_0}^{X_0}(x)^T)^{-1}\xi\|' \leq e^{-\gamma t_0}\|\xi\|'.$$

Note also that, provided the initial metric $\|.\|$ is chosen in such a way that $\|X_0(x)\|_x = 1$ for every x in $\mathcal{M}$, one has, for every $t_0 \in \mathbb{R}$,

$$\forall \xi \in E_0^*(X_0, x), \quad \|(d\varphi_{t_0}^{X_0}(x)^T)^{-1}\xi\|' = \|\xi\|'.$$

In other words, we have constructed a metric adapted to the dynamics of $\varphi_t^{X_0}$. Recall that this new metric is a priori only continuous. Nevertheless, we may use it to define stable and unstable cones. We fix a small parameter $\alpha > 0$ and we introduce:

$$C^{ss}(\alpha) := \left\{(x,\xi) \in T^*\mathcal{M}\backslash 0 : \|\xi_u + \xi_0\|_x' \leq \alpha \|\xi_s\|_x'\right\},$$
$$C^{u}(\alpha) := \left\{(x,\xi) \in T^*\mathcal{M}\backslash 0 : \alpha\|\xi_u + \xi_0\|_x' \geq \|\xi_s\|_x'\right\}.$$

In the following, α is always chosen small enough to ensure that $C^{ss}(\alpha) \cap C^u(\alpha) = \emptyset$. We have the following properties, for every $t \geq 0$,

$$\forall (x,\xi) \in C^{ss}(\alpha), \quad \|(d\varphi_{-t}^{X_0}(x)^T)^{-1}(\xi_u + \xi_0)\|_x' \leq e^{-t\gamma}\alpha \|d\varphi_t^{X_0}(x)^T(\xi_s)\|',$$
$$\forall (x,\xi) \in C^{u}(\alpha), \quad \alpha e^{-t\gamma}\|(d\varphi_t^{X_0}(x)^T)^{-1}(\xi_u + \xi_0)\|_x' \geq \|d\varphi_{-t}^{X_0}(x)^T(\xi_s)\|'.$$

In particular, the cone $C^u(\alpha)$ (resp. $C^{ss}(\alpha)$) is stable under the forward (resp. backward) flow of $\varphi_{X_0}^t$.

Proposition 3.1. *From the continuity of the Anosov splitting, one knows that, for every $\alpha > 0$, there exists a neighborhood $\mathcal{U}_\alpha(X_0) \subset \mathcal{A}$ of X_0 such that, $\forall X \in \mathcal{U}_\alpha(X_0)$,*

$$E_u^*(X) \oplus E_0^*(X)\backslash 0 \subset C^u(\alpha) \text{ and } E_s^*(X)\backslash 0 \subset C^{ss}(\alpha).$$

The following result will be useful in our analysis:

Lemma 3.2. *Let $X_0 \in \mathcal{A}$ and let $\alpha > 0$ be small enough to ensure $C^{ss}(\alpha) \cap C^u(\alpha) = \emptyset$. There exist a neighborhood $\mathcal{U}_\alpha(X_0)$ of X_0 in the C^∞ topology and $T_\alpha > 0$ (both depending on α) such that*

$$\forall X \in \mathcal{U}_\alpha(X_0),\ \forall t \geq T_\alpha,\ \Phi_{-t}^X(C^{ss}(\alpha)) \subset C^{ss}(\alpha),\ \text{and } \Phi_X^t(C^u(\alpha)) \subset C^u(\alpha).$$

Proof. To begin with, let us first note that we could have defined an adapted norm $\|.\|'_X$ for every vector field X close enough to X_0. We would like to verify that all these norms are *uniformly* equivalent – see equation (3.3) below. For that purpose, we set $\tilde{f}(x,\xi) = f(x, \xi/\|\xi\|_x)\|\xi\|_x$ with f defined in the appendix, which is independent of X. By compactness of $S^*\mathcal{M}$, there exists some constant $C > 0$ such that

$$\forall (x,\xi) \in T^*\mathcal{M}, \quad C^{-1}\|\xi\|_x \leq \tilde{f}(x,\xi) \leq C\|\xi\|_x.$$

Combining this with[2] (A.6) and (A.7), one can verify that, provided $\gamma > 0$ is chosen small enough in the definition of $\|.\|'_X$, one has, for every $(x,\xi) \in T^*\mathcal{M}$ and $X \in \mathcal{U}_\alpha(X_0)$

$$\|\xi_0(X)\| + C^{-2}\left(\|\xi_u(X)\| + \|\xi_s(X)\|\right) \leq \|\xi\|'_X \leq \|\xi_0(X)\| + C^2\left(\|\xi_u(X)\| + \|\xi_s(X)\|\right),$$

where $\xi = \xi_0(X) + \xi_u(X) + \xi_s(X)$ is the Anosov decomposition associated with X and $\mathcal{U}_\alpha(X_0) \subset \mathcal{A}$ some small neighborhood of X_0. Note that $C > 0$ is independent of $X \in \mathcal{U}_\alpha(X_0)$. By continuity of the Anosov decomposition with respect to X, there is $C > 0$ such that for every $(x,\xi) \in T^*\mathcal{M}$ and for every $X \in \mathcal{U}(X_0)$,

$$C^{-1}\|\xi\| \leq \|\xi\|'_X \leq C\|\xi\|. \tag{3.3}$$

Let us now prove our Lemma. We only discuss the case of $C^u(\alpha)$ as the other case is similar. First of all, thanks to the continuity of the unstable and stable directions (with respect to X) and thanks to (3.3), one can find $\alpha_1 > 0$ and an open neighborhood $\mathcal{U}_\alpha(X_0)$ of X_0 such that, for every X in $\mathcal{U}_\alpha(X_0)$, one has $C^u_X(\alpha_1) \subset C^u(\alpha)$, where $C^u_X(\alpha_1)$ is the cone of aperture α_1 built from X instead[3] of X_0. Up to shrinking the neighborhood of X_0 a little bit more and by a similar argument, we can also find $\alpha_2 > 0$ such that, for every X in $\mathcal{U}_\alpha(X_0)$, one has $C^u(\alpha_2) \subset C^u_X(\alpha_1)$. Observe now that, if $(x,\xi) \in C^u(\alpha)$, then there exists $T_\alpha > 0$ such that $\Phi^{T_\alpha}_{X_0}(C^u(\alpha)) \subset C^u(\alpha_2/2)$. Hence, up to shrinking $\mathcal{U}_\alpha(X_0)$ one more time, we can deduce that, for every $X \in \mathcal{U}_\alpha(X_0)$ and for every $(x,\xi) \in C^u(\alpha)$, one has $\Phi^{T_\alpha}_X(x,\xi) \in C^u(\alpha_2) \subset C^u_X(\alpha_1)$. We deduce that, for every $t \geq T_\alpha$, $\Phi^t_X(C^u(\alpha)) \subset C^u_X(\alpha_1) \subset C^u(\alpha)$, which concludes the proof. □

3.2. **Escape functions.** In order to study analytical properties of Anosov flows, we shall make use of the microlocal tools developped by Faure-Sjöstrand [FaSj] and Dyatlov-Zworski [DyZw1]. One of the key ingredients of these spectral constructions is the existence of an escape function:

Lemma 3.3 (Escape functions). *There exists a function $f \in C^\infty(T^*\mathcal{M}, \mathbb{R}_+)$ which is 1-homogeneous for $\|\xi\|_x \geq 1$, a constant $c_0 > 0$ and a constant $\tilde{\alpha}_0 > 0$ (small enough) such that the following properties hold:*

1. *$f(x,\xi) = \|\xi\|_x$ for $\|\xi\|_x \geq 1$ and $(x,\xi) \notin C^{uu}(\tilde{\alpha}_0) \cup C^{ss}(\tilde{\alpha}_0)$,*
2. *for every $N_0, N_1 > 0$ and $0 < \alpha_0 < \tilde{\alpha}_0$, there exist $\alpha_1 < \alpha_0$ and a neighborhood $\mathcal{U}(X_0)$ of X_0 in the $\mathcal{C}^\infty$-topology for which one can construct, for any X in $\mathcal{U}(X_0)$, a smooth function*

$$m_X^{N_0,N_1} : T^*\mathcal{M} \to [-2N_0, 2N_1]$$

[2]Note also that the stable/unstable bundles depend continuously on X .

[3]It means that we replace $\|.\|'$ by $\|.\|'_X$ and the components $\xi_{0/u/s}$ of X_0 by the ones of X.

with the following requirements

- $m_X^{N_0,N_1}$ *is* 0*-homogeneous for* $\|\xi\|_x \geq 1$,
- $m_X^{N_0,N_1}(x,\xi/\|\xi\|_x) \geq N_1$ *on* $C^{ss}(\alpha_1)$, $m_X^{N_0,N_1}(x,\xi/\|\xi\|_x) \leq -N_0$ *on* $C^{uu}(\alpha_1)$ *and* $m_X^{N_0,N_1}(x,\xi/\|\xi\|_x) \geq N_1/2$ *in a small vicinity of* $E_0^*(X_0)$,
- $m_X^{N_0,N_1}(x,\xi/\|\xi\|_x) \geq \frac{N_1}{4} - 2N_0$ *outside* $C^{uu}(\alpha_0)$
- *there exist* $R \geq 1$ *such that, for every* $X \in \mathcal{U}(X_0)$ *and for every* (x,ξ) *outside a small vicinity of* $E_0^*(X_0)$ *(independent of* X*), one has*

$$\|\xi\|_x \geq R \quad \Longrightarrow \quad X_H(G_X^{N_0,N_1})(x,\xi) \leq -2c_0 \min\{N_0,N_1\}, \tag{3.4}$$

where

$$G_X^{N_0,N_1}(x,\xi) := m_X^{N_0,N_1}(x,\xi)\ln(1+f(x,\xi)), \tag{3.5}$$

and where R *can be chosen equal to* 1 *on* $C^{uu}(\alpha_1) \cup C^{ss}(\alpha_1)$.

- *there exists a constant* $C_{N_0,N_1} > 0$ *such that, for every* $X \in \mathcal{U}(X_0)$,

$$\|\xi\|_x \geq R \quad \Longrightarrow \quad X_H(G_X^{N_0,N_1})(x,\xi) \leq C_{N_0,N_1}, \tag{3.6}$$

(3) *Moreover,*

$$X \in C^\infty(\mathcal{M};T^*\mathcal{M}) \to m_X^{N_0,N_1} \in \mathcal{C}^\infty(T^*\mathcal{M},[-2N_0,2N_1])$$

is a smooth function.

Under this form, this Lemma was proved in [FaSj, Lemma 1.2] (or Lemma [DyZw1, Lemma C.1]). For our purpose, the only inputs with the statements from these references is that we need the escape function to depend smoothly on the vector field X and the conic neighborhoods must be chosen uniformly w.r.t. X. We postpone the proof of this Lemma to Appendix A. Note that, compared with the construction of [FaSj], we do not have decay of the escape function $G_X^{N_0,N_1}$ in a small vicinity of the flow direction but this will be compensated by the ellipticity of the principal symbols in these directions – see e.g. the proof of Proposition 6.1 below. We could have chosen $f(x,\xi)$ to depend on X and in that manner, we would get $X_H(G_X^{N_0,N_1}) \leq 0$ for every ξ large enough even near the flow direction – see [FaSj]. Despite the fact that $f(x,\xi)$ is not equal to $\|\xi\|_x$ in a vicinity of E_u^* and of E_s^*, we emphasize that $C^{-1}\|\xi\|_x \leq f(x,\xi) \leq C\|\xi\|_x$ for $|\xi| \geq 1$ (for some uniform constant $C > 0$).

3.3. **Pollicott-Ruelle spectrum.** Consider a smooth complex vector bundle $E \to \mathcal{M}$ equipped with a flat connection $\nabla : \Omega^0(\mathcal{M},E) \to \Omega^1(\mathcal{M},E)$, where $\Omega^k(\mathcal{M},E) = C^\infty(\mathcal{M};\Lambda^k(T^*\mathcal{M})\otimes E)$. This connection induces a representation

$$\rho : \pi_1(\mathcal{M}) \to \mathrm{GL}(\mathbb{C}^r) \tag{3.7}$$

by taking $\rho([\gamma])$ to be the parallel transport with respect to ∇ along a representative γ of $[\gamma] \in \pi_1(\mathcal{M})$. We also denote by $\mathcal{E}$ the graded vector bundle

$$\mathcal{E} := \bigoplus_{k=0}^n \mathcal{E}^k, \quad \mathcal{E}^k := \wedge^k(T^*\mathcal{M}) \otimes E.$$

Associated with this connection is a twisted exterior derivative d^∇ acting on the space $\Omega(\mathcal{M},E) = \oplus_{k=0}^n \Omega^k(\mathcal{M},E)$. Since ∇ is flat, one has $d^\nabla \circ d^\nabla = 0$. As before, we fix a smooth Riemannian metric g on $\mathcal{M}$ and a smooth hermitian structure $\langle .,.\rangle_E$ on E. This induces a scalar product on $\Omega(\mathcal{M},E)$ by setting, for every $(\psi_1,\psi_2)\in\Omega^k(\mathcal{M},E)$,

$$\langle\psi_1,\psi_2\rangle_{L^2}:=\int_{\mathcal{M}}\langle\psi_1,\psi_2\rangle_{\mathcal{E}^k}\mathrm{dvol}_g.$$

We set $L^2(\mathcal{M},\mathcal{E})$ (or $L^2(\mathcal{M})$ if there is no ambiguity) to be the completion of $\Omega(\mathcal{M},E)$ for this scalar product. The set of De Rham currents valued[4] in E is denoted by $\mathcal{D}'(\mathcal{M},E)$.

Given $X\in\mathcal{A}$, we define the twisted Lie derivative

$$\mathbf{X}:=i_Xd^{\nabla}+d^{\nabla}i_X:\Omega(\mathcal{M},E)\to\Omega(\mathcal{M},E). \tag{3.8}$$

The differential operator $-i\mathbf{X}$ has diagonal principal symbol given by

$$\sigma(-i\mathbf{X})(x,\xi)=H(x,\xi)\mathrm{Id}_{\mathcal{E}} \tag{3.9}$$

(recall $H(x,\xi)=\xi(X(x))$). Note that $\mathbf{X}$ preserves $\Omega^k(\mathcal{M},E)$ for each k. Also, since $[\mathbf{X},i_X]=0$, it also preserves sections of the bundle (depending smoothly on X)

$$\mathcal{E}_0:=\mathcal{E}\cap\ker i_X=\bigoplus_{k=0}^{n-1}\underbrace{\mathcal{E}^k\cap\ker i_X}_{:=\mathcal{E}_0^k}. \tag{3.10}$$

It was shown in [BuLi, FaSj, GLP, DyZw1] that this differential operator has a discrete spectrum when acting on convenient Banach spaces of currents. Let us recall this result using the microlocal framework from [FaSj, DyZw1]. Using [Zw1, Th. 8.6] and letting $N_0,N_1>0$ be two positive parameters, we set

$$\mathbf{A}_h(N_0,N_1,X):=\exp\left(\mathrm{Op}_h\left(G_X^{N_0,N_1}\mathrm{Id}_{\mathcal{E}}\right)\right),$$

where Op_h is a semiclassical quantization procedure on $\mathcal{M}$ [Zw1, Th. 14.1]. We then define the (semiclassical) *anisotropic Sobolev spaces*:

$$\forall 0<h\leq 1,\quad \mathcal{H}_h^{m_X^{N_0,N_1}}(\mathcal{M},\mathcal{E}):=\mathbf{A}_h(N_0,N_1,X)^{-1}L^2(\mathcal{M};\mathcal{E}),$$

where we used the subscript X to remind the dependence of these spaces on the vector field X. These spaces are related to the usual semiclassical Sobolev spaces $H_h^k(\mathcal{M};\mathcal{E}):=(1+h^2\Delta_{\mathcal{E}})^{-k/2}L^2(\mathcal{M};\mathcal{E})$ as follows ($\Delta_{\mathcal{E}}$ is some positive Laplacian on $\mathcal{E}$)

$$H_h^{2N_1}(\mathcal{M},\mathcal{E})\subset\mathcal{H}_h^{m_X^{N_0,N_1}}(\mathcal{M},\mathcal{E})\subset H_h^{-2N_0}(\mathcal{M},\mathcal{E}), \tag{3.11}$$

with continuous injections. Stated in the case of a general smooth vector bundle E, the main results from [FaSj, Th. 1.4-5, § 5] and [DyZw1, Prop. 3.1-3] read as follows:

Proposition 3.4. *Let X be an element in $\mathcal{U}(X_0)$ where $\mathcal{U}(X_0)$ is the neighborhood of Lemma 3.3. Then, there exists $C_X>0$ (depending continuously[5] on $X\in\mathcal{A}$) such that,*

[4]Observe that E' can be identified with E via the Hermitian structure.

[5]Even if not explicitely written in [FaSj], this observation can be deduced from paragraph 3.2 of this reference and from Lemma 3.3 above.

for any $0 < h \leq 1$ *and for any* N_0, N_1*, the resolvent*

$$(\mathbf{X}+\lambda)^{-1} = \int_0^{+\infty} e^{-t\mathbf{X}} e^{-t\lambda} dt : \mathcal{H}_h^{m_X^{N_0,N_1}}(\mathcal{M},\mathcal{E}) \to \mathcal{H}_h^{m_X^{N_0,N_1}}(\mathcal{M},\mathcal{E})$$

is holomorphic in $\{\mathrm{Re}(\lambda) > C_X\}$ *and has a meromorphic extension to*[6]

$$\{\mathrm{Re}(\lambda) > C_X - c_0 \min\{N_0, N_1\}\},$$

where $c_0 > 0$ *is the constant from Lemma 3.3. The poles of this meromorphic extension are called the Pollicott-Ruelle resonances and the range of the residues are the corresponding generalised resonant states. Moreover, the poles and residues of the meromorphic extension are intrinsic and do not depend on the choice of escape function used to define the anisotropic Sobolev space.*

This result should be understood as follows. In these references, $(\mathbf{X}+\lambda) : \mathcal{D}(\mathbf{X}) \to \mathcal{H}_h^m$ is shown to be a family of Fredholm operators of index 0 depending analytically on λ in the region $\{\mathrm{Re}(\lambda) > C_X - \frac{c_0}{2} \min\{N_0, N_1\}\}$. Then, the poles of the meromorphic extension are the eigenvalues of $-\mathbf{X}$ on $\mathcal{H}_h^{m_X^{N_0,N_1}}(\mathcal{M},\mathcal{E})$. We shall briefly rediscuss the proofs of [FaSj, DyZw1] in Proposition 6.1 below as we will need to control the continuity of $(\mathbf{X}+\lambda)^{-1}$ with respect to $X \in \mathcal{A}$. We also refer to the recent work of Guedes-Bonthonneau for related results [GB].

