Université Pierre et Marie Curie

École Doctorale Paris Centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

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Sur quelques résultats d'analyse harmonique dans les L^p non commutatifs

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Soutenue le 24 septembre 2009 devant le jury composé de :

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Remerciements

Je tiens avant tout à remercier Gilles Pisier pour le temps et les mathématiques qu'il a partagés avec moi. Avoir été initié par lui à la recherche mathématique a été un honneur et une grande chance ; j'ai beaucoup profité de chacune des très nombreuses discussions que nous avons eues, et la richesse de son enseignement dépasse largement ce qui peut transparaître à la lecture de cette thèse.

Je remercie également Roland Speicher et Quanhua Xu d'avoir accepté d'être rapporteurs de mon travail. Merci tout particulièrement à Quanhua Xu qui m'a invité de nombreuses fois à Besançon, et avec qui j'ai eu de nombreux échanges.

C'est aussi un honneur pour moi que Claire Anantharaman-Delaroche, Philippe Biane, Dario Cordero-Erausquin, Georges Skandalis et Quanhua Xu aient accepté de faire partie du jury. C'est un plaisir d'y retrouver Georges Skandalis, qui m'a fait faire mes premiers pas dans les algèbres d'opérateurs lors de son cours à l'ENS.

Pendant cette thèse j'ai partagé mon temps entre deux équipes, l'Équipe d'Analyse Fonctionnelle de l'Institut de Mathématiques de Jussieu et le Département de Mathématiques et Applications à l'École Normale Supérieure. Je tiens à remercier tous leurs membres qui ont rendu ces années si agréables.

Lors de ma thèse j'ai été accueilli de nombreuses fois au laboratoire de mathématiques de l'Université de Besançon. J'ai une pensée pour tous ses membres. J'ai également pu profiter à plusieurs reprises de l'hospitalité du département de mathématiques de Texas A&M University, que je remercie pour les excellentes conditions de séjour et de travail offertes.

Mais il n'y a pas que les maths dans la vie. Merci à tous ceux, familles ou amis, qui, à Paris, en montagne, ou ailleurs, m'ont entouré pendant ces années.

Enfin et surtout, merci à Marielle pour m'avoir soutenu quand j'étais là, attendu quand je ne l'étais pas, et avec qui tout est si facile. Merci.

Résumé

Résumé

Cette thèse présente quelques résultats d'analyse harmonique sur les L^p non commutatifs. La première partie consiste en l'étude des isométries complètes préservant l'unité entre sous-espaces d'espaces L^p non commutatifs contenant l'unité. Le résultat principal est une généralisation non commutative d'un théorème de Rudin et Plotkin et affirme que si p n'est pas un entier pair, une telle isométrie complète est toujours la restriction d'un isomorphisme entre les algèbres de von Neumann engendrées. La deuxième partie traite des inégalités de Haagerup en probabilités libres. Le résultat principal énonce une inégalité de Haagerup à coefficients opérateurs renforcée pour les normes p (avec p entier pair ou $p = \infty$) sur l'espace engendré par les mots holomorphes de longueur donnée dans l'algèbre de von Neumann des groupes libres. Des résultats analogues sont montrés dans le cadre plus général des variables $\mathscr{R}$-diagonales libres. La preuve, de nature combinatoire, est basée sur l'étude de certaines partitions non croisées et de processus de symétrisation de partitions. La troisième partie de cette thèse caractérise les matrices de Hankel à valeurs opérateurs dans les espaces L^p non commutatifs à valeurs vectorielles. En annexe sont présentées quelques remarques et questions sur les fonctions opérateur-Lipschitz.

Mots-clefs

Espaces L^p non commutatifs et Espaces L^p non commutatifs à valeurs vectorielles, isométries complètes, inégalités de Haagerup, opérateurs de Hankel, fonctions opérateur Lipschitz.

On certain results of harmonic analysis in non-commutative L^p-spaces

Abstract

This thesis presents some results of harmonic analysis in non-commutative L^p-spaces. The first part consists in the study of complete isometries between unital subspaces of non-commutative L^p spaces. The main result states that when p is not an even integer such a complete isometry is the restriction of an isomorphism between the von Neumann algebras generated by theses subspaces. The second part is devoted to the Haagerup inequalities in free probability. The main result is a strong Haagerup inequality with operator coefficient for the p norms (with p an even integer or $p = \infty$) on the space generated by the holomorphic words of a given length in the von Neumann algebra of the free groups. Analogous results are proved in the more genral setting of free $\mathscr{R}$-diagonal operators. The proof, of combinatorial nature, is based on the study of certain non-crossing partitions and on a symmetrization process of partitions. In the third part of this thesis we characterize the bounded Hankel matrices in the vector valued non-commutative L^p spaces. In the appendix we present some questions and remarks on operator Lipschitz functions.

Keywords

Non-commutative L^p spaces and vector-valued non-commutative L^p spaces, complete isometries, Haagerup inequalities, Operator Lipschitz functions.opérateur Lipschitz.

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Introduction

Cette thèse s'inscrit dans la théorie de l'analyse non commutative. Cette branche des mathématiques a clairement son inspiration dans la physique quantique, mais s'est développée depuis des années. Elle s'insère dans une démarche générale importante des mathématiques récentes : le remplacement des fonctions (ou des scalaires) par des opérateurs sur des espaces de Hilbert. La première conséquence devient le caractère non commutatif du produit. Le fait remarquable dans ces théories est que souvent, une fois que l'on a établi un dictionnaire correct entre la théorie commutative et la théorie non-commutative, la plupart des énoncés commutatifs se transposent en énoncés non-commutatifs, qui restent (souvent) vrais, mais avec des preuves différentes.

On s'intéresse ici à la théorie des espaces d'opérateurs (ou espaces de Banach non commutatifs) et aux espaces L^p non commutatifs. Dans cette introduction je commencerai par donner quelques rappels sur ces théories, avant de donner une description détaillée des différents chapitres de cette thèse.

0.1 Quelques rappels

0.1.1 Espaces d'opérateurs

La théorie des espaces d'opérateurs est, comme son nom l'indique, la théorie des espaces vectoriels constitués d'opérateurs sur un espace de Hilbert. Elle s'insère dans la démarche présentée ci-dessus dans laquelle les fonctions ou les scalaires sont remplacés par des opérateurs. De ce point de vue, les espaces d'opérateurs sont parfois considérés comme les analogues non commutatifs des espaces de Banach, puisqu'ils consistent en des espaces vectoriels normés dans lesquels on autorise des combinaisons linéaires à valeurs opérateurs. Pour une introduction détaillée, le lecteur est invité à lire [44].

Les applications entre espaces d'opérateurs intéressantes à étudier sont les applications *complètement bornées.*

Du fait que $M_n(B(H)) \simeq M_n(\mathbb{C}) \otimes B(H)$ s'identifie canoniquement à $B(H^{\oplus n})$, à tout espace d'opérateurs est naturellement associée une famille de normes $\|\cdot\|_n$ sur $M_n(E) \simeq M_n(\mathbb{C}) \otimes E$ pour $n \in \mathbb{N}$, $n \geq 1$. Une application linéaire $u : E \to F$ entre espaces d'opérateurs est dite complètement bornée si les applications $u^{(n)} = \mathrm{id}_n \otimes u : M_n(E) \to M_n(F)$ sont bornées uniformément en n, et on note $\|u\|_{cb} = \sup_n \|u^{(n)}\|$.

Dans la catégorie des espaces d'opérateurs, les objets sont les espaces d'opérateurs et les morphismes sont les applications complètement bornées. On y identifie deux espaces d'opérateurs s'ils sont complètement isométriques. La donnée d'un espace d'opérateurs abstrait est donc simplement la donnée d'un espace vectoriel E et d'une suite de normes $\|\cdot\|_m$ sur $M_m \otimes E$.

Remarque 1. Les espaces d'opérateurs sont une structure intermédiaire entre les espaces de Banach et les algèbres d'opérateurs. Par cela, on entend que tout espace d'opérateurs est un espace de Banach (si on ne se souvient que de la norme sur E), et que réciproquement tout espace de Banach peut être muni en général de plusieurs (une infinité) structures d'espaces d'opérateurs. De même, (par définition !), toute C^*-algèbre est un espace d'opérateurs si on ne se souvient que de la suite des normes $\|\cdot\|_n$. Il est peut-être moins immédiat de se rendre compte que pour un espace d'opérateurs abstrait donné, il peut exister plusieurs réalisations de cet espace comme une partie d'un $B(H)$ telles que les C^*-algèbres engendrées sont distinctes (ne sont pas isomorphes). Un exemple frappant est donné par les espaces d'opérateurs engendrés par les générateurs (resp. les générateurs et leurs inverses) dans la C^*-algèbre réduite du groupe libre à r générateurs. Inversement, dans certains cas connaître l'espace d'opérateurs permet de retrouver l'algèbre d'opérateurs engendrée. Ce genre de résultat est obtenu à l'aide du théorème de factorisation des applications complètement bornées qui permet de relier des propriétés d'espaces d'opérateurs à des morphismes d'algèbres d'opérateurs.

L'étude des applications complètement bornées a déjà une histoire assez importante, mais c'est seulement à partir des années 90 que la théorie des espaces d'opérateurs (comme espaces de Banach non commutatifs) s'est vraiment développée, avec les travaux de Ruan ([49]) qui caractérisent de façon abstraite les espaces d'opérateurs parmi les espaces E munis d'une suite de normes sur $M_n(E)$ pour $n = 1, 2, \dots$. Cette caractérisation permet en particulier l'introduction de techniques d'analyse dans cette théorie, comme la notion de quotient, de dualité, d'interpolation complexe, etc... pour des espaces d'opérateurs.

0.1.2 Espaces L^p non commutatifs

Dans la théorie de l'intégration non commutative, les fonctions sont remplacées par des *opérateurs* sur un espace de Hilbert, et les mesures sont remplacées par des *traces*. Plus précisément :

Définition 2. Soit $\mathcal{M}$ une algèbre de von Neumann, et τ une trace semi-finie, fidèle et normale sur $\mathcal{M}$, et soit $1 \leq p < \infty$. L'application $x \mapsto (\tau(|x|^p))^{1/p}$, définie sur l'ensemble des $x \in \mathcal{M}$ tels que $\tau(|x|^p) < \infty$, est une norme. L'espace obtenu par complétion est appelé espace L^p non commutatif et noté $L^p(M, \tau)$ ou bien $L^p(\tau)$.

L'espace $L^\infty(\mathcal{M}, \tau)$ est simplement $\mathcal{M}$ muni de sa norme en tant qu'algèbre de von Neumann.

On est parfois amené à travailler avec des C^*-algèbres munies de traces. On peut bien sûr dans ce cas aussi définir les espaces L^p associés, mais il coïncident avec les espaces L^p qui correspondent à l'algèbre de von Neumann engendrée par la représentation GNS.

L'espace L^p non commutatif peut-être le plus simple est la classe de Schatten, qui correspond au cas où $\mathcal{M} = B(\ell_2)$, τ est la trace usuelle. L'espace L^p associé est noté S_p, et il correspond à l'ensemble des opérateurs $x \in B(\ell_2)$ tels que $Tr(|x|^p) < \infty$. Les analogues de dimension finie qui correspondent à $\mathcal{M} = M_n(\mathbb{C})$ sont notés S_n^p.

De même qu'on définit les espaces d'opérateurs comme les sous-espaces linéaires d'algèbres de von Neumann, il est naturel d'étudier les sous-espaces d'espaces L^p non commutatifs $L^p(\mathcal{M}, \tau)$. Comme $M_n(\mathcal{M})$ est muni d'une trace semi-finie normale et fidèle $\mathrm{Tr}_n \otimes \tau$, on peut aussi parler d'analogues d'applications complètement bornées dans ce cadre là.

Remarque 3. On peut noter que d'après le travail présenté dans le chapitre 1 de cette thèse, le lien entre structure de « sous-espace d'espace L^p » et « algèbre de von Neumann engendrée » est plus simple lorsque $1 \leq p < \infty$ et $p \notin 2\mathbb{N}$ que dans le cas $p = \infty$: le théorème principal énonce en effet que dans ce cas, la donnée de la famille des normes p

sur les matrices à coefficients dans un sous-espace unital d'une algèbre de von Neumann détermine entièrement l'algèbre de von Neumann engendrée (voir la description détaillée dans la partie 0.2.1).

On aurait ainsi pu, pour tout p, développer une théorie des sous-espaces des espaces L^p non commutatifs, parallèle à la théorie des espaces d'opérateurs. Mais Pisier a inclus cette étude dans la théorie des espaces d'opérateurs.

Comme le cas $p = \infty$ correspond à la norme d'opérateurs, les espaces L^∞ non commutatifs sont par définition des espaces d'opérateurs. Dans le cas $p = 1$, l'espace $L^1(\mathcal{M}, \tau)$ s'identifie au prédual de l'algèbre de von Neumann, et est donc également muni d'une structure d'espace d'opérateur par les axiomes de Ruan. Comme, pour $1 < p < \infty$, $L^p(\mathcal{M})$ s'identifie isométriquement à l'interpolé complexe entre $L^1(\mathcal{M})$ et $L^\infty(\mathcal{M})$, on définit une structure d'espace d'opérateurs sur $L^p(\mathcal{M})$ par l'interpolation complexe des espaces d'opérateurs.

Le fait remarquable (et heureux) est que cette structure abstraite d'espace d'opérateur sur les espaces L^p non commutatifs est compatible avec la structure naïve décrite précédemment, à savoir que Pisier a montré que pour une application linéaire entre sous-espaces d'espaces L^p non commutatifs, sa norme complètement bornée coïncide avec $\sup_n \|\mathrm{id}_n \otimes u : L^p(M_n \otimes M) \to L^p(M_n \otimes M)\|$.

Plus généralement Pisier a développé ([42]) dans le cadre non-commutatif la notion d'espace L^p à valeurs vectorielles pour des algèbres de von Neumann hyperfinies.

0.1.3 Espaces L^p non commutatifs à valeurs vectorielles

Là encore, pour définir $L^p(M; E)$ il faut que E soit muni d'une structure d'espace d'opérateurs, et alors $L^p(M; E)$ est à nouveau un espace d'opérateurs. Cette structure est encore définie par interpolation, et parmi ses propriétés importantes on peut noter :

- Une application linéaire entre espaces d'opérateurs $u : E \to F$ est complètement bornée si et seulement les applications $u \otimes id : S_p^n(X) \to S_p^n(Y)$ pour $n \geq 1$ sont uniformément bornées. Plus précisément,

$$\left\|u \otimes id : S_p^n(X) \to S_p^n(Y)\right\| = \|u \otimes id : M_n(X) \to M_n(Y)\|$$

- (Théorème de Fubini) Complètement isométriquement, $S_p^n\big(L^p(\mathcal{M}, \tau)\big) \simeq L^p\big(M_n \otimes A, \mathrm{tr}_n \otimes \tau\big)$.

0.2 Contenu de la thèse

Cette thèse contient trois chapitres et une annexe, rédigés en anglais. Les deux premiers chapitres sont des articles écrits pendant ma thèse. Le chapitre 1 décrit un travail, intitulé *Complete isometries between subspaces of noncommutative L^p-spaces*, que j'ai effectué en début de thèse, et qui a été accepté dans Journal of Operator Theory. Dans le chapitre 2 je présente un travail intitulé *Strong Haagerup inequalities with operator coefficients*, et qui a été accepté à Journal of Functional Analysis. Le chapitre 3 décrit les matrices de Hankel (et matrices de Hankel généralisées) qui sont bornées dans les espaces L^p non commutatifs à valeurs vectorielles. L'annexe A traite de fonctions opérateur-Lipschitz : je présente essentiellement ce qui est connu sur le sujet et pose quelques questions. J'inclus ci-dessous des descriptions détaillées en français (chaque chapitre commence par une description analogue en anglais).

0.2.1 Chapitre 1

Le résultat principal de ce chapitre est un analogue non-commutatif d'un *théorème d'équimesurabilité* dû à Plotkin et Rudin. Ce résultat appartient au domaine de l'étude des isométries entre espaces de Banach. Les isométries entre espaces L^p (commutatifs) ont été tout d'abord étudiées par Banach puis Lamperti. L'étude des isométries entre sous-espaces d'espaces L^p remonte au moins aux travaux de Forelli dans les années 1960 ([16]), mais le résultat le plus général dans cette direction est dû indépendamment à Plotkin [46] et à Rudin [50] ; voir aussi Hardin [21].

L'étude d'isométries entre espaces L^p non commutatifs a également été l'objet d'étude d'un certain nombre de mathématiciens : pour les algèbres de von Neumann avec une trace, la caractérisation de ces isométries a été donnée par Yeadon [52]. Pour le cas non tracial, voir [51] et [25]. Les isométries complètes entre espaces L^p non commutatifs ont été étudiées récemment par Junge, Ruan et Sherman dans [25].

Dans ce chapitre on étudie le cas non encore traité, à savoir les isométries entre sous-espaces d'espaces L^p non commutatifs, et les théorèmes principaux sont très similaires aux résultats commutatifs.

Commençons par rappeler le théorème de Plotkin et Rudin :

Théorème 4 (Plotkin, Rudin)**.** Soit $0 < p < \infty$ avec $p \notin 2\mathbb{N}$, et soient μ et ν deux mesures de probabilité (sur des espaces mesurés arbitraires). Soit $n \in \mathbb{N}$ et $f_1, \dots, f_n \in L^p(\mu)$, $g_1, \dots, g_n \in L^p(\nu)$.

Supposons que pour tous $z_1, \dots, z_n \in \mathbb{C}$,

$$\int |1 + z_1 f_1 + \cdots + z_n f_n|^p \, d\mu = \int |1 + z_1 g_1 + \cdots + z_n g_n|^p \, d\nu. \tag{1}$$

Alors $(f_1, \dots, f_n)$ et $(g_1, \dots, g_n)$ sont équimesurables (dans un vocabulaire probabiliste, les variables aléatoires à valeurs dans $\mathbb{C}^n$ $(f_1, \dots, f_n)$ et $(g_1, \dots, g_n)$ sont équidistribuées).

Le principal résultat présenté dans ce chapitre est un résultat similaire dans le cadre non-commutatif (avec quelques conditions techniques). Comme souvent, la modification principale à apporter à cet énoncé est qu'il faut remplacer les combinaisons linéaires à coefficients scalaires par des coefficients opérateurs dans (1) ; c'est-à-dire qu'on parle d'isométries complètes plutôt que d'isométries.

Au moins en ce qui concerne des opérateurs bornés, le fait que deux familles d'opérateurs aient la même $*$-distribution est équivalent au fait qu'elles engendrent des algèbres de von Neumann isomorphes.

Le principal résultat montré dans ce chapitre est donc le théorème suivant :

Théorème 5. Soient $(\mathcal{M}, \tau)$ et $(\mathcal{N}, \widetilde{\tau})$ des algèbres de von Neumann équipées de traces normales, finies et normalisées, et $0 < p < \infty$, $p \notin 2\mathbb{N}$. Soit $u : E \to L^p(\mathcal{N}, \widetilde{\tau})$ une application linéaire définie sur un sous-espace $E \subset L^p(\mathcal{M}, \tau)$ contenant $1_{\mathcal{M}}$ et telle que $u(1_{\mathcal{M}}) = 1_{\mathcal{N}}$.

Supposons de plus que $E \subset L^\infty(\mathcal{M}) (= \mathcal{M})$.

Si u est une isométrie complète, $u(E) \subset L^\infty(\mathcal{N})$ et u s'étend en un isomorphisme préservant la trace entre l'algèbre de von Neumann $VN(E)$ engendrée par E et l'algèbre de von Neumann $VN(u(E))$ engendrée par $u(E)$.

La preuve de ce théorème se décompose en deux étapes logiquement distinctes : la première consiste à prouver le résultat dans le cas où l'on suppose $u(E) \subset L^\infty(\mathcal{N})$; la seconde consiste à prouver que si u est 2-isométrique, alors nécessairement $u(E) \subset L^\infty(\mathcal{N})$.