Remark 1. *For technical reasons appearing later in the analysis of the wave-front set of the Schwartz kernel of* $(\mathbf{X}+\lambda)^{-1}$*, we use a semiclassical parameter* h *and a semiclassical quantization, even though the operator* $\mathbf{X}+\lambda$ *is not semiclassical. For this Proposition, one could just fix* $h = 1$ *but some statement for* $h \to 0$ *will be used later on in the proof of Proposition 6.3.*

Remark 2. *In the following, we will take* $N_0 = N_1$ *and thus we will omit the index* N_1 *in* $G_X^{N_0,N_1}$*,* $m_X^{N_0,N_1}$ *and* $\mathbf{A}_h(N_0, N_1, X)$*.*

4. Twisted Ruelle zeta function and variation formula

In this section, we shall introduce the Ruelle zeta function and derive a formula[7] for its variation with respect to the vector field $X \in \mathcal{A}$. More precisely, we consider a smooth 1-parameter family $\tau \in (-1,1) \mapsto X_\tau \in \mathcal{A}$ on $\mathcal{M}$ and we fix a representation $\rho : \pi_1(\mathcal{M}) \to \mathrm{GL}(\mathbb{C}^r)$. We define the *Ruelle zeta function* of (X_τ, ρ) as in [Fr4] by the converging product[8]

$$\zeta_{\tau,\rho}(\lambda) := \prod_{\gamma_\tau \in \mathcal{P}_\tau} \det(1 - \varepsilon_{\gamma_\tau} \rho([\gamma_\tau]) e^{-\lambda \ell(\gamma_\tau)}) \tag{4.1}$$

for $\mathrm{Re}(\lambda) > \Lambda_\tau$ (for some $\Lambda_\tau > 0$), where $\mathcal{P}_\tau$ is the set of primitive periodic orbits of X_τ, $[\gamma_\tau]$ represents the class of γ_τ in $\pi_1(\mathcal{M})$, and $\ell(\gamma_\tau)$ denotes the period of the orbit γ_τ. Recall also that $\varepsilon_{\gamma_\tau}$ is the orientation index of the closed orbit. To justify the convergence, it suffices to

[6] The proof in [FaSj] was given in great details for $h = 1$ and one can verify that the region for the meromorphic extension can be chosen uniformly for $0 < h \leq 1$.

[7] Similar method is also used in [FRZ] for Selberg zeta function on surfaces of constant curvature.

[8] As we shall consider families $\tau \mapsto X_\tau$, if no confusion is possible we will use the index (or the exponent) τ instead of X_τ in the various quantities $\varphi_t^{X_\tau}$, $\zeta_{X_\tau,\rho}$, etc.

combine the fact that for a fixed Hermitian product $\langle\cdot,\cdot\rangle_E$ on E, there is $C>0$ depending only on $(\nabla, E, \langle\cdot,\cdot\rangle_E)$ such that $||\rho([\gamma_\tau])||_{E\to E}\leq e^{C\ell(\gamma_\tau)}$, together with Margulis bound [Ma] on the growth of periodic orbits

$$|\{\gamma\in\mathcal{P}_\tau:\ \ell(\gamma_\tau)\leq T\}|\sim\frac{e^{Th^\tau_{\rm top}}}{Th^\tau_{\rm top}}\quad\text{as}\quad T\to+\infty \tag{4.2}$$

where $h^\tau_{\rm top}$ denotes the topological entropy of the flow φ^τ_t of X_τ at time $t=1$.

4.1. **Variation of lengths of periodic orbits.** The first ingredient is the following consequence of the structural stability of Anosov flows:

Lemma 4.1. *Assume that $X_0\in\mathcal{A}$. There exists a neighborhood $\mathcal{U}(X_0)$ of X_0 such that $\tau\mapsto X_\tau\in\mathcal{U}(X_0)$ is a smooth family of Anosov flows on $\mathcal{M}$. Moreover, there is a smooth family $\tau\mapsto h_\tau\in C^0(\mathcal{M},\mathcal{M})$ of homeomorphisms defined near $\tau=0$ such that $h_\tau(\gamma_0)=\gamma_\tau$ for each $\gamma_0\in\mathcal{P}_0$, the map $\tau\mapsto\ell(\gamma_\tau)=\ell(h(\gamma_0))$ is C^1 near 0 for each $\gamma_0\in\mathcal{P}_0$, and*

$$\partial_\tau\ell(\gamma_\tau)=-\int_{\gamma_\tau}q_\tau$$

if $\partial_\tau X_\tau=q_\tau X_\tau+X_\tau^\perp$, with $X_\tau^\perp\in C^0(\mathcal{M};E_u(X_\tau)\oplus E_s(X_\tau))$.

Proof. We consider the Anosov vector field X_0. Following [DMM, App. A], we introduce the space $C_{X_0}(\mathcal{M},\mathcal{M})$ of continuous functions h from $\mathcal{M}$ to $\mathcal{M}$ which are C^1 along X_0. This means that, for all x in $\mathcal{M}$, the map $t\mapsto h\circ\varphi^t_{X_0}(x)$ is C^1 and the map $x\mapsto\frac{d}{dt}\left(h\circ\varphi^{X_0}_t(x)\right)_{t=0}=:D_{X_0}h(x)\in T\mathcal{M}$ is continuous. Building on earlier arguments of Moser and Mather for Anosov diffeomorphisms, de la Llave, Marco and Moriyon proved the structural stability theorem of Anosov via an implicit function theorem [DMM, App. A].

Proposition 4.2 (De la Llave-Marco-Moriyon [DMM])**.** *With the previous conventions, there exists an open neighborhood $\mathcal{U}(X_0)$ of X_0 in $\mathcal{A}$ and a $\mathcal{C}^\infty$ map*

$$S:X\in\mathcal{U}(X_0)\mapsto(h_X,\theta_X)\in C_{X_0}(\mathcal{M},\mathcal{M})\times C^0(\mathcal{M},\mathbb{R}),$$

where $S(X_0)=(\mathrm{Id},1)$ and

$$\partial_t(h_X(\varphi^0_t(x)))|_{t=0}=\theta_X(x)X(h_X(x)),\quad\forall x\in\mathcal{M}$$

if φ^0_t is the flow of X_0. Moreover, h_X is a homeomorphism of $\mathcal{M}$ for each X.

We take a connected component of the curve X_τ lying in $\mathcal{U}(X_0)$, which amounts to consider X_τ for $|\tau|<\delta$ with $\delta>0$ small enough. Writing the flow of X_τ by φ^τ_t and $h_\tau:=h_{X_\tau}$, $\theta_\tau:=\theta_{X_\tau}$, this result can be rewritten in an integrated version:

$$\forall x\in\mathcal{M},\quad h_\tau(\varphi^0_t(x))=\varphi^\tau_{\int_0^t\theta_\tau\circ\varphi^0_s(x)ds}(h_\tau(x)).$$

Fix now a primitive closed orbit γ_0 of the flow φ^0_t (with period $\ell(\gamma_0)$) and fix a point x_0 on this orbit. From the previous formula, one has

$$h_\tau(x_0)=\varphi^\tau_{\int_0^{\ell(\gamma_0)}\theta_\tau\circ\varphi^0_s(x_0)ds}\left(h_\tau(x_0)\right).$$

In particular, the period of the closed orbit for X_τ equals

$$\ell(\gamma_\tau) = \int_{\gamma_0} \theta_\tau \in C^\infty((-\delta,\delta),\mathbb{R}_+^*).$$

Let us now compute its derivative by differentiating $h_\tau(x_0) = \varphi^\tau_{\ell(\gamma_\tau)}(h_\tau(x_0))$ at $\tau = 0$:

$$\left(\frac{\partial h_\tau}{\partial \tau}(x_0)\right)_{|\tau=0} = \frac{\partial}{\partial \tau}\varphi^\tau_{\ell(\gamma_0)}(x_0)_{|\tau=0} + \partial_\tau \ell(\gamma_\tau)_{|\tau=0} X_0(x_0) + d\varphi^0_{\ell(\gamma_0)}(x_0)\cdot\left(\frac{\partial h_\tau}{\partial \tau}(x_0)\right)_{|\tau=0}. \tag{4.3}$$

Let $\beta_{x_0} : T_{x_0}\mathcal{M} \to \mathbb{R}$ be defined such that, if $V \in T_{x_0}\mathcal{M}$, then $V = \beta_{x_0}(V)X_0(x_0) + V^\perp$ where $V^\perp \in E_{u,x_0}(X_0) \oplus E_{s,x_0}(X_0)$. Pairing (4.3) with β_{x_0}, we get

$$\partial_\tau \ell(\gamma_\tau)_{|\tau=0} = -\beta_{x_0}\left(\frac{\partial}{\partial \tau}\varphi^\tau_{\ell(\gamma_0)}(x_0)_{|\tau=0}\right). \tag{4.4}$$

Since β_{x_0} is $d\varphi^0_{\ell(\gamma_0)}(x_0)$ invariant, we have

$$\begin{aligned}\beta_{x_0}\left(\frac{\partial}{\partial \tau}\varphi^\tau_{\ell(\gamma_0)}(x_0)_{|\tau=0}\right) &= \beta_{x_0}\left(\left(d\varphi^0_{\ell(\gamma_0)}(x_0)\right)^{-1}\cdot\frac{\partial}{\partial \tau}\varphi^\tau_{\ell(\gamma_0)}(x_0)_{|\tau=0}\right)\\ &= \int_0^{\ell(\gamma_0)} \frac{d}{dt}\beta_{x_0}\left(\left(d\varphi^0_t(x_0)\right)^{-1}\cdot\frac{\partial}{\partial \tau}\varphi^\tau_t(x_0)_{|\tau=0}\right)dt. \end{aligned}\tag{4.5}$$

On the other hand, we have

$$\frac{\partial}{\partial t}\left(\left(d\varphi^0_t(x_0)\right)^{-1}\cdot\frac{\partial}{\partial \tau}\varphi^\tau_t(x_0)_{|\tau=0}\right) = d\varphi^0_t(x_0)^{-1}\frac{\partial^2}{\partial s\partial \tau}\left(\varphi^0_{-s}\circ\varphi^\tau_{t+s}(x_0)\right)_{|(s,\tau)=0},$$

and $\frac{\partial}{\partial s}(\varphi^0_{-s}\circ\varphi^\tau_{t+s}(x_0)) = -X_0(\varphi^0_{-s}\circ\varphi^\tau_{t+s}(x_0)) + X_\tau(\varphi^0_{-s}\circ\varphi^\tau_{t+s}(x_0)) + \mathcal{O}(s)$. Hence, one finds

$$\frac{\partial}{\partial t}\left(\left(d\varphi^0_t(x_0)\right)^{-1}\cdot\frac{\partial}{\partial \tau}\varphi^\tau_t(x_0)_{|\tau=0}\right) = \left(d\varphi^0_t(x_0)\right)^{-1}\cdot\left(\frac{\partial X^\tau}{\partial \tau}(\varphi^0_t(x_0))\right)_{|\tau=0}. \tag{4.6}$$

By (4.4)-(4.6) and by the invariance of the Anosov splitting, we get the desired equation (the same argument works at each τ instead of $\tau = 0$). □

Remark 3. *A consequence Lemma 4.1 is that, for every $\gamma_0 \in \mathcal{P}_0$, one has*

$$\frac{\ell(\gamma_0)}{2} \le \ell(\gamma_\tau) \le 2\ell(\gamma_0),$$

provided that $\mathcal{U}(X_0)$ is chosen small enough (independently of the closed orbit).

4.2. **Variation of Ruelle zeta function in the convergence region.** We start with the following result which is a consequence of Lemma 4.1.

Lemma 4.3. *Under the above assumptions, there exist $\tau_0 > 0$ and $C_0 > 0$ such that $X_\tau \in \mathcal{U}(X_0)$ for every $\tau \in (-\tau_0, \tau_0)$ and such that the map*

$$\tau \in (-\tau_0,\tau_0) \mapsto \zeta_{\tau,\rho}(.) \in \mathrm{Hol}(\Omega_0)$$

is of class C^1 *where* $\Omega_0 := \{\mathrm{Re}(\lambda) > C_0\}$. *Moreover, for every* $\tau \in (-\tau_0, \tau_0)$

$$\zeta_{\tau,\rho}(\lambda) = \zeta_{0,\rho}(\lambda) \exp\left(-\lambda \int_0^\tau \sum_{\gamma_{\tau'}} \frac{\ell^\sharp(\gamma_{\tau'})}{\ell(\gamma_{\tau'})} \left(\int_{\gamma_{\tau'}} q_{\tau'}\right) e^{-\lambda \ell(\gamma_{\tau'})} \varepsilon_{\gamma_{\tau'}} \operatorname{Tr}(\rho([\gamma_{\tau'}])) d\tau'\right),$$

where the sum runs over all closed orbits of $X_{\tau'}$, $\ell^\sharp(\gamma_{\tau'})$ *is the period of the primitive orbit generating* $\gamma_{\tau'}$, $\varepsilon_{\gamma_{\tau'}}$ *is the orientation index*[9] *of* $\gamma_{\tau'}$ *and*

$$\int_{\gamma_{\tau'}} q_{\tau'} = \int_0^{\ell(\gamma_{\tau'})} q_{\tau'} \circ \varphi_t^{\tau'} dt.$$

Proof. The fact that $\lambda \mapsto \zeta_{\tau,\rho}(\lambda)$ is holomorphic in some half plane $\{\mathrm{Re}(\lambda) > C_\tau\}$ was already discussed. The fact that C_0 can be chosen uniformly in τ follows from Lemma 4.1 and Remark 3 together with (4.2) at $\tau = 0$. Let us now compute the derivative with respect to the parameter τ. For that purpose, we compute the derivative of each term in the sum defining $\log \zeta_{\tau,\rho}(.)$. Precisely, we write

$$\partial_\tau \left(\log\det\left(\mathrm{Id} - \varepsilon_{\gamma_\tau} e^{-\lambda \ell(\gamma_\tau)} \rho([\gamma_\tau])\right)\right) = \lambda \partial_\tau \ell(\gamma_\tau) \sum_{k=1}^{+\infty} e^{-k\lambda \ell(\gamma_\tau)} \varepsilon_{\gamma_\tau}^k \operatorname{Tr}(\rho([\gamma_\tau])^k).$$

The same kind of considerations as above allows to verify that the sum of this quantity over all primitive orbits is a continuous map from $(-\tau_0, \tau_0)$ to $\mathrm{Hol}(\Omega_0)$. Hence, the map $\tau \in (-\tau_0, \tau_0) \mapsto \ln \zeta_{\tau,\rho}(., \tau) \in \mathrm{Hol}(\Omega_0)$ is C^1 with a derivative given by

$$\partial_\tau \log \zeta_{\tau,\rho}(\lambda) = \lambda \sum_{\gamma \in \mathcal{P}_\tau} \partial_\tau \ell(\gamma_\tau) \sum_{k=1}^{+\infty} e^{-\lambda k \ell(\gamma_\tau)} \varepsilon_{\gamma_\tau}^k \operatorname{Tr}(\rho([\gamma_\tau])^k).$$

It remains to integrate this expression between 0 and τ and use Lemma 4.1. □

One of the technical issue with the formula of Lemma 4.3 is that q_τ is in general C^0 (or Hölder), and it makes it difficult to relate it with distributional traces as in [GLP, DyZw1]. To bypass this problem we introduce an invertible smooth bundle map $S_\tau : T\mathcal{M} \to T\mathcal{M}$ such that $S_\tau(X_0) = X_\tau$ and

$$\forall 0 \leq k \leq n, \quad A_\tau^{(k)} := \partial_\tau(\wedge^k S_\tau)\left(\wedge^k S_\tau^{-1}\right) : \wedge^k(T\mathcal{M}) \to \wedge^k(T\mathcal{M}). \tag{4.7}$$

Our next Lemma allows to express the variation of the Ruelle zeta function in terms of this bundle map $A_\tau^{(k)}$ instead of the continuous function q_τ:

Lemma 4.4. *With the conventions of Lemma 4.3, one has, for every* $\tau \in (-\tau_0, \tau_0)$, *for every closed orbit* γ_τ *and for every* $x \in \gamma_\tau$,

$$q_\tau(x) = -\frac{1}{\det(\mathrm{Id} - P(\gamma_\tau))} \sum_{k=0}^n (-1)^k \operatorname{Tr}\left(A_\tau^{(k)}(x) \wedge^k d\varphi_{\ell(\gamma_\tau)}^\tau(x)\right),$$

where $P(\gamma_\tau) = d\varphi_{\ell(\gamma_\tau)}^\tau(x)|_{E_u(X_\tau)\oplus E_s(X_\tau)}$ *is the linearized Poincaré map at* $x \in \gamma_\tau$.

[9] For a nonprimitive orbit $k.\gamma$, his is equal to $\varepsilon_{k.\gamma} = \varepsilon_\gamma^k$.

Proof. Fix τ_1 in $(-\tau_0,\tau_0)$ and x belonging to a closed orbit γ_{τ_1}. Write

$$\frac{\det\left(\mathrm{Id}-S_\tau S_{\tau_1}^{-1}d\varphi^{\tau_1}_{\ell(\gamma_{\tau_1})}(x)\right)}{\det\left(\mathrm{Id}-P(\gamma_{\tau_1})\right)}=\frac{\det\left(\mathrm{Id}-d\varphi^{\tau_1}_{\ell(\gamma_{\tau_1})}(x)-(S_\tau-S_{\tau_1})S_{\tau_1}^{-1}d\varphi^{\tau_1}_{\ell(\gamma_{\tau_1})}(x)\right)}{\det(\mathrm{Id}-P(\gamma_{\tau_1}))}.$$

We now differentiate this expression at $\tau=\tau_1$. We have

$$(S_\tau-S_{\tau_1})S_{\tau_1}^{-1}=(\tau-\tau_1)\left(\frac{dS_\tau}{d\tau}\right)_{|\tau=\tau_1}S_{\tau_1}^{-1}+\mathcal{O}((\tau-\tau_1)^2).$$

Oberve now that $\left(\frac{dS_\tau}{d\tau}\right)_{|\tau=\tau_1}S_{\tau_1}^{-1}(X_{\tau_1})=\left(\frac{dX_\tau}{d\tau}\right)_{|\tau=\tau_1}$. Hence, one finds

$$q_{\tau_1}=-\frac{d}{d\tau}\left(\frac{\det\left(\mathrm{Id}-S_\tau S_{\tau_1}^{-1}d\varphi^{\tau_1}_{\ell(\gamma_{\tau_1})}(x)\right)}{\det\left(\mathrm{Id}-P(\gamma_{\tau_1})\right)}\right)_{|\tau=\tau_1}$$

by using the decomposition $\mathbb{R}X_{\tau_1}\oplus E_s(X_{\tau_1})\oplus E_u(X_{\tau_1})$. On the other hand,

$$\det\left(\mathrm{Id}-S_\tau S_{\tau_1}^{-1}d\varphi^{\tau_1}_{\ell(\gamma_{\tau_1})}(x)\right)=\sum_{k=0}^n(-1)^k\mathrm{Tr}\left(\wedge^k\left(S_\tau S_{\tau_1}^{-1}d\varphi^{\tau_1}_{\ell(\gamma_{\tau_1})}(x)\right)\right).$$

Differentiating this expression at $\tau=\tau_1$, this yields

$$q_{\tau_1}=-\frac{1}{\det\left(\mathrm{Id}-P(\gamma_{\tau_1})\right)}\sum_{k=0}^n(-1)^k\mathrm{Tr}\left(\frac{d}{d\tau}\left(\wedge^k\left(S_\tau S_{\tau_1}^{-1}d\varphi^{\tau_1}_{\ell(\gamma_{\tau_1})}(x)\right)\right)_{|\tau=\tau_1}\right),$$

from which the conclusion follows. □

Combining Lemma 4.3 and Lemma 4.4, we get

Corollary 4.5. *With the conventions of Lemma 4.3, one has, for every $\tau\in(-\tau_0,\tau_0)$ and for $\lambda\in\Omega_0$*

$$\frac{\zeta_{\tau,\rho}(\lambda)}{\zeta_{0,\rho}(\lambda)}=\exp\left(-\lambda\int_0^\tau\sum_{k=0}^n(-1)^k\sum_{\gamma_{\tau'}}\frac{\ell^\sharp(\gamma_{\tau'})}{\ell(\gamma_{\tau'})}\frac{\left(\int_{\gamma_{\tau'}}\mathrm{Tr}\left(A^{(k)}_{\tau'}\wedge^k d\varphi^{\tau'}_{\ell(\gamma_{\tau'})}\right)\right)}{|\det(\mathrm{Id}-P(\gamma_{\tau'}))|e^{\lambda\ell(\gamma_{\tau'})}}\mathrm{Tr}(\rho([\gamma_{\tau'}]))d\tau'\right).$$

5. Variation formula in the non-convergent region

We recall that [GLP, DyZw1] show that $\zeta_{\tau,\rho}(\lambda)$ admits a meromorphic continuation $\lambda\in\mathbb{C}$. This was achieved by relating the Ruelle zeta function to some *flat trace* of some operator. We will use similar ideas to rewrite $\frac{\zeta_{\tau,\rho}(\lambda)}{\zeta_{0,\rho}(\lambda)}$ in terms of flat traces by analysing

$$F^{(k)}_\tau(\lambda):=\sum_{\gamma_\tau}\frac{\ell^\sharp(\gamma_\tau)}{\ell(\gamma_\tau)}\frac{\left(\int_{\gamma_\tau}\mathrm{Tr}\left(A^{(k)}_\tau\wedge^k d\varphi^\tau_{\ell(\gamma_\tau)}(x)\right)\right)e^{-\lambda\ell(\gamma_\tau)}}{|\det(\mathrm{Id}-P(\gamma_\tau))|}\mathrm{Tr}(\rho([\gamma_\tau])).\qquad(5.1)$$

Note that, in these references, the meromorphic extension was proved under some orientability hypothesis but this assumption can be removed by introducing the orientation index in the definition of the Ruelle zeta function as we did.