La preuve de la première étape est de nature combinatoire. La démarche est de calculer l'$*$-distribution de familles d'opérateurs dans E à partir des normes p de matrices à coefficients dans E. Par le calcul fonctionnel, si $\|x\| < 1/3$, on commence par développer en séries entières $|1+x|^p = (1+x+x^*+x^*x)^{p/2} = \sum_{n\geq 0} \binom{p/2}{n}(x+x^*+x^*x)^n$, ce qui permet d'exprimer la norme $\|1+x\|_p$ à partir des $*$-moments de x. Il s'agit ensuite d'aller dans l'autre sens, c'est-à-dire d'exprimer les $*$-moments d'éléments à partir des normes p. Ce n'est pas possible lorsque p est un entier pair puisque dans ce cas la précédente somme ne contient qu'un nombre fini de termes, mais lorsque $p \notin 2\mathbb{N}$, je décris une méthode combinatoire pour le faire. Pour cela on prend dans le développement précedent pour x des matrices bien choisies à valeurs dans E. L'outil principal est l'observation que pour tout n, et dès que $m \geq n/2$ il existe des matrices $a_1, \dots a_n$ dans $M_m(\mathbb{C})$ telles que

$$\mathrm{Tr}_m(a_{\sigma(1)}a_{\sigma(2)}\dots a_{\sigma(n)}) = \begin{cases} 1 & \text{si } \sigma \text{ est une permutation circulaire de } \{1,2,\dots,n\}. \\ 0 & \text{pour une autre permutation } \sigma. \end{cases}$$

Cette observation permet au passage de donner une preuve élémentaire d'un résultat obtenu à l'aide matrices de aléatoires dans [10] (Lemme 1.4 de cette thèse).

La deuxième étape logique est plus analytique et est prouvée dans le partie 1.2.4. Le résultat principal est le suivant :

Théorème 6. Si $a \in L^p(\mathcal{M})$ et $b \in L^p(\mathcal{N})$ sont tels que $a^2 = 0$ et $b^2 = 0$, et tels que $\|1+za\|_p = \|1+zb\|_p$ pour tout $z \in \mathbb{C}$, alors pour tout $n \in 2\mathbb{N} \cup \{\infty\}$, $a \in L^n(\mathcal{M})$ si et seulement si $b \in L^n(\mathcal{N})$ et de plus $\|a\|_n = \|b\|_n$.

La preuve se fait par récurrence sur n et est basée à nouveau sur le développement en séries entières de $\|1+x\|_p^p$: il s'agit d'exprimer $\|a\|_n$ à partir des normes $\|1+za\|_p$ et des normes $\|a\|_k$ pour $k < n$ pairs. Mais les opérateurs n'étant pas bornés, il n'y a pas de convergence en norme mais en mesure (l'analogue non commutatif de la convergence presque sûre). Il s'agit donc d'utiliser les outils de convergence d'intégration non-commutative : lemme de Fatou et théorème de convergence dominée. La principale difficulté rencontrée est pour l'application du lemme de Fatou, puisqu'il faut prouver la positivité de certains opérateurs. Curieusement c'est par des méthodes d'études qualitatives d'équations différentielles que cette positivité est montrée (Proposition 1.20 et Lemma 1.21).

Voici comment se décompose le chapitre 1 : la première partie est une introduction en anglais, qui reprend la description qui vient d'être faite. Ensuite, pour introduire les idées principales de la preuve, je présente une preuve (simple) du Théorème 5 dans le cas particulier où $E^* = E$ et $u(x^*) = u(x)^*$ pour tout $x \in E$. Dans la partie 1.2 sont prouvés les principaux points techniques de la preuve. Ensuite dans la partie 1.3 le théorème principal ainsi que ses reformulations et conséquences directes sont établis. Enfin dans une dernière partie je présente d'autres conséquences des résultats de la partie 1.2, principalement dans le cas où l'espace E est muni d'une structure additionnelle (par exemple cas où E auto-adjoint ou stable par multiplication, ou encore cas des espaces H^p non commutatifs).

0.2.2 Chapitre 2

Ce chapitre a pour objet les inégalités de Haagerup en probabilités libres.

Soient F_r le groupe libre à r générateurs et $|\cdot|$ la longueur associée à l'ensemble de ces générateurs et de leurs inverses. On note $C^*_\lambda(F_r)$ la C^*-algèbre réduite du groupe libre, c'est-à-dire la C^*-algèbre engendrée par la représentation régulière gauche λ de F_r sur $\ell^2(F_r)$. Dans [19], Haagerup a démontré le résultat suivant, connu depuis sous le nom d'inégalité de Haagerup : pour toute fonction $f : F_r \to \mathbb{C}$ à support dans l'ensemble des mots de longueur d,

$$\left\| \sum_{g \in F_r} f(g)\lambda(g) \right\|_{C^*_\lambda(F_r)} \leq (d+1)\|f\|_2. \tag{2}$$

Cette inégalité a eu beaucoup d'applications et de généralisations. Par exemple l'un de ses intérêts est qu'elle donne un critère effectif pour construire des opérateurs bornés dans $C^*_\lambda(F_r)$, puisqu'elle implique en particulier que pout tout $f : F_r \to \mathbb{C}$

$$\left\| \sum_{g \in F_r} f(g)\lambda(g) \right\|_{C^*_\lambda(F_r)} \leq 2\sqrt{\sum_{g \in F_r} (|g|+1)^4 |f(g)|^2},$$

où la norme « de Sobolev » $\sqrt{\sum_{g \in F_r} (|g|+1)^4 |f(g)|^2}$ est bien plus facile à calculer que la norme d'opérateur de $\lambda(f) = \sum f(g)\lambda(g)$. Les groupes pour lesquels des inégalités du même type sont vraies (avec une certaine puissance de $(d+1)$ à la place du terme $(d+1)$) sont appelés les groupes à propriété RD [22] et ont été beaucoup étudiés.

Dans une autre direction les inégalités de Haagerup ont aussi été étendues à la théorie des espaces d'opérateurs. Cela concerne les mêmes inégalités dans lesquelles on remplace la fonction f par une fonction à valeurs opérateurs (matricielles). Cette question a tout d'abord été étudiée par Haagerup et Pisier dans [20], et le résultat le plus complet a été présenté par Buchholz dans [8]. L'un de ses intérêts est qu'il donne une interprétation du terme $(d+1)$ de l'inégalité classique. En effet, pour l'inégalité à coefficients opérateurs, le terme $(d+1)\|f\|_2$ est remplacé par une somme de $d+1$ différentes normes de f, toutes dominées par $\|f\|_2$ dans le cas scalaire. Plus précisément si S est l'ensemble des générateurs de F_r et de leurs inverses, une fonction $f : F_r \to M_n(\mathbb{C})$ dont le support est contenu dans l'ensemble des mots de longueur d peut être vue comme une famille $(a_{h_1,\ldots,h_d})_{(h_1,\ldots,h_d) \in S^d}$ de matrices de la façon suivante : $a_{(h_1,\ldots,h_d)} = f(h_1 h_2 \ldots h_d)$ si $|h_1 \ldots h_d| = d$ et $a_{(h_1,\ldots,h_d)} = 0$ sinon.

Cette famille de matrices $a = (a_h)_{h \in S^d}$ peut être vue de plusieurs façons comme une plus grosse matrice, pour chaque décomposition de $S^d \simeq S^l \times S^{d-l}$. Si les matrices a_h sont vues comme des opérateurs sur l'espace de Hilbert $H = \mathbb{C}^n$, on note M_l l'opérateur de $H \otimes \ell^2(S)^{\otimes d-l}$ dans $H \otimes \ell^2(S)^{\otimes l}$ dont la décomposition en blocs est donnée par

$$M_l = \left(a_{(s,t)}\right)_{s \in S^l, t \in S^{d-l}}.$$

Avec ces notations, le résultat de Buchholz est

Théorème 7 ([8],Theorem 2.8)**.** Si le support de $f : F_r \to M_n(\mathbb{C})$ est contenu dans l'ensemble des mots de longueur d, et si l'on définit $(a_h)_{h \in S^d}$ et M_l pour $0 \leq l \leq d$ comme précédemment, alors

$$\left\| \sum_{g \in S^d} f(g) \otimes \lambda(g) \right\|_{M_n \otimes C^*_\lambda(F_r)} \leq \sum_{l=0}^{d} \|M_l\|.$$

Pour les normes dans les espaces L^p, les mêmes inégalités ont été prouvées dans [33] (mais avec des constantes qui ne sont pas bornées quand $d \to \infty$). Voir aussi les travaux [48] et [24].

Plus récemment et en direction des probabilités libres, Kemp et Speicher [27] ont fait l'étonnante découverte que, bien qu'en toute généralité la constante $(d+1)$ soit optimale dans (2), si l'on restreint à des fonctions (à valeurs scalaires) dont le support est contenu dans l'ensemble W_d^+ des mots de longueur d en les générateurs $g_1, \ldots, g_r$ mais *pas leurs inverses*, alors le terme $(d+1)$ dans (2) peut être remplacé par un terme de l'ordre de $\sqrt{d}$.

Théorème 8 ([27],Theorem 1.4)**.** Si $f : F_r \to \mathbb{C}$ a son support contenu dans W_d^+, alors

$$\left\| \sum_{g \in W_d^+} f(g)\lambda(g) \right\|_{C^*_\lambda(F_r)} \leq \sqrt{e}\sqrt{d+1}\|f\|_2.$$

Dans le chapitre 2 je généralise (et améliore) ces inégalités avec des coefficients opérateurs. De même que pour la généralisation de l'inégalité de Haagerup (2), l'inégalité que j'obtiens donne une interprétation du terme $\sqrt{d+1}$: pour des coefficients opérateurs, ce terme est remplacé par la combinaison ℓ^2 des normes des matrices M_l introduites ci-dessus.

Théorème 9. Pour $d \in \mathbb{N}$, soit $W_d^+ \subset F_\infty$ l'ensemble des mots de longueur d en les générateurs g_i (mais pas leurs inverses). Pour $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ soit $g_k = g_{k_1} \ldots g_{k_d} \in W_d^+$.

Soit $a = (a_k)_{k \in \mathbb{N}^d}$ une famille (à support fini) de matrices, et pour $0 \leq l \leq d$ notons $M_l = \left(a_{(k_1,\ldots,k_l),(k_{l+1},\ldots,k_d)}\right)_{(k_1,\ldots,k_l)\in\mathbb{N}^l,(k_{l+1},\ldots,k_d)\in\mathbb{N}^{d-l}}$ la matrice par blocs correspondante. Alors

$$\left\| \sum_{k \in \mathbb{N}^d} a_k \otimes \lambda(g_k) \right\| \leq 4^5 \sqrt{e} \left(\sum_{l=0}^{d} \|M_l\|^2 \right)^{1/2}. \tag{3}$$

Même dans le cas scalaire ce résultat est une amélioration du résultat de Kemp et Speicher.

De même que dans [27], la même preuve s'applique au cadre plus général des éléments $\mathscr{R}$-diagonaux $*$-libres. On obtient aussi des inégalités similaires pour les normes dans les espaces L^p non-commutatifs lorsque p est un entier pair :

Théorème 10. Soient c un opérateur $\mathscr{R}$-diagonal et $(c_k)_{k\in\mathbb{N}}$ une famille de copies $*$-libres de c sur un C^*-espace de probabilité $(\mathcal{A}, \tau)$. Soient $(a_k)_{k\in\mathbb{N}^d}$ une famille à support fini de matrices, et $M_l = \left(a_{(k_1,\ldots,k_l),(k_{l+1},\ldots,k_d)}\right)$ pour $0 \leq l \leq d$ les matrices par blocs correspondantes.

Pour $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ notons $c_k = c_{k_1} \ldots c_{k_d}$.

Alors pour tout $p \in 2\mathbb{N} \cup \{\infty\}$.

$$\left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_p \leq 4^5 \|c\|_2^{d-2} \|c\|_p^2 e \sqrt{1 + \frac{2d}{p}} \left(\sum_{l=0}^{d} \|M_l\|_p^2 \right)^{1/2}. \tag{4}$$

L'idée générale de la preuve du Théorème 2.4 est la même que dans [27] : on prouve d'abord le résultat pour $p = 2m \in 2\mathbb{N}$; et pour cela on utilise les cumulants libres, qui permettent d'exprimer les moments de variables en termes de partitions non croisées (voir la partie 2.1.2 pour des définitions). La plus grande partie du chapitre 2 (partie 2.1) est de nature combinatoire et consiste d'une part en l'étude d'une certaine classe de partitions non-croisées et d'autre part en la définition et l'étude d'un processus de symétrisation de partitions.

Dans le cas présent (longueur d et variables $\mathscr{R}$-diagonales) les partitions qui interviennent sont les partitions de $[2dm] = \{1, 2, \ldots, 2dm\}$ qui appartiennent à l'ensemble $NC^*(d, m)$ défini de la façon suivante. Découpons $[2dm]$ en $2m$ intervalles $J_1, \ldots, J_{2m}$ de longueur d : $J_k = \{(k-1)d + 1, \ldots, kd\}$. Pour tout $i \in [2dm]$ on note k_i l'entier compris entre 1 et $2m$ tel que $i \in J_{k_i}$. Avec ces notations une partition π appartient à $NC^*(d, m)$ si elle est non croisée, si tous ses blocs sont de taille paire et si pour tous $i < j$ deux éléments consécutifs d'un bloc de π, $k_i \neq k_j \mod 2$. L'un des buts du chapitre 2 est d'étudier $NC^*(d, m)$ (section 2.1.2). La partie de $NC^*(d, m)$ constituée de partitions dont les blocs sont tous de taille 2 est notée $NC^*_2(d, m)$; sa combinatoire est bien connue puisque $NC^*_2(d, m)$ est naturellement en bijection avec les chaînes croissantes de d partitions non-croisées de $[m]$ (pour l'ordre du raffinement). Pour étudier $NC^*(d, m)$ je suis amené à définir une projection $\widetilde{\mathcal{P}} : NC^*(d, m) \to NC^*_2(d, m)$. J'obtiens en particulier le résultat suivant :

Théorème 11. Pour toute partition $\sigma \in NC^*_2(d, m)$ il y a au plus 4^{2m} partitions $\pi \in NC^*(d, m)$ telles que $\widetilde{\mathcal{P}}(\pi) = \sigma$.

De plus pour une telle partition π, il y a au plus $4m$ éléments $i \in [2dm]$ tels que le bloc de π auquel i appartient n'est pas une paire.

Le point important dans ce théorème est que les bornes 4^{2m} et $4m$ obtenues ne dépendent pas de d ; une façon d'interpréter ce résultat est donc de dire que $NC^*(d, m)$ est proche de $NC^*_2(d, m)$ uniformément en d. Comme conséquence on obtient en particulier le fait que le cardinal de $NC^*(d, m)$ est inférieur à $\big(16e(d+1)\big)^m$.

Plus généralement, si $[2N]$ est décomposé en k intervalles $S_1, \ldots, S_k$, on peut s'intéresser (dans la partie 2.1.4) à l'ensemble (que je note ici $NC(S_1, \ldots, S_k)$) des partitions non croisées de $[2N]$ dont les blocs sont de cardinal pair et n'ont pas deux éléments dans le même intervalle S_i pour $1 \leq i \leq k$. On note encore $NC_2(S_1, \ldots, S_k)$ l'ensemble des telles partitions avec des blocs de taille 2. On peut également définir une projection naturelle $Q : NC(S_1, \ldots, S_k) \to NC_2(S_1, \ldots, S_k)$, et dans ce cadre-là,

Lemme 12. Pour toute partition $\sigma \in NC_2(S_1, \ldots, S_k)$, il y a au plus 4^{k-2} partitions $\pi \in NC(S_1, \ldots, S_k)$ telles que $Q(\pi) = \sigma$. De plus pour un tel $\pi \in NC(S_1, \ldots, S_k)$ il y a au plus $2k - 4$ éléments de $i \in [2N]$ tels que le bloc de π auquel i appartient n'est pas une paire.

D'autre part dans le calcul de la norme p (pour $p = 2m$) de $\sum_k a_k \otimes c_k$ on est également naturellement amené à étudier les expressions $S(a, \pi, d, m)$ pour des partitions π de l'intervalle $\{1, 2, \ldots, 2dm\} = [2dm]$, définies par

$$S(a, \pi, d, m) = \sum_{(k_1, \ldots, k_{2m}) \prec \pi} Tr(a_{k_1} \widetilde{a}^*_{k_2} \ldots a_{k_{2m-1}} \widetilde{a}^*_{k_{2m}}),$$

où si $k \in \mathbb{N}^{2dm}$ on note $k \prec \pi$ si $k_i = k_j$ pour tous i, j dans un même bloc de la partition π.

Pour majorer ces expressions $S(a, \pi, d, m)$, je suis amené à définir et à étudier des applications de symétrisation de partitions. Pour un entier N, je définis $2N$ telles applications $P_1, \ldots, P_{2N}$ sur l'ensemble de partitions de $\{1, \ldots, 2N\} = [2N]$. L'observation de départ est qu'en appliquant une inégalité de type Cauchy-Schwarz à $S(a, \pi, d, m)$, on obtient une majoration en terme de ces même expressions pour d'autres partitions, qui sont précisément les images de π par les applications de symétrisation :

Lemme 13. Si π est une partition de $[2dm]$ et $a = (a_k)_{k \in \mathbb{N}^N}$ est comme ci-dessus, alors pour tout entier $1 \leq i \leq m$

$$|S(a, \pi, d, m)| \leq (S(a, P_{id}(\pi), d, m))^{1/2} \left(S(a, P_{(m+i)d}(\pi), d, m)\right)^{1/2}.$$

Cette inégalité peut bien sûr être itérée, et on est donc amené à étudier le « processus de symétrisation » qui consiste à partir d'une partition π et à lui appliquer successivement des applications de symétrisation P_{id}. Je montre alors qu'en choisissant de façon judicieuse (par exemple au hasard) les application P_{id} et en partant d'une partition dans $NC^*(d, m)$, on obtient à partir d'un nombre fini d'étapes sur une partition qui est complètement symétrique (invariante par toutes les applications P_{id}) et pour laquelle l'expression $S(a, \pi, d, m)$ est exactement la norme p (à la puissance p) d'une des matrices M_l. On obtient donc qu'il existe des $\mu_l \geq 0$ avec $\sum \mu_l = 1$ tels que

$$|S(a, \pi, d, m)| \leq \prod_{l=0}^{d} \|M_l\|_{2m}^{2m\mu_l}.$$

Le point délicat est de déterminer exactement les exposants μ_l. Cela revient à trouver des invariants combinatoires pour les applications de symétrisation. Ces invariants s'avèrent être liés à la projection $\widetilde{\mathcal{P}} : NC^*(d, m) \to NC_2^*(d, m)$. Plus précisément, comme $NC_2^*(d, m)$ s'identifie aux chaînes croissantes de longueur d de partitions non croisées de $[m]$, $\widetilde{\mathcal{P}}$ induit une application $NC^*(d, m) \to NC(m)^d$ ($\pi \mapsto (\pi^1, \ldots, \pi^k)$), et je montre que la quantité $|\pi^k|$ (pour tout $1 \leq k \leq d$) est un invariant des application P_{id} dans le sens où pour tout i si $\pi' = P_{id}\pi$ et $\pi'' = P_{id+md}\pi$, alors $|\pi'^k| + |\pi''^k| = 2|\pi'^k|$. On en déduit donc que $\mu_l = (|\pi^{k+1}| - |\pi^k|)/(m-1)$ avec la convention que $|\pi^0| = 1$ et $|\pi^{d+1}| = m$. À partir de là il suffit d'appliquer les résultats d'Edelman [11] qui décrivent l'ensemble des chaînes croissantes dans $NC(m)$ pour conclure.

En adaptant les méthodes décrites ci-dessus, j'obtiens également les résultats suivants. Soit c un opérateur $\mathscr{R}$-diagonal et $(c_k)_{k \in \mathbb{N}}$ une famille de copies $*$-libres de c. Pour $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{1, *\}^d$ et $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ on note $c_{k,\varepsilon} = c_{k_1}^{\varepsilon_1} \ldots c_{k_d}^{\varepsilon_d}$. On établit tout d'abord une extension de l'inégalité de Haagerup pour l'espace engendré par les $c_{k,\varepsilon}$ pour k, ε vérifiant $k_i = k_{i+1} \Rightarrow \varepsilon_i = \varepsilon_{i+1}$, c'est-à-dire pour lesquels $\lambda(g)_{k,\varepsilon}$ a pour longueur d. On note I_d l'ensemble de tels (k, ε).

Théorème 14. Soit $(a_{(k,\varepsilon)})_{(k,\varepsilon) \in (\mathbb{N} \times \{1,*\})^d}$ une famille à support fini de matrices telle que $a_{(k,\varepsilon)} = 0$ si $(k, \varepsilon) \notin I_d$. Pour $0 \leq l \leq d$, notons M_l la matrice obtenue comme précédemment à partir de $(a_{(k,\varepsilon)})$ pour la décomposition $(\mathbb{N} \times \{1, *\})^d = (\mathbb{N} \times \{1, *\})^l \times (\mathbb{N} \times \{1, *\})^{d-l}$.

Alors pour tout $p \in 2\mathbb{N} \cup \{\infty\}$

$$\left\| \sum_{(k,\varepsilon) \in (\mathbb{N} \times \{1,*\})^d} a_{k,\varepsilon} \otimes c_{k,\varepsilon} \right\|_p \leq 4^5 \|c\|_p^2 \|c\|_2^{d-2} (d+1) \max_{0 \leq l \leq d} \|M_l\|_p.$$

De façon similaire pour des opérateurs auto-adjoints j'obtiens :

Théorème 15. Soit μ une mesure symétrique à support compact sur $\mathbb{R}$ et soit c un opérateur auto-adjoint dont la distribution est μ.

Soient $(c_k)_{k \in \mathbb{N}}$ des copies libres de c et $(a_{k_1,\ldots,k_d})_{k_1,\ldots,k_d \in \mathbb{N}}$ une famille à support fini de matrices telle que $a_{k_1,\ldots,k_d} = 0$ si $k_i = k_{i+1}$ pour un $1 \leq i < d$. Alors pour $p \in 2\mathbb{N} \cup \{\infty\}$

$$\left\| \sum_{(k_1,\ldots,k_d) \in \mathbb{N}^d} a_{k_1,\ldots,k_d} \otimes c_{k_1} \ldots c_{k_d} \right\|_p \leq 4^5 \|c\|_p^2 \|c\|_2^{d-2} (d+1) \max_{0 \leq l \leq d} \|M_l\|_p.$$

Dans le cas semi-circulaire ce résultat est bien connu : il est dû à Bożejko [7], et a été reprouvé par des méthodes combinatoires par Biane et Speicher, Theorem 5.3.4 de [4]. La preuve que je présente s'inspire de cette preuve et utilise certains de ses résultats.

Ces résultats ne sont pas complètement satisfaisants dans la mesure où l'on s'attendrait à pouvoir remplacer le terme $(d+1)\max_{0\leq l\leq d}\|M_l\|$ par $\sum_{l=0}^d \|M_l\|$. Le principal obstacle pour obtenir un tel résultat est l'absence d'invariant combinatoire pour les applications de symétrisation sur l'ensemble des partitions qui apparaissent (par exemple dans le cas auto-adjoint) dans le calcul des moments de variables de la forme $c_{k_1}\dots c_{k_d}$ pour $k_1\neq k_2$, $k_2\neq k_3$... Les partitions qui interviennent sont celles de l'ensemble $NC(d,m)$ (défini dans la partie 2.1.4), ou bien encore $NC(J_1,\dots,J_{2m})$ avec les notations précédentes. Pour être plus précis on peut bien décrire un invariant de ces applications, mais il manque une interprétation simple comme celle qui les relie aux chaînes croissantes d'éléments de $NC(m)$ dans le cas de $NC^*(d,m)$.

0.2.3 Chapitre 3

L'objet de ce chapitre est l'étude des matrices de Hankel dans les espaces L^p à valeurs vectorielles $S^p[E]$ pour un espace d'opérateurs E. Le résultat principal est une caractérisation de la norme de telles matrices en termes d'espaces de Besov de fonctions analytiques $B_p^s(E)_+$ (les espaces de Besov B_{p+}^s sont des sous-espaces d'espaces L^p commutatifs, voir la partie 3.2 pour une définition précise). La conséquence surprenante de ce résultat est que ces normes ne dépendent que de la structure d'espace de Banach de E, alors que les normes dans $S^p[E]$ dépendent de la structure d'espace d'opérateurs de E. Le résultat principal est le suivant :

Pour toute série formelle $\varphi=\sum_{n\in\mathbb{N}}\widehat{\varphi}(n)z^n$ où les $\widehat{\varphi}(k)$ appartiennent à un espace d'opérateurs E, la matrice de Hankel Γ_φ est définie par

$$\Gamma_\varphi = \left(\widehat{\varphi}(j+k)\right)_{j,k\geq 0}.$$

Théorème 16. Une matrice de Hankel $(a_{j+k})_{j,k\geq 0}$ appartient à $S^p[E]$ si et seulement si la série formelle $\sum_{n\geq 0} a_n z^n$ appartient à $B_p^{1/p}(E)_+$.

Plus précisément il y a une constante $C>0$ telle que pour tout espace d'opérateurs E, toute série formelle $\varphi=\sum_{n\in\mathbb{N}}\widehat{\varphi}(n)z^n$ où $\widehat{\varphi}(k)\in E$ et tout $1\leq p<\infty$,

$$C^{-1}\|\varphi\|_{B_p^{1/p}(E)_+}\leq \|\Gamma_\varphi\|_{S_p[E]}\leq Cp\,\|\varphi\|_{B_p^{1/p}(E)_+}.$$

Remarque 17. Après avoir écrit la version finale de cette thèse, j'ai réalisé qu'en modifiant légèrement la preuve de ce théorème, on peut obtenir une constante de l'ordre de $\sqrt{p}$ à la place de p à droite dans l'inégalité précédente. De plus cet ordre de grandeur est optimal. Cette amélioration n'est pas incluse ici ; le lecteur intéressé est invité à lire l'article qui ne tardera pas être écrit pour plus de détails.

Comme souvent dans des résultats sur les espaces L^p non-commutatifs, ce théorème est prouvé à l'aide de la méthode d'interpolation complexe. Pour $p=1$ on peut prouver ce résultat directement. Une première idée naturelle pour prouver ce théorème pour un p quelconque serait d'obtenir quelque chose pour $p=\infty$. Les matrices de Hankel à valeurs vectorielles pour $p=\infty$ sont décrites complètement par le théorème de Nehari (et sa version vectorielle), qui affirme que pour $E\subset B(\ell^2)$ et $p=\infty$, Γ_φ appartient à $B(\ell^2)\otimes E$ si et seulement s'il existe une fonction $\psi\in L^\infty(\mathbb{T};B(\ell^2))$ telle que $\widehat{\psi}(k)=\widehat{\varphi}(k)$ pour tout $k>0$. Mais pour des espaces d'opérateurs non injectifs, le fait de relier une telle fonction

ψ à quelque chose qui ne dépend que de E me semble difficile. Une autre idée naturelle aurait été d'interpoler entre $p = 1$ et $p = 2$ puisque souvent les cas $p = 2$ sont évidents. Mais je tiens à souligner que même dans le cas où $p = 2$, le Théorème 16 est non trivial (et je ne vois pas de façon directe de prouver le cas $p = 2$).

On est donc amené à emprunter un chemin détourné. Celui-ci consiste à passer d'un problème à un seul paramètre p à un problème à plusieurs (3 ici) paramètres, de façon à « créer de la place » pour pouvoir interpoler. On fait cela en étudiant les matrices de Hankel généralisées.

Pour α, β réels (ou complexes) la matrice de Hankel généralisée de symbole φ est définie par

$$\Gamma_\varphi^{\alpha,\beta} = \Big((1+j)^\alpha (1+k)^\beta \widehat{\varphi}(j+k)\Big)_{j,k\geq 0}.$$

Le théorème que l'on prouve caractérise les matrices de Hankel généralisées qui appartiennent à $S^p[E]$ à condition que $\alpha + 1/2p > 0$ et $\beta + 1/2p > 0$.

Théorème 18. Soient $1 \leq p \leq \infty$ et $\alpha, \beta > -1/2p$. Alors pour toute série formelle $\varphi = \sum_{n\geq 0} \widehat{\varphi}(n) z^n$ avec $\widehat{\varphi}(n) \in E$, $\Gamma_\varphi^{\alpha,\beta} \in S_p[E]$ si et seulement si $\varphi \in B_p^{1/p+\alpha+\beta}(E)_+$.

Plus précisément, pour tout $M > 0$, il existe une constante $C = C_M$ (dépendant uniquement de M) telle que pour tout espace d'opérateurs E, toute telle φ, tout $1 \leq p \leq \infty$ et tous $\alpha, \beta \in \mathbb{R}$ tels que $-1/2p < \alpha, \beta < M$,

$$C^{-1} \|\varphi\|_{B_p^{1/p+\alpha+\beta}(E)_+} \leq \left\|\Gamma_\varphi^{\alpha,\beta}\right\|_{S_p[E]} \leq \frac{C}{\sqrt{(\alpha + 1/2p)(\beta + 1/2p)}} \|\varphi\|_{B_p^{1/p+\alpha+\beta}(E)_+}. \quad (5)$$

Ces résultats étendent des résultats antérieurs de Peller dans le cas scalaire ou dans le cas où $E = S^p$ ([40]). Dans le cas scalaire, le résultat de Peller exprime donc que le sous-espace de S^p constitué des matrices de Hankel (ou des matrices de Hankel généralisées) est isomorphe à un espace de Besov. Le cas où $E = S^p$ montre que cet isomorphisme est en fait un isomorphisme complet. Le Théorème 18 exprime que cet isomorphisme a la propriété plus forte d'être *régulier* dans le sens de [41].

Les résultats de ce chapitre sont plus à considérer comme des remarques sur les preuves de Peller plutôt que des résultats complètement nouveaux, dans la mesure où chacune des étapes de la preuve est semblable à une étape d'une des preuves de Peller qui sont données dans le chapitre 6 de [40] (mais il y a quand même du travail, puisque par exemple contrairement au cas complètement borné, le cas $p = 2$ n'est pas trivial alors que Peller interpole entre $p = 1$ et $p = 2$). Par souci de complétude je donne cependant des preuves détaillées de tous les résultats énoncés. L'organisation de ce chapitre est la suivante : je commence par faire des rappels (sans preuve) sur les opérateurs réguliers puis (avec preuve) sur les espaces de Besov. Dans une dernière partie je donne une preuve du Théorème 18 : par dualité on est ramené à prouver que l'application $\varphi \to \Gamma_\varphi^{\alpha,\beta}$ est régulière de l'espace de Besov dans S^p. Le cas $p = 1$ ou $p = \infty$ est montré directement, et le cas intermédiaire en est déduit par un argument d'interpolation.

0.2.4 Annexe A

Dans l'annexe A je fais quelques remarques et pose des questions sur les fonctions opérateur-Lipschitz.

Le type de questions auxquelles on s'intéresse est : étant données une norme sur une algèbre de von Neumann et une fonction continue $f : \mathbb{R} \to \mathbb{R}$, la fonction $A \mapsto f(A)$ (définie pour un opérateur auto-adjoint A par le calcul fonctionnel continu) est-elle Lipschitzienne pour la norme que l'on considère ? (On s'intéresse principalement à la norme d'opérateur et aux normes L^p).

Plus précisément je m'intéresse dans cette annexe à la question de savoir : quelle est la condition sur l'algèbre de von Neuman et sa norme pour que toute fonction Lipschitzienne $f : \mathbb{R} \to \mathbb{R}$ reste Lipschitzienne étendue aux opérateurs ? On sait depuis longtemps que pour la norme d'opérateurs, il existe des fonctions qui sont Lipschitziennes mais pas opérateur-Lipschitziennes, la plus simple étant la fonction $f(t) = |t|$. Pour les normes p, la question a été longtemps ouverte jusqu'à ce que très récemment Potapov et Sukochev [47] y répondent par l'affirmative : tout fonction Lipschitzienne est operateur Lipschitzienne pour la norme p dès que $p < \infty$. Je présente une preuve de leur résultat (c'est une preuve simplifiée par rapport à la première version qui est apparue ; la même simplification a aussi été remarquée indépendamment par Potapov et Sukochev).

Cela semble clore définitivement la question, mais je voudrais mentionner le fait que la dépendance que l'on obtient entre la constante de Lipschitz de f et sa constante opérateur-Lipschitz est intéressante à étudier (en fonction de p si $1 < p < \infty$, ou bien en fonction de la dimension n pour la norme d'opérateurs sur M_n), et reste encore ouverte pour $1 < p < \infty$.

Chapter 1

Complete isometries between subspaces of noncommutative L^p-spaces

Introduction

The study of isometries between Banach spaces has been an active area of research in the theory of Banach spaces for a long time, see for example the survey [15]. The isometries between L^p spaces were first described by Banach, with a final proof given by Lamperti. The study of isometries between subspaces of L^p-spaces, goes back at least to the 1960's with Forelli's work [16], but the most general result is due independently to Plotkin in a series of articles in the 1970′s [46] and to Rudin in [50]; see also Hardin [21]. The reader is refered the [28, Chapter 2] for a survey.

The study of isometries between whole noncommutative L^p spaces has already interested a few mathematicians, and the final characterization (in the tracial case) was given by Yeadon in [52]. Recent results were also obtained for non semifinite von Neumann algebras ([51] and [25]). The study of complete isometries between noncommutative L^p spaces has also been more recently studied by Junge, Ruan and Sherman in [25].

In this chapter we will be interested in the study of complete isometries between subspaces of noncommutative L^p spaces, and the main results are close analogues of the result for isometries in subspaces of classical L^p spaces.

We first recall Plotkin's and Rudin's theorem:

Theorem 1.1 (Plotkin, Rudin). *Let $0 < p < \infty$ and $p \neq 2, 4, 6, 8, \ldots$. Let μ and ν be two probability measures (on arbitrary measure spaces Ω and Ω'). Let finally n be a positive integer and $f_1, \ldots f_n \in L^p(\mu)$, $g_1, \ldots g_n \in L^p(\nu)$.*

Assume that for all complex numbers $z_1, \ldots z_n \in \mathbb{C}$,

$$\int |1 + z_1 f_1 + \ldots z_n f_n|^p \, d\mu = \int |1 + z_1 g_1 + \ldots z_n g_n|^p \, d\nu. \tag{1.1}$$

Then $(f_1, \ldots f_n)$ and $(g_1, \ldots g_n)$ form two equimeasurable families. Probabilistically, this means that the $\mathbb{C}^n$-valued random variables $(f_1, \ldots f_n)$ and $(g_1, \ldots g_n)$ have the same distribution.

The following theorem was also proved by Rudin in his paper [50]. It had previously been proved in weaker forms by Forelli ([16] and [17]).

Theorem 1.2 (Rudin). *Let μ and ν be as above, and $0 < p < \infty$, $p \neq 2$. Let $M \subset L^p(\mu)$ be a (complex) unital algebra (with respect to the point-wise product), and $A : M \to L^p(\nu)$ a unital linear isometry: $A(1) = 1$ and*

$$\int |f|^p d\mu = \int |A(f)|^p d\nu \qquad \forall f \in M.$$

- *Then for all $f, g \in M$:*

$$A(fg) = A(f)A(g) \qquad \forall f, g \in M$$

  and

$$\|A(f)\|_\infty = \|f\|_\infty.$$

- *If moreover $M \subset L^\infty$ or $p \neq 4, 6, 8, \dots$, then for all n and $f_1, \dots f_n \in M$, $(f_1, \dots f_n)$ and $(Af_1, \dots Af_n)$ are equimeasurable.*

In this chapter similar results are proved in the noncommutative setting (with some additional boundedness conditions). The commutative L^p-spaces have to be replaced by noncommutative spaces $L^p(\mathcal{M}, \tau)$ associated to a von Neumann algebra $\mathcal{M}$ with a finite normalized trace τ, and isometries are replaced by complete isometries. Let us briefly introduce the vocabulary.

In the whole chapter $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \widetilde{\tau})$ are von Neumann algebras equipped with normal faithful finite (*n.f.f.*) traces. The units of $\mathcal{M}$ and $\mathcal{N}$ are denoted by $1_{\mathcal{M}}$ and $1_{\mathcal{N}}$ or simply by 1. The traces will always be assumed to be normalized: $\tau(1) = 1$.

When n is an integer, the set of $\mathcal{M}$-valued $n \times n$ matrices is denoted by $M_n(\mathcal{M})$, is identified with the tensor product $M_n \otimes \mathcal{M}$ and is provided with a normal faithful tracial state $\tau^{(n)} \overset{\text{def}}{=} \mathrm{tr}_n \otimes \tau$. Here tr_n denotes the normalized trace on M_n:

$$\mathrm{tr}_n(a) = \frac{1}{n}\mathrm{Tr}(a) = \frac{1}{n} \sum_{1 \leq j \leq n} a_{j,j}.$$

The unit of $M_n(\mathcal{M})$ is $1_n \otimes 1_{\mathcal{M}}$ and will be denoted simply by 1 when no confusion is possible.

Let $0 < p < \infty$. If $x \in \mathcal{M}$, the "p-norm" of x is denoted by $\|x\|_p$ and is equal to

$$\|x\|_p = \|x\|_{L^p(\tau)} \overset{\text{def}}{=} \left(\tau\left(|x|^p\right)\right)^{1/p}.$$

In the same way, if $x \in M_n(\mathcal{M})$, $\|x\|_p$ denotes the quantity $\|x\|_{L^p(\tau^{(n)})}$. Remark that $\|\cdot\|_p$ is a norm only if $p \geq 1$. In this case, $L^p(\mathcal{M}, \tau)$ is defined as the completion of $\mathcal{M}$ with respect to the norm $\|\cdot\|_p$ (see the survey [45] for more details, see also section 1.2.2). If $p = \infty$, $L^\infty(\mathcal{M}, \tau)$ is just $\mathcal{M}$ with the operator norm. The space $L^p(\mathcal{M}, \tau)$ will be denoted by $L^p(\mathcal{M})$ or $L^p(\tau)$ when no confusion is possible.

As usual, the main modification one has to bring in order to deal with the noncommutativity is the fact that one has to allow operator coefficients instead of scalar coefficients in (1.1).

In the whole chapter, we will try to use the following notation: unless explicitly specified, small letters x or y will stand for elements of the von Neumann algebras $\mathcal{M}$ or $\mathcal{N}$, a, b will stand for finite complex-valued matrices viewed as matricial coefficients. Operators written with capital letters will be matrices with coefficients in $\mathcal{M}$ or $\mathcal{N}$. The letters z and λ (resp. s and t) will denote complex (resp. real) numbers. In a typical equation like

$$S = \sum_k z_k a_k \otimes x_k \in M_n(\mathcal{M}),$$

it should thus be clear to which set all the z_k, a_k and x_k belong.

At least as far as bounded operators are concerned, the fact that two families of noncommutative random variables (*i.e.* elements of the L^p-spaces) are equimeasurable can be expressed by requesting that their $*$-distributions are the same. Let us recall the definition of the distribution of noncommutative random variables. If $(x_i)_{i\in I} \subset \mathcal{M}$ is a family of operators in $\mathcal{M}$, its distribution with respect to τ is the linear form on the free algebra generated by elements indexed by I that maps a polynomial $P((X_i)_{i\in I})$ in non commuting variables to $\tau(P\big((x_i)_{i\in I}\big)$. Its $*$-distribution is the distribution of $(x_i, x_i^*)_{i\in I}$. The fact that two families of *bounded* operators have the same $*$-distributions is known to be equivalent to saying that they generate isomorphic tracial von Neumann algebras (Lemma 1.25).

The main result of this chapter is the following theorem:

Theorem 1.3. *Let $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \widetilde{\tau})$ be von Neumann algebras equipped with faithful normal finite normalized traces. Let $E \subset L^p(\mathcal{M}, \tau)$ be a subspace of $L^p(\mathcal{M}, \tau)$, and let $u : E \to L^p(\mathcal{N}, \widetilde{\tau})$ be a linear map. Denote by $\mathrm{id} \otimes u : M_n \otimes E \to M_n \otimes L^p(\widetilde{\tau})$ the natural extension of u to $M_n(E)$. Fix $0 < p < \infty$ such that p is not an even integer.*

Assume that the following boundedness condition holds: $E \subset L^\infty(\mathcal{M}) (= \mathcal{M})$.

Assume that for all $n \in \mathbb{N}$ and all $X \in M_n(E)$, the following equality between the p-"norms" holds:

$$\|1_n \otimes 1_{\mathcal{M}} + X\|_p = \|1_n \otimes 1_{\mathcal{N}} + (\mathrm{id} \otimes u)(X)\|_p. \tag{1.2}$$

Let $VN(E)$ denote the von Neumann subalgebra generated by E in $\mathcal{M}$. Then $u(E) \subset L^\infty(\mathcal{N})$ and u extends to a von Neumann algebra isomorphism $u : VN(E) \to VN(u(E))$ that preserves the traces, and this extension is unique.

In particular, if E is an algebra, then u agrees with the multiplicative structure of E: if $x, y \in E$, then $u(xy) = u(x)u(y)$. Moreover, if $x \in E$ and $x^* \in E$, then $u(x^*) = u(x)^*$.

First some remarks: as in the commutative case, the condition $p \notin 2\mathbb{N}$ is crucial. Indeed in the simplest case when $p = 2n$ and $E = \mathbb{C}X$ is one-dimensional, with $X^* = X$ and $Y = u(X) = Y^*$, then condition (1.2) holds as soon as the distributions of X and Y coincide on every polynomial of degree less than $2n$, which does not imply that the distributions agree on every polynomial.

It is also easy to see that it is necessary to allow matrix coefficients to appear in (1.2), and that the theorem does not hold when (1.2) is assumed only for $x \in E$. A simple example is when $E = \mathcal{M} = \mathcal{N} = M_n$ equipped with its normalized trace tr_n and u is the transposition map $u : (a_{ij}) \to (a_{ji})$. Then u is isometric for every p-norm but is not a morphism of algebras.