5.1. **Reformulation via distributional traces.** Let us start with a brief reminder on *flat traces*. First, if M is a compact manifold and $\Gamma \subset T_0^*M$ a closed conic subset, we define, following Hörmander [Hö, Section 8.2], the space

$$\mathcal{D}'_\Gamma(M) := \{u \in \mathcal{D}'(M); \mathrm{WF}(u) \subset \Gamma\}.$$

Its topology is described using sequences in [Hö, Def. 8.2.2.], we will recall it later. Denote by Δ the diagonal in $\mathcal{M}\times\mathcal{M}$ and by $N^*\Delta \subset T_0^*(\mathcal{M}\times\mathcal{M})$ the conormal bundle to the diagonal. If $E \to \mathcal{M}$ is a vector bundle over $\mathcal{M}$, the Atiyah–Bott flat trace of a $K \in \mathcal{D}'_\Gamma(\mathcal{M}\times\mathcal{M}; E\otimes E^*)$ with $\Gamma \cap N^*\Delta = \emptyset$ is defined by

$$\mathrm{Tr}^\flat(K) := \langle \mathrm{Tr}(\iota_\Delta^* K), 1\rangle$$

where $i_\Delta : \mathcal{M} \to \mathcal{M}\times\mathcal{M}$ is the natural inclusion map $i(x) := (x, x)$ and Tr denotes the local trace of endomorphisms $\mathrm{End}(E) = E \otimes E^*$, so that $\mathrm{Tr}(i_\Delta^* K) \in \mathcal{D}'(\mathcal{M})$.

Lemma 5.1. *For each closed conic subset $\Gamma \subset T^*(\mathcal{M}\times\mathcal{M})$ satisfying $\Gamma \cap N^*\Delta = \emptyset$, the flat trace $\mathrm{Tr}^\flat$ is a sequentially continuous linear form*

$$\mathrm{Tr}^\flat : \mathcal{D}'_\Gamma(\mathcal{M}\times\mathcal{M}; E\otimes E^*) \to \mathbb{C}$$

with respect to the topology of $\mathcal{D}'_\Gamma(\mathcal{M}\times\mathcal{M}; E\otimes E^*)$.

Proof. This follows directly from continuity of the pullback from $\mathcal{D}'_\Gamma(\mathcal{M}\times\mathcal{M}; E\otimes E^*) \mapsto \mathcal{D}'_{i_\Delta^*\Gamma}(\mathcal{M})$ [Hö, Theorem 8.2.4] and continuity of the pairing against 1. ☐

For an operator $B : C^\infty(\mathcal{M}; E) \to \mathcal{D}'(\mathcal{M}; E)$ with Schwartz kernel K_B satisfying $K_B \in \mathcal{D}'_\Gamma(\mathcal{M}\times\mathcal{M}; E\otimes E^*)$ for some Γ with $\Gamma \cap N^*\Delta = \emptyset$, we write

$$\mathrm{Tr}^\flat(B) := \mathrm{Tr}^\flat(K_B).$$

Then, by a slight extension of the Guillemin trace formula [GS, p. 315], we have

$$\mathrm{Tr}^\flat\left(A_\tau^{(k)} e^{-t\mathbf{X}_\tau}|_{\Omega^k(M,E)}\right) = \sum_{\gamma_\tau} \frac{\ell^\sharp(\gamma_\tau)}{\ell(\gamma_\tau)} \frac{\int_{\gamma_\tau} \mathrm{Tr}\left(A_\tau^{(k)} \wedge^k d\varphi^\tau_{\ell(\gamma_\tau)}\right)}{|\det(\mathrm{Id} - P(\gamma_\tau))|} \mathrm{Tr}(\rho([\gamma_\tau]))\delta(t - \ell(\gamma_\tau)), \quad (5.2)$$

in $\mathcal{D}'(\mathbb{R}_{>0})$, where this equality holds for every τ such that $X_\tau \in \mathcal{U}(X_0)$ and where the sum runs over all closed orbits. Here, we choose $t_0 > 0$ so that there is some $C > 0$ uniform in τ (τ is also close enough to 0) such that $\min_{x\in\mathcal{M}} d_g(x, \varphi^\tau_{t_0}(x)) \geq C$ and define the meromorphic family of operators (well-defined by Proposition 3.4)

$$Q_\tau(\lambda) := e^{-t_0\mathbf{X}_\tau}(-\mathbf{X}_\tau - \lambda)^{-1}. \quad (5.3)$$

By the same arguments as in [DyZw1, § 4], we obtain that $\mathrm{Tr}^\flat(A_\tau^{(k)} Q_\tau(\lambda)|_{\mathcal{E}^k})$ is well-defined for each small τ as a meromorphic function in $\lambda \in \mathbb{C}$ and

$$\text{if } \mathrm{Re}(\lambda) > C_0, \quad F_\tau^{(k)}(\lambda) = -e^{-\lambda t_0}\mathrm{Tr}^\flat\left(A_\tau^{(k)} Q_\tau(\lambda)|_{\mathcal{E}^k}\right) \quad (5.4)$$

with $C_0 > 0$ given by Lemma 4.3.

5.2. **Proof of Theorem 2.** The proof of Theorem 2 will follow directly from Corollary 4.5 and the following

Theorem 4. *Assume that $X_0 \in \mathcal{A}$ is such that $\mathbf{X}_0$ has no Ruelle resonance at $\lambda = 0$ and let $\mathcal{Z} \subset \mathbb{C}$ be a simply connected open subset containing 0 and a point inside the region $\{\mathrm{Re}(\lambda) > C_{X_0}\}$ and such that $\mathbf{X}_0$ has no Ruelle resonance in $\overline{\mathcal{Z}}$. Then, there exists a neighborhood $\mathcal{U}(X_0) \subset \mathcal{A}$ of X_0 such that*
1) the operator $(-\mathbf{X} - \lambda)^{-1}$ of Proposition 3.4 is holomorphic in $\mathcal{Z}$ for all $X \in \mathcal{U}(X_0)$.
2) if $\tau \mapsto X_\tau \in \mathcal{U}(X_0)$ is a smooth map with $X_\tau|_{\tau=0} = X_0$, then $\tau \mapsto \mathrm{Tr}^\flat(A_\tau^{(k)} Q_\tau(\lambda)|_{\mathcal{E}^k})$ is continuous with values in $\mathrm{Hol}(\mathcal{Z})$, with $A_\tau^{(k)}$ defined by (4.7).

Take $B_k(X_0, \epsilon) := \{X \in \mathcal{A}; \|X - X_0\|_{C^k} \leq \epsilon\}$ contained in the neighborhood $\mathcal{U}(X_0)$ of Theorem 4, for some $k \in \mathbb{N}, \epsilon > 0$, and for $X \in B_k(X_0, \epsilon)$ define $X_\tau := X_0 + \tau(X - X_0)$ for $\tau \in (-\delta, 1 + \delta)$ with $\delta > 0$ small so that $X_\tau \in B_k(X_0, \epsilon)$. Now each $\mathbf{X}_\tau$ has no resonances in $\mathcal{Z}$ and 2) in Theorem 4 with (5.4) show that $\tau \mapsto F_\tau^{(k)}(\lambda)$ can be extended as a continuous family of functions in $\mathrm{Hol}(\mathcal{Z})$ for $\tau \in [0, 1]$. Corollary 4.5 then shows that $\zeta_{\tau,\rho}(\lambda)/\zeta_{0,\rho}(\lambda)$ admits a holomorphic extension in $\mathcal{Z}$ with $\zeta_{\tau,\rho}(0) = \zeta_{0,\rho}(0)$. Thus $\zeta_{X,\rho}(0) = \zeta_{X_0,\rho}(0)$. The proof of Theorem 4 will be given in the next section.

6. Continuity of the resolvent and Proof of Theorem 4

The purpose of this section is to prove the properties of the Schwartz kernel of the resolvent that were used in the proof of Theorem 2. We are interested in the continuity with respect to τ of the flat trace of the operator

$$Q_\tau(\lambda) := e^{-t_0 \mathbf{X}_\tau}(-\mathbf{X}_\tau - \lambda)^{-1} \tag{6.1}$$

where we recall that we chose $t_0 > 0$ so that there is some $C > 0$ uniform in τ (here τ is close enough to 0) such that

$$\min_{x \in \mathcal{M}} d_g(x, \varphi_{t_0}^\tau(x)) \geq C$$

where d_g is the Riemannian distance induced by a metric g. The arguments used here are variations on the microlocal proofs of Faure-Sjöstrand in [FaSj] and Dyatlov-Zworski in [DyZw1]. The continuity with respect to the resolvent also follows from Butterley-Liverani [BuLi]. For $k \in \mathbb{R}$, we will write $\Psi_h^k(\mathcal{M}; \mathcal{E})$ for the space of semi-classical pseudo-differential operators [Zw1, Chapter 14.2] (on sections of $\mathcal{E}$) with symbols in the class $S_h^k(T^*\mathcal{M}; \mathcal{E})$ defined by: $a_h \in S_h^k(T^*\mathcal{M}; \mathcal{E})$ if $a_h \in C^\infty(T^*\mathcal{M}; \mathrm{End}(\mathcal{E}))$ satisfies $|\partial_x^\alpha \partial_\xi^\beta a_h(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{k - |\beta|}$ with $C_{\alpha\beta}$ independent of h. As mentionned before, we also take a semi-classical quantisation Op_h mapping $S_h^k(T^*\mathcal{M}; \mathcal{E})$ to $\Psi_h^k(\mathcal{M}; \mathcal{E})$. The operators in the class $\Psi^k(\mathcal{M}; \mathcal{E}) := \Psi_{h_0}^k(\mathcal{M}; \mathcal{E})$ for some fixed small $h_0 > 0$ are called pseudo-differential operators. We introduce the family of h-pseudodifferential operators:

$$P_X(h, \lambda) := \mathbf{A}_h(N_0, X)(-h\mathbf{X} - h\lambda)\mathbf{A}_h(N_0, X)^{-1}. \tag{6.2}$$

6.1. **Continuity of the resolvent for families of Anosov flows.** For the first part of Theorem 4 we prove:

Proposition 6.1. *Let X_0 and $\mathcal{Z}$ chosen as in Theorem 4. There exist a neighborhood $\mathcal{U}(X_0)$ of X_0, $h_0 > 0$ and $C > 0$ such that, for every $0 < h < h_0$, and for every $X \in \mathcal{U}(X_0)$, the map $\lambda \in \mathcal{Z} \mapsto P_X(h,\lambda)^{-1} \in \mathcal{L}(L^2, L^2)$ is holomorphic and*

$$\forall \lambda \in \mathcal{Z}, \quad \left\| P_X(h,\lambda)^{-1} \right\|_{H^1_h \to L^2} \leq C h^{-1-4N_0}. \tag{6.3}$$

Moreover, for every $0 < h < h_0$, the following map is continuous

$$X \in \mathcal{U}(X_0) \mapsto P_X(h,\lambda)^{-1} \in \mathrm{Hol}\left(\mathcal{L}(H^1_h, L^2)\right).$$

Proof. In order to prove this Proposition, we need to review the proofs from [FaSj, p.340-345] – see also paragraph 5 from this reference or [DyZw1] for a semiclassical formulation as described here. Note already from Proposition 3.4 that, for every $X \in \mathcal{U}(X_0)$, $\lambda \in \mathcal{Z} \mapsto P_X(h,\lambda)^{-1} \in \mathcal{L}(L^2, L^2)$ is meromorphic.

Recall from [FaSj, Lemma 5.3] that

$$P_X(h,\lambda) = \mathrm{Op}_h\left(\left(-iH_X - h\lambda + h\left\{H_X, G_X^{N_0}\right\}\right)\mathrm{Id}\right) + \mathcal{O}_X(h) + \mathcal{O}_{m_X^{N_0}}(h^2), \tag{6.4}$$

where $H_X(x,\xi) = \xi(X(x))$ and where the remainders are understood as bounded operator on $L^2(\mathcal{M};\mathcal{E})$. Only the second remainder depends on the choice of the order function, and both remainders can be made uniform in terms of $X \in \mathcal{U}(X_0)$ thanks to Lemma 3.3. Following [FaSj, § 3.3], one can introduce an operator $\hat{\chi}_0 = \mathrm{Op}_h(\chi_0 \mathrm{Id})$ in $\Psi^0_h(\mathcal{M};\mathcal{E})$ depending only on X_0 with $\chi_0 \geq 0$ and so that (c_0 is the constant from Lemma 3.3)

$$\forall (x,\xi) \in T^*\mathcal{M}, \quad \left\{H_X, G_X^{N_0}\right\} - \chi_0(x,\xi)^2 \leq -2c_0 N_0 \tag{6.5}$$

Remark 4. *Note that we have some flexibility in the choice of the operator $\widehat{\chi}_0$. Besides the fact that it belongs to $\Psi^0_h(\mathcal{M},\mathcal{E})$, the only requirements we shall need are*

- *$\chi_0^2 = C_{N_0} + 2c_0 N_0$ (inside a small conic neighborhood of $E_0^*(X_0)$, where $C_{N_0} > 0$ is the uniform constant from (3.6)),*
- *outside a slightly larger conic neighborhood of $E_0^*(X_0)$, $\mathrm{supp}(\chi_0)$ is contained in $\{\|\xi\| \leq 3R/2\}$ where R is the parameter from Lemma 3.3,*
- *χ_0 satisfies (6.5) in $\{\|\xi\| \leq R\}$.*

Next we let $\hat{\chi}_1 = \mathrm{Op}_h(\chi_1 \mathrm{Id}) \in \Psi^0_h(\mathcal{M})$ with $\chi_1 \in C^\infty_0(T^*\mathcal{M}, \mathbb{R}_+)$ satisfying $\mathrm{supp}(\chi_1) \subset \{\|\xi\| \leq 3R/2\}$, and $\chi_1(x,\xi) = 1$ for $\|\xi\| \leq R$, and we define[10]

$$\hat{\chi} := \hat{\chi}_1^* \hat{\chi}_1 + h \hat{\chi}_0^* \hat{\chi}_0 \in \Psi^0_h(\mathcal{M};\mathcal{E}). \tag{6.6}$$

Following [FaSj, p. 344] (with the addition of a semiclassical parameter), one can verify that, for $0 < h \leq h_0$ small enough,

$$(P_X(h,\lambda) - \hat{\chi})^{-1} : L^2(\mathcal{M},\mathcal{E}) \to L^2(\mathcal{M},\mathcal{E})$$

[10] The operator $\hat{\chi}_1^* \hat{\chi}_1$ is not necessary for this proof but will be useful for the wavefront set analysis later.

is bounded for $\mathrm{Re}(\lambda) > C_0 - c_0 N_0$, where C_0 is some positive constant that can be chosen uniformly in terms of $X \in \mathcal{U}(X_0)$. Moreover, their proof yields a uniform upper bound: there is $C > 0$ such that

$$\forall X \in \mathcal{U}(X_0),\ \forall 0 < h < 1, \quad \left\| (P_X(h,\lambda) - \hat{\chi})^{-1} \right\|_{L^2 \to L^2} \leq C h^{-1}. \tag{6.7}$$

By adding a constant $s \in [-1,1]$ to the order function $m_X^{N_0}$, the same argument as above works and we can pick the operators $\hat{\chi}_0$ and $\hat{\chi}_1$ independently of $s \in [-1,1]$. Since the consideration of $P_\tau(h,\lambda) - \hat{\chi}$ acting on $H_h^s(\mathcal{M};\mathcal{E})$ is equivalent to its conjugation by $\mathrm{Op}_h((1+f)^s)$, it implies that

$$h(P_X(h,\lambda) - \hat{\chi})^{-1} : H_h^s(\mathcal{M};\mathcal{E}) \to H_h^s(\mathcal{M};\mathcal{E}) \tag{6.8}$$

is uniformly bounded in (λ, X, h) for all (X,λ) as before and all $h > 0$ small. In order to study the continuity, we first write

$$\begin{aligned}(P_X(h,\lambda) - \hat{\chi})^{-1} &= (P_{X_0}(h,\lambda_0) - \hat{\chi})^{-1} \\ &+ (P_X(h,\lambda) - \hat{\chi})^{-1} \left(P_{X_0}(h,\lambda_0) - P_X(h,\lambda)\right) (P_{X_0}(h,\lambda_0) - \hat{\chi})^{-1}.\end{aligned}$$

Thanks to the Calderón-Vaillancourt Theorem [Zw1, Th. 5.1], one knows that

$$\|P_{X_0}(h,\lambda_0) - P_X(h,\lambda)\|_{H_h^1 \to L^2} \leq C\|X - X_0\|_{C^k} + h|\lambda - \lambda_0|$$

for some $k \geq 1$ large enough (depending only on the dimension of $\mathcal{E}$) and for some $C > 0$ independent of h, X and λ. Hence, combined with (6.8), we find that the map $(X,\lambda) \mapsto (P_X(h,\lambda) - \hat{\chi})^{-1} \in \mathcal{L}(H_h^1, L^2)$ is continuous.