However, it is unclear whether the theorem holds if (1.2) is only assumed for every $x \in M_n(E)$ for a *fixed* n (even for $n = 2$).

When $p = 2m$ is an even integer, the situation is different: it is possible to show that if (1.2) holds for $n = m$, then (1.2) holds for any n. See Theorem 1.28.

The techniques used in the proof of Theorem 1.3 do not allow to state the result when E is a general subspace of $L^p(\mathcal{M})$ (*i.e.* not necessarily made of bounded operators). Indeed the proof relies on Lemma 1.25 which says that the $*$-distribution of a family of bounded operators characterizes the von Neumann algebra they generate. This result is known to be false for unbounded operators even in the commutative case (it is the moment problem). Moreover the proof relies on the expansion in power series of operators of the form $|1+x|^p$, which allows to compute the $*$-distribution of operators (Lemma 1.7). At first sight this seems to require that the operator x is bounded. However it is possible to get some results of this kind for unbounded operators using a noncommutative version of dominated convergence theorem from [12]: see Lemma 1.12. It is also immediate to see that Theorem 1.3 still holds if the boundedness condition is replaced by the assumption that $E \cap L^\infty(\mathcal{M})$ (or even $E \cap L^\infty + u^{(-1)}(u(E) \cap L^\infty)$ by Theorem 1.16) is dense in E.

In the case when E is self-adjoint and u is assumed to map a self-adjoint operator to a self-adjoint operator (which is *a posteriori* always true, see Lemma 1.32), Theorem 1.3 can be deduced from the commutative Theorem 1.1. Although it is contained in the general case, this special case is proved in the first section of this chapter, since the proof uses the same idea as in the general case but with simpler computations.

In the second section of this chapter the main technical results are proved. The first one establishes the link between the trace of products of operators and p-norms of linear combinations of these operators (Lemma 1.7 for bounded operators and Lemma 1.12 for the general case). The second one (Theorem 1.16) proves that in the setting of Theorem 1.3, if $E \subset L^\infty(\mathcal{M})$ then $u(E) \subset L^\infty(\mathcal{N})$.

In section 1.3 the main theorem (analogous to Theorem 1.1) is derived from Lemma 1.7 (Theorem 1.23 and Theorem 1.3) and also reformulated in the operator space setting (Corollary 1.27). We also derive an approximation result and we discuss the necessity of taking matrices of arbitrary size in (1.2) (but this question is mainly left open).

In a last part, some other consequences of the results of section 1.2 are established, dealing with maps defined on subspaces of L^p which have an additional algebraic structure (e.g. self-adjoint, or stable by multiplication...). In particular a noncommutative analogue of Rudin's Theorem 1.2 is derived. We end the chapter with some comments and questions.

1.1 Self-adjoint case

In this section we prove the special case explained in the introduction as a consequence of the commutative theorem.

Let $p \in \mathbb{R}^+ \setminus 2\mathbb{N}$, $E \subset \mathcal{M}$ and $u : E \to \mathcal{N}$ be as in Theorem 1.3. Assume furthermore that E is self-adjoint (if $x \in E$, $x^* \in E$) and that $u(x^*) = u(x)^*$ for $x \in E$.

Let us sketch the proof in this special case: for any self-adjoint operators $x_1, \dots x_n$ in E, denote $y_k = u(x_k)$. Then for any self-adjoint matrices $a_1, \dots a_n$, since $\sum_k a_k \otimes x_k$ and $\sum_k a_k \otimes y_k$ are self-adjoint, they generate commutative von Neumann algebras, and Rudin's theorem can be applied to deduce that they have the same distribution. The conclusion thus follows from Lemma 1.25 and from the following linearization result (and the fact that E is spanned by self-adjoint operators):

Lemma 1.4. *Let $x_1, \dots x_n \in \mathcal{M}$ and $y_1, \dots y_n \in \mathcal{N}$ be self-adjoint operators. Assume that for all m and all self-adjoint $m \times m$ matrices $a_1 \dots a_n$, the operators $a_1 \otimes x_1 + \dots a_n \otimes x_n$ and $a_1 \otimes y_1 + \dots a_n \otimes y_n$ have the same distribution with respect to the traces $\mathrm{tr}_m \otimes \tau$ and $\mathrm{tr}_m \otimes \widetilde{\tau}$:*

$$\mathrm{dist}(a_1 \otimes x_1 + \dots a_n \otimes x_n) = \mathrm{dist}(a_1 \otimes y_1 + \dots a_n \otimes y_n) \tag{1.3}$$

Then $(x_1, \dots x_n)$ *and* $(y_1, \dots y_n)$ *have the same distribution.*

Independently of our work, this lemma was obtained in [10] using random matrices, and was used to give a new formulation of Connes's embedding problem.

Here we provide a different and elementary proof that consists in exhibiting specific matrices $a_1, \dots a_n$. The idea is the same as in the proof of the general case of Theorem 1.3, but here the computations are simpler.

In fact the result of Collins and Dykema is apparently slightly stronger than the one stated above in the sense that they only assume that (1.3) holds for any self-adjoint matrices a_i with a spectrum included in $[c, d]$ for some fixed real numbers $c < d$. But it is not hard to deduce their result from the one above. More precisely, $m \in \mathbb{N}$ and $c < d$ being fixed, if one only assumes that (1.3) holds for any self-adjoint matrices a_i of size m with a spectrum included in $[c, d]$, then it holds for any self-adjoint matrices $a_i \in M_m$ (without restriction on the spectrum). Indeed, if $c < \lambda < d$ and $a_i \in M_m$ are arbitrary, then for $t \in \mathbb{R}$ small enough, the matrices $\lambda 1_m + t a_i$ all have spectrum in $[c, d]$; and the distribution of $\sum (1_m + t a_i) \otimes x_i$ for infinitely many different values of t is enough to determine the distribution of $\sum a_i \otimes x_i$.

Proof of Lemma 1.4. Let m be an integer and take $(i_1, \dots i_m) \in \{1, 2, \dots n\}^m$. We want to prove that

$$\tau(x_{i_1} x_{i_2} \dots x_{i_n}) = \widetilde{\tau}(y_{i_1} y_{i_2} \dots y_{i_n}).$$

Relabeling and repeating if necessary the x_i's and y_i's, it is enough to prove it when $m = n$ and $i_k = k$ for all k. We are left to prove that

$$\tau(x_1 x_2 \dots x_n) = \widetilde{\tau}(y_1 y_2 \dots y_n). \tag{1.4}$$

Take $z_1, \dots z_n \in \mathbb{C}$ and consider the $n \times n$ self-adjoint matrices $a_k = z_k e_{k,k+1} + \overline{z_k} e_{k+1,k}$ if $k < n$ and $a_n = z_n e_{n,1} + \overline{z_n} e_{1,n}$; the expression $(\mathrm{tr}_n \otimes \tau) \left((\sum_{k=1}^n a_k \otimes x_k)^n \right)$ can be viewed as a polynomial in the z_j's and the $\overline{z_j}$'s, and the coefficient in front of $z_1 z_2 \dots z_n$ is equal to $\tau(x_1 x_2 \dots x_n)$. This is not hard to check from the trace property of τ and from the fact that for a permutation σ on $\{1; 2 \dots n\}$,

$$\mathrm{tr}_n (e_{\sigma(1),\sigma(1)+1 \mod n} e_{\sigma(2),\sigma(2)+1 \mod n} \cdots e_{\sigma(n),\sigma(n)+1 \mod n})$$

is nonzero if and only if σ is a circular permutation, in which case it is equal to $1/n$.

Thus (1.4) holds, and this concludes the proof. ☐

Remark 1.5. The following property for n-uples $a_1, \dots a_n$ of $m \times m$ matrices is the key combinatorial property used in the proof above and will later on be considered in this chapter:

$$\mathrm{tr}_m (a_{\sigma(1)} a_{\sigma(2)} \dots a_{\sigma(n)}) = \begin{cases} 1 & \text{for a circular permutation } \sigma \text{ on } \{1; 2; \dots n\}. \\ 0 & \text{for another permutation } \sigma. \end{cases} \tag{1.5}$$

A permutation σ is said to be circular if there is an integer k such that $\sigma(j) = j + k \mod n$ for all $1 \le j \le n$.

As noted in the proof above (and it was the main combinatorial trick in the proof), the matrices $a_j = n^{1/n} e_{j,j+1 \mod n} \in M_n$ have the property (1.5). But in fact in section 1.4, it will be interesting to find n matrices $a_1 \dots a_n$ with the same property but with smaller size. And this is possible with matrices of size m for $m \ge n/2$:

If $n = 2m$, then the following choice of the $a_j \in M_m$ $(j = 1 \dots n)$ works:

$$\begin{aligned} a_{2j-1} &= e_{j,j} && \text{for } j = 1 \dots m \\ a_{2j} &= e_{j,j+1} && \text{for } j = 1 \dots m-1 \\ a_{2m} &= m e_{m,1} \end{aligned}$$

If $n = 2m - 1$, then the following choice of the $a_j \in M_m$ $(j = 1 \dots n)$ works:

$$\begin{aligned} a_{2j-1} &= e_{j,j} && \text{for } j = 1 \dots m-1 \\ a_{2j} &= e_{j,j+1} && \text{for } j = 1 \dots m-1 \\ a_{2m-1} &= m e_{m,1}. \end{aligned}$$

1.2 Expression of the moments in term of the p-norms

In this section, we prove that the trace of a product of finitely many operators or of their adjoints can be computed from the p-norm of the linear (matrix-valued) combinations of these operators. The main results are Lemma 1.7 for bounded operators and its refinement Lemma 1.12 for unbounded operators. We also prove that a map u as in Theorem 1.3 maps a bounded operator to a bounded operator (Theorem 1.16).

1.2.1 Case of bounded operators

First suppose we are given $x_1, x_2, \dots x_n$ elements of the von Neumann algebra $\mathcal{M}$ (here the x_i's are *bounded operators*), and $\varepsilon_1, \dots \varepsilon_n \in \{1, *\}$. If x is an element of a von Neumann algebra and $\varepsilon \in \{1, *\}$, let x^ε denote x if $\varepsilon = 1$ and x^* if $\varepsilon = *$ (for a complex number z, $z^* = \overline{z}$).

For clearness, we adopt the following (classical) notation: for every $z = (z_1, \dots z_n) \in \mathbb{C}^n$ and $k = (k_1, \dots k_n) \in \mathbb{N}^n$, we write $z^k = \prod_j z_j^{k_j}$ and $\bar{z}^k = \prod_j \bar{z}_j^{k_j}$. In the same way, one writes $z^\varepsilon = \prod_j z_j^{\varepsilon_j}$. If f is a formal series $f(z) = \sum_{k,l \in \mathbb{N}^n} a_{k,l} z^k \bar{z}^l$, we denote $f(z)[z^k \bar{z}^l] = a_{k,l}$. We will also denote by $\binom{\beta}{n}$ the generalized binomial coefficient defined, for $\beta \in \mathbb{C}$ and $n \in \mathbb{N}$, by:

$$\binom{\beta}{n} = \beta(\beta - 1) \dots (\beta - n + 1)/n!. \tag{1.6}$$

Pick n matrices $a_1, \dots a_n$ with complex coefficients (say of size m). The a_j's will soon be assumed to satisfy (1.5). For all $z = (z_1, z_2, \dots z_n) \in \mathbb{C}^n$, denote by $S_z \in M_m(\mathcal{M})$ the matrix

$$S_z = S(z_1, \dots z_n) = 1 + \sum_{j=1}^{n} z_j a_j^{\varepsilon_j} \otimes x_j. \tag{1.7}$$

The following combinatorial lemma justifies the choice of the a_j's:

Lemma 1.6. *Denote by $\alpha(\varepsilon)$ or simply α the number of indices $1 \le j \le n$ such that $\varepsilon_j = *$ and $\varepsilon_{j+1} = 1$ (again if $j = n$, $\varepsilon_{n+1} = \varepsilon_1$).*

If the a_j's satisfy (1.5) *and S_z is defined by* (1.7), *then for any integer k,*

$$\tau^{(m)}\left((S_z^* S_z - 1)^k [z^\varepsilon]\right) = \tau(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}) k \binom{\alpha}{n-k} \tag{1.8}$$

Proof. Recall that

$$S_z^*S_z - 1 = \sum_{j\le n} z_j a_j^{\varepsilon_j} \otimes x_j + \sum_{j\le n} \overline{z_j} a_j^{\varepsilon_j *} \otimes x_j^* + \sum_{i,j\le n} \overline{z_i} z_j a_i^{\varepsilon_i *} a_j^{\varepsilon_j} \otimes x_i^* x_j.$$

For one of the terms of $\sum_{j\le n} z_j a_j^{\varepsilon_j} \otimes x_j$ to bring a contribution to the coefficient of $\prod_j z_j^{\varepsilon_j}$ in $(S_z^*S_z-1)^k$, it is necessary that $\varepsilon_j = 1$, and then $z_j a_j^{\varepsilon_j} \otimes x_j = z_j^{\varepsilon_j} a_j \otimes x_j^{\varepsilon_j}$. In the same way, for one of the terms of $\sum_{j\le n} \overline{z_j} a_j^{\varepsilon_j *} \otimes x_j^*$ to bring a contribution, it is necessary that $\varepsilon_j = *$ and then $\overline{z_j} a_j^{\varepsilon_j *} \otimes x_j^* = z_j^{\varepsilon_j} a_j \otimes x_j^{\varepsilon_j}$. Last, for one of the terms of $\sum_{i,j\le n} \overline{z_i} z_j a_i^{\varepsilon_i *} a_j^{\varepsilon_j} \otimes x_i^* x_j$ to have a nonzero contribution, the values of ε_i and ε_j must be $\varepsilon_i = *$ and $\varepsilon_j = 1$, and then $\overline{z_i} z_j a_i^{\varepsilon_i *} a_j^{\varepsilon_j} \otimes x_i^* x_j = z_i^{\varepsilon_i} z_j^{\varepsilon_j} a_i a_j \otimes x_i^{\varepsilon_i} x_j^{\varepsilon_j}$. Thus if one denotes $y_j = x_j^{\varepsilon_j}$,

$$\tau^{(m)}\left((S_z^*S_z-1)^k\right)[z^\varepsilon] = \tau^{(m)}\left(\left(\sum_{1\le j\le n} z_j^{\varepsilon_j} a_j \otimes y_j + \sum_{i,j,\varepsilon_i = * \text{ and } \varepsilon_j = 1} z_i^{\varepsilon_i} z_j^{\varepsilon_j} a_i a_j \otimes y_i y_j\right)^k\right)[z^\varepsilon].$$

Developing and using the assumption (1.5) on the a_j's, one gets

$$\tau^{(m)}\left((S_z^*S_z-1)^k\right)[z^\varepsilon] = \sum_{l=1}^n C_l \tau\left(y_l y_{l+1} \dots y_{l-1}\right), \tag{1.9}$$

where the indices have to be understood modulo n and where C_l denotes the number of ways of writing formally the word $y_l y_{l+1} \dots y_{l-1}$ (which is of length n) as a concatenation of k "elementary bricks" of the form y_j (for $1 \le j \le n$) or $y_j y_{j+1}$ with $\varepsilon_j = *$ and $\varepsilon_{j+1} = 1$. Each of these bricks has length 1 or 2. If α_l denotes the number of apparitions of the subsequence $*, 1$ in the sequence $\varepsilon_l, \varepsilon_{l+1 \mod n}, \dots \varepsilon_{l-1 \mod n}$ (not cyclically this time!), then for C_l to be non zero it is necessary that $k \le n \le k + \alpha_l$. In that case C_l is equal to the number of ways of choosing the $n-k$ bricks of size 2 among the α_j possible, the other bricks being of size 1. Thus $C_l = \binom{\alpha_l}{n-k}$. The fact that τ is a trace then allows to write (1.9) as

$$\begin{aligned}\tau^{(m)}\left((S_z^*S_z-1)^k\right)[z^\varepsilon] &= \tau\left(y_1 y_2 \dots y_n\right) \sum_l \binom{\alpha_l}{n-k} \\ &= \tau\left(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}\right) \sum_l \binom{\alpha_l}{n-k}.\end{aligned}$$

It remains to notice that $\alpha_l = \alpha - 1$ if $\varepsilon_{l-1} = *$ and $\varepsilon_l = 1$ (which is the case for α different values of l), and that $\alpha_l = \alpha$ otherwise (for the $n - \alpha$ remaining values of l). The preceding equation then becomes

$$\tau^{(m)}\left((S_z^*S_z-1)^k\right)[z^\varepsilon] = \tau\left(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}\right)\left(\alpha\binom{\alpha-1}{n-k} + (n-\alpha)\binom{\alpha}{n-k}\right).$$

Equation (1.8) follows from the elementary equality

$$\alpha\binom{\alpha-1}{n-k} + (n-\alpha)\binom{\alpha}{n-k} = k\binom{\alpha}{n-k}. \qquad \square$$

Note that the above proof only uses combinatorial arguments, it therefore also holds with minor modifications when the assumption $x_j \in \mathcal{M}$ is replaced by $x_j \in L^n(\mathcal{M}, \tau)$ for all j.

The following lemma establishes the link between the p-norm of S_z and the trace of the product of the $x_j^{\varepsilon_j}$.

Lemma 1.7. *Let* $0 < p < \infty$. *Let* $a_1, \dots a_n$ *be matrices* satisfying (1.5), *and, remembering* (1.7), *define the function* $\varphi : \mathbb{R}^n \to \mathbb{C}$ *by*

$$\varphi(r_1, \dots r_n) = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \left\| S\left(r_1 e^{i\theta_1}, \dots r_n e^{i\theta_n}\right) \right\|_p^p \prod_j \exp(-i\theta_j)^{\varepsilon_j} d\theta_1 \dots d\theta_n.$$

Then φ *is indefinitely differentiable on a neighborhood of* 0, *and if* α *is defined as in Lemma 1.6*

$$\frac{d^{(n)}}{dr_1 \dots dr_n} \varphi(0, \dots 0) = \lim_{r_1, \dots r_n \to 0} \frac{1}{r_1 \dots r_n} \varphi(r_1, \dots r_n) = \tau(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}) \sum_{k=0}^{\alpha} (n-k) \binom{p/2}{n-k} \binom{\alpha}{k}. \quad (1.10)$$

Proof. The idea of the proof is the following: S_z is a small perturbation of the unit, which allows to write $|S_z|^p$ as a converging series. Equation (1.10) follows from the identification of the term in front of $\prod_j z_j^{\varepsilon_j}$. First write:

$$\begin{aligned} S_z^* S_z &= 1 + \sum_{j \le n} z_j a_j^{\varepsilon_j} \otimes x_j + \sum_{j \le n} \overline{z_j} a_j^{\varepsilon_j *} \otimes x_j^* + \sum_{i,j \le n} \overline{z_i} z_j a_i^{\varepsilon_i *} a_j^{\varepsilon_j} \otimes x_i^* x_j \\ &= 1 + \sum_{1 \le j \le n^2 + 2n} C_j. \end{aligned}$$

In the last line, we denoted by C_j the $n^2 + 2n$ terms that appear on the preceding line. Remark that if $\sup |z_j| = \delta \le 1$, then $\|C_j\| \le \delta K$ where $K = \max_j \left(\|a_j\| \|x_j\|, \|a_j\|^2 \|x_j\|^2\right)$.

By the functional calculus for bounded operators, for z small enough (*i.e.* $\|1 - S_z^* S_z\| < 1$), one has:

$$(S_z^* S_z)^{p/2} = \sum_{k \ge 0} \binom{p/2}{k} (S_z^* S_z - 1)^k = \sum_{k \ge 0} \binom{p/2}{k} \sum_{1 \le j_1, \dots j_k \le n^2 + 2n} C_{j_1} \dots C_{j_k}. \quad (1.11)$$

The series above converges absolutely and uniformly when $\delta = \sup |z_j|$ is small enough, *i.e.* $K(n^2 + 2n)\delta < 1$. Indeed, $\|C_{j_1} \dots C_{j_k}\| \le \delta^k K^k$, and since $\binom{p/2}{k}$ tends to 0 as $k \to \infty$, one has

$$\sum_{k \ge 0} \sum_{1 \le j_1, \dots j_k \le n^2 + 2n} \sup_{|z_j| \le \delta \forall j} \left\| \binom{p/2}{k} C_{j_1} \dots C_{j_k} \right\| \le \sum_{k \ge 0} (n^2 + 2n)^k \left| \binom{p/2}{k} \right| \delta^k K^k < \infty.$$

We can thus reorder the terms of the sum (1.11) along powers of z_j and $\overline{z}_j$:

$$|S_z|^p = \sum_{k,l \in \mathbb{N}^n} z_1^{k_1} \dots z_n^{k_n} \overline{z}_1^{l_1} \dots \overline{z}_n^{l_n} D_{k,l}, \quad (1.12)$$

where $D_{k,l}$ are some operators in $M_m \otimes \mathcal{M}$, which are in fact some polynomials in $a_1^{\varepsilon_1} \otimes x_1 \dots a_n^{\varepsilon_n} \otimes x_n$ and their adjoints. Taking the trace $\tau^{(m)}$ on both sides of (1.12) , one gets

$$\|S_z\|_p^p = \sum_{k,l\in\mathbb{N}^n} \lambda_{k,l}\, z_1^{k_1}\dots z_n^{k_n}\,\bar{z}_1^{l_1}\dots\bar{z}_n^{l_n}. \tag{1.13}$$

In this sum, we wrote $k=(k_1,\dots k_n)$ and $l=(l_1,\dots l_n)$. The coefficient $\lambda_{k,l}$ is equal to

$$\lambda_{k,l} = \sum_{j\leq |l|+|k|} \binom{p/2}{j} \tau^{(m)}\left((S_z^*S_z-1)^j\right)[z^k\bar{z}^l].$$

If E is defined as the set of indices $(k,l)\in\mathbb{N}^n\times\mathbb{N}^n$ such that $k_j-l_j=1$ if $\varepsilon_j=1$ and $k_j-l_j=-1$ if $\varepsilon_j=*$, then for $r_1,\dots r_n$ small enough, we are allowed to exchange the series and the integral in the definition of $\varphi(r_1,\dots r_n)$ and we get the following expression of φ as a converging power series:

$$\varphi(r_1,\dots r_n) = \sum_{(k,l)\in E} \lambda_{k,l}\, r_1^{k_1+l_1}\dots r_n^{k_n+l_n}.$$

The two left-hand sides of (1.10) are thus equal to λ_{k^0,l^0} where $k_j^0=1$ if $\varepsilon_j=1$, $k_j^0=0$ else, and $l_j^0=1-k_j^0$. In other words, λ_{k^0,l^0} is the coefficient of $\prod_j z_j^{\varepsilon_j}$ in (1.13):

$$\frac{d^{(n)}}{dr_1\dots dr_n}\varphi(0,\dots 0) = \lim_{r_1,\dots r_n\to 0}\frac{1}{r_1\dots r_n}\varphi(r_1,\dots r_n) = \lambda_{k^0,l^0}, \tag{1.14}$$

with

$$\lambda_{k^0,l^0} = \sum_{j\in\mathbb{N}} \binom{p/2}{j} \underbrace{\tau^{(m)}\left((S_z^*S_z-1)^j\right)[z^\varepsilon]}_{\stackrel{\text{def}}{=}\gamma_j}. \tag{1.15}$$

But from Lemma 1.6,

$$\gamma_j = \tau\left(x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_n^{\varepsilon_n}\right) j \binom{\alpha}{n-j}.$$

Putting this equation together with (1.14) and (1.15), we finally get (1.10), which proves the Lemma. □

Remark 1.8. In the case when $p/2$ is an integer (*i.e.* p is an even integer), the same result holds in a more general setting, when the x_i's are not bounded but are in the noncommutative L^p space associated to $(\mathcal{M},\tau)$. Indeed, then the sum on the right-hand side of (1.11) makes sense as a finite sum of elements which all are in $L^1(\mathcal{M},\tau)$. Indeed, from Hölder's inequality, a product of k elements of L^p is in $L^{p/k}$. This allows to take the trace in (1.11) and to follow the rest of the proof.

Of course when p is different from an even integer, the proof does not apply for unbounded operators: it is indeed unclear what sense should be given to the series (1.11), and more importantly taking the trace to get (1.13) makes no sense. However, using a noncommutative dominated convergence theorem from [12], it is possible to modify the proof and get similar results with unbounded operators.

1.2.2 Unbounded case

The reader is refered to [12] for all facts and definitions on measure topology and generalized s-numbers. Just recall that if $(\mathcal{M}, \tau)$ is a von Neumann algebra with a *n.f.f.* normalized trace, the t-th singular number of a closed densely defined (possibly unbounded) operator Y affiliated with $\mathcal{M}$ is defined as

$$\mu_t(Y) = \inf\left\{\|YE\|, E \text{ is a projection in } \mathcal{M} \text{ with } \tau(1-E) \le t\right\}.$$

The map $t \to \mu_t(Y)$ is non-increasing and vanishes on $t \geq 1$, and $\mu_t(Y) < \infty$ if $t > 0$.

Moreover the measure topology makes the set of τ-measurable operators affiliated with $\mathcal{M}$ a topological $*$-algebra in which a sequence (Y_n) converges to Y if and only if $\mu_t(Y - Y_n) \to 0$ for all $t > 0$. More precisely, the following inequalities hold for any positive real numbers $s, t > 0$ and any (closed densely defined) operators T and S affiliated with $\mathcal{M}$ (Lemma 2.5 in [12]):

$$\begin{aligned} \mu_t(\lambda T) &= |\lambda|\mu_t(T) \text{ for any } \lambda \in \mathbb{C} &\quad (1.16)\\ \mu_t(T) &= \mu_t(|T|) = \mu_t(T^*) &\quad (1.17)\\ \mu_{t+s}(T+S) &\le \mu_t(T) + \mu_s(S) &\quad (1.18)\\ \mu_{t+s}(TS) &\le \mu_t(T)\mu_s(S). &\quad (1.19)\end{aligned}$$

Another property from [12, Lemma 2.5] is the fact that $\mu_s(f(T)) = f(\mu_s(T))$ for any operator $T \geq 0$ and any continuous increasing function on $\mathbb{R}$ with $f(0) = 0$. As a consequence, for any continuous function f on $\mathbb{R}$ with $f(0) = 0$ and any self-adjoint T affiliated with $\mathcal{M}$,

$$\mu_t(f(T)) \le \sup_{|u| \le \mu_t(T)} |f(u)| \tag{1.20}$$

For any $0 < p \leq \infty$, the noncommutative L^p-space $L^p(\mathcal{M}, \tau)$ is identified with the set of closed densely defined operators Y affiliated with $\mathcal{M}$ such that the function $t \mapsto \mu_t(Y)$ is in $L^p([0,1], dt)$. Moreover, the p-norm of this function is equal to $\|Y\|_p$.

We now fix $0 < p < \infty$.

The first fact we prove is the following lemma, which basically says that when the x_j's are unbounded operators affiliated with $\mathcal{M}$, the development in power series of $|S_z|^p$ (1.12) still holds, but in the measure topology instead of the norm topology.

Lemma 1.9. *Let X be a closed densely defined operator affiliated with a von Neumann algebra $(\mathcal{M}, \tau)$. For $r > 0$, denote by Y_r the operator*

$$Y_r = (1 + rX)^*(1 + rX) - 1 = rX + rX^* + r^2X^*X.$$

Then as $r \to 0$, the following convergence holds in the measure topology:

$$\frac{1}{r^n}\left(|1 + rX|^p - \sum_{j=0}^{n} \binom{p/2}{j} Y_r^j\right) \to 0. \tag{1.21}$$

Proof. We first claim that for all $t > 0$, $\sup_{r<1} \mu_t(Y_r/r) < \infty$. Indeed, from (1.18), we have:

$$\begin{aligned} \mu_{2t}(Y_r/r) &\le \mu_t(X + X^*) + \mu_t(rX^*X)\\ &= \mu_t(X + X^*) + r\mu_t(X^*X)\end{aligned}$$

The claim follows from the fact that $\mu_t(x) < \infty$ for all closed densely defined operator x.

Fix now $t > 0$ and take $M = \sup_{r<1} \mu_t(Y_r/r)$. Then (Proposition 2.2 in [12]) if $E = E_{[-M,M]}(Y_r/r)$ and $r < 1$, we have $\tau(1-E) \leq t$ and by the functional calculus, since Y and E commute and are self-adjoint,

$$(1+Y_r)^{p/2}E = (E+Y_rE)^{p/2}E = \sum_{j\geq 0}\binom{p/2}{j}(Y_rE)^jE = \sum_{j\geq 0}\binom{p/2}{j}Y_r^jE.$$

The previous series converges in the operator norm topology if $rM < 1$, since in that case, $\|Y_rE\| \leq rM < 1$. Then

$$\begin{aligned}\left\|\frac{1}{r^n}\left(|S_{rz}|^p - \sum_{j=0}^n\binom{p/2}{j}Y_r^j\right)E\right\| &= \left\|\frac{1}{r^n}\left((1+Y_r)^{p/2} - \sum_{j=0}^n\binom{p/2}{j}Y_r^j\right)E\right\| \\ &= \left\|\sum_{j\geq n+1} r^{-n}\binom{p/2}{j}(Y_rE)^j\right\| \\ &\leq \sum_{j\geq n+1}\left|\binom{p/2}{j}\right| r^{-n}(Mr)^j \to 0.\end{aligned}$$

This proves that $\mu_t\left(r^{-n}\left(|1+rX|^p - \sum_{j=0}^n\binom{p/2}{j}Y_r^j\right)\right)$ tends to zero as $r \to 0$ for every $t > 0$. This concludes the proof. □

Let us denote by Q_n the linear projection from the space of complex polynomial $\mathbb{C}[r]$ to the subspace $\mathbb{C}_n[r]$ of the polynomials of degree at most n given by: $Q_n(r^k) = r^k$ if $r \leq n$ and $Q_n(r^k) = 0$ if $k > n$. If V is any vector space over the field of complex numbers, this projection naturally extends to the space of polynomials with coefficients in V (this extension if simply the tensor product map $\mathrm{id} \otimes Q_n$ if one identifies the space of polynomials with coefficients in V with the tensor product $V \otimes \mathbb{C}[r]$). For simplicity this extension will still be denoted by Q_n. The following result follows from Lemma 1.9:

Corollary 1.10. *Let X and Y_r be as above (for any $r > 0$). Then as $r \to 0$,*

$$\frac{1}{r^n}\left(|1+rX|^p - Q_n\left(\sum_{j=0}^n\binom{p/2}{j}Y_r^j\right)\right) \to 0.$$

Proof. It follows immediately from Lemma 1.9 and from the fact that if T is affiliated with $(\mathcal{M}, \tau)$ and $k > n$, then

$$\frac{1}{r^n}r^kT \to 0 \text{ in the measure topology as } r \to 0.$$ □

The next step is to get a domination result necessary to apply Fack and Kosaki's dominated convergence theorem. More precisely, we prove:

Lemma 1.11. *With the same notation as above, there are constants C and K depending only on p and n such that for all $r < 1$ and all $0 < t \leq 1$,*

$$\mu_t\left(\frac{1}{r^n}\left(|1+rX|^p - Q_n(\sum_{j=0}^n\binom{p/2}{j}Y_r^j)\right)\right) \leq C(\mu_{t/K}(X)^n + \mu_{t/K}(X)^p) \qquad (1.22)$$

Proof. Denote by $m_{t,r}$ the left-hand side of (1.22):

$$m_{t,r} \stackrel{\text{def}}{=} \mu_t\left(\frac{1}{r^n}\left(|1+rX|^p - Q_n(\sum_{j=0}^n \binom{p/2}{j} Y_r^j)\right)\right).$$

Fix an integer K such that $K \geq 2n2^{2n+1}$ and $K \geq 3 \times 2^{2n+1}$. Define a real number s by $s = t/K$. To prove that $m_{t,r} \leq C(\mu_s(X)^n + \mu_s(X)^p)$, we consider two cases, depending on the value of $r\mu_s(X)$.

First assume that $r\mu_s(X) \geq 1$.

Note that there are some real numbers $\lambda_{k,\varepsilon}$ indexed by the integers $k \geq 0$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots \varepsilon_k) \in \{1, *\}^k$ such that for any $r > 0$ (and any n),

$$Q_n\left(\sum_{j=0}^n \binom{p/2}{j} Y_r^j\right) = \sum_{k=0}^n \sum_{\varepsilon \in \{1,*\}^k} \lambda_{k,\varepsilon} r^k X^{\varepsilon_1} X^{\varepsilon_2} \ldots X^{\varepsilon_k}.$$

Thus, using (1.18) 2^{n+1} times, one gets

$$r^n m_{t,r} \quad \leq \quad \mu_{t/2^{n+1}}(|1+rX|^p) + \sum_{k=0}^n \sum_{\varepsilon \in \{1,*\}^k} |\lambda_{k,\varepsilon}| r^k \mu_{t/2^{n+1}}(X^{\varepsilon_1} X^{\varepsilon_2} \ldots X^{\varepsilon_k})$$

Since $t/2^{n+1} \geq t/K = s$, we have that

$$\begin{aligned}
\mu_{t/2^{n+1}}(|1+rX|^p) &\leq \mu_s(|1+rX|^p) \\
&= \mu_s(|1+rX|)^p \\
&= \mu_s(1+rX)^p \\
&\leq (1+r\mu_s(X))^p \\
&\leq 2^p(r\mu_s(X))^p \\
&\leq \begin{cases} r^n 2^p \mu_s(X)^p & \text{if } p \geq n \\ r^n 2^p \mu_s(X)^n & \text{if } p \leq n \end{cases}
\end{aligned}$$

In these computations, the fact that $\mu_s(f(T)) = f(\mu_s(T))$ for any operator $T \geq 0$ and any continuous increasing function on $\mathbb{R}$ with $f(0) = 0$ was used, together with the assumption $1 \leq r\mu_s(X)$.

From (1.19) and (1.17), we get, for $0 \leq k \leq n$,

$$\mu_{t/2^{n+1}}(X^{\varepsilon_1} X^{\varepsilon_2} \ldots X^{\varepsilon_k}) \leq \mu_{t/(k2^{n+1})}(X)^k.$$

Since $t/(k2^{n+1}) \geq t/K = s$, we have that

$$\frac{1}{r^n} r^k \mu_{t/2^{n+1}}(X^{\varepsilon_1} X^{\varepsilon_2} \ldots X^{\varepsilon_k}) \leq r^{k-n} \mu_s(X)^k \leq \mu_s(X)^n.$$

This concludes the proof that $m_{t,r} \leq C\left(\mu_s(X)^p + \mu_s(X)^n\right)$ for some C, in the case when $r\mu_s(X) \geq 1$.

Let us now assume that $r\mu_s(X) < 1$. We want to prove that in that case, there is a constant C not depending on r and t such that

$$r^n m_{r,t} \leq C r^n \mu_s(X)^n. \tag{1.23}$$

In the same way as above, write

$$|1+rX|^p - Q_n(\sum_{j=0}^{n} \binom{p/2}{j} Y_r^j) =$$
$$|1+rX|^p - \sum_{j=0}^{n} \binom{p/2}{j} Y_r^j + \sum_{k=n+1}^{2n} \sum_{\varepsilon \in \{1,*\}^k} \widetilde{\lambda}_{k,\varepsilon} r^k X^{\varepsilon_1} X^{\varepsilon_2} \dots X^{\varepsilon_k},$$

for some real numbers $\widetilde{\lambda}_{k,\varepsilon}$ depending neither on r nor on t.

Again, using (1.18), one gets

$$r^n m_{r,t} \leq \mu_{t/2^{2n+1}} \left(|1+rX|^p - \sum_{j=0}^{n} \binom{p/2}{j} Y_r^j \right) +$$
$$\sum_{k=n+1}^{2n} \sum_{\varepsilon \in \{1,*\}^k} \widetilde{\lambda}_{k,\varepsilon} r^k \mu_{t/2^{2n+1}} (X^{\varepsilon_1} X^{\varepsilon_2} \dots X^{\varepsilon_k}).$$

The second term is easy to dominate using that $r\mu_s(X) < 1$ and that if $n < k \leq 2n$, then $t/k2^{2n+1} \geq t/K = s$:

$$\mu_{t/2^{2n+1}}(X^{\varepsilon_1} X^{\varepsilon_2} \dots X^{\varepsilon_k}) \leq \mu_{t/k2^{2n+1}}(X)^k \leq \mu_s(X)^k \leq r^{n-k} \mu_s(X)^n.$$

For the first term, we use (1.20) for $T = Y_r$ and $f(x) = (1+x)^{p/2} - \sum_{k=0}^{n} \binom{p/2}{k} x^k$ (if $x \geq -1$, and say $f(x) = f(-1)$ else). Indeed, we have that $f(x) = o(x^n)$ as $x \to 0$, in particular there is a constant C_1 such that $|f(x)| \leq C_1 |x|^n$ if $|x| \leq 3$. If one proves that $\mu_{t/2^{2n+1}}(Y_r) \leq 3r\mu_s(X)$, we thus have that

$$\mu_{t/2^{2n+1}} \left(|1+rX|^p - \sum_{j=0}^{n} \binom{p/2}{j} Y_r^j \right) \leq C_1 3^n r^n \mu_s(X)^n,$$

which would complete the proof of (1.23).

We are left to prove that $\mu_{t/2^{2n+1}}(Y_r) \leq 3r\mu_s(X)$. But since $t/2^{2n+1} \geq 3t/K$, we have that $\mu_{t/2^{2n+1}}(Y_r) \leq \mu_{3s}(Y_r)$, and thus using (1.18), one gets

$$\begin{aligned} \mu_{t/2^{2n+1}}(Y_r) \leq \mu_{3s}(r^2 X^* X + rX + rX^*) &\leq r^2 \mu_s(X^* X) + r\mu_s(X) + r\mu_s(X^*) \\ &= (r\mu_r(X))^2 + 2r\mu_s(X) \leq 3r\mu_s(X). \quad \square \end{aligned}$$

It is now possible to use Fack and Kosaki's dominated convergence theorem [12, Theorem 3.6] to prove the main result of this part, which is the unbounded version of Lemma 1.7:

Lemma 1.12. *Let $0 < p < \infty$. Assume that $x_1 \dots x_n \in L^p(\mathcal{M}, \tau)$ and take $\varepsilon_1, \dots \varepsilon_n \in \{1, *\}$. If $x_j \in L^n(\mathcal{M}, \tau)$ for all j, then the trace $\tau(x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n})$ "can be computed from the p-norms" of linear combinations of the x_i's with coefficients in M_m for some m.*

More precisely, let $m \in \mathbb{N}$ and $a_1, \dots a_n \in M_m$ satisfying (1.5) *(as explained in Remark 1.5, such a choice of the a_j's can be achieved for any $m \geq n/2$). Define α as in Lemma 1.6 and denote $\forall z \in \mathbb{C}^n$*

$$S_z = 1 + \sum_{j=1}^{n} z_j a_j^{\varepsilon_j} \otimes x_j.$$

Let $Z = (Z_1, \dots Z_n)$ be a (classical) $\mathbb{C}^n$-valued random variable where the Z_j's are uniformly distributed in $\{\exp(2ik\pi/3), k = 1, 2, 3\}$ and independent. Denote by $\mathbb{E}$ the expected value with respect to Z. Then

$$\frac{1}{r^n}\mathbb{E}\left[\|S_{rZ}\|_p^p \prod_{j=1}^n \overline{Z_j}^{\varepsilon_j}\right] \xrightarrow{r\to 0} \tau(x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_n^{\varepsilon_n})\sum_{k=0}^{\alpha}(n-k)\binom{p/2}{n-k}\binom{\alpha}{k}. \tag{1.24}$$

Proof. The first step of the proof consists in using Corollary 1.10 and Lemma 1.11 in the von Neumann algebra $(M_m(\mathcal{M}), \tau^{(m)})$ in order to apply [12, Theorem 3.6]. Fix $z \in \mathbb{C}^n$, and denote $Y_r = S_{rz}^* S_{rz} - 1$ (the dependence of Y_r on z is implicit). If $T_r = \frac{1}{r^n}\left(|S_{rz}|^p - Q_n\left(\sum_{j=0}^n \binom{p/2}{j} Y_r^j\right)\right)$, then from Corollary 1.10, T_r converges to 0 in measure, and from Lemma 1.11, T_r is dominated in the following way: there are positive constants C and K such that for any $0 < t \leq 1$ and any $0 < r < 1$,

$$\mu_t(T_r) \leq C(\mu_{t/K}(X)^p + \mu_{t/K}(X)^n), \tag{1.25}$$

where $X = \sum_{j=1}^n z_j a_j^{\varepsilon_j} \otimes x_j$. In particular, $X \in L^p(M_m(\mathcal{M}), \tau^{(m)}) \cap L^n(M_m(\mathcal{M}), \tau^{(m)})$. To deduce that

$$\frac{1}{r^n}\tau^{(m)}\left(|S_{rz}|^p - \sum_{j=0}^n \binom{p/2}{j} Y_r^j\right) \to 0, \tag{1.26}$$

it is thus sufficient to prove that the domination term $C(\mu_{t/K}(X)^p + \mu_{t/K}(X)^n)$ is (as a function of t), in $L^1(\mathbb{R}_+, dt)$ (see [12, Theorem 3.6]). But this follows from the fact that, since $X \in L^p(M_m(\mathcal{M}), \tau^{(m)})$ (resp. $X \in L^n(M_m(\mathcal{M}), \tau^{(m)})$), the function $t \mapsto \mu_t(X)$ is in $L^p(\mathbb{R}_+, dt)$ (resp. $L^n(\mathbb{R}_+, dt)$. This proves (1.26).