Next, as in [FaSj, p. 344], one can construct $E_X(h,\lambda) \in \Psi_h^{-1}(\mathcal{M};\mathcal{E})$ whose principal symbol is supported in a conic neighborhood of $E_0^*(X_0)$ so that

$$(P_X(h,\lambda) - \hat{\chi}) E_X(h,\lambda) = \mathrm{Id} + S_X(h,\lambda), \quad E_X(h,\lambda)(P_X(h,\lambda) - \hat{\chi}) = \mathrm{Id} + T_X(h,\lambda)$$

with $S_X(h,\lambda)$ and $T_X(h,\lambda)$ both in $\Psi_h^0(\mathcal{M};\mathcal{E})$ such that the support of their principal symbols intersects $\mathrm{supp}(\chi_0) \cup \mathrm{supp}(\chi_1)$ inside a compact region of $T^*\mathcal{M}$ which is independent of (X,λ). Note that all these pseudodifferential operators depend continuously in (X,λ) (these are just parametrices in the elliptic region). Then,

$$K_X(h,\lambda) := \hat{\chi}(P_X(h,\lambda) - \hat{\chi})^{-1} = \hat{\chi} E_X(h,\lambda) - \hat{\chi} T_X(h,\lambda)(P_X(h,\lambda) - \hat{\chi})^{-1} \tag{6.9}$$

is compact as $\hat{\chi} E_X(h,\lambda) \in \Psi_h^{-1}(\mathcal{M};\mathcal{E})$ and $\hat{\chi} T_X(h,\lambda) \in \Psi_h^{-1}(\mathcal{M},\mathcal{E})$.

This operator (viewed as an element of $\mathcal{L}(H_h^1, H_h^1)$) depends continuously on (X,λ). Moreover, from our upper bound on the modulus of continuity of $(X,\lambda) \mapsto (P_X(h,\lambda) - \hat{\chi})^{-1}$, we get

$$\|K_X(h,\lambda) - K_{X_0}(h,\lambda_0)\|_{H_h^1 \to H_h^1} \leq \frac{1}{h^2} \omega(|\lambda - \lambda_0|, ||X - X_0||_{C^k}),$$

where $\omega(x,y)$ is independent of (h,X,λ) and verifies $\omega(x,y) \to 0$ as $(x,y) \to 0$. With this family of compact operators, we get the identity (as meromorphic operators in λ on H_h^1)

$$P_X(h,\lambda)^{-1} = (P_X(h,\lambda) - \hat{\chi})^{-1} (\mathrm{Id} + K_X(h,\lambda))^{-1}. \tag{6.10}$$

Now, from the definition of $\mathcal{Z}$, we know that, for every $\lambda \in \mathcal{Z}$, $(\mathrm{Id} + K_{X_0}(h,\lambda))$ is invertible in $\mathcal{L}(H_h^1, H_h^1)$. Thus, by continuity of the inverse map, we can then conclude that this remains true for any $||X - X_0||_{C^k}$ small enough uniformly for $\lambda \in \mathcal{Z}$ (as of $P_{X_0}(h,\lambda)$ remains invertible for λ in $\overline{\mathcal{Z}}$). The neighborhood depends a priori on h but, as all the operators $P_X(h,\lambda)$ are conjugated for different values of h, it can be made uniform in h. It now only remains to verify the upper bound on the norm of the resolvent. For that purpose, we can fix $h = h_0 > 0$ with h_0 small enough. The above proof shows that $P_X(h_0,\lambda)$ is uniformly bounded (for $X \in \mathcal{U}(X_0)$ and $\lambda \in \overline{\mathcal{Z}}$) as an operator from $H^1_{h_0}$ to L^2. Then, we write for $X \in \mathcal{U}(X_0)$

$$P_X(h,\lambda) = \frac{h}{h_0}\mathbf{A}_h(N_0,X)\mathbf{A}_{h_0}(N_0,X)^{-1}P_X(h_0,\lambda)\mathbf{A}_{h_0}(N_0,X)\mathbf{A}_h(N_0,X)^{-1}.$$

We observe that

$$\left\|\mathbf{A}_h(N_0,X)\mathbf{A}_{h_0}(N_0,X)^{-1}\right\|_{L^2\to L^2} + \left\|\mathbf{A}_{h_0}(N_0,X)\mathbf{A}_h(N_0,X)^{-1}\right\|_{H^1_h\to H^1_{h_0}} \lesssim h^{-2N_0},$$

from which we can deduce the expected upper bound on the norm of the resolvent. □

6.2. **Wavefront set of the Schwartz kernel of the resolvent.** The next part consists in bounding locally uniformly in (τ,λ) the Schwartz kernel of the operator $Q_\tau(\lambda)$ defined in (6.1).

First, let us introduce a bit of terminology. Let M be a compact manifold (in practice, we take $M = \mathcal{M}$ or $M = \mathcal{M}\times\mathcal{M}$). We refer for example to [DyZw1, Appendix C.1] for a summary of the notion of wavefront set $\mathrm{WF}(A) \subset T^*_0M$ (resp. $\mathrm{WF}(u) \subset T^*_0M$) of an operator $A \in \Psi^k(M)$ (resp. of a distribution $u \in \mathcal{D}'(M)$). For $\Gamma \subset T^*_0M$ a closed conic set, we say that a family $u_\tau \in \mathcal{D}'(M)$ with $\tau \in [\tau_1,\tau_2] \subset \mathbb{R}$ is bounded in $\mathcal{D}'_\Gamma$ if it is bounded in $\mathcal{D}'$ and for each τ-independent $A \in \Psi^0(M)$ with $\mathrm{WF}(A)\cap\Gamma = \emptyset$,

$$\forall N \in \mathbb{N}, \exists C_{N,A} > 0, \forall \tau \in [\tau_1,\tau_2], \quad ||A(u_\tau)||_{H^N} \leq C_{N,A}.$$

This can also be described in terms of Fourier transform in charts (see [DyZw1, Appendix C.1]). Similarly, we refer to [DyZw1, Appendix C.2] for a summary on the semi-classical wavefront set $\mathrm{WF}_h(A) \subset \overline{T^*M}$ (resp. $\mathrm{WF}_h(u) \subset \overline{T^*M}$) of an operator $A = \mathrm{Op}_h(a_h) \in \Psi^k_h(M)$ (resp. of a h-tempered family of distributions $u_h \in \mathcal{D}'(M)$); here $\overline{T^*M}$ denotes the fiber-radially compactified cotangent bundle (see [Va, Section 2.1]). For $\Gamma \subset \overline{T^*\mathcal{M}}$ a closed set (not necessarily conic), we say that a family of h-tempered distributions $u_{h,\tau}$ (in the sense $\sup_\tau ||u_{\tau,h}||_{H^{-N}(M)} = \mathcal{O}(h^{-N})$ for some $N > 0$) is bounded in $\mathcal{D}'_\Gamma$ if for each τ-independent $A \in \Psi^0_h(M)$ with $\mathrm{WF}_h(A)\cap\Gamma = \emptyset$,

$$\forall N \in \mathbb{N}, \exists C_{N,A} > 0, \forall \tau \in [\tau_1,\tau_2], \quad ||A(u_{h,\tau})||_{H^N(M)} \leq C_{N,A}h^N.$$

This can also be described in terms of the semiclassical Fourier transform in charts (see [DyZw1, Appendix C.2]).

We recall from [Hö, Definition 8.2.2] the topology of $\mathcal{D}'_\Gamma(M)$: a sequence $u_\tau \in \mathcal{D}'_\Gamma(M)$ converges to u_{τ_0} in $\mathcal{D}'_\Gamma(M)$ as $\tau \to \tau_0$ if $u_\tau \to u_{\tau_0}$ in $\mathcal{D}'(M)$ and $(u_\tau)_\tau$ is bounded in $\mathcal{D}'_\Gamma$.

We note that all these properties hold the same way for sections of vector bundles.

Next, we recall a result which is essentially Lemma 2.3 in [DyZw1] characterising the wave-front set of a family $K_\tau \in \mathcal{D}'(\mathcal{M}\times\mathcal{M};\mathcal{E}\otimes\mathcal{E}')$, but uniformly in the parameter τ. We shall use a semi-classical parameter $h>0$ for this characterisation.

Lemma 6.2. *Let $K_\tau \in \mathcal{D}'(\mathcal{M}\times\mathcal{M};\mathcal{E}\otimes\mathcal{E}')$ be an h-independent bounded family depending on $\tau\in[\tau_1,\tau_2]$ and let $\mathcal{K}_\tau$ be the associated operator on $\mathcal{M}$. Let $\Gamma\subset T^*(\mathcal{M})\times T^*(\mathcal{M})$ be a fixed closed conic set, independent of τ. Assume that for each point $(y,\eta,z,-\zeta)\in(T^*\mathcal{M}\times T^*\mathcal{M})\backslash\Gamma$ with $||(\eta,\zeta)||\in[2R,4R]$ (for some $R>0$), there are small relatively compact neighborhoods U of (z,ζ) and V of (y,η) in $T^*\mathcal{M}$ such that, for all family $f_h\in C^\infty(\mathcal{M};\mathcal{E})$ independent of τ satisfying $||f_h||_{L^2}=1$ and $\mathrm{WF}_h(f_h)\subset U$, then for each τ-independent $B_h\in\Psi^0_h(\mathcal{M},\mathcal{E})$ microlocally supported inside V, we have $\mathrm{WF}_h(B_h\mathcal{K}_\tau f_h)\cap V=\emptyset$ uniformly in τ, i.e.*

$$\forall N\in\mathbb{N},\exists C_{N,B}>0,\forall\tau,\forall h\in(0,1)\quad ||B_h\mathcal{K}_\tau f_h||_{L^2}\leq C_{N,B}h^N. \tag{6.11}$$

Then, one has $(K_\tau)_\tau$ is a bounded family of distributions in $\mathcal{D}'_\Gamma(\mathcal{M}\times\mathcal{M};\mathcal{E}\otimes\mathcal{E}')$.

Proof. The proof is readily the same as [DyZw1, Lemma 2.3] by just adding the τ dependence and we note that it suffices to fix $||(\eta,\zeta)||\in[2R,4R]$ for some $R>0$ instead of considering all (η,ζ). □

6.2.1. *Main technical result.* We shall now prove that the kernel of the resolvent is uniformly bounded in $\mathcal{D}'_\Gamma(\mathcal{M}\times\mathcal{M};\mathcal{E}\otimes\mathcal{E}')$, where Γ is a closed cone that does not intersect the conormal $N^*\Delta$ of the diagonal.

Proposition 6.3. *There exist a small neighborhood $\mathcal{U}(X_0)$ of X_0 in the C^∞-topology and a closed conic set $\Gamma\subset T^*_0(\mathcal{M}\times\mathcal{M})$ not intersecting $N^*\Delta$ such that, for every $\tau\mapsto X_\tau$ as in 2) of Theorem 4,*

$$(\tau,\lambda)\in[-\delta,\delta]\times\overline{\mathcal{Z}}\mapsto Q_\tau(\lambda)(.,.)\in\mathcal{D}'_\Gamma(\mathcal{M}\times\mathcal{M},\mathcal{E}\otimes\mathcal{E}')$$

is bounded, where $\delta>0$ is small enough to ensure that $X_\tau\in\mathcal{U}(X_0)$ for all $\tau\in[-\delta,\delta]$.

Proof. Thanks to Proposition 6.1, we already know that the Schwartz kernel of $Q_\tau(\lambda)$ is uniformly bounded on $\mathcal{D}'(\mathcal{M}\times\mathcal{M};\mathcal{E}\otimes\mathcal{E}')$ and $Q_\tau(\lambda)\to Q_{\tau_0}(\lambda_0)$ in this space as $(\tau,\lambda)\to(\tau_0,\lambda_0)$ for $|\tau_0|\leq\delta,\lambda_0\in\overline{\mathcal{Z}}$. Hence, it only remains to show that the family is bounded in $\mathcal{D}'_\Gamma(\mathcal{M}\times\mathcal{M},\mathcal{E}\otimes\mathcal{E}')$. We shall use the criteria of Lemma 6.2 to get a bound on the kernel of the resolvent and, up to some details of presentation, we will follow partly [DyZw1] by combining with [FaSj] and we shall verify that everything is bounded uniformly in the parameter τ.

We take some $R>0$ larger than the R appearing in Lemma 3.3 and we fix some point (z,ζ) in $T^*\mathcal{M}$ such that $2R\leq\|\zeta\|\leq 4R$. Let U be a small enough neighborhood of (z,ζ) in $T^*\mathcal{M}$ so that $U_{t_0,\delta}:=\bigcup_{|\tau|\leq\delta}\Phi^\tau_{t_0}(U)$ satisfies $\overline{U}\cap\overline{U_{t_0,\delta}}=\emptyset$ where the existence of U is guaranteed by the choice of t_0. We also fix R large enough so that $\overline{U_{t_0,\delta}}\cap\{\|\xi\|\leq 3R/2\}=\emptyset$ for each (z,ζ) with $\|\zeta\|\in[2R,4R]$. Let $f_h\in C^\infty(\mathcal{M};\mathcal{E})$ be a family independent of τ such that $\mathrm{WF}_h(f_h)\subset U$ and $\|f_h\|_{L^2}=1$. Define

$$\tilde{f}_h(\tau):=he^{-t_0\mathbf{X}_\tau}f_h$$

which verifies that $\mathrm{WF}_h(\tilde{f}_h(\tau)) \subset \overline{U_{t_0,\delta}}$ uniformly in τ, thus not intersecting $\overline{U}$. Let

$$u_h(\tau,\lambda) = (-h\mathbf{X}_\tau - h\lambda)^{-1}\tilde{f}_h(\tau),$$

where $|\tau| \leq \delta$ for some small $\delta > 0$ and where λ varies in $\mathcal{Z}$.

We now conjugate the operators with $\mathbf{A}_h(N_0,\tau)$ in order to work with the more convenient operator $P_\tau(h,\lambda)$ defined in (6.2) (with $X = X_\tau$), i.e.

$$P_\tau(h,\lambda)\tilde{u}_h(\tau,\lambda) = \tilde{F}_h(\tau), \quad \text{with}$$
$$\tilde{u}_h(\tau,\lambda) := \mathbf{A}_h(N_0,\tau)u_h(\tau,\lambda), \quad \tilde{F}_h(\tau) := \mathbf{A}_h(N_0,\tau)\tilde{f}_h(\tau).$$

Observe that $\mathrm{WF}_h(\tilde{F}_h(\tau)) \subset \overline{U_{t_0,\delta}}$ uniformly in τ (as the order functions used to define $\mathbf{A}_h(N_0,\tau)$ are uniform in τ– see Lemma 3.3) and that $\|\tilde{F}_h(\tau)\|_{H^1_h} \lesssim \|\tilde{f}_h(\tau)\|_{H_h^{2N_0+1}} \lesssim h$, where the involved constants are still uniform for (τ,λ) in the allowed region. From the resolvent bound from Proposition 6.1, one has, uniformly in (τ,λ), $\|\tilde{u}_h(\tau,\lambda)\|_{L^2} \lesssim h^{-4N_0}$. In order to apply Lemma 6.2, we just need to verify that $\mathrm{WF}_h(\tilde{u}_h(\tau,\lambda)) \cap U = \emptyset$ uniformly in (τ,λ) thanks to the uniformity of $\mathbf{A}_h(N_0,\tau)$ in (τ,λ). For that purpose, we fix a family $(B_h)_{0<h\leq 1} \subset \Psi_h^0(\mathcal{M})$ whose semiclassical wavefront set is contained in $\overline{U}$. We also need to use the operator (with $\hat{\chi}$ defined in (6.6)) and functions

$$P_\tau^\chi(h,\lambda) := P_\tau(h,\lambda) - \hat{\chi}, \quad \tilde{u}_h^\chi(\tau,\lambda) := P_\tau^\chi(h,\lambda)^{-1}\tilde{F}_h(\tau)$$

where we recall that $P_\tau^\chi(h,\lambda)$ is invertible on $L^2(\mathcal{M})$ for $\lambda \in \overline{\mathcal{Z}}$ and that the norm of the inverse $||P_\tau^\chi(h,\lambda)^{-1}||_{L^2\to L^2} = \mathcal{O}(h^{-1})$ uniformly for (τ,λ) in the allowed region.

We start with the simplest part of phase space where the operator $P_\tau(h,\lambda)$ is elliptic, i.e. we suppose that $(z,\zeta) \in T_0^*\mathcal{M}$ does not belong to the cone

$$C^{us}(\alpha) := \left\{(x,\xi) \in T^*\mathcal{M}\backslash 0 : \alpha\|\xi_u + \xi_s\|' \geq \|\xi_0\|'\right\},$$

for some small $\alpha > 0$ with the conventions of Section 3.1; here and below, the cones are defined with respect to the Anosov decomposition of the vector field X_0. The operator $P_\tau(h,\lambda)$ is elliptic outside $C^{us}(\alpha)$ uniformly for τ small enough. We can then use the fact that $\mathrm{WF}_h(B_h)$ is contained in a region where the principal symbol of $P_\tau(h,\lambda)$ is uniformly (in (τ,λ)) bounded away from 0. This allows us to write, for every $N \geq 1$,

$$B_h = \tilde{B}_h^N(\tau,\lambda)P_\tau(h,\lambda) + \mathcal{O}_{L^2\to L^2}(h^N)$$

where $\tilde{B}_h^N(\tau,\lambda) \in \Psi_h^0(\mathcal{M})$ and where the constant in the remainder are uniform in (τ,λ) in the allowed region. Note that $\tilde{B}_h^N(\tau,\lambda)$ depends on (τ,λ) but, as these two parameters remain bounded, $\mathrm{WF}_h(\tilde{B}_h^N(\tau,\lambda)) \subset U$ uniformly in (τ,λ). Gathering these informations, we get

$$\|B_h\tilde{u}_h(\tau,\lambda)\|_{L^2} \leq \|\tilde{B}_h^N(\tau,\lambda)\tilde{F}_h(\tau)\|_{L^2} + \mathcal{O}(h^N)\|\tilde{u}_h(\tau,\lambda)\|_{L^2}.$$

Since $\mathrm{WF}_h(\tilde{F}_h(\tau)) \subset \overline{U_{t_0,\delta}}$ (uniformly in τ) does not intersect $\overline{U}$, we find that, for every $N \geq 1$, there exists $C_N > 0$ such that, for every (τ,λ) in the allowed region, $\|B_h\tilde{u}_h(\tau,\lambda)\|_{L^2} \leq C_N h^{N-4N_0}$. Therefore $\mathrm{WF}_h(\tilde{u}_h(\tau,\lambda)) \cap \overline{U} = \emptyset$ uniformly in (τ,λ).