Now replace z in (1.26) by the random variable Z defined above, multiply by $\prod_{j=1}^n \overline{Z_j}^{\varepsilon_j}$ and take the expected value. Since z is no longer fixed, Y_r is denoted by $Y_r(z)$ to remember that Y_r depends on z. Since Z only takes a finite number of values, equation (1.26) then becomes:

$$\frac{1}{r^n}\mathbb{E}\left[\tau^{(m)}\left(\prod_{j=1}^n \overline{Z_j}^{\varepsilon_j}|S_{rZ}|^p - Q_n(\sum_{j=0}^n \binom{p/2}{j}\prod_{j=1}^n \overline{Z_j}^{\varepsilon_j} Y_r(Z)^j)\right)\right] \xrightarrow{r\to 0} 0.$$

Note that $Q_n(\sum_{j=0}^n \binom{p/2}{j} Y_r(z)^j)$ is, as a function of $z = (z_1, \dots z_n)$, a polynomial in the $2n$ variables z_i and $\overline{z_j}$ with coefficients in $L^1(M_m \otimes \mathcal{M}, \tau^{(m)})$ (this follows from Hölder's inequality and from the fact that $x_j \in L^n(\mathcal{M}, \tau)$). Moreover, if $P(z_1, \dots z_n)$ is such a polynomial, *i.e.* $P(z) = \sum_{k,l\in\mathbb{N}^n, |k|+|l|\leq n} X_{k,l} z^k \overline{z}^l$, then

$$\mathbb{E}\left[\prod_{j=1}^n \overline{Z_j}^{\varepsilon_j} P(Z)\right] = X_{k^0,l^0},$$

where $k^0 \in \mathbb{N}^n$ and $l^0 \in \mathbb{N}^n$ are again defined by $k_j^0 = 1$ if $\varepsilon_j = 1$, $k_j^0 = 0$ else, and $l_j^0 = 1 - k_j^0$. If D_{k^0,l^0} denotes the coefficient in front of $z^{k^0}\overline{z}^{l^0}$ in $\sum_{j=0}^n \binom{p/2}{j} Y_1(z)^j$, then one has:

$$\begin{aligned}\mathbb{E}\left[Q_n(\sum_{j=0}^n \binom{p/2}{j}\prod_{j=1}^n \overline{Z_j}^{\varepsilon_j} Y_r(Z)^j)\right] &= \mathbb{E}\left[Q_n(\sum_{j=0}^n \binom{p/2}{j}\prod_{j=1}^n \overline{Z_j}^{\varepsilon_j} Y_1(rZ))\right] \\ &= r^n D_{k^0,l^0}.\end{aligned}$$

Taking the trace $\tau^{(m)}$, dividing by r^n and taking the limit as $r \to 0$ in (1.26), one gets

$$\frac{1}{r^n}\mathbb{E}\left[\prod_{j=1}^n \overline{Z_j}^{\varepsilon_j} \|S_{rZ}\|_p^p\right] \xrightarrow{r\to 0} \tau^{(m)}(D_{k^0,l^0}).$$

This shows (1.24) since from Lemma 1.6,

$$\tau^{(m)}(D_{k^0,l^0}) = \tau(x_1^{\varepsilon_1}x_2^{\varepsilon_2}\dots x_n^{\varepsilon_n})\sum_{k=0}^{\alpha}(n-k)\binom{p/2}{n-k}\binom{\alpha}{k}. \qquad \square$$

1.2.3 Boundedness on $E \cap L^2$ of isometries on $E \subset L^p$

In this subsection and in the next one, we study how isometric properties for one p-norm imply boundedness (and isometric) properties for the q-norms for $q \neq p$.

Here we first show that a unital map which is isometric between subspaces of noncommutative L^p-spaces for $1 \leq p < \infty$ is also isometric for the 2-norm. This is a noncommutative analogue of [16, Proposition 1], where the author proves that a unital isometry between subspaces of commutative probability L^p-spaces is also an isometry for the L^2-norm, and our proof is inspired by Forelli's proof.

Then in Theorem 1.16, we will prove that any unital and 2-isometric map between subspaces of noncommutative L^p-spaces for $0 < p < \infty$, $p \notin 2\mathbb{N}$ is also isometric for the $2n$-norm for any $n \in \mathbb{N} \cup \{\infty\}$.

Theorem 1.13. *Let $(\mathcal{M},\tau)$ and $(\mathcal{N},\widetilde{\tau})$ be as in the introduction, and $1 \leq p < \infty$.*
Let $x \in L^p(\mathcal{M},\tau)$ and $y \in L^p(\mathcal{N},\widetilde{\tau})$ such that for any $z \in \mathbb{C}$,

$$\|1 + zx\|_p = \|1 + zy\|_p. \tag{1.27}$$

Then $\|x\|_2 < \infty$ if and only if $\|y\|_2 < \infty$, and $\|x\|_2 = \|y\|_2$.

The following Lemma will be used; its proof was communicated to me by Pisier.

Lemma 1.14. *Let A be a bounded operator on a Hilbert space H, and $p \geq 1$. Then*

$$|1 + A|^p + |1 - A|^p + |1 + A^*|^p + |1 - A^*|^p \geq 4 \tag{1.28}$$

Proof. By the operator convexity of the function $t \to t^r$ for $1 \leq r \leq 2$ and by an induction argument, it is enough to prove (1.28) for $p = 1$. For convenience we denote by $C = |1 + A| + |1 - A| + |1 + A^*| + |1 - A^*|$.

By [3, Corollary 1.3.7], for any operator B on H, the following operator on $H \oplus H$ is positive:

$$\begin{pmatrix} |B| & B^* \\ B & |B^*| \end{pmatrix}.$$

Replacing B respectively by $1 + A$, $1 - A$, $1 + A^*$ and $1 - A^*$ and adding the four resulting positive operators, we get that the following operator is also positive:

$$\begin{pmatrix} C & 4 \\ 4 & C \end{pmatrix}.$$

It is classical that this implies that $C \geq 4$ (see for example [3, Theorem 1.3.3]). $\square$

Remark 1.15. The Lemma is stated for bounded operators, but by approximation it also applies to closed densely defined unbounded operators.

The inequality (1.28) does not hold for $0 < p < 1$ (take $A = 1$). But if one could find a finitely supported probability measure ν on $\mathbb{C} \setminus \{0\}$ such that

$$\int (|1 + zA|^p + |1 + zA^*|^p)\, d\nu(z) \geq 2, \tag{1.29}$$

then one would be able to get the conclusion of Theorem 1.13 also for the values of p for which (1.29) holds.

Proof of Theorem 1.13. If $\|x\|_2 = \|y\|_2 = \infty$, there is nothing to prove.

If $\|x\|_2 < \infty$ and $\|y\|_2 < \infty$, then the fact that $\|x\|_2 = \|y\|_2$ follows from Lemma 1.12 with $n = 2$, $m = 1$ and $(\varepsilon_1, \varepsilon_2) = (*, 1)$ (and hence $\alpha = 1$). Indeed, by the hypothesis (1.27), the left-hand side in equation (1.24) does not change if one takes $x_1 = x_2 = x \in L^p(\mathcal{M})$ or $x_1 = x_2 = y \in L^p(\mathcal{N})$. Therefore the right-hand sides are also equal:

$$\tau(x^*x)\left(2\binom{p/2}{2} + \binom{p/2}{1}\right) = \tau(y^*y)\left(2\binom{p/2}{2} + \binom{p/2}{1}\right).$$

This implies that $\|x\|_2^2 = \tau(x^*x) = \tau(y^*y) = \|y\|_2^2$ since $2\binom{p/2}{2} + \binom{p/2}{1} = p^2/4 \neq 0$. Hence we only have to prove that if $\|x\|_2 < \infty$, then $\|y\|_2 < \infty$.

Denote by $C(x)$ and $C(y)$ the following operators:

$$C(x) = \sum_{\omega \in \{1, i, -1, -i\}} (|1 + \omega x|^p + |1 + \omega x^*|^p - 2)$$
$$C(y) = \sum_{\omega \in \{1, i, -1, -i\}} (|1 + \omega y|^p + |1 + \omega y^*|^p - 2)$$

By Lemma 1.14, $C(x)$ and $C(y)$ are positive operators, and by Lemma 1.9, $C(rx)/r^2$ (resp. $C(ry)/r^2$) converges in measure to $p^2(x^*x + xx^*)$ (resp. $p^2(y^*y + yy^*)$) as $r \to 0$. Moreover, for any r, the hypothesis (1.27) implies that

$$\tau(C(rx)) = 2 \sum_{\omega \in \{1, i, -1, -i\}} \|1 + \omega r x\|_p^p - 8 = \widetilde{\tau}(C(ry)).$$

By Fatou's Lemma ([12, Theorem 3.5]), we thus have that

$$2p^2\|y\|_2^2 = \widetilde{\tau}\left(p^2(y^*y + yy^*)\right) \leq \liminf_{r \to 0} \tau(C(rx)/r^2).$$

We now use Fack and Kosaki's dominated convergence theorem [12, Theorem 3.6] to prove that $\tau(C(rx))/r^2 \to 2p^2\|x\|_2^2$. By the domination Lemma 1.11 and the property (1.18) of singular numbers, there are constants $C, K > 0$ such that

$$\mu_t(C(rx)/r^2) \leq C(\mu_{t/K}(x)^2 + \mu_{t/K}(x)^p).$$

As in the proof of Lemma 1.12, this is enough to deduce that

$$\lim_{r \to 0} \tau(C(rx)/r^2) = \tau\left(\lim_{r \to 0} C(rx)/r^2\right) = 2p^2\|x\|_2^2.$$

This concludes the proof. □

1.2.4 Boundedness on $E \cap L^{2n}$ for all n of 2-isometries on $E \subset L^p$

Here we prove that a unital 2-isometric map between unital subspaces of noncommutative L^p-spaces maps a bounded operator to a bounded operator. The general idea is to prove by induction on n that such a map is also an isometry for the q-norm when $q = 2n$ and to make n grow to ∞. The idea of the proof is similar to the proof of Theorem 1.13. The precise statement is:

Theorem 1.16. *Let $0 < p < \infty$, p not an even integer. Let $x \in L^p(\mathcal{M}, \tau)$ and $y \in L^p(\mathcal{N}, \widetilde{\tau})$ such that, for any $a \in M_2(\mathbb{C})$*

$$\|1 + a \otimes x\|_p = \|1 + a \otimes y\|_p .$$

Then for any $n \in \mathbb{N}^* \cup \{\infty\}$, $x \in L^{2n}(\mathcal{M})$ if and only if $y \in L^{2n}(\mathcal{N})$, and when this holds $\|x\|_{2n} = \|y\|_{2n}$.

The Theorem is proved with the use of a classical 2 by 2 matrix trick, Fatou's Lemma and expansions in power series of operators of the form $|1 + a|^p$ for a satisfying $a^2 = 0$. More precisely, for such an a, we derive an expression of the following form (Corollary 1.19 and Lemma 1.20):

$$|1 + a|^p + |1 - a|^p + |1 + a^*|^p + |1 - a^*|^p \simeq \sum_{n=0}^{N} \lambda_n |a|^{2n} + \lambda_n |a^*|^{2n}$$

and are able to use a qualitative study of differential equations (Lemma 1.21) to prove the positivity (or negativity) of the difference of the two above terms.

Lemma 1.17. *Let a be an element of a $*$-algebra such that $a^2 = 0$. Then if one denotes by a_1, a_2, a_3, a_4 the expressions*

$$\begin{aligned} a_1 &= a^*a + a + a^* \\ a_2 &= a^*a - a - a^* \\ a_3 &= aa^* + a + a^* \\ a_4 &= aa^* - a - a^* \end{aligned}$$

then for any integer $m \geq 1$,

$$\sum_{j=1}^{4} a_j^m = 2P_m(a^*a) + 2P_m(aa^*),$$

where the polynomial P_m is defined by

$$P_m(X) = \left(\frac{X + \sqrt{X^2 + 4X}}{2} \right)^m + \left(\frac{X - \sqrt{X^2 + 4X}}{2} \right)^m \quad \text{for any } X \in \mathbb{R}^+. \tag{1.30}$$

Proof. We can assume that a is a free element satisfying $a^2 = 0$, so that there are well defined polynomials A_m, B_m, C_m and D_m in $\mathbb{R}[X]$ such that

$$(a_1)^m = A_m(a^*a) + aB_m(a^*a) + C_m(a^*a)a^* + aD_m(a^*a)a^*.$$

Thus we can write

$$\sum_{j=1}^{4} a_j^m = 2P_m(a^*a) + 2P_m(aa^*)$$

with $P_m = A_m + XD_m \in \mathbb{R}[X]$.

It is easy to check that the sequences of polynomials $(A_m)_m$ and $(D_m)_m$ (and hence (P_m)) satisfy the following induction relations:

$$\begin{aligned} A_{m+2}(X) &= X\,(A_{m+1}(X) + A_m(X)) \quad \text{if } m \geq 0 \\ D_{m+2}(X) &= X\,(D_{m+1}(X) + D_m(X)) \quad \text{if } m \geq 1 \\ P_{m+2}(X) &= X\,(P_{m+1}(X) + P_m(X)) \quad \text{if } m \geq 1 \end{aligned}$$

But the right-hand side of (1.30) also satisfies the same relation, it is therefore enough (and trivial) to check that equality (1.30) holds for $m = 1$ and $m = 2$. ☐

Lemma 1.18. *Let a be a closed densely defined operator affiliated with a von Neumann algebra $(\mathcal{M}, \tau)$ such that $a^2 = 0$. Let $a_i, i = 1, 2, 3, 4$ be as in Lemma 1.17. Then for any continuous function $f : [-1, \infty) \to \mathbb{R}$,*

$$\sum_{j=1}^{4} f(a_i)^m = 2f\left(\frac{a^*a + \sqrt{(a^*a)^2 + 4a^*a}}{2}\right) + 2f\left(\frac{a^*a - \sqrt{(a^*a)^2 + 4a^*a}}{2}\right) + \\ 2f\left(\frac{aa^* + \sqrt{(aa^*)^2 + 4aa^*}}{2}\right) + 2f\left(\frac{aa^* - \sqrt{(aa^*)^2 + 4aa^*}}{2}\right) - 4f(0). \quad (1.31)$$

Proof. Lemma 1.17 implies that (1.31) holds when f is a polynomial. By continuity of the continuous functional calculus (with respect to the measure topology when a is unbounded), (1.31) thus holds for any continuous f. ☐

Corollary 1.19. *Let $0 < p < \infty$ and a, as above, satisfying $a^2 = 0$. Then*

$$|1 + a|^p + |1 - a|^p + |1 + a^*|^p + |1 - a^*|^p = 2\psi(a^*a) + 2\psi(aa^*) - 4,$$

where ψ is the function $\psi : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\psi(t) = \left(1 + \frac{t + \sqrt{t^2 + 4t}}{2}\right)^{p/2} + \left(1 + \frac{t - \sqrt{t^2 + 4t}}{2}\right)^{p/2}.$$

Proof. This is immediate since with the notation above, $|1 + a|^p = (1 + a_1)^{p/2}$, $|1 - a|^p = (1 + a_2)^{p/2}$, $|1 + a^*|^p = (1 + a_3)^{p/2}$ and $|1 - a^*|^p = (1 + a_4)^{p/2}$ ☐

Let us study the function ψ.

Proposition 1.20. *The following properties hold for ψ:*

1. *ψ is a solution to the following differential equation on $\mathbb{R}^+$:*

$$(t^2 + 4t)y'' + (t + 2)y' - \frac{p^2}{4}y = 0. \quad (1.32)$$

2. *ψ has an expansion in power series around 0, more precisely for $|t| < 4$,*

$$\psi(t) = \sum_{n \geq 0} \frac{2}{(2n)!} \prod_{k=0}^{n-1} \left(\frac{p^2}{4} - k^2\right) t^n = \sum_{n \geq 0} \lambda_n t^n. \quad (1.33)$$

3. For any $0 < p < \infty$, for any $t \in \mathbb{R}^+$ and any $N \in \mathbb{N}$,

$$\psi(t) - \sum_{n=0}^{N} \lambda_n t^n \begin{cases} \geq 0 & \text{if } p \geq 2N \text{ or } \lfloor N - \frac{p}{2} \rfloor \text{ is odd.} \\ \leq 0 & \text{otherwise.} \end{cases} \tag{1.34}$$

In this proposition, for a real number t, the symbol $\lfloor t \rfloor$ denotes the largest integer smaller than or equal to t.

Proof. Checking (1) is just an easy computation, the details are left to the reader. It is also easy to see that ψ has an expansion in power series around 0, and (2) follows from the fact that both left-hand and right-hand sides of (1.33) satisfy (1.32) and have value 2 at $t = 0$.

Let us prove (3). Let us fix p and N. As a function of t, the left-hand side of (1.34) satisfies the following differential equation:

$$(t^2 + 4t)y'' + (t + 2)y' - \frac{p^2}{4} y = (2N + 1)(2N + 2)\lambda_{N+1} t^N.$$

Moreover, (2) shows that the left-hand side of (1.34) and its derivative has the same sign as λ_{N+1} when t is small (with $t > 0$).

Note also that $\lambda_{N+1} \geq 0$ if $p \geq 2N$ or if $\lfloor N - \frac{p}{2} \rfloor$ is odd, that $\lambda_{N+1} \leq 0$ else.

The fact (3) thus follows from Lemma 1.21 applied to $\pm\left(\psi(t) - \sum_{n=0}^{N} \lambda_n t^n\right)$ depending on the sign of λ_{N+1}. $\square$

Lemma 1.21. *Let a, b, c and d be continuous functions on $\mathbb{R}^+$ such that for any $t > 0$,*

$$a(t) > 0$$
$$c(t) < 0$$
$$d(t) > 0$$

Let y be a C^2 function on $\mathbb{R}^+$ solution of $ay'' + by' + cy = d$, and $t_0 > 0$ such that $y(t_0) > 0$ and $y'(t_0) > 0$. Then $y(t) > 0$ for any $t \geq t_0$.

Proof. We prove that $y'(t) > 0$ for any $t \geq t_0$. Assume that it is not true, and take $t_1 = \min\{t > t_0, y'(t) = 0\}$. Since $y'(t_1) = 0$ and $y'(t) > 0$ if $t_0 < t < t_1$, we have that $y''(t_1) \leq 0$.

On the other hand, since $y' \geq 0$ on (t_0, t_1), $y(t_1) \geq y(t_0) > 0$. Thus $y''(t_1) = (d(t_1) - c(t_1)y(t_1))/a(t_1) > 0$, which is a contradiction. $\square$

It is now possible to derive the main result of this part:

Lemma 1.22. *Let $0 < p < \infty$ and $(\lambda_n)_{n \geq 0} \in \mathbb{R}^{\mathbb{N}}$ defined by (1.33). Take $a \in L^p(\mathcal{M}, \tau)$ such that $a^2 = 0$, and fix an integer $N > 0$ with $\lambda_N \neq 0$.*

1. If $\|a\|_{2N-2} < \infty$

$$\|a\|_{2N}^{2N} \leq \liminf_{t \to 0} \frac{1}{2\lambda_N t^{2N}} \left(\|1 + ta\|_p^p + \|1 - ta\|_p^p - 2 - 2 \sum_{n=1}^{N-1} \lambda_n t^{2n} \|a\|_{2n}^{2n} \right) \tag{1.35}$$

2. Moreover, if $\|a\|_{2N} < \infty$, the previous inequality becomes an equality. More precisely,

$$\|a\|_{2N}^{2N} = \lim_{t\to 0} \frac{1}{2\lambda_N t^{2N}} \left(\|1+ta\|_p^p + \|1-ta\|_p^p - 2 - 2\sum_{n=1}^{N-1} \lambda_n t^{2n} \|a\|_{2n}^{2n} \right) \tag{1.36}$$

Proof. The first fact is a consequence of the properties of ψ and of Fatou's lemma.

Denote by $b(t,a)$ the following (unbounded) operator affiliated with $\mathcal{M}$:

$$b(t,a) = \frac{1}{\lambda_N t^{2N}} \left(\psi(t^2 a^* a) - \sum_{n=0}^{N-1} \lambda_n t^{2n} (a^*a)^n \right).$$

In this equation, $(a^*a)^0$ is equal to $1_{\mathcal{M}}$. Note that the operators $b(t,a)$ are affiliated with the commutative von Neumann algebra generated by a^*a, which is isomorphic to the space of (classes of) bounded measurable functions on some probability space (Ω,μ).

Then (1.33) implies that $b(t,a) \to (a^*a)^N$ in the measure topology as $t \to \infty$ (in fact the convergence holds almost surely if the operators are viewed as functions on Ω). But (1.34) also implies that $b(t,a) \geq 0$. Thus one can apply Fatou's lemma to conclude that

$$\|a\|_{2N}^{2N} = \tau((a^*a)^N) \leq \liminf_{t\to 0} \tau(b(t,a)). \tag{1.37}$$

Replace a by a^* in the preceding inequality, and add the two equations to get (using $\|a^*\|_q = \|a\|_q$ for any real q)

$$2\|a\|_{2N}^{2N} \leq \liminf_{t\to 0} \frac{1}{\lambda_N t^{2N}} \tau \left(\psi(t^2 a^*a) + \psi(t^2 aa^*) - \sum_{n=0}^{N-1} \lambda_n t^{2n} ((a^*a)^n + (aa^*)^n) \right).$$

Applying Corollary 1.19 and the linearity of the trace yields to the desired conclusion (since $(aa^*)^n$ and $(a^*a)^n$ belong to $L^1(\mathcal{M})$ for $n \leq N-1$).

To prove the second fact, we prove that if $\|a\|_{2N} < \infty$, then equality holds in (1.37). But this follows from the (classical) dominated convergence theorem since

$$\left| \psi(t) - \sum_{n=0}^{N} \lambda_n t^n \right| \leq C(t^N + t^{p/2})$$

for some constant C not depending on $t \in \mathbb{R}$. □

The proof of Theorem 1.16 follows:

Proof of Theorem 1.16. First note that the statement for $n = \infty$ follows from the one for $n \in \mathbb{N}$, since $\|x\|_\infty = \lim_{n\to\infty} \|x\|_{2n}$. So we focus on the case when n is a positive integer.

The idea is to construct operators related to x and y of zero square by putting then in a corner of a 2 by 2 matrix, and then to use Lemma 1.22. So let us denote $a(x)$ and $a(y)$ the operators

$$a(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in M_2(L^p(\mathcal{M})) \simeq L^p(M_2(\mathcal{M}))$$

$$a(y) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in M_2(L^p(\mathcal{M})) \simeq L^p(M_2(\mathcal{M}))$$

Note that $a(x)^2 = 0$, that for any $q \in \mathbb{R} \cup \{\infty\}$, $\|a(x)\|_q = 2^{-1/q}\|x\|_q$, and that the same holds for y. Moreover $\|1+ta(x)\|_p = \|1+ta(y)\|_p$ for any $t \in \mathbb{R}$. It is thus enough to prove that if $\|a(x)\|_{2n} < \infty$, then $\|a(y)\|_{2n} < \infty$ and $\|a(y)\|_{2n} = \|a(x)\|_{2n}$. We prove this by induction on n.

So take $N > 0$, assume that the aforementioned statement holds for any $n < N$ (note that we assume nothing if $N = 1$). Suppose that $\|a(x)\|_{2N} < \infty$. Then by induction hypothesis for any $n < N$, $\|y\|_{2n} = \|x\|_{2n}$. Thus the right-hand side of (1.35) is the same when a is replaced by $a(y)$ or by $a(x)$. But for $a = a(x)$, it is equal, by (1.36), to $\|a(x)\|_{2N}$. Hence (1.35) proves that $\|a(y)\|_{2N} \leq \|a(x)\|_{2N} < \infty$.

Applying (1.36) again with $a(y)$ yields to $\|a(y)\|_{2N} = \|a(x)\|_{2N}$. □

1.3 Proof of Theorem 1.3

In this section we develop some consequences of Lemma 1.7. We are given $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \widetilde{\tau})$ two von Neumann algebras with normal faithful tracial states.

Let $x_1, \dots x_n \in \mathcal{M}$ and $y_1, \dots y_n \in \mathcal{N}$. The noncommutative analogue (in the bounded case) of Theorem 1.1 is:

Theorem 1.23. *Let $0 < p < \infty$ such that $p \neq 2, 4, 6 \dots$ is not an even integer. Suppose that for all $m \in \mathbb{N}$ and all $a_1 \dots a_n \in M_m$,*

$$\|1 + \sum a_i \otimes x_i\|_p = \|1 + \sum a_i \otimes y_i\|_p.$$

Then the n-uples $(x_1, \dots x_n)$ and $(y_1, \dots y_n)$ have the same $*$-distributions. More precisely, for all $P \in \mathbb{C}\langle X_1, \dots X_{2n}\rangle$ polynomial in $2n$ non commuting variables,

$$\tau\left(P(x_1, \dots x_n, x_1^*, \dots x_n^*)\right) = \widetilde{\tau}\left(P(y_1, \dots y_n, y_1^*, \dots y_n^*)\right). \tag{1.38}$$

This theorem relies on Lemma 1.7 and on the following Lemma:

Lemma 1.24. *Let $N, \alpha \in \mathbb{N}$ be integers such that $N \geq 1$ and $\alpha \leq N/2$. Then if p is a positive number such that $p \notin 2\mathbb{N}$ or $p \geq 2(N - \alpha)$, then*

$$\sum_{k=0}^{\alpha} (N-k) \binom{p/2}{N-k} \binom{\alpha}{k} \neq 0.$$

Proof. Take α, N and p as in the Lemma. Since $(N-k)\binom{p/2}{N-k} = p/2 \binom{p/2-1}{N-k-1}$, showing the Lemma is the same as showing that

$$\sum_{k=0}^{\alpha} \binom{p/2 - 1}{N-k-1} \binom{\alpha}{k} \neq 0. \tag{1.39}$$

For every real number β, let us consider the left-hand side of (1.39) where $p/2 - 1$ is replaced by β. Since $\binom{\beta}{n}$ is a polynomial function in β of degree n, the expression $P(\beta) \stackrel{\text{def}}{=} \sum_{k=0}^{\alpha} \binom{\beta}{N-k-1}\binom{\alpha}{k}$ is a polynomial in β of degree $N-1$. To prove that it takes nonzero values for $\beta = p/2 - 1$, we show that it has $N-1$ roots different from $p/2 - 1$. More precisely, we show that if β is an integer such that $-\alpha \leq \beta \leq N - \alpha - 2$, then $P(\beta) = 0$.

First if β is an integer between 0 and $N - \alpha - 2$ included, then for any $0 \leq k \leq \alpha$, it is immediate to check from the definition (1.6) that $\binom{\beta}{N-k-1} = 0$, which implies $P(\beta) = 0$.

The second fact to check is that if l is an integer such that $1 \leq l \leq \alpha$, then $P(-l) = 0$. Let us fix such an l. Then writing $\binom{-l}{N-k-1} = (-1)^{N-k-1}\binom{N-k+l-2}{l-1}$ we get

$$P(-l) = \sum_{k=0}^{\alpha} \binom{-l}{N-k-1}\binom{\alpha}{k} = \sum_{k=0}^{\alpha} \binom{\alpha}{k}(-1)^{N-k-1}\binom{N-k+l-2}{l-1}.$$

It only remains to note that l and N being fixed, $\binom{N-k+l-2}{l-1}$ is (as a function of k) a polynomial of degree $l-1 < \alpha$. The equality $P(-l) = 0$ arises from the fact that if $1 \leq i < \alpha$,

$$\sum_{k=0}^{\alpha} \binom{\alpha}{k}(-1)^k k^i = 0.$$

□

Theorem 1.23 follows:

Proof of Theorem 1.23. By linearity it is enough to prove (1.38) when P is a monomial. The fact to be proved is that for every finite sequence $i_1, \dots i_N$ of indices between 1 and n, and for every sequence $\varepsilon_1, \dots \varepsilon_N \in \{1, *\}$,

$$\tau\left(\prod_k x_{i_k}^{\varepsilon_k}\right) = \widetilde{\tau}\left(\prod_k y_{i_k}^{\varepsilon_k}\right).$$

But from lemma 1.7, if α is the number of indices k such that $\varepsilon_k = *$ and $\varepsilon_{k+1 \mod N} = 1$, we have

$$\tau\left(\prod_k x_{i_k}^{\varepsilon_k}\right)\sum_{k=0}^{\alpha}(N-k)\binom{p/2}{N-k}\binom{\alpha}{k} = \widetilde{\tau}\left(\prod_k y_{i_k}^{\varepsilon_k}\right)\sum_{k=0}^{\alpha}(N-k)\binom{p/2}{N-k}\binom{\alpha}{k}.$$

This implies that $\tau\left(\prod_k x_{i_k}^{\varepsilon_k}\right) = \widetilde{\tau}\left(\prod_k y_{i_k}^{\varepsilon_k}\right)$ since from Lemma 1.24 if $p \notin 2\mathbb{N}$

$$\sum_{k=0}^{\alpha}(N-k)\binom{p/2}{N-k}\binom{\alpha}{k} \neq 0.$$

□

Theorem 1.3 is an immediate consequence of Theorem 1.23, Theorem 1.16 and of the following well-known lemma:

Lemma 1.25. *Let $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \widetilde{\tau})$ be two von Neumann algebras equipped with faithful normal tracial states, and let $(x_i)_{i\in I} \in \mathcal{M}$ and $(y_i)_{i\in I} \in \mathcal{N}$ be noncommutative random variables that have the same $*$-distribution. Then the von Neumann algebras generated respectively by the x_i's and the y_i's are isomorphic, with a normal isomorphism sending x_i on y_i and preserving the trace.*

Proof of Theorem 1.3. Let $(x_i)_{i\in I}$ be a family spanning E. If $y_i = u(x_i)$ for any $i \in I$, then Theorem 1.16 shows that $\|y_i\|_\infty < \infty$, which is equivalent to the fact that $y_i \in \mathcal{N}$. By Theorem 1.23, the families (x_i, x_i^*) and (y_i, y_i^*) have the same distribution and so by Lemma 1.25, u extends to a trace preserving isomorphism between the von Neumann algebras generated by the x_i's and y_i's respectively. □

It is also possible to get some approximation results using ultraproducts:

1.3.1 Approximation results

Corollary 1.26. *Let $(\mathcal{M}_\alpha, \tau_\alpha)_{\alpha\in A}$ be a net of von Neumann algebras equipped with normal faithful normalized traces. Let I be a set, and for all α, let $(x_i^\alpha)_{i\in I} \in {\mathcal{M}_\alpha}^I$ such that for all $i \in I$, the net $(x_i^\alpha)_\alpha$ is uniformly bounded, i.e. $\sup_\alpha \|x_i^\alpha\| < \infty$. Assume that there is a family $(y_i)_{i\in I}$ in a von Neumann algebra $(\mathcal{N}, \widetilde{\tau})$ and a $p \notin 2\mathbb{N}$ such that for all integer n and all finitely supported family $(a_i)_{i\in I} \in M_n$, the following holds:*

$$\lim_\alpha \|1 + \sum_i a_i \otimes x_i^\alpha\|_p = \|1 + \sum_i a_i \otimes y_i\|_p. \tag{1.40}$$

Then the net $\left((x_i^\alpha)_i\right)_\alpha$ converges in $*$-distribution to $(y_i)_i$. Moreover (1.40) *holds with p replaced by any $0 < q < \infty$.*

Proof. Indeed let $\mathcal{U}$ be any ultraproduct on A finer that the net (α), and for $i \in I$ consider x_i the image of $(x_i^\alpha)_{\alpha\in A}$ in the von Neumann ultraproduct $\mathcal{M} = \prod_\mathcal{U} \mathcal{M}_\alpha$. If $\mathcal{M}$ is equipped with the tracial state $\tau = \lim_\mathcal{U} \tau_\alpha$, then the assumption (1.40) implies that for all m and all $a_i \in M_m$,

$$\|1 + \sum_i a_i \otimes x_i\|_p = \|1 + \sum_i a_i \otimes y_i\|_p.$$

Lemma 1.7 implies that $(x_i)_i$ and $(y_i)_i$ have the same $*$-distribution. This exactly means that $(x_i^\alpha)_i$ converges in $*$-distribution to $(y_i)_i$ as $\alpha \in \mathcal{U}$.

Since this holds for any ultrafilter $\mathcal{U}$ finer than the net (α), this proves the convergence in $*$-distribution of the net $\left((x_i^\alpha)_i\right)_\alpha$ to $(y_i)_i$. The fact that (1.40) then holds with p replaced by any $0 < q < \infty$ is immediate. □

Theorem 1.3 can also be reformulated in the operator space setting:

1.3.2 Reformulation in the operator space setting

Let $\mathcal{M} \subset B(H)$ be a von Neumann algebra equipped with a normal faithful trace τ satisfying $\tau(1) = 1$. Let E be a linear subspace of $\mathcal{M}$. There are several “natural” operator space structures on E:

For all $1 \leq p \leq \infty$, the noncommutative L^p-spaces $L^p(\mathcal{M}, \tau)$ are equipped with a natural operator space structure (see [44, chapter 7]). (when $p = \infty$, $L^p(\mathcal{M}, \tau)$ is the von Neumann algebra $\mathcal{M}$ with its obvious operator space structure).

Then the linear embedding $E \subset L^p(\mathcal{M}, \tau)$ allows to define, for all $1 \leq p \leq \infty$, an operator space structure on E, which we denote by $O_p(E)$.

In this setting, Theorem 1.3 states that if E is a linear subspace of $\mathcal{M}$ containing the unit and if $1 \leq p < \infty$ and $p \notin 2\mathbb{N}$, then the operator space structure $O_p(E)$ together with the unit entirely determines the von Neumann algebra generated by E and the trace on it. In particular it determines all of the other operator space structures $O_q(E)$ for all $1 \leq q \leq \infty$.

More precisely:

Corollary 1.27. *Let $1_\mathcal{M} \in E \subset \mathcal{M}$ be as above, $(\mathcal{N}, \widetilde{\tau})$ be another von Neumann algebra equipped with a normal faithful tracial state, $u : E \to \mathcal{N}$ be a unit preserving linear map and $1 \leq p < \infty$ with $p \notin 2\mathbb{N}$.*

If $u : O_p(E) \to L^p(\mathcal{N}, \widetilde{\tau})$ is a complete isometry, then u uniquely extends to an isomorphism between the von Neumann subalgebras generated by E and its image; moreover u is then trace preserving. In particular, for all $1 \leq q \leq \infty$, $u : O_q(E) \to L^q(\mathcal{N}, \widetilde{\tau})$ is a complete isometry.

Proof. The proof is a reformulation of Theorem 1.3 once we know the two following results from the theory of noncommutative vector valued L^p-spaces developed in [42]:

A map $u : X -> Y$ between two operator spaces is completely isometric if and only if for all n, the map $u \otimes id : S_p^n(X) \to S_p^n(Y)$ is an isometry (Lemma 1.7 in [42]). More precisely, for any $n \in \mathbb{N}$ and any $1 \leq p \leq \infty$,

$$\left\| u \otimes id : S_p^n(X) \to S_p^n(Y) \right\| = \| u \otimes id : M_n(X) \to M_n(Y) \|$$

The second result is Fubini's theorem, which states that isometrically (and even completely isometrically, but this is of no use here) $S_p^n\big(L^p(\mathcal{M},\tau)\big) \simeq L^p\big(M_n \otimes A, \mathrm{tr}_n \otimes \tau\big)$. See Theorem 1.9 in [42].

These two results together prove that the hypotheses in Corollary 1.27 imply those in Theorem 1.3, and thus the result is proved. □

1.3.3 On the necessity of taking matrices of arbitrary size

Here we discuss the necessity of taking matrices of arbitrary size in Theorem 1.3. In view of Theorem 1.3 a natural question is thus:

Let $p \in \mathbb{R}$. Consider the class $\mathcal{E}_{p,1}$ of all linear maps u between subspaces of noncommutative L^p spaces constructed on von Neumann algebras equipped with a *n.f.f.* normalized trace. Is there an integer n such that for any such $u : E \to F$, if (1.2) holds for all $x \in M_n(E)$, then it holds for any m and any $x \in M_m(E)$? The smallest such integer will be denoted by $n_{p,1}$.

A similar question is:

Let $p \in \mathbb{R}$. Consider the class $\mathcal{E}_p$ of all linear maps u between subspaces of noncommutative L^p spaces constructed on von Neumann algebras equipped with a normal semifinite faithful normalized trace. Is there an integer n such that for any such $u \in \mathcal{E}_p$, if u is n-isometric, then u is completely isometric? The smallest such integer will be denoted by n_p.

As was noted in the introduction, the transposition map from M_n to M_n ($n \geq 2$) shows that, except for $p = 2$, we necessarily have $n_{p,1} > 1$ and $n_p > 1$.

When $p \notin 2\mathbb{N}$, it is not clear whether $n_{p,1} < \infty$ (or $n_p < \infty$).

In the opposite direction, as announced in the introduction, when $p = 2m \in 2\mathbb{N}$, then it is not hard to prove that $n_{p,1} \leq m$ and $n_p \leq m$.

Theorem 1.28. *Let $p = 2m \in 2\mathbb{N}$. Let $(\mathcal{M},\tau)$, $(\mathcal{N},\widetilde{\tau})$ be as in Theorem 1.3.*

Let $E \subset L^p(\mathcal{M},\tau)$ be a subspace and $u : E \to L^p(\mathcal{N},\widetilde{\tau})$ be a linear map.

Assume that for all $x \in M_m(E)$, the following equality between the p-norms holds:

$$\forall x \in M_m(E), \quad \|1_m \otimes 1_{\mathcal{M}} + x\|_{2m} = \|1_m \otimes 1_{\mathcal{N}} + (\mathrm{id} \otimes u)(x)\|_{2m}. \tag{1.41}$$

Then in fact this equality holds for $x \in M_n(E)$ for every $n \in \mathbb{N}$:

$$\|1_n \otimes 1_{\mathcal{M}} + x\|_{2m} = \|1_n \otimes 1_{\mathcal{N}} + (\mathrm{id} \otimes u)(x)\|_{2m}.$$

Theorem 1.29. *Let $p = 2m \in 2\mathbb{N}$. Let $(\mathcal{M},\tau)$, $(\mathcal{N},\widetilde{\tau})$ be (exceptionally) von Neumann algebras with normal faithful* semifinite *traces.*

Let $E \subset L^p(\mathcal{M},\tau)$ be a subspace and $u : E \to L^p(\mathcal{N},\widetilde{\tau})$ be a linear map.

If u is m-isometric (i.e. $\|x\|_{L^p(\tau^{(m)})} = \|(\mathrm{id} \otimes u)(x)\|_{L^p(\widetilde{\tau}^{(m)})}$ for any $x \in M_m(E)$), then u is completely isometric.

Remark 1.30. Note that Theorem 1.28 is not a formal consequence of Theorem 1.29. Indeed, when $1 \notin E$, assuming (1.41) for any $x \in M_m(E)$ is stronger that assuming that $u : E \to L^p(\widetilde{\tau})$ is m-isometric, and is weaker than assuming that the map $\widetilde{u} : span(1, E) \to L^p(\widetilde{\tau})$ that extends u by $\widetilde{u}(1) = 1$ is m-isometric. We therefore give a proof of the two results.

We first provide the proof of Theorem 1.29 which is simpler:

Proof of Theorem 1.29. Assume that u is m-isometric. By Lemma 1.25 it clearly suffices to prove that if $x_1, \dots x_{2m} \in E$ and $y_i = u(x_i)$, then

$$\tau(x_1^* x_2 x_3^* x_4 \dots x_{2m-1}^* x_{2m}) = \widetilde{\tau}(y_1^* y_2 y_3^* y_4 \dots y_{2m-1}^* y_{2m}).$$

But this is easy to get if one takes $a_1 \dots a_{2m} \in M_m$ satisfying (1.5) and one applies $\|x\|_{2m} = \|(\mathrm{id} \otimes u)(x)\|_{2m}$ to $x = x(z_1, \dots z_{2m}) \in M_m(E)$ defined by

$$x = \sum_{j=1}^{m} \overline{z_{2j-1}} a_{2j-1}^* \otimes x_{2j-1} + z_{2j} a_{2j} \otimes x_{2j-1}$$

for any $(z_1, \dots z_{2m}) \in \mathbb{C}^{2m}$.