Then, since $P_\tau^\chi(h,\lambda)$ is elliptic in $\{||\xi|| \leq R\}$ and outside $C^{us}(\alpha)$, the same ellipticity argument shows that uniformly for (τ,λ) in the allowed region we have

$$\|\zeta\| \in [2R, 4R] \Longrightarrow \mathrm{WF}_h(\tilde{u}_h^\chi(\tau,\lambda)) \cap \mathrm{WF}_h(\hat{\chi}) \subset \{\|\xi\| > R\} \tag{6.12}$$

while, if $(z,\zeta) \notin C^{us}(\alpha)$, then $\mathrm{WF}_h(\tilde{u}_h^\chi(\tau,\lambda)) \cap \overline{U} = \emptyset$ uniformly in (τ,λ). Observe that

$$\tilde{u}_h(\tau,\lambda) = \tilde{u}_h^\chi(\tau,\lambda) - P_\tau(h,\lambda)^{-1}\hat{\chi}\tilde{u}_h^\chi(\tau,\lambda),$$

hence if we can prove that $\hat{\chi}\tilde{u}_h^\chi(\tau,\lambda) = \mathcal{O}_{L^2}(h^N)$ for all N uniformly in (τ,λ), then it is equivalent to prove the wave front properties for $\tilde{u}_h^\chi(\tau,\lambda)$ or for $\tilde{u}_h(\tau,\lambda)$ thanks to resolvent bound of Proposition 6.1.

It now remains to deal with the part of phase space where the symbol of $P_\tau(h,\lambda)$ is not elliptic. We start with the regularity/smallness near $E_s^*(X_{\tau_0})$ for large $\|\xi\|$.

Lemma 6.4. *Let $(z,\zeta) \in C^{us}(\alpha_0)$ satisfying $\|\zeta\| \in [2R,4R]$ for $\alpha_0 > 0$ small. There exist $\alpha_1 < \alpha_0$ small enough and $R > 0$ large enough such that, if $\mathcal{U} := C^{ss}(\alpha_1) \cap \{\|\xi\| \geq 8R\}$, then, for each $B_h \in \Psi_h^0(\mathcal{M};\mathcal{E})$ independent of (τ,λ) with $\mathrm{WF}_h(B_h) \subset \mathcal{U}$, we have: for each $N > 0$ there is C_N such that for all τ close enough to τ_0 and $\lambda \in \mathcal{Z}$*

$$\|B_h\tilde{u}_h(\tau,\lambda)\|_{L^2}^2 \leq C_N h^N, \quad \left\|B_h\tilde{u}_h^\chi(\tau,\lambda)\right\|_{L^2}^2 \leq C_N h^N.$$

Proof. Recall that (z,ζ) is the point around which the sequence $(f_h)_{0<h\leq 1}$ is microlocalized. To deal with this case, we will make use of the radial propagation estimates from [Va, DyZw1], the only difference being that we need to verify the uniformity in the parameter τ. First of all, we write that, uniformly in (τ,λ),

$$\forall v \in C^\infty(\mathcal{M};\mathcal{E}), \quad \|B_h v\|_{L^2}^2 = \langle \mathrm{Op}_h(b(h))v, v\rangle + \mathcal{O}(h^N)\|v\|_{L^2}^2, \tag{6.13}$$

where $b(h) = \sum_{j=0}^N h^j b_j$ are symbols supported in $\mathcal{U}$.

We now fix a nondecreasing smooth function $\tilde{\chi}_1$ on $\mathbb{R}$ which is equal to 1 on $[N_1,+\infty)$ and to 0 on $(-\infty, N_1/4 - N_0]$. Take $\alpha_1 < \alpha_0$ small, and using Remark 7 we set

$$\chi_\tau(x,\xi) := \tilde{\chi}_1\left(\tilde{m}_\tau^{N_0,N_1}(x,\xi)\right).$$

For $\|\xi\|_x \geq 1$, we have $\chi_\tau \equiv 0$ outside $C^{ss}(\alpha_0)$, $\chi_\tau \equiv 1$ on $C^{ss}(\alpha_1)$ and $\{H_\tau, \chi_\tau\} \leq 0$. We will use this smooth function in order to microlocalize our operators near $C^{ss}(\alpha_1)$ at infinity (the radial source). After possibly shrinking $\mathcal{U}$ (by adjusting α_1, R) and thanks to (A.6), we may suppose that there exist $R_0 < \tilde{R}_0$ such that $f(x,\xi) \geq \tilde{R}_0$ on $\mathcal{U}$ and $f(x,\xi) \leq R_0$ on $\overline{U_{t_0,\delta}}$. We fix $\tilde{\chi}_2$ to be a nondecreasing smooth function on $\mathbb{R}$ which is equal to 1 near $[\ln(1+\tilde{R}_0),+\infty)$ and to 0 near $(-\infty, \ln(1+R_0)]$. We set

$$\chi_2(x,\xi) = \tilde{\chi}_2(\ln(1 + f(x,\xi))).$$

With these conventions, one has $\chi_2 \equiv 1$ in a neighborhood of $\mathcal{U}$, $\chi_2 \equiv 0$ in a neighborhood of $\overline{U_{t_0,\delta}}$ and $\{H_\tau, \chi_2\}(x,\xi) \leq 0$ for $\|\xi\|_x \geq 1$ such that $(x,\xi) \in C^{ss}(\alpha_0)$, for all τ near τ_0. We now define $A_h(\tau) = A_h^*(\tau)$ in $\Psi_h^0(M;\mathcal{E})$ with principal symbol $a_\tau := \chi_\tau\chi_2\mathrm{Id}$ and

$\mathrm{WF}_h(A_h(\tau)) \subset \mathrm{supp}(a_\tau)$, thus $\mathrm{WF}_h(A_h(\tau)) \cap \overline{U_{t_0,\delta}} = \emptyset$ uniformly for (τ, λ) in the allowed region. From the composition rules for pseudo-differential operators,

$$A_h(\tau)P_\tau(h,\lambda) + P_\tau(h,\lambda)^* A_h(\tau) = h \operatorname{Op}_h \left(\left(-\{H_\tau, a_\tau\} - 2a_\tau \left(\mathrm{Re}(\lambda) + \left\{ H_\tau, G_\tau^{N_0} \right\} \right) \right) \mathrm{Id} \right) + \mathcal{O}_{\Psi_h^0(\mathcal{M},\mathcal{E})}(h^2).$$

Note that the remainder has semiclassical wavefront set contained in $\cup_\tau \mathrm{supp}(a_\tau)$ uniformly in (τ, λ). Then, from our construction

$$\left(-\{H_\tau, a_\tau\} - 2a_\tau (\mathrm{Re}(\lambda) + \left\{ H_\tau, G_\tau^{N_0} \right\}) - b_0 \right) \mathrm{Id} \geq 0.$$

Note that we got the positivity of the symbol provided that we choose N_0 large enough in a manner that depends only on b_0 and $\mathcal{Z}$ (recall that $\left\{ H_\tau, G_\tau^{N_0} \right\} \leq -c_0 N_0$ for every $\|\xi\|_x \geq 1$ when $(x,\xi) \in C^{ss}(\alpha_1)$). We can then use the Garding inequality proved in [DyZw3, Proposition E.35]: combining with (6.13), we get for all v in $C^\infty(\mathcal{M};\mathcal{E})$

$$\|B_h v\|_{L^2}^2 \leq h^{-1} 2\mathrm{Re}(\langle A_h(\tau)P_\tau(h,\lambda)v, v\rangle_{L^2}) + h\langle R_h(\tau,\lambda)v, v\rangle_{L^2} + \mathcal{O}(h^N)\|v\|_{L^2}^2,$$

where $R_h(\tau,\lambda) \in \Psi_h^0(\mathcal{M};\mathcal{E})$ satisfies $\mathrm{WF}_h(R_h(\tau,\lambda)) \subset \mathcal{V}$ with $\mathcal{V}$ a small neighborhood of $\cup_\tau \mathrm{supp}(a_\tau)$ in $\overline{T^*M}$ uniform in (τ,λ). Then, for all v in $C^\infty(\mathcal{M};\mathcal{E})$ and uniformly in (τ,λ), one has

$$\|B_h v\|_{L^2}^2 \leq 2h^{-1}\|A_h(\tau)P_\tau(h,\lambda)v\|_{L^2}\|v\|_{L^2} + h\langle R_h(\tau,\lambda)v, v\rangle_{L^2} + \mathcal{O}(h^N)\|v\|_{L^2}^2.$$

This is a kind of weakened version of the radial estimates (near the source) from [Va, DyZw1] which holds uniformly in (τ,λ). Using that $\|\tilde{u}_h(\tau,\lambda)\|_{L^2} \leq Ch^{-4N_0}$ uniformly in (τ,λ), we find by letting[11] $v \to \tilde{u}_h(\tau,\lambda)$ that, for all $N > 0$, there is $C_N > 0$ so that

$$\|B_h \tilde{u}_h(\tau,\lambda)\|_{L^2}^2 \leq 2h^{-1-4N_0}\|A_h \tilde{F}_h\|_{L^2} + h\langle R_h(\tau,\lambda)\tilde{u}_h(\tau,\lambda), \tilde{u}_h(\tau,\lambda)\rangle_{L^2} + C_N h^{N-8N_0}.$$

Using the facts that $\mathrm{WF}_h(\tilde{F}_h(\tau)) \subset \overline{U_{t_0,\delta}}$ and $\mathrm{WF}_h(A_h(\tau)) \cap \overline{U_{t_0,\delta}} = \emptyset$ uniformly in (τ,λ) we obtain that, for every $N \geq 1$, there exists $C_N > 1$ such that $\|A_h(\tau)\tilde{F}_h(\tau)\|_{L^2} \leq C_N h^{N+1}$ uniformly in (τ,λ). Hence, one has, uniformly in (τ,λ),

$$\|B_h \tilde{u}_h(\tau,\lambda)\|_{L^2}^2 \leq h\langle R_h(\tau,\lambda)\tilde{u}_h(\tau,\lambda), \tilde{u}_h(\tau,\lambda)\rangle_{L^2} + C_N h^{N-8N_0}.$$

We can now reiterate this procedure with $B_h^* B_h$ replaced by $h^{\frac{1}{2}} R_h(\tau,\lambda)$ which has $\mathrm{WF}_h(R_h(\tau,\lambda)) \subset \mathcal{V}$, thus not intersecting $\overline{U_{t_0,\delta}}$. After a finite number of steps, we find $\|B_h \tilde{u}_h(\tau,\lambda)\|_{L^2} \leq C_N h^{\frac{N}{2}-4N_0}$ uniformly in (τ,λ). The case with $\tilde{u}_h^\chi(\tau,\lambda)$ is exactly the same by using that $\mathrm{WF}_h(\hat{\chi}) \cap \{\|\xi\| \geq 8R\} \cap C^{us}(\alpha_0) = \emptyset$. Hence, $P_\tau(h,\lambda)$ coincide with $P_h^\chi(\tau,\lambda)$ microlocally in the region $\{\|\xi\| \geq 8R\}$ where we do the analysis. □

For each $\alpha_0 > \alpha_1 > 0$ small, and for each $(z,\zeta) \in C^{us}(\alpha_0) \setminus C^{uu}(\alpha_1)$ satisfying $||\zeta|| \in [2R, 4R]$, there exist an open neighborhood U of (z,ζ) and a uniform time $T_1 > 0$ such that $\Phi_{-T_1}^\tau(\overline{U}) \subset \mathcal{U}$ (defined in Lemma 6.4). Take now $B_h^{(1)} \in \Psi_h^0(\mathcal{M};\mathcal{E})$ with $\mathrm{WF}_h(B_h^{(1)}) \subset U$. As $(z,\zeta) \in C^{us}(\alpha_0)$ (hence not in the trapped set of the flows Φ_t^τ, given by $E_0^*(X_\tau)$), by taking U and δ small enough we can suppose that, for every $t \in [0, T_1]$ and for any τ small, $\Phi_{-t}^\tau(\overline{U}) \cap \overline{U_{t_0,\delta}} = \emptyset$. Hence, by propagation of singularities [DyZw1, Prop. 2.5] for the operator $iP_\tau(h,\lambda)$ and by the regularity near the radial source (Lemma 6.4), one knows that

[11]We can use [DyZw3, Lemma E.47] to justify the convergence in the inequality.

$\|B_h^{(1)}\tilde{u}_h(\tau,\lambda)\|_{L^2} \leq C_N h^N$ for all N with C_N uniform in (τ,λ) (in the allowed region). Note that due to the compactness of $\mathrm{WF}_h(B_h)$, evaluating $\|B_h u_h\|_{L^2}$ or $\|B_h\tilde{u}_h\|_{L^2}$ is equivalent. Here, we notice that, due to the facts that we just use propagation for a uniform finite time and that the Hamiltonian flow Φ_t^τ is smooth in τ, the proof of [DyZw1, Prop. 2.5] can be repeated uniformly for τ close enough to 0. This concludes the case where $(z,\zeta)\notin C^{uu}(\alpha_1)$. Note that the same argument also works for $\tilde{u}_h^\chi$ as we can apply propagation of singularities [DyZw1, Prop. 2.5] with the operator $iP_\tau^\chi(h,\lambda)$ as well (using that $\chi_1^2\geq 0$).

We now discuss the case where the sequence $(f_h)_{0<h\leq 1}$ is microlocalized near $(z,\zeta)\in C^{uu}(\alpha_1)$ with $\|\zeta\|\in[2R,4R]$. In that case, we will need to use the auxiliary sequence $(\tilde{u}_h^\chi(\tau,\lambda))_{0<h\leq 1}$. First, we see similarly that there is a uniform time $T_2>0$ such that for each $(x,\xi)\in C^{us}(\alpha_0)\setminus C^{uu}(\alpha_1)$ satisfying $\|\xi\|\in[R/2,3R/2]$ and for every τ close enough to 0, $\Phi^\tau_{-T_2}(x,\xi)\in\mathcal{U}$. One more time, we can apply propagation of singularities as in [DyZw1, Prop. 2.5] and Lemma 6.4 to $\tilde{u}_h^\chi(\tau,\lambda)$ with the operator $P_h^\chi(\tau,\lambda)$. From that, we deduce that, uniformly in (τ,λ), $\mathrm{WF}_h(\tilde{u}_h^\chi(\tau,\lambda))\cap V=\emptyset$ for V a small neighborhood of (x,ξ). Thus, one has, uniformly in (τ,λ),

$$\left(\mathrm{WF}_h(\tilde{u}_h^\chi(\tau,\lambda))\cap \mathrm{WF}_h(\hat{\chi})\right)\subset\left(C^{uu}(\alpha_1)\cup\{\|\xi\|\leq R/2\}\right). \tag{6.14}$$

Combining with (6.12), we get uniformly in (τ,λ)

$$\left(\mathrm{WF}_h(\tilde{u}_h^\chi(\tau,\lambda))\cap \mathrm{WF}_h(\hat{\chi})\right)\subset\left(C^{uu}(\alpha_1)\cap\{\|\xi\|\in[R,3R/2]\}\right). \tag{6.15}$$

If α_1 is chosen small enough, then, for each $(x,\xi)\in C^{uu}(\alpha_1)$ with $\|\xi\|\in[R,3R/2]$, there is a uniform time $T_3>0$ (with respect to τ) such that $\Phi^\tau_{-T_3}(x,\xi)\in\{\|\xi\|\leq R/2\}$. We now combine propagation of singularities as above with the elliptic estimate (6.12). From the above, we conclude that, uniformly in (τ,λ),

$$\mathrm{WF}_h(\tilde{u}_h^\chi(\tau,\lambda))\cap \mathrm{WF}_h(\hat{\chi})=\emptyset. \tag{6.16}$$

As expected, we find that $\tilde{u}_h^\chi(\tau,\lambda)=\tilde{u}_h(\tau,\lambda)+\mathcal{O}_{L^2}(h^N)$ uniformly in (τ,λ). Hence, it remains to show that, if B_h is microlocalized inside a neighborhood U of $(z,\zeta)\in C^{uu}(\alpha_1)$ with $\|\zeta\|\in[2R,4R]$, then $B_h\tilde{u}_h^\chi(\tau,\lambda)=\mathcal{O}(h^N)$ uniformly in (τ,λ). For that purpose, it is sufficient to combine propagation of singularities [DyZw1, Prop. 2.5] with the elliptic estimate (6.12) as before. Indeed, as above and up to shrinking U a little bit, there is $T_4>0$ such that $\Phi^\tau_{-T_4}(\overline{U})\subset\{\|\xi\|\leq R/2\}$ uniformly in τ and such that $\Phi^\tau_{-t}(\overline{U})\cap\overline{U_{t_0,\delta}}=\emptyset$ for every $0\leq t\leq T_4$. □

We conclude the proof of 2) in Theorem 4 by combining Lemma 5.1, the sequential continuity of $(\tau,\lambda)\mapsto Q_\tau(\lambda)$ in $\mathcal{D}'_\gamma(\mathcal{M}\times\mathcal{M};\mathcal{E}\otimes\mathcal{E}')$ from Proposition 6.3 : this shows that for every $0\leq k\leq n$ the map

$$(\tau,\lambda)\in[-\delta,\delta]\times\overline{\mathcal{Z}}\mapsto \mathrm{Tr}^\flat\left(A_\tau^{(k)}Q_\tau(\lambda)|_{\mathcal{E}^k}\right)\in\mathbb{C} \tag{6.17}$$

is continuous. Finally, by an application of the Cauchy formula and by Proposition 6.1, one can verify that, for every $\tau\in[-\delta,\delta]$ and for every $0\leq k\leq n$,

$$\lambda\in\mathcal{Z}\mapsto \mathrm{Tr}^\flat\left(A_\tau^{(k)}Q_\tau(\lambda)|_{\mathcal{E}^k}\right)$$

is an holomorphic function using Cauchy's formula and the continuity of (6.17).

Finally, let us remark that the arguments of this section combined with [DyZw1, §4] also show the following

Proposition 6.5. *Suppose that X_0 is an Anosov vector field and that the representation ρ_0 (induced by the connection) is such that $\mathbf{X}_0$ has no resonance at $\lambda = 0$. Then, the maps*

$$X \mapsto \zeta_{X,\rho_0}(0) \quad \textit{and} \quad \rho \mapsto \zeta_{X_0,\rho}(0)$$

are continuous near X_0 (resp. ρ_0).

Note that we only treated the case where X varies. Yet, the same argument holds when we vary ρ and when we fix X_0 as it only modifies $\mathbf{X}_0$ by subprincipal symbols.

7. Fried conjecture in dimension 3 and some cases in dimension 5

7.1. **The kernel of X at** $\lambda = 0$**.** In this section, we will analyze when 0 is not a resonance for the operator $\mathbf{X}$ of (3.8) associated to a vector field $X \in \mathcal{A}$. We define

$$C^k := \ker(\mathbf{X}|_{\mathcal{E}^k})^p, \quad C_0^k := C^k \cap \ker i_X$$

where $p \geq 1$ is the smallest integer so that $\ker(\mathbf{X}^{(k)})^p = \ker(\mathbf{X}^{(k)})^{p+1}$, and where here we mean the kernel on the anisotropic spaces. By [DaRi, Th. 2.1], the complex

$$0 \xrightarrow{d^\nabla} C^0 \xrightarrow{d^\nabla} C^1 \xrightarrow{d^\nabla} \dots \xrightarrow{d^\nabla} C^n \xrightarrow{d^\nabla} 0. \tag{7.1}$$

is quasi–isomorphic to the twisted De Rham complex $(\Omega^\bullet(\mathcal{M}, E), d^\nabla)$ hence the cohomology of (7.1) coincides with the twisted De Rham cohomology. We will denote by $H^k(\mathcal{M};\rho)$ the twisted de Rham cohomology of degree k with ρ the representation associated with the flat bundle (E, ∇).

We say that $X \in \mathcal{A}$ is a *contact Anosov flow* if there is $\alpha \in \Omega^1(\mathcal{M})$ such that $i_X\alpha = 1$, $i_X d\alpha = 0$ and $d\alpha$ is symplectic on $\ker\alpha$. The dimension of $\mathcal{M}$ will be denoted $n = 2n_0 + 1$ in that case. In particular, one has $\mathbf{X}\alpha = 0$ and $\mathbf{X}d\alpha = 0$, and $\mathbf{X}\mu = 0$ if $\mu = \alpha \wedge d\alpha^{n_0}$. To begin with, we notice a few commutation relations that will be extensively used. For all $u \in \mathcal{D}'(\mathcal{M};\mathcal{E})$

$$\mathbf{X}i_X u = i_X\mathbf{X}u, \quad \mathbf{X}(\alpha \wedge u) = \alpha \wedge \mathbf{X}u, \quad \mathbf{X}(u \wedge d\alpha) = (\mathbf{X}u) \wedge d\alpha. \tag{7.2}$$

The Koszul complex is naturally associated with our problem

$$0 \xrightarrow{i_X} C^{2n_0+1} \xrightarrow{i_X} C^{2n_0} \xrightarrow{i_X} \dots \xrightarrow{i_X} C^1 \xrightarrow{i_X} C^0 \xrightarrow{i_X} 0,$$

and in the contact case there is a dual complex

$$0 \xrightarrow{\wedge\alpha} C^0 \xrightarrow{\wedge\alpha} C^1 \xrightarrow{\wedge\alpha} \dots \xrightarrow{\wedge\alpha} C^{2n_0} \xrightarrow{\wedge\alpha} C^{2n_0+1} \xrightarrow{\wedge\alpha} 0.$$

Lemma 7.1. *For $X \in \mathcal{A}$, the complex $(C^\bullet, i_X)$ is acyclic. If in addition X is contact with contact form α, $(C^\bullet, \wedge\alpha)$ is acyclic and we have a decomposition :*

$$\forall 0 \leq k \leq 2n_0 + 1, \quad C^k = (C_0^{k-1} \wedge \alpha) \oplus C_0^k.$$

Proof. If $u \in C^k \cap \ker(i_X) = C_0^k$, then $i_X \Pi_0 (\theta \wedge u) = \Pi_0 (\theta(X) u) = u$ if $\theta \in \Omega^1(\mathcal{M})$ satisfies $\theta(X) = 1$ and Π_0 is the projector on $C^\bullet$. Thus $(C^\bullet, i_X)$ is acyclic. According to (7.2), $\alpha \wedge u$ belongs to C^{k+1} whenever u belongs to C^k. For $u \in C^k \cap \ker(\wedge\alpha)$, one has $\alpha \wedge (i_X u) = \alpha(X) u = u$. Hence, $(C^\bullet, \wedge\alpha)$ is acyclic. For $u \in C^k$, we can write $u = \alpha \wedge i_X u + (u - \alpha \wedge i_X u)$ with $u - \alpha \wedge i_X u \in C_0^k$, and if $u \in C_0^k$ satisfies $\alpha \wedge u = 0$, then $u = i_X(\alpha \wedge u) = 0$. □

From the contact structure, we can also deduce the following duality property:

Lemma 7.2. *Suppose that $X \in \mathcal{A}$ is contact, then for every $0 \leq k \leq n_0$,*

$$C_0^k \simeq C_0^{2n_0 - k}, \quad C^k \simeq C^{2n_0+1-k}.$$

Proof. The bundle $N := \ker \alpha$ is smooth and $\omega := d\alpha$ is symplectic on N. The form ω induces a non-degenerate pairing G on $\Lambda^k N^*$ for each $k \in [1, 2n_0]$, invariant by X. Following [Li, Ya], we can define a (smooth) Hodge star operator $\star : \Lambda^k N^* \to \Lambda^{2n_0-k} N^*$

$$\beta_1 \wedge \star\beta_2 := G(\beta_1, \beta_2)\omega^{n_0}/n_0!.$$

One can check from $L_X G = 0$ and $L_X \omega = 0$ (L_X the Lie derivative) that $\mathbf{X}\star = \star\mathbf{X}$, and thus $\star : C_0^k \to C_0^{2n_0-k}$ is an isomorphism since $\star\star = \mathrm{Id}$. It remains to use Lemma 7.1 to obtain $C^k \simeq C^{2n_0+1-k}$. □

Proposition 7.3. *Suppose that $X \in \mathcal{A}$ is contact on $\mathcal{M}$ with dimension $2n_0 + 1$. The following statements are equivalent:*

(1) *$C^{n_0-1} = 0$ and $H^{n_0}(\mathcal{M}, \rho) = 0$,*
(2) *$C^{n_0} = 0$,*
(3) *For all $0 \leq k \leq 2n_0 + 1$, $C^k = 0$.*

Suppose that $X \in \mathcal{A}$ (not necessarily contact) on a 3-manifold $\mathcal{M}$ and that X preserves some smooth volume form. Then $(\Omega^\bullet(\mathcal{M}, E), d^\nabla)$ is acyclic with $C^0 = 0$ if and only if $\forall 0 \leq k \leq 3, C^k = 0$.

Proof. The statement (3) $\Longrightarrow$ (1) follows from the quasi–isomorphism between $\left(C^\bullet, d^\nabla\right)$ and $\left(\Omega^\bullet(\mathcal{M}, E), d^\nabla\right)$. Let us show (1) $\Longrightarrow$ (2). Since $C^{n_0-1} = 0$, we have $C^{n_0+2} = 0$ by Lemma 7.2. Moreover, by Poincaré duality, $H^{n_0}(\mathcal{M}, \rho) = H^{n_0+1}(\mathcal{M}, \rho) = 0$. Then, still from the quasi–isomorphism, we have that $d^\nabla : C^{n_0} \mapsto C^{n_0+1}$ is an isomorphism. We can now use the acyclicity of $(C^\bullet, i_X)$ and the same argument shows $i_X : C^{n_0+1} \mapsto C^{n_0}$ is an isomorphism. So, combined with Lemma 7.1, this shows that $\mathbf{X}_{|C^{n_0}} = i_X d^\nabla + d^\nabla i_X = i_X d^\nabla : C^{n_0} \mapsto C^{n_0}$ is an isomorphism. However, by our definition, $\mathbf{X}_{|C^{n_0}}$ is nilpotent. Thus, $C^{n_0} = C^{n_0+1} = 0$. To show (2) $\Longrightarrow$ (3), from Lemmas 7.1 and 7.2, it suffices to show that $C_0^{n_0} = C_0^{n_0-1} = 0$ implies $C_0^k = 0$ for every $0 \leq k \leq n_0 - 2$. By [Ya, Cor. 2.7], $u \mapsto u \wedge (d\alpha)$ maps $C_0^k \to C_0^{k+2}$ injectively[12] if $k \leq n_0 - 1$, thus we have $\dim C_0^{n_0} \geq \dim C_0^{n_0-2} \geq \ldots$ and $\dim C_0^{n_0-1} \geq \dim C_0^{n_0-3} \geq \ldots$, which shows that (2) $\Longrightarrow$ (3).

In case $n = 3$ (i.e., $n_0 = 1$), the proof of the converse sense is the same as before. For the direct sense, we cannot use Lemma 7.2. But we still have $C^0 = C^3 = 0$ since X preserves

[12]This follows from surjectivity of the map $u \in C^\infty(\mathcal{M}; \mathcal{E}_0^{n-k-2}) \mapsto u \wedge d\alpha \in C^\infty(\mathcal{M}; \mathcal{E}_0^{n-k})$.

some smooth volume form μ. The rest of the proof is exactly the same as (1) $\Longrightarrow$ (2) given before. $\square$

Lemma 7.4. *Assume $X \in \mathcal{A}$ preserves a smooth volume form μ and assume (E, ∇) is a bundle with flat unitary connection. Let u be an element of C^0 such that $\mathbf{X}u = 0$. Then $u \in C^\infty(\mathcal{M}; E)$ and $d^\nabla u = 0$.*

Proof. Note that $\mathbf{X}^* = -\mathbf{X}$ on $C^\infty(\mathcal{M}; E)$, since $\mathbf{X}\mu = 0$ and that for $v_1, v_2 \in C^\infty(\mathcal{M}; E)$,

$$\langle \mathbf{X}v_1, v_2\rangle_{L^2} = \int_{\mathcal{M}} \langle \mathbf{X}v_1, v_2\rangle_E \mu = \int_{\mathcal{M}} \mathbf{X}(\langle v_1, v_2\rangle_E)\mu - \int_{\mathcal{M}} \langle v_1, \mathbf{X}v_2\rangle_E \mu = -\langle v_1, \mathbf{X}v_2\rangle_{L^2}.$$

Hence, we can apply [DyZw2, Lemma 2.3] and deduce that $u \in C^\infty(\mathcal{M}; E)$. Now we use the argument of [FRS, Lemma 3]. We can lift u to its universal cover $\widetilde{\mathcal{M}}$ to get a bounded $\pi_1(\mathcal{M})$ equivariant $\widetilde{u} \in C^\infty(\widetilde{\mathcal{M}}; \mathbb{C}^r)$ satisfying $\widetilde{u}(\widetilde{\varphi}_t(x)) = \widetilde{u}(x)$ for all $x \in \mathcal{M}$ and $\widetilde{\varphi}_t$ is the lifted flow on $\widetilde{\mathcal{M}}$. This implies $d\widetilde{u}_{\varphi_{-t}(x)} = (d\widetilde{\varphi}_t)^T_{\varphi_{-t}(x)} d\widetilde{u}_x$. For $x \in \mathcal{M}$ assume that $d\widetilde{u}_x \notin E_s^* \oplus E_0^*$, then as $t \to +\infty$ we get $|d\widetilde{u}_{\varphi_{-t}(x)}|_{\mathbb{C}^r} \to +\infty$, but $|d\widetilde{u}|_{\mathbb{C}^r} \in L^\infty$ thus a contradiction. The same argument by letting $t \to -\infty$ tells us that $d\widetilde{u}_x \in E_u^* \oplus E_0^*$ thus $d\widetilde{u}_x \in E_0^*(x)$. But $d\widetilde{u}(X) = 0$, thus $d\widetilde{u}(x) = 0$. Then $d^\nabla u = \nabla u = 0$ on $\mathcal{M}$. $\square$

7.2. **Proof of Theorem 1 - Fried conjecture in dimension 3.** We start with the first statement in Theorem 1. Let X_0 be an Anosov vector field preserving a smooth volume form μ and ∇ be a flat unitary connection on a Hermitian bundle E inducing an acyclic representation ρ. By Lemma 7.4, we find $C^0 = 0$ and by Proposition 7.3, we obtain $C^k = 0$ for all $k \in [0, 3]$. Then Theorem 2 shows that $\zeta_{X,\rho}(0) = \zeta_{X_0,\rho}(0)$ for all X in a neighborhood $\mathcal{U}(X_0) \subset \mathcal{A}$ of X_0.

Let us show the second part of Theorem 1. It suffices to show that there is a sequence $X_n \in \mathcal{A}$ such that $X_n \to X_0$ in $C^\infty(\mathcal{M}; T^*\mathcal{M})$ and such that $|\zeta_{X_n,\rho}(0)|^{-1} = \tau_\rho(\mathcal{M})$. Sanchez-Morgado [Sa2, Th. 1] (based on [Sa1, Ru, Fr5]) showed that transitive analytic Anosov vector fields X satisfy $|\zeta_{X,\rho}(0)|^{-1} = \tau_\rho(\mathcal{M})$ if there is a closed orbit γ of X so that $\ker(\rho([\gamma]) - \varepsilon_\gamma^j \mathrm{Id}) = 0$ for each $j \in \{0, 1\}$. Among other things including the spectral construction of [Ru], Sanchez-Morgado's argument relied crucially on the existence (for Anosov transitive flows on 3-manifolds) of a Markov partition [Rat, p. 885] whose rectangles have boundaries in $W^u(\gamma) \cup W^s(\gamma)$ for any fixed closed orbit γ. Recall that, for Anosov transitive flows, $W^{u/s}(\gamma)$ is everywhere dense in $\mathcal{M}$.

If the monodromy property is satisfied for some orbit γ of X_0, then, for all vector fields X in a small neighborhood $\mathcal{U}(X_0)$, there is a periodic orbit γ_X of X in the same free homotopy class and the corresponding flow is topologically transitive by the strong structural stability Theorem 4.2. Therefore, the results of Sanchez-Morgado applies for any X in $\mathcal{U}(X_0)$ provided that it satisfies some analyticity property. The conclusion of the proof is then given by the following when there exists a closed orbit γ such that the monodromy property of [Sa2] is verified.

Proposition 7.5. *There exists a real analytic structure on $\mathcal{M}$ compatible with the C^∞ structure and a sequence $(X_n)_n \subset \mathcal{A}$ of analytic Anosov vector fields such that $X_n \to X_0$ in the C^∞ topology.*

Proof. By Whitney [Wh, Th. 1 p. 654, Lemma 24 p. 668] (see also [Hi, Th. 7.1 p. 118]), there exists a C^∞ embedding σ of $\mathcal{M}$ into $\mathbb{R}^N$ for some $N \in \mathbb{N}$ such that $\sigma(\mathcal{M})$ is a real analytic submanifold of $\mathbb{R}^N$. It follows from such embedding that the manifold $\mathcal{M}$ inherits some analytic structure compatible with the C^∞ structure of $\mathcal{M}$ since $\mathcal{M}$ is diffeomorphic to some analytic submanifold of $\mathbb{R}^N$. The tangent bundle $T\mathcal{M} \mapsto \mathcal{M}$ also inherits the real analytic structure from $\mathcal{M}$ which makes it a real analytic bundle in the sense of [KrPa, Def. 2.7.8 p. 57]. Therefore by the Grauert–Remmert Theorem [Hi, Th. 5.1 p. 65], the space of analytic maps $\mathcal{M} \mapsto T\mathcal{M}$ is everywhere dense in $C^\infty(\mathcal{M}, T\mathcal{M})$ for the strong C^∞-topology. In particular, a vector field X on $\mathcal{M}$ is understood as a smooth map $\mathcal{M} \mapsto T\mathcal{M}$ transverse to the fibers of $T\mathcal{M}$ which is C^1 stable. Hence any analytic map $\mathcal{M} \mapsto T\mathcal{M}$ sufficiently close to X in the C^1 topology will be transverse to the fibers of $T\mathcal{M}$ and its image in $T\mathcal{M}$ can be realized as the graph of a real analytic section $\tilde{X}$ of $T\mathcal{M}$ (see also [CiEl, Cor. 5.49 p. 106] for similar results). □

It now remains to discuss when we only suppose that ρ is acyclic and that $H^1(\mathcal{M}, \mathbb{R}) \neq \{0\}$. In that case, one knows from [Pl, Th. 2.1] that X_0 has a closed orbit γ_0 which is homologically nontrivial. It may happen that no closed orbit verifies the monodromy condition of [Sa2]. Yet, we can fix a closed one form $\alpha_0 \in H^1(\mathcal{M}, \mathbb{R})$ such that $\int_{\gamma_0} \alpha_0 \neq 0$. Then, we define $\nabla_s = \nabla + is\alpha_0 \wedge$ (with $s \in \mathbb{R}$) which still induces a unitary representation. Recall that, for $s = 0$, 0 is not a resonance of $\mathbf{X}_0$ according to Lemma 7.4 and to Proposition 7.3. Thus, for s small enough, ∇_s also remains acyclic thanks to the finite dimensional Hodge theory [BiZh, (1.6)] or to [DaRi, Th. 2.1] combined with the fact that 0 is still not a resonance of $\mathbf{X}_0 + is\alpha_0(X_0)$ by the arguments[13] used to prove Proposition 6.1. One can verify that, for $s \neq 0$ small enough, the monodromy condition of [Sa2] is verified. Hence, for every $s \neq 0$ small enough, one has $|\zeta_{X_0,\rho_s}(0)|^{-1} = \tau_{\rho_s}(\mathcal{M})$. By Proposition 6.5 and by continuity of the map $\rho \mapsto \tau_\rho(\mathcal{M})$, we can conclude that $|\zeta_{X_0,\rho}(0)|^{-1} = \tau_\rho(\mathcal{M})$.

7.3. **Fried conjecture near hyperbolic metrics in dimension $n = 5$ - Proof of Theorem 3.** We refer to [Fr2, BuOl, Ju] for backgrounds on Ruelle/Selberg zeta functions for hyperbolic manifolds. Let $M = \Gamma\backslash\mathbb{H}^{n_0+1}$ be a smooth oriented compact (n_0+1)-dimensional hyperbolic manifold with $n_0 \geq 2$ and $SM = \Gamma\backslash S\mathbb{H}^{n_0+1}$ its unit tangent bundle, where here $\Gamma \subset \mathrm{SO}(n_0+1, 1)$ is a co-compact discrete subgroup with no torsion. We consider a unitary representation $\rho : \pi_1(M) \to U(r)$ for $r \in \mathbb{N}$, and since $\pi_1(SM) \simeq \pi_1(M)$ if $n_0 + 1 \geq 3$, ρ induces a representation $\widetilde{\rho} : \pi_1(SM) \to U(r)$. By considering functions w on $\mathbb{H}^{n+1}$ with values in $\mathbb{R}^r$ that are Γ-equivariant (i.e., $\forall \gamma \in \Gamma$, $\gamma^* w = \rho(\gamma) w$), we obtain a rank r vector bundle $E \to M$ equipped with a unitary flat connection ∇, and similarly by using $\widetilde{\rho}$ we obtain a bundle $\widetilde{E}$ and a flat connection $\widetilde{\nabla}$ on SM.

We let X be the vector field of the geodesic flow on $\mathcal{M} := SM$, and following the previous sections, this induces an operator on section of $\widetilde{\mathcal{E}} := \oplus_k \wedge^k T^*(SM) \otimes \widetilde{E}$

$$\mathbf{X} : \Omega(SM; \widetilde{E}) \to \Omega(SM, \widetilde{E}), \quad \mathbf{X} := i_X d^{\widetilde{\nabla}} + d^{\widetilde{\nabla}} i_X.$$

[13] The proof is even simpler in this case as adding $is\alpha_0(X_0)$ only modifies the operator by a subsprincipal symbol.

and we write $\mathbf{X}^{(k)} := \mathbf{X}|_{\Omega_0^k(SM;\widetilde{E})}$ where $\Omega_0^k(SM;\widetilde{E}) := \Omega^k(SM;\widetilde{E}) \cap \ker i_X$.