Indeed, $\|x\|_{2m}^{2m}$ is a polynomial in the complex numbers z_j and $\overline{z_j}$, the coefficient in front of $z_1 z_2 \dots z_{2m}$ is $\tau(x_1^* x_2 x_3^* x_4 \dots x_{2m-1}^* x_{2m})$. □

Proof of Theorem 1.28. Roughly, the idea of the proof is the same as the previous one: the $2m$ norm of $1 + \sum a_j \otimes x_j$, depends, as a function of the x_j's, only on a finite number of moments of the x_j's. And Lemma 1.7 shows that these moments can be computed from the $2m$-norm of $1 + y$ when y describes the set of $m \times m$ matrices with values in the linear space generated by the x_j's.

But the description of these particular moments is not as simple as in Theorem 1.29, and the computations are more complicated.

Take $x \in M_n(E)$, say $x = 1 + \sum_{j=1}^N a_j \otimes x_j$ where $a_j \in M_n$ and $x_j \in E$. Denote by $y_j = u(x_j)$. First compute

$$\|1 + x\|_{2m}^{2m} = \tau^{(n)}(\left(1 + x + x^* + x^* x\right)^m)$$

The same kind of enumeration as in the proof of Lemma 1.6 shows that for any integer j,

$$\tau^{(n)}\left(\left(x + x^* + x^* x\right)^j\right) = \sum_{k=j}^{2j} \sum_{(\varepsilon_1, \dots, \varepsilon_k) \in \{1, *\}^k} \frac{j}{k} \binom{\alpha(\varepsilon)}{k-j} \tau^{(m)}\left(x^{\varepsilon_1} x^{\varepsilon_2} \dots x^{\varepsilon_k}\right).$$

Multiplying the above equation by $\binom{m}{j}$ and summing on j yields to

$$\tau^{(n)}\left(\left(1 + x + x^* + x^* x\right)^m\right) = \sum_{k=0}^{2m} \sum_{\substack{\varepsilon_1, \dots, \varepsilon_k \in \{1, *\} \\ i_1, \dots i_k \in \{1, \dots N\}}} \mathrm{tr}_n\left(a_{i_1}^{\varepsilon_1} \dots a_{i_k}^{\varepsilon_k}\right) \tau\left(x_{i_1}^{\varepsilon_1} \dots x_{i_k}^{\varepsilon_k}\right) \sum_{0 \le j \le k} \frac{j}{k} \binom{m}{j} \binom{\alpha(\varepsilon)}{k-j}. \quad (1.42)$$

But the assumption (1.41) together with Lemma 1.7 (and Remark 1.8) imply that for any $k \le 2m$, any $\varepsilon \in \{1, *\}^k$ and any $(i_1, \dots i_k) \in \{1, \dots N\}^k$,

$$\tau\left(x_{i_1}^{\varepsilon_1} \dots x_{i_k}^{\varepsilon_k}\right) \sum_{0 \le j \le k} \frac{j}{k} \binom{m}{j} \binom{\alpha(\varepsilon)}{k-j} = \widetilde{\tau}\left(y_{i_1}^{\varepsilon_1} \dots y_{i_k}^{\varepsilon_k}\right) \sum_{0 \le j \le k} \frac{j}{k} \binom{m}{j} \binom{\alpha(\varepsilon)}{k-j}.$$

Remembering (1.42), we get that

$$\|1+x\|_{L^p(\tau^{(n)})} = \|1+u^{(n)}(x)\|_{L^p(\widetilde{\tau}^{(n)})}.$$

Since this holds for any n and any $x \in M_n(E)$, we have the desired conclusion. □

Now we discuss the case of $p \notin 2\mathbb{N}$. We are unable to determine whether $n_p < \infty$ (or $n_{p,1} < \infty$), but we are able to show that the assertion $n_{p,1} < \infty$ is related to an assertion concerning the $*$-distributions of single matricial operators, which we detail below.

If $(x_i)_{i\in I} \in \mathcal{M}^I$ and $(y_i)_{i\in I} \in \mathcal{N}^I$ are two families of operators in von Neumann algebras with *n.f.f.* traces, the same arguments as in Lemma 1.4 show that these families have the same $*$-distribution if for any integer n, and any (finitely supported) family $(a_i)_{i\in I} \in M_n^I$,

$$* -\mathrm{dist}(\sum_{i\in I} a_i \otimes x_i) = * - \mathrm{dist}(\sum_{i\in I} a_i \otimes y_i). \tag{1.43}$$

It is also natural to ask: is there an integer n such that (1.43) for all $a_i \in M_n$ imply that (x_i) and (y_i) have the same $*$-distribution? In the same way as above, the smallest such integer will be denoted by N. (If such integer does not exist, take $N = \infty$).

Since (1.43) implies that $\|1 + \sum a_i \otimes x_i\|_p = \|\sum 1 + a_i \otimes y_i\|_p$, Theorem 1.23 shows that when p is not an even integer, $N \leq n_{p,1}$. To show that $n_{p,1} = \infty$, it would thus be enough to show $N = \infty$.

1.4 Other Applications

In this section we prove some other consequences of Lemma 1.7 and Lemma 1.12. In particular we prove a noncommutative (weaker) version of Rudin's Theorem 1.2: Theorem 1.31. A result of the same kind (dealing with bounded operators only) and using the same ideas has already been developed in [30]. The main difference is that in [30], the author stays at the Banach space level (as opposed to the operator space level, *i.e.* he does not allow matrix coefficients in (1.44)).

Theorem 1.31. *Let $(\mathcal{M},\tau)$ and $(\mathcal{N},\widetilde{\tau})$ be as in Theorem 1.3. Let $0 < p < \infty$ and $p \neq 2, 4$. Let $M \subset L^p(\mathcal{M},\tau)$ be a subalgebra (not necessarily self-adjoint) of $L^p(\mathcal{M},\tau)$ containing $1_{\mathcal{M}}$, and let $u : M \to L^p(\mathcal{N},\widetilde{\tau})$ be a linear map such that $u(1) = 1$.*

Assume that $u^{(2)} = \mathrm{id}\otimes u : M_2(M) \to M_2(L^p(\mathcal{N},\widetilde{\tau}))$ is an isometry for the p-"norms":

$$\forall a \in M_2(M) \ \ \|a\|_p = \|u^{(2)}(a)\|_p. \tag{1.44}$$

Assume moreover that $M \subset L^4(\mathcal{M},\tau)$.

Then for all $a, b \in M$

$$u(ab) = u(a)u(b).$$

Proof. The proof is based on Lemma 1.12. Theorem 1.16 implies that $u(M) \subset L^4(\mathcal{N},\widetilde{\tau})$.

If $a, b \in M$, note that

$$\|u(ab) - u(a)u(b)\|_2^2 = \widetilde{\tau}(u(b)^*u(a)^*u(a)u(b)) + \widetilde{\tau}(u(ab)^*u(ab)) \\ - \widetilde{\tau}(u(b)^*u(a)^*u(ab)) - \widetilde{\tau}(u(ab)^*u(a)u(b)). \tag{1.45}$$

Apply Lemma 1.12 with $n = 4$, $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (*, *, 1, 1)$ (so that with the notation of Lemma 1.7, $\alpha = 1$), $m = n/2 = 2$ and with $(x_1, x_2, x_3, x_4) = (b, a, a, b)$ on the one hand and $(x_1, x_2, x_3, x_4) = (u(b), u(a), u(a), u(b))$ on the other hand. One gets:

$$\tau(b^* a^* ab)\left(4\binom{p/2}{4} + 3\binom{p/2}{3}\right) =$$

$$\widetilde{\tau}(u(b)^* u(a)^* u(a) u(b))\left(4\binom{p/2}{4} + 3\binom{p/2}{3}\right).$$

But $4\binom{p/2}{4} + 3\binom{p/2}{3} = p^2(p/2-1)(p/2-2)/24 \neq 0$ if $p \neq 0, 2, 4$. Thus,

$$\widetilde{\tau}(u(b)^* u(a)^* u(a) u(b)) = \tau(b^* a^* ab).$$

The same argument yields to

$$\widetilde{\tau}(u(ab)^* u(a) u(b)) = \tau((ab)^* ab) = \tau(b^* a^* ab),$$

$$\widetilde{\tau}(u(b)^* u(a)^* u(ab)) = \tau(b^* a^* ab),$$

$$\widetilde{\tau}(u(ab)^* u(ab)) = \tau(b^* a^* ab).$$

Thus, remembering (1.45), one gets

$$\|u(ab) - u(a)u(b)\|_2^2 = 0.$$

Since $\widetilde{\tau}$ is supposed to be faithful, this implies $u(ab) = u(a)u(b)$. □

For unital isometries defined on self-adjoint subspaces, the situation is also nice:

Lemma 1.32. *Let $1 \leq p < \infty$ and $p \neq 2$. Let $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \widetilde{\tau})$ be as in Theorem 1.3. Let $E \subset L^p(\tau)$ be a unital and self-adjoint subspace (i.e. $x \in E \Rightarrow x^* \in E$) and $u : E \to L^p(\widetilde{\tau})$ a unital isometric map.*

Then for any $x \in E$ such that $\|x\|_2 < \infty$, $u(x^*) = u(x)^*$.

Proof. The proof is of the same kind of the one above: take $x \in E \cap L^2(\mathcal{M})$, and first apply Theorem 1.13 to show that $\|u(x)\|_2, \|u(x^*)\|_2 < \infty$. Then the proof consists in applying Lemma 1.12 in order to prove that $\|u(x^*) - u(x)^*\|_2^2 = 0$. The details are not provided. □

When the unital completely isometric map u in Theorem 1.3 is defined on the whole L^p space, we recover some very special cases of known results by Yeadon [52, Theorem 2] for isometries and Junge, Ruan and Sherman [25, Theorem 2] for 2-isometries:

Theorem 1.33. *Let $p \in \mathbb{R}^+$, $p \neq 2$. Let $u : L^p(\mathcal{M}, \tau) \to L^p(\mathcal{N}, \widetilde{\tau})$ be a linear map such that $u(1_\mathcal{M}) = 1_\mathcal{N}$.*

- *If $p \geq 1$ or u maps self-adjoint operators to self-adjoint operators, and if u is isometric, then u maps $\mathcal{M}$ into $\mathcal{N}$ and preserves the trace, the adjoint and the Jordan product: for any $a, b \in \mathcal{M}$*

$$\widetilde{\tau}(u(a)) = \tau(a),\ u(a^*) = u(a)^* \text{ and } u(ab + ba) = u(a)u(b) + u(b)u(a)$$

- *If u is 2-isometric (i.e. $u^{(2)}$ is isometric) and $p \neq 2$, then the image of $\mathcal{M}$ is a von Neumann algebra and the restriction of u to $\mathcal{M}$ is a von Neumann algebra trace preserving isomorphism.*

Proof. We start by the first point. Take u as above. Note that by Lemma 1.32, if $p \geq 1$ then u preserves the adjoint.

For any $a \in \mathcal{M}$ such that $a^* = a$, apply the commutative Theorem 1.2 to the unital isometric map u from the commutative unital subalgebra of $L^p(\mathcal{M})$ generated by a into the commutative unital subalgebra of $L^p(\mathcal{N})$ generated by $u(a)$. One gets that $\|u(a)\|_\infty < \infty$ and that $u(a^2) = u(a)^2$ for any self-adjoint $a \in M$. By polarization, this implies that for any self-adjoint $a, b \in M$,

$$u(ab + ba) = u(a)u(b) + u(b)u(a).$$

By linearity this equality extends to any $a, b \in M$, and $\|a\|_\infty < \infty$. The fact that u preserves the trace is an application of Lemma 1.12 with $n = 1$.

Assume now that u is 2-isometric (with $0 < p \neq 2 < \infty$). By Theorem 1.16 and Lemma 1.32, u (and hence $u^{(2)}$) preserves the adjoint map. We can apply the isometric case above for $u^{(2)}$. Thus $u^{(2)}$ is a trace preserving Jordan map. If $a, b \in \mathcal{M}$, the equation $u^{(2)}(\widetilde{a}\widetilde{b} + \widetilde{b}\widetilde{a}) = u^{(2)}(\widetilde{a})u^{(2)}(\widetilde{b}) + u^{(2)}(\widetilde{b})u^{(2)}(\widetilde{a})$ for

$$\widetilde{a} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \text{ and } \widetilde{b} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

yields to $u(ab) = u(a)u(b)$ (and $u(ba) = u(b)u(a)$). Thus u is a $*$-isomorphism from the von Neumann algebra $\mathcal{M}$ onto its image, and it preserves the trace. This implies, by Lemma 1.25, that $u(\mathcal{M})$ is a von Neumann subalgebra of $\mathcal{N}$ and that u is a von Neumann algebra isomorphism. □

1.4.1 Applications to noncommutative H^p spaces

The main result also applies in the setting of noncommutative H^p spaces (see [6]).

Definition 1.34. Let $(\mathcal{M}, \tau)$ be, as above, a von Neumann algebra with a faithful normal normalized trace. A tracial subalgebra A of $(\mathcal{M}, \tau)$ is a weak-$*$ closed unital subalgebra such that the conditional expectation $\phi_A : \mathcal{M} \to A \cap A^*$ satisfies $\phi_A(ab) = \phi_A(a)\phi_A(b)$ for any $a, b \in A$.

A noncommutative H^p space is the closure, denoted $[A]_p$, of a tracial subalgebra A in $L^p(\mathcal{M}, \tau)$.

Since by definition, a noncommutative H^p space is a unital subspace of $L^p(\mathcal{M}, \tau)$ in which the subset of bounded operators is dense, Theorem 1.3 automatically implies the following, which gives a beginning of answer to a question raised in [6] and [30]:

Theorem 1.35. *Let $p \notin 2\mathbb{N}$.*

A unital complete isometry between noncommutative H^p spaces extends to an isomorphism between the von Neumann algebras they generate.

Moreover, if a non commutative H^p space is unitally completely isometric to a unital subspace E of a noncommutative L^p space, then E is a noncommutative H^p space.

For a 2-isometric map between noncommutative H^p-spaces, we also get something:

Theorem 1.36. *Let $p \in \mathbb{R}$, $p \neq 2, 4$.*

Let u be a 2-isometric and unital map from a noncommutative H^p space $[A]_p$ into a noncommutative L^p space $L^p(\mathcal{N}, \widetilde{\tau})$.

Then the image of A is a subalgebra B of $L^p(\mathcal{N})$ such that for any $a, b \in A$,

$$u(ab) = u(a)u(b).$$

Moreover, $B \cap B^* = u(A \cap A^*)$ is a von Neumann algebra such that the restriction to B of the conditional expectation $\Phi_B : L^p(\mathcal{N}) \to L^p(B \cap B^*)$ satisfies, for any $a \in M$

$$\Phi_B(u(a)) = u(\Phi_A(a)) \in N \cap B^*.$$

In the case when p is not an even integer, B is contained in $\mathcal{N}$.

Proof. The fact that $B \stackrel{\text{def}}{=} u(A)$ is contained in $\mathcal{N}$ when $p \notin 2\mathbb{N}$ follows from Theorem 1.16.

Theorem 1.31 implies that B is an algebra and that $u(ab) = u(a)u(b)$.

The fact that $B \cap B^* = u(A \cap A^*)$ is immediate from Lemma 1.32, and Theorem 1.33 shows that $u(A \cap A^*)$ is a von Neumann algebra.

Recall that the conditional expectation $\Phi_B : \mathcal{N} \to B \cap B^*$ coincides with the orthogonal projection from $L^2(\mathcal{N})$ to $L^2(B \cap B^*)$. To check the last equation

$$\Phi_B(u(a)) = u(\Phi_A(a))$$

we thus have to show that for any $x \in B \cap B^*$,

$$\widetilde{\tau}\,(xu(a)) = \widetilde{\tau}\,(xu(\Phi_A(a)))\,.$$

Write $x = u(b)$ for $b \in \mathcal{M}$. The above equation arises from the definition of $\Phi_A(a)$, from the multiplicativity of u and from the fact that, by Lemma 1.12, $\widetilde{\tau} \circ u = \tau$:

$$\begin{aligned}
\widetilde{\tau}\,(xu(a)) &= \widetilde{\tau}\,(u(ba)) \\
&= \tau\,(ba) \\
&= \tau\,(b\Phi_A(a)) \\
&= \widetilde{\tau}\,(u(b\Phi_A(a))) \\
&= \widetilde{\tau}\,(xu(\Phi_A(a)))\,.
\end{aligned}$$

This concludes the proof. □

We end this chapter with some additional remarks and questions related Yeadon's result. The main theorem of [52] in particular contains the following:

Lemma 1.37. *Let u be an isometry from an L^p-space $L^p(\mathcal{M})$ constructed on a von Neumann algebra $(\mathcal{M}, \tau)$ with a* n.f.f. *trace to an L^p-space $L^p(\mathcal{N})$ constructed on a von Neumann algebra $(\mathcal{N}, \sigma)$ with a normal* semifinite *faithful trace.*

Then if $u(1) = b$ is positive, then b commutes with $u(L^p(\mathcal{M}))$ and has full support.

Remark 1.38. This also holds if $\mathcal{N}$ does not carry a semifinite trace and $L^p(\mathcal{N})$ is Haagerup's generalized L^p space (see [25, Theorem 3.1]).

This fact allows to reduce the general case to the unital case. Here are the details: if one denotes by $b = u(1) \geq 0 \in L^p(\mathcal{M})$, by $s \in \mathcal{N}$ the support projection of b, and by $\widetilde{\mathcal{N}}$ the von Neumann subalgebra of $s\mathcal{N}s$ generated by $b^{-1}u(\mathcal{M})$, then $\widetilde{\mathcal{N}}$ carries a *n.f.f.* trace given by

$$\widetilde{\tau}(x) = \begin{cases} \sigma(b^p x) & \text{if } \sigma \text{ was a semifinite trace on } \mathcal{N} \\ tr(b^p x) & \text{in Haagerup's construction} \end{cases}$$

(in Haagerup's construction, tr is the trace functional on $L^1(\mathcal{N})$).

The assumption that u is an isometry exactly means that (the unital linear map) $b^{-1} \cdot u$ is an isometry from $L^p(\mathcal{M}, \tau)$ to $L^p(\widetilde{\mathcal{N}}, \widetilde{\tau})$.

Thus Yeadon's Lemma 1.37 (resp. with the preceding remark) and Theorem 1.33 of this chapter are enough to recover Yeadon's result (resp. Junge, Ruan and Sherman's result with the restriction that the first L^p-space be semifinite). Of course, all this is not surprising at all since Lemma 1.37 contains most of the results from [52], and we therefore do not provide more details. But this leads naturally to the question: to what extend Lemma 1.37 can be generalized when u is only defined on a unital subspace of $L^p(\mathcal{M}, \tau)$?

As justified above, it is natural to wonder whether the same result holds for isometries between subspaces of noncommutative L^p spaces. More precisely, let $1 \in E \in L^p(\mathcal{M}, \tau)$ be a unital subspace of a noncommutative L^p space with τ a *n.f.f.* trace. Let $u : E \to L^p(\mathcal{N})$ be an isometry between E and a subspace of an arbitrary noncommutative L^p space such that $u(1) \geq 0$. Then is it true that $u(1)$ commutes with $u(E)$ and has full support in $u(E)$? (that is: if s is the support projection of $u(1)$, then $su(x) = u(x)s = u(x)$ for any $x \in E$). As noted above, this would allow to use all the results of this chapter for u and would have several interesting consequences.

It should be noted that Yeadon's proof (as well as the generalization in [25]) consists in applying the equality condition in Clarkson's inequality for projections with disjoint supports. This of course is not possible for a general subspace of $L^p(\mathcal{M})$ since it may not contain any nontrivial projection.

Chapter 2

Strong Haagerup inequalities with operator coefficients

Introduction

Let F_r be the free group on r generators and $|\cdot|$ the length function associated to this set of generators and their inverses. The left regular representation of F_r on $\ell^2(F_r)$ is denoted by λ, and the C^*-algebra generated by $\lambda(F_r)$ is denoted by $C^*_\lambda(F_r)$. In [19] (Lemma 1.4), Haagerup proved the following result, now known as the Haagerup inequality: for any function $f : F_r \to \mathbb{C}$ supported by the words of length d,

$$\left\| \sum_{g \in F_r} f(g)\lambda(g) \right\|_{C^*_\lambda(F_r)} \leq (d+1)\|f\|_2. \tag{2.1}$$

This inequality has many applications and generalizations. It indeed gives a useful criterion for constructing bounded operators in $C^*_\lambda(F_r)$ since it implies in particular that for $f : F_r \to \mathbb{C}$

$$\left\| \sum_{g \in F_r} f(g)\lambda(g) \right\|_{C^*_\lambda(F_r)} \leq 2\sqrt{\sum_{g \in F_r} (|g|+1)^4 |f(g)|^2},$$

and the so-called Sobolev norm $\sqrt{\sum_{g \in F_r} (|g|+1)^4 |f(g)|^2}$ is much easier to compute that the operator norm of $\lambda(f) = \sum f(g)\lambda(g)$. The groups for which the same kind