We define the *dynamical zeta function* of X acting on $\Omega_0^k(SM;\widetilde{E})$ by

$$Z_{\mathbf{X}^{(k)}}(\lambda) = \exp\Big(- \sum_{\gamma \in \mathcal{P}} \sum_{j=1}^{\infty} \frac{1}{j} \frac{e^{-\lambda j \ell(\gamma)} \mathrm{Tr}(\widetilde{\rho}(\gamma)^j) \mathrm{Tr}(\wedge^k P(\gamma)^j)}{|\det(1 - P(\gamma)^j)|} \Big) \tag{7.3}$$

where $\mathcal{P}$ denotes the set primitive closed geodesics and $P(\gamma)$ is the linearized Poincaré map of the geodesic flow along this geodesic. Note that $\mathcal{P}$ is parametrized by the conjugacy classes of primitive elements in the group Γ. It is known [GLP, DyZw1] that $Z_{\mathbf{X}^{(k)}}(\lambda)$ has an analytic continuation to $\lambda \in \mathbb{C}$ and its zeros are the Ruelle resonances of $\mathbf{X}^{(k)}$ on SM with multiplicities.

Let $K = \mathrm{SO}(n_0+1)$ be the compact subgroup of $G := \mathrm{SO}(n_0+1, 1)$ so that $\mathbb{H}^{n_0+1} = G/K$ and we can identify $S\mathbb{H}^{n_0+1} = G/H$ where $H := \mathrm{SO}(n_0) \subset K$ is the stabilizer of a spacelike element in $\mathbb{R}^{n_0+1,1}$. We have $M = \Gamma\backslash G/K$ as locally symmetric spaces of rank 1 and $SM = \Gamma\backslash G/H$.

Let us define $\xi_p : \mathrm{SO}(n_0) \to \mathrm{GL}(S^p\mathbb{R}^{n_0})$ to be the canonical (unitary) representation of $\mathrm{SO}(n_0)$ into the space $S^p\mathbb{R}^{n_0}$ of symmetric tensors of order p on $\mathbb{R}^{n_0}$. This representation decomposes into irreducible representations of $\mathrm{SO}(n_0)$

$$\xi_p = \sum_{2q \le p} \sigma_{p-2q}$$

where $\sigma_r : \mathrm{SO}(n_0) \to \mathrm{GL}(S_0^r\mathbb{R}^{n_0})$ is the canonical representation of $\mathrm{SO}(n_0)$ into the space of trace-free symmetric tensors of order r. We also define $\nu_l : \mathrm{SO}(n_0) \to \mathrm{GL}(\Lambda^l\mathbb{R}^{n_0})$ to be the canonical (unitary) representation of $\mathrm{SO}(n_0)$ on l-forms.

For each primitive closed geodesic γ on M (i.e. primitive closed orbit on SM), there is an associated conjugacy class in Γ, with a representative that we still denote by $\gamma \in \Gamma$ and whose axis in $\mathbb{H}^{n+1}$ descends to the geodesic γ. There is also a neighborhood of the geodesic in M that is isometric to a neighborhood of the vertical line $\{z = 0\}$ in the upper half-space $\mathbb{H}^{n_0+1} = \mathbb{R}^+_{z_0} \times \mathbb{R}^{n_0}_z$ quotiented by the elementary group generated by

$$(z_0, z) \mapsto e^{\ell(\gamma)}(z_0, m(\gamma)z),$$

where $m(\gamma) \in \mathrm{SO}(n_0)$ and $\ell(\gamma) > 0$ being the length of γ. The linear Poincaré map along this closed geodesic on $E_s \oplus E_u$ is conjugate to the map

$$P(\gamma) : (w_s, w_u) \mapsto (e^{-\ell(\gamma)} m(\gamma) w_s, e^{\ell(\gamma)} m(\gamma) w_u) \tag{7.4}$$

where we identify E_s and E_u with $\mathbb{R}^{n_0}$.

To any irreducible unitary representation μ of $\mathrm{SO}(n_0)$ and the representation ρ of $\pi_1(M)$ being fixed, we can define a *Selberg zeta function* $Z_{S,\mu}(\lambda)$ by

$$Z_{S,\mu}(\lambda) := \exp\Big(- \sum_{\gamma \in \mathcal{P}} \sum_{j=1}^{\infty} \frac{\mathrm{Tr}(\tilde{\rho}(\gamma)^j)\mathrm{Tr}(\mu(m(\gamma)^j))e^{-\lambda j \ell(\gamma)}}{j \det(1 - P_s(\gamma)^j)} \Big) \tag{7.5}$$

where the sum is over all primitive closed geodesics and $P_s(\gamma_0) = P(\gamma_0)|_{E^s}$ is the contracting part of $P(\gamma)$. This series converges uniformly for $\mathrm{Re}(\lambda) > n_0$. For any unitary representation

μ of $\mathrm{SO}(n_0)$, we can also define $Z_{S,\mu}(\lambda)$ by the formula (7.5), and if $\mu = \sum_{q=1}^p \mu_q$ is a decomposition into irreducible representations, $Z_{S,\mu}(\lambda) = \prod_{q=1}^p Z_{S,\mu_q}(\lambda)$. By [BuOl, Theorem 3.15], $Z_{S,\mu}(\lambda)$ has a meromorphic continuation to $\lambda \in \mathbb{C}$, and if n_0+1 if odd, the only zeros and poles are contained in $\mathrm{Re}(\lambda) \in [0, n_0]$.

Proposition 7.6. *In the region of convergence* $\mathrm{Re}(\lambda) > n_0$*, we have for* $k \in [0, n_0]$

$$Z_{\mathbf{X}^{(k)}}(\lambda) = \prod_{p=0}^{\infty}\prod_{q=0}^{\infty}\prod_{l=0}^{k} Z_{S,\nu_l\otimes\nu_{k-l}\otimes\sigma_p}(\lambda + 2(q-l) + p + n_0 + k) \tag{7.6}$$

Proof. To factorise $Z_{\mathbf{X}^{(k)}}(\lambda)$ with some Selberg zeta functions, we compute for $j \in \mathbb{N}$

$$\begin{aligned} |\det(1 - P(\gamma)^j)|^{-1} =& e^{-n_0 j\ell(\gamma)} \det(1 - e^{-j\ell(\gamma)} m(\gamma)^j)^{-1} \det(1 - P_s(\gamma)^j)^{-1} \\ =& e^{-n_0 j\ell(\gamma)} \det(1 - P_s(\gamma)^j)^{-1} \sum_{r=0}^{\infty} e^{-rj\ell(\gamma)} \mathrm{Tr}(\xi_r(m(\gamma)^j)) \end{aligned}$$

where we used $\det(1-B)^{-1} = \sum_{r=0}^{\infty} \mathrm{Tr}(S^r B)$ with $S^r B$ the action of B on symmetric tensors on $\mathbb{R}^{n_0}$ if $B \in \mathrm{End}(\mathbb{R}^{n_0})$ with $|B| < 1$. Now we can use

$$\begin{aligned} \sum_{r=0}^{\infty} e^{-rj\ell(\gamma)} \mathrm{Tr}(\xi_r(m(\gamma)^j)) &= \sum_{r=0}^{\infty}\sum_{2q\le r} e^{-rj\ell(\gamma)} \mathrm{Tr}(\sigma_{r-2q}(m(\gamma)^j)) \\ &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} e^{-(p+2q)j\ell(\gamma)} \mathrm{Tr}(\sigma_p(m(\gamma)^j)) \end{aligned}$$

Now we also have $\mathrm{Tr}(\wedge^k P(\gamma)^j) = \sum_{l=0}^k e^{j(2l-k)\ell(\gamma)} \mathrm{Tr}(\nu_l(m(\gamma)^j) \otimes \nu_{k-l}(m(\gamma)^j))$. Combining all this, we thus get

$$Z_{\mathbf{X}^{(k)}}(\lambda) = \exp\Big(- \sum_{\gamma\in\mathcal{P}}\sum_{j=1}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{l=0}^{k} \frac{1}{j} \frac{e^{-(\lambda+n_0+p+2(q-l)+k)j\ell(\gamma)}\mathrm{Tr}(\widetilde{\rho}(\gamma)^j)\mathrm{Tr}(\mu_{l,k,p}(m(\gamma)^j)}{|\det(1 - P_s(\gamma)^j)|}\Big)$$

with $\mu_{l,k,p} := \nu_l \otimes \nu_{k-l} \otimes \sigma_p$. This gives the result. Note that the products in (7.6) converge for $\mathrm{Re}(\lambda) > 0$. $\square$

We notice that in each $\mathrm{Re}(\lambda) > -N$ for $N > 0$ fixed, there is only finitely many Selberg type functions in the factorisation (7.6) whose exponent of convergence is on the right of 0, this means that only finitely many Selberg terms can bring a zero to $Z_{\mathbf{X}^{(k)}}(\lambda)$ in $\mathrm{Re}(\lambda) > -N$. In particular at $\lambda = 0$, only the terms l, k, q, p with

$$2(q-l) + p + k \le 0 \tag{7.7}$$

can contribute to a zero (or a pole) there. Theorem 3 follows directly from Theorem 1, Fried formula 1.2 for hyperbolic manifolds [Fr2] and the following:

Proposition 7.7. *Let* $M = \Gamma\backslash\mathbb{H}^3$ *be a smooth compact oriented hyperbolic manifold and let* ρ *be a unitary representation of* $\pi_1(M)$*. The multiplicity* $m_k(0) := \mathrm{RankRes}_{\lambda=0}(-\mathbf{X}_k - \lambda)^{-1}$

of 0 *as a Ruelle resonance for* $\mathbf{X}_k$ *are given by*

$$m_0(0) = \dim H^0(M;\rho), \quad m_1(0) = 2\dim H^1(M,\rho),$$
$$m_2(0) = 2(\dim H^1(M,\rho) + \dim H^0(M;\rho)), \quad m_{4-k}(0) = m_k(0)$$

where $H^k(M;\rho)$ *is the twisted de Rham cohomology of degree* k *associated to* ρ*.*

Proof. For $k = 0$, from (7.6) and (7.7), we see that only the term $Z_{S,\sigma_0}(\lambda+2)$ can contribute to a zero to the dynamical zeta function $Z_{\mathbf{X}^{(0)}}(\lambda)$. By Selberg trace formula [BuOl, Corollary 5.1], $Z_{S,\sigma_0}(\lambda+2)$ has a zero of order $\dim\ker\Delta_0$ where $\Delta_0 = (d^\nabla)^* d^\nabla$ on sections of the flat Hermitian bundle (E,∇) associated to ρ.

For $k = 1$, the condition (7.7) reduces to the following cases to analyse: $q = 0$, $l = 1$, $p = 0, 1$. For $p = 0$, the only term to consider is $Z_{S,\nu_1}(\lambda+1)$, the Selberg zeta function on 1-forms. As explained in Section 5.3 of [BuOl], ν_1 decomposes into two irreducibles $\nu_1^+ \oplus \nu_1^-$ and by [BuOl, Proposition 5.6], each irreducible brings a zero of order $-\dim H^0(M,\rho) + \dim H^1(M,\rho)$ at $\lambda = 0$: the contribution to $Z_{\mathbf{X}^{(1)}}(\lambda)$ at $\lambda = 0$ coming from $Z_{S,\nu_1}(\lambda+1)$ is a zero or pôle with order $-2\dim H^0(M,\rho) + 2\dim H^1(M,\rho)$. Next the term $p = 1$: we need to look at $Z_{S,\nu_1\otimes\sigma_1}(\lambda+2)$. First we decompose $\sigma_1 \otimes \nu_1 = \nu_1 \otimes \nu_1$ into irreducibles: $\nu_1 \otimes \nu_1 = \sigma_0 \oplus \nu_2 \oplus \sigma_2$. Since $\nu_2 \simeq \nu_0$ is equivalent to the trivial representation, $Z_{S,\sigma_0\oplus\nu_2}(\lambda+2) = (Z_{S,\sigma_0}(\lambda+2))^2$ has a zero of order $2\dim H^0(M,\rho)$ at $\lambda = 0$. Now, for $Z_{S,\sigma_2}(\lambda+2)$ we can use Proposition B.1, which gives that the order of $Z_{S,\sigma_2}(\lambda+2)$ at $\lambda = 0$ is $\dim(\ker\nabla^*\nabla - 2) \cap \ker D^*$ where ∇ is the twisted covariant derivative on $S_0^2 T^*M \otimes E$ and D^* the divergence operator. But by Bochner identity [DFG, Equation (2.4)], $\nabla^*\nabla \geq 3$ and thus $\dim(\ker\nabla^*\nabla - 2) \cap \ker D^* = 0$. We conclude that the order at $\lambda = 0$ of $Z_{\mathbf{X}^{(1)}}(\lambda)$ is $2\dim H^1(M,\rho)$.

For $k = 2$, if $l = 2$ one has to consider $(p,q) = (0,0)$, $(p,q) = (0,1)$, $(p,q) = (1,0)$, $(p,q) = (2,0)$. First $(p,q) = 0$, one get the term $Z_{S,\nu_0}(\lambda)$ since $\nu_2 \simeq \nu_0$, and this has a zero of order $\dim H^0(M,\rho)$ at $\lambda = 0$. For $(p,q) = (0,1)$, $Z_{S,\nu_0}(\lambda+2)$ has a zero of order $\dim H^0(M,\rho)$ at $\lambda = 0$. For $(p,q) = (1,0)$, we get the term $Z_{S,\sigma_1}(\lambda+1)$ which has a zero of order $-2\dim H^0(M,\rho) + 2\dim H^1(M,\rho)$ as discussed above. For $(p,q) = (2,0)$, we get $Z_{S,\sigma_2}(\lambda+2)$ which has no zero at $\lambda = 0$ as above. Now for $l = 1$, only $(p,q) = (0,0)$ could contribute, and we get the terms $Z_{S,\nu_1\otimes\nu_1}(\lambda+2)$ which, as shown above, has a zero of order $2\dim H^0(M,\rho)$. This ends the proof. □

Remark 5. *We remark that such a result could alternatively be obtained using the works* [DFG, KuWe]*, with the advantage of knowing the presence of Jordan blocks. The work* [DFG] *also directly implies that in all dimension* $n_0 + 1 \geq 4$*, one always has* $m_1(0) = \dim H^1(M;\rho)$ *for* $M = \Gamma\backslash\mathbb{H}^{n_0+1}$ *co-compact. However, for higher degree forms, and* $n_0 \geq 4$*, it turns out that* $m_k(0)$ *could a priori be non-topological: for example, when* $n_0 = 4$*, some computations based on Proposition 7.6 and Selberg formula for irreducible representations as used above shows that when* $\dim\ker(\Delta_0 - 4) = j > 0$*, these* j *elements in the kernel contribute to* $m_3(0)$*.*

Appendix A. Proof of Lemma 3.3

A.1. **Family of order functions.** In this paragraph, we fix the aperture of the cones $\alpha_0 > 0$ small enough to ensure that $C^{ss}(\alpha_0) \cap C^u(\alpha_0) = \emptyset$ and we fix some small parameter $\delta > 0$.

We construct an order function for every X in a small enough neighborhood of X_0. For that purpose, we closely follow the lines of [FaSj, Lemma 2.1]. We fix $T'_{\alpha_0} > T_{\alpha_0}$ where T_{α_0} is given by Lemma 3.2. The time T'_{α_0} will be determined later on in a way that depends only on α_0. For our construction, we also let $m_0(x,\xi) \in \mathcal{C}^\infty(S^*\mathcal{M},[0,1])$ to be equal to 1 on $C^u(\alpha_0)$ and to 0 on $C^{ss}(\alpha_0)$. Then, we set

$$m_X(x,\xi) := \frac{1}{2T'_{\alpha_0}} \int_{-T'_{\alpha_0}}^{T'_{\alpha_0}} m_0 \circ \tilde{\Phi}_t^X(x,\xi)dt. \tag{A.1}$$

Note that m_X *depends smoothly on* X as we chose T'_{α_0} independently of X near X_0. First of all, we note that

$$\tilde{X}_H m_X(x,\xi) = \frac{1}{2T'_{\alpha_0}} \left(m_0 \circ \tilde{\Phi}^X_{T'_{\alpha_0}}(x,\xi) - m_0 \circ \tilde{\Phi}^X_{-T'_{\alpha_0}}(x,\xi) \right), \tag{A.2}$$

where $\tilde{X}_H$ is the vector field of $\tilde{\Phi}_t^X$. We also observe that, for every (x,ξ) inside $S^*\mathcal{M}$, the set

$$\mathcal{I}_{X_0}(x,\xi) := \left\{ t \in \mathbb{R} : \tilde{\Phi}_t^{X_0}(x,\xi) \in S^*\mathcal{M}\backslash \left(C^u(\alpha_0/2) \cup C^{ss}(\alpha_0/2)\right) \right\}$$

is an interval whose length is bounded by some constant $T''_{\alpha_0} > 0$. Fix now a point $(x,\xi) \in S^*\tilde{M}$ and a vector field which is close enough to X_0 (to be determined). If $\tilde{\phi}_X^t(x,\xi) \in C^u(\alpha_0)$ for every $t \in \mathbb{R}$, then the set

$$\tilde{\mathcal{I}}_X(x,\xi) := \left\{ t \in \mathbb{R} : \tilde{\Phi}_t^X(x,\xi) \in S^*\mathcal{M}\backslash \left(C^u(\alpha_0) \cup C^{ss}(\alpha_0)\right) \right\}$$

is empty and the same holds if $\tilde{\Phi}_t^X(x,\xi) \in C^{ss}(\alpha_0)$ for every $t \in \mathbb{R}$. Hence, it remains to bound the length of $\tilde{\mathcal{I}}_X(x,\xi)$ when the orbit of (x,ξ) crosses $S^*\mathcal{M}\backslash (C^u(\alpha_0) \cup C^{ss}(\alpha_0))$ and we may suppose without loss of generality that $(x,\xi) \in S^*\mathcal{M}\backslash (C^u(\alpha_0) \cup C^{ss}(\alpha_0))$. Up to the fact that we may have to decrease a little bit the size of the set $\mathcal{U}_{\alpha_0}(X_0)$ appearing in Lemma 3.2, we have that $\tilde{\Phi}^X_{T''_{\alpha_0}}(x,\xi)$ belongs to $C^u(\alpha_0)$. Hence, thanks to Lemma 3.2, one finds that, for every $t \geq T''_{\alpha_0} + T_{\alpha_0}$, one has $\tilde{\Phi}_t^X(x,\xi) \in C^u(\alpha_0)$. The same holds in backward times. Hence, the diameter of $\tilde{\mathcal{I}}_X(x,\xi)$ is uniformly bounded by $2(T_{\alpha_0} + T''_{\alpha_0})$ and we pick $T'_{\alpha_0} = \frac{T_{\alpha_0}+T''_{\alpha_0}}{\delta}$ for $\delta < 1$.

We set

$$\mathcal{O}^u(X) = \tilde{\Phi}^X_{T'_{\alpha_0}}(S^*\mathcal{M}\backslash C^{ss}(\alpha_0)) \text{ and } \mathcal{O}^{ss}(X) = \tilde{\Phi}^X_{-T'_{\alpha_0}}(S^*\mathcal{M}\backslash C^u(\alpha_0)).$$

Let us now discuss the properties of m_X for X belonging to $\mathcal{U}_{\alpha_0}(X_0)$:

(1) If $(x,\xi) \in \mathcal{O}^u(X)$, then $\tilde{\Phi}^X_{-T'_{\alpha_0}}(x,\xi) \notin C^{ss}(\alpha_0)$. Hence, from the definition of T'_{α_0}, one has $\tilde{\Phi}^X_{T'_{\alpha_0}}(x,\xi) \in C^u(\alpha_0)$ and, from (A.2), one deduce that $\tilde{X}_H m_X \geq 0$ on $\mathcal{O}^u(X)$. Similarly, one has

$$m_X(x,\xi) = \frac{1}{2T'_{\alpha_0}} \left(\int_{-T'_{\alpha_0}}^{-T'_{\alpha_0}+2(T_{\alpha_0}+T''_{\alpha_0})} m_0 \circ \tilde{\Phi}_t^X(x,\xi)dt + \int_{-T'_{\alpha_0}+2(T_{\alpha_0}+T''_{\alpha_0})}^{T'_{\alpha_0}} m_0 \circ \tilde{\Phi}_t^X(x,\xi)dt \right),$$

from which one can infer

$$\forall (x,\xi) \in \mathcal{O}^u(X), \quad m_X(x,\xi) \geq 1 - \frac{T_{\alpha_0} + T''_{\alpha_0}}{T'_{\alpha_0}} = 1 - \delta.$$

(2) Reasoning along similar lines, one also finds that, for every $(x,\xi)\in\mathcal{O}^{ss}(X)$, $\tilde{X}_H m_X \geq 0$ and

$$m_X(x,\xi)\leq\delta.$$

(3) Let (x,ξ) be an element of $S^*\mathcal{M}\backslash(\mathcal{O}^u(X)\cup\mathcal{O}^{ss}(X))$. In that case, one has $\tilde{\Phi}^X_{-T'_{\alpha_0}}(x,\xi)\in C^{ss}(\alpha_0)$ and $\tilde{\Phi}^X_{T'_{\alpha_0}}(x,\xi)\in C^u(\alpha_0)$. Thus, one finds

$$\tilde{X}_H m_X(x,\xi)=\frac{1}{2T'_{\alpha_0}}\left(m_0\circ\tilde{\Phi}^X_{T'_{\alpha_0}}(x,\xi)-m_0\circ\tilde{\Phi}_X^{-T'_{\alpha_0}}(x,\xi)\right)=\frac{1}{2T'_{\alpha_0}}>0. \tag{A.3}$$

(4) Let now $(x,\xi)\in S^*\mathcal{M}\backslash C^u(\alpha_0)$. Write

$$m_X(x,\xi)\leq\frac{1}{2}+\frac{1}{2T'_{\alpha_0}}\int_{-T'_{\alpha_0}}^0 m_0\circ\tilde{\Phi}^X_t(x,\xi)dt\leq\frac{1+\delta}{2}.$$

Let us conclude this construction with the following useful observation:

Lemma A.1. *Let $\alpha_0>0$ be small enough to ensure that $C^u(\alpha_0)\cap C^{ss}(\alpha_0)=\emptyset$. Then, there exists $0<\alpha_1<\alpha_0$ and a neighborhood $\mathcal{U}_{\alpha_0}(X_0)$ of X_0 in $\mathcal{A}$ such that, for every $X\in\mathcal{U}_{\alpha_0}(X_0)$,*

$$C^u(\alpha_1)\cap S^*\mathcal{M}\subset\mathcal{O}^u(X)\quad\text{and}\quad C^{ss}(\alpha_1)\cap S^*\mathcal{M}\subset\mathcal{O}^{ss}(X).$$

Proof. We only treat the case of $\mathcal{O}^u(X)$, the other one being similar. First of all, we note that by construction (with $\gamma>0$ as in Section 3.1)

$$\Phi^{X_0}_{T'_{\alpha_0}}\left(C^u\left(\alpha_0 e^{-\gamma T'_{\alpha_0}}/2\right)\right)\subset C^u(\alpha_0/2).$$

Hence, up to the fact that we may have to shrink the above neighborhood $\mathcal{U}_{\alpha_0}(X_0)$ a little bit, one can verify that, for every $X\in\mathcal{U}_{\alpha_0}(X_0)$,

$$\Phi^{X}_{T'_{\alpha_0}}\left(C^u\left(\alpha_0 e^{-\gamma T'_{\alpha_0}}/2\right)\right)\subset C^u(\alpha_0)\subset T^*\mathcal{M}\backslash C^{ss}(\alpha_0),$$

which concludes the proof by taking $\alpha_1=\alpha_0 e^{-\gamma T'_{\alpha_0}}/2$. □

Remark 6. *In all the construction so far, we could have defined the cones $C^{uu}(\alpha)$ and $C^s(\alpha)$ and a decaying order function $\tilde{m}_X(x,\xi)$ which is close to 0 on $C^s(\alpha)$ and close to 1 on $C^{uu}(\alpha)$.*

A.2. **Definition of the escape function.** We start with the construction of the function $f(x,\xi)\in\mathcal{C}^\infty(T^*M,\mathbb{R}_+)$. For $\|\xi\|_x\geq 1$, it will be 1-homogeneous and equal to $\|\xi\|_x$ outside the cones $C^{uu}(\tilde{\alpha}_0)$ and $C^{ss}(\tilde{\alpha}_0)$ for $\tilde{\alpha}_0>0$ small enough (to be determined). Following the proof of [DyZw1, Lemma C.1] (see also [GBWe, Lemma 2.2]), we set, for (x,ξ) near $C^{ss}(\tilde{\alpha}_0/2)$ and $\|\xi\|_x\geq 1$,

$$f(x,\xi):=\exp\left(\frac{1}{T_1}\int_0^{T_1}\ln\|(d\varphi_t^{X_0}(x)^T)^{-1}\xi\|_{\varphi^t_{X_0}(x)}dt\right).$$

Recall that, for every ξ in $E_s^*(X_0,x)$, one has $\|(d\varphi_t^{X_0}(x)^T)^{-1}\xi\|\leq Ce^{-\beta t}\|\xi\|$ for every $t\geq 0$ (where C,β are some uniform constants). Hence, if we set $T_1=2\frac{\ln C}{\beta}$, we find that, for every

$(x,\xi)\in E_s^*(X_0)$ with $\|\xi\|_x\geq 1$, $X_{H_0}f(x,\xi)\leq -f(x,\xi)\frac{\beta}{2}$. Similarly, picking T_1 large enough, we set, for (x,ξ) near $C^{uu}(\tilde{\alpha}_0/2)$ and $\|\xi\|_x\geq 1$,

$$f(x,\xi):=\exp\left(\frac{1}{T_1}\int_0^{T_1}\ln\|(d\varphi_t^{X_0}(x)^T)^{-1}\xi\|_{\varphi_{X_0}^t(x)}dt\right),$$

and we find that $X_{H_0}f(x,\xi)\geq f(x,\xi)\frac{\beta}{2}$ on $E_u^*(X_0)$. By continuity, we find that there exists some (small enough) $\tilde{\alpha}_0>0$ such that, for every $\|\xi\|_x\geq 1$,

$$(x,\xi)\in C^{ss}(\tilde{\alpha}_0/2)\Rightarrow X_{H_0}f(x,\xi)\leq -f(x,\xi)\frac{\beta}{2}, \tag{A.4}$$

and

$$(x,\xi)\in C^{uu}(\tilde{\alpha}_0/2)\Rightarrow X_{H_0}f(x,\xi)\geq f(x,\xi)\frac{\beta}{2}. \tag{A.5}$$

As the function $f(x,\xi)$ is 1-homogeneous, we can find a neighborhood $\mathcal{U}(X_0)$ of X_0 in the $\mathcal{C}^\infty$-topology such that, for every X in $\mathcal{U}(X_0)$ and for every $\|\xi\|_x\geq 1$,

$$(x,\xi)\in C^{ss}(\tilde{\alpha}_0/2)\Rightarrow X_{H}f(x,\xi)\leq -f(x,\xi)\frac{\beta}{4}, \tag{A.6}$$

and

$$(x,\xi)\in C^{uu}(\tilde{\alpha}_0/2)\Rightarrow X_{H}f(x,\xi)\geq f(x,\xi)\frac{\beta}{4}. \tag{A.7}$$

Finally, we note that there exists some uniform constant $C>0$ such that, for every X in $\mathcal{U}(X_0)$ and for $\|\xi\|_x\geq 1$,

$$-Cf(x,\xi)\leq X_Hf(x,\xi)\leq Cf(x,\xi) \tag{A.8}$$

We are now ready to construct our family of escape functions $G_X^{N_0,N_1}(x,\xi)$:

$$G_X^{N_0,N_1}(x,\xi):=m_X^{N_0,N_1}(x,\xi)\ln(1+f(x,\xi)),$$

with $m_X^{N_0,N_1}\in\mathcal{C}^\infty(T^*M,[-2N_0,2N_1])$ which is 0-homogeneous for $\|\xi\|_x\geq 1$. In order to construct this function, we will make use of the order functions defined in paragraph A.1 as in [FaSj, p. 337-8]. Before doing that, let us observe that

$$X_HG_X^{N_0,N_1}(x,\xi)=X_H(m_X^{N_0,N_1})(x,\xi)\ln(1+f(x,\xi))+m_X^{N_0,N_1}(x,\xi)\frac{X_Hf(x,\xi)}{1+f(x,\xi)}. \tag{A.9}$$

We now fix a small enough neighborhood $\mathcal{U}(X_0)$ of X_0 so that f enjoys (A.6) and (A.7) for all X in $\mathcal{U}(X_0)$ and so that we can apply the results of paragraph A.1. Following [FaSj], we set, for $\|\xi\|_x\geq 1$,

$$m_X^{N_0,N_1}(x,\xi):=N_1\left(2-m_X\left(x,\frac{\xi}{\|\xi\|_x}\right)-\tilde{m}_X\left(x,\frac{\xi}{\|\xi\|_x}\right)\right)-2N_0\tilde{m}_X\left(x,\frac{\xi}{\|\xi\|_x}\right), \tag{A.10}$$

where we used the conventions of paragraph A.1 and Remark 6. First, notice that, by construction, $X_H(m_X^{N_0,N_1})\leq 0$ for $\|\xi\|_x\geq 1$. Recall that the order functions m_X and $\tilde{m}_X$ depends on the parameters $\alpha_0>0$ and $\delta>0$ and that they depend smoothly on X. Here, we fix $0<\delta<\min(1/2,\frac{\min(N_0,N_1)}{4(N_0+N_1)})$ and $0<\alpha_0<\tilde{\alpha}_0/2$. We then find that $m_X^{N_0,N_1}(x,\xi/\|\xi\|_x)\geq N_1$ on $\mathcal{O}^{ss}(X)$, $m_X^{N_0,N_1}(x,\xi/\|\xi\|_x)\leq -N_0$ on $\mathcal{O}^{uu}(X)$ and $m_X^{N_0,N_1}(x,\xi/\|\xi\|_x)\geq N_1/2$ in a small vicinity of $E_0^*(X_0)$. We also have that $m_X^{N_0,N_1}(x,\xi/\|\xi\|_x)\geq N_1/4-2N_0$ for (x,ξ)

outside $C^{uu}(\alpha_0)$. We now fix α_1 to be the aperture of the cone appearing in Lemma A.1. This allows to verify the first three requirements of $m_X^{N_0,N_1}$.

Remark 7. *We could also have defined*

$$\tilde{m}_X^{N_0,N_1}(x,\xi) := N_1\left(1 - m_X\left(x, \frac{\xi}{\|\xi\|_x}\right)\right) - N_0\tilde{m}_X\left(x, \frac{\xi}{\|\xi\|_x}\right).$$

We still have $\tilde{m}_X^{N_0,N_1}(x,\xi) \geq N_1$ *on* $\mathcal{O}^{ss}(X)$, $\tilde{m}_X^{N_0,N_1}(x,\xi) \leq \frac{N_1}{4} - N_0$ *outside* $C^{ss}(\alpha_0)$.

Finally, combining $X_H(m_X^{N_0,N_1}) \leq 0$ with (A.9) for $||\xi|| \geq 1$, we immediately get the upper bound (3.6). It now remains to verify the decay property (3.4). For that purpose, we shall use the conventions of paragraph A.1 and set, for every $X \in \mathcal{U}(X_0)$,

$$\tilde{\mathcal{O}}^{uu}(X) = \mathcal{O}^{uu}(X) \cap \mathcal{O}^{u}(X),\ \tilde{\mathcal{O}}^{0}(X) = \mathcal{O}^{s}(X) \cap \mathcal{O}^{u}(X),\ \text{and}\ \tilde{\mathcal{O}}^{ss}(X) = \mathcal{O}^{ss}(X) \cap \mathcal{O}^{s}(X),$$

which contains respectively $C^{uu}(\alpha_1)$, $C^{u}(\alpha_1) \cap C^{s}(\alpha_1)$ and $C^{ss}(\alpha_1)$ for $\alpha_1 > 0$ small enough (see Lemma A.1). Note also that $\tilde{\mathcal{O}}^{0}(X)$ is contained inside $C^{u}(\alpha_0) \cap C^{s}(\alpha_0)$ which is a small vicinity of $E_0^*(X_0)$. Based on (A.9), we can now establish (3.4) except in this small cone around the flow direction. Outside $\tilde{\mathcal{O}}^{uu}(X) \cup \tilde{\mathcal{O}}^{0}(X) \cup \tilde{\mathcal{O}}^{ss}(X)$, it follows from (A.3) and (A.9). Inside $\tilde{\mathcal{O}}^{uu}(X)$ and $\tilde{\mathcal{O}}^{ss}(X)$, it follows from (A.6), (A.7) and (A.9).

Appendix B. Selberg zeta function on trace-free symmetric tensors

Proposition B.1. *Let n be even and $M = \Gamma\backslash\mathbb{H}^{n+1}$ be a compact hyperbolic manifold. Let $\rho : \pi_1(M) \to U(V_\rho)$ be a finite dimensional unitary representation and let σ_m be the irreducible unitary representation of* $\mathrm{SO}(n)$ *into the space $S_0^m\mathbb{R}^n$ of trace-free symmetric tensors of order $m \geq 1$ on $\mathbb{R}^n$. Then the Selberg zeta function $Z_{S,\sigma_m}(s)$ on M associated to σ_p and ρ is holomorphic and the order of its zeros are given by*

$$\mathrm{ord}_{s_0} Z_{S,\sigma_m}(s) = \begin{cases} \dim\ker(\nabla^*\nabla - n^2/4 - m + (s_0 - n/2)^2) \cap \ker D^* & \text{if } s_0 \neq n/2 \\ 2\dim\ker(\nabla^*\nabla - n^2/4 - m) \cap \ker D^* & \text{if } s_0 = n/2 \end{cases}$$

where ∇ is the twisted Levi-Civita covariant derivative on $S_0^m T^*M \otimes E$, $E \to M$ being the flat bundle over M obtained from the representation ρ, and $D^* = -\mathrm{Tr} \circ \nabla$ is the divergence operator.

Proof. We follow [BuOl, Theorem 3.15]. First we need to view σ_m as the restriction of a sum of irreducibles representations of $\mathrm{SO}(n+1)$ as in Section 1.1.2 [BuOl]: it is not difficult to check that

$$\sigma_m = (\Sigma_m - \Sigma_{m-1})|_{\mathrm{SO}(n)}$$

where Σ_m denotes the irreducible unitary representation of $\mathrm{SO}(n+1)$ into the space $S_0^m\mathbb{R}^{n+1}$. By Section 1.1.3 of [BuOl], there is a $\mathbb{Z}^2$-graded homogeneous vector bundle $V_{\sigma_m} = V_{\Sigma_m}^+ \oplus V_{\Sigma_m}^-$ over $\mathbb{H}^{n+1}$ with $V_{\Sigma_m}^+ = S_0^m\mathbb{R}^{n+1}$ and $V_{\Sigma_m}^- = S_0^{m-1}\mathbb{R}^{n+1}$, and we define the bundle $V_{M,\rho\otimes\sigma_m} = \Gamma\backslash(V_\rho \otimes V_{\sigma_m})$ over M. Denoting $E \to M$ the bundle over M obtained from V_ρ by quotienting by Γ and $S_0^m T^*M$ the bundle of trace-free symmetric tensors of order m on M, the bundle $V_{M,\rho\otimes\sigma_m}$ is isomorphic to the bundle $\mathcal{E} := (S_0^m T^*M \oplus S_0^{m-1}T^*M) \otimes E$. There is a differential operator $A_{\sigma_m}^2$ on $\mathcal{E}$ constructed from the Casimir operator that has eigenvalues in correspondence with the zeros/poles of $Z_{S,\sigma_m}(s)$, it is given $A^2_{\sigma_m} = -\Omega - c(\sigma_m)$ where Ω is the Casimir operator and $c(\sigma) = n^2/4 - |\mu(\sigma_m)|^2 - 2\mu(\sigma).\rho_{\rm so(n)}$ with $\mu(\sigma_m)$ the highest weight of σ and $\rho_{\rm so(n)} = (\frac{n}{2}-1, \frac{n}{2}-2, \dots, 0)$. Here we have $\mu(\sigma_m) = (m, 0, \dots, 0)$ thus

$$c(\sigma_m) = \frac{n^2}{4} - m(m+n-2).$$

We then obtain the formula

$$A^2_{\sigma_m} = (\Delta_m - c(\sigma_m)) \oplus (\Delta_{m-1} - c(\sigma_m))$$

where $\Delta_m = \nabla^*\nabla - m(m+n-1)$ is the Lichnerowicz Laplacian on (twisted) trace-free symmetric tensors of order m on M (see for instance [Ha1, Section 5]). Now we have by [Ha1, Lemma 5.2] that $D^*\Delta_m = \Delta_{m-1}D^*$ if D^* is the divergence operator defined by $D^*u = -{\rm Tr}(\nabla u)$, and whose adjoint is $D = \mathcal{S}\nabla$ is the symmetrised covariant derivative. This gives $\Delta_m D = D\Delta_{m-1}$, but since D is elliptic with no kernel by [HMS, Proposition 6.6], it has closed range and D gives an isomorphism

$$D : \ker(\Delta_{m-1} - c(\sigma_m) - s) \to \ker(\Delta_m - c(\sigma_m) - s) \cap (\ker D^*)^\perp$$

for each $s \in \mathbb{R}$. In particular, one obtains that for each $s \in \mathbb{R}$

$$\dim\ker(\Delta_m - c(\sigma_m) - s) - \dim\ker(\Delta_{m-1} - c(\sigma_m) - s) = \dim(\ker(\Delta_m - c(\sigma_m) - s) \cap \ker D^*).$$

Now by [BuOl, Theorem 3.15], the function $Z_{S,\sigma_m}(s)$ has a zero at s of order

$$2\dim(\dim(\ker(\Delta_m - c(\sigma_m) \cap \ker D^*)) \text{ if } s = \tfrac{n}{2}$$
$$\dim(\dim(\ker(\Delta_m - c(\sigma_m) \cap \ker D^*)) \text{ if } s \neq \tfrac{n}{2}.$$

□

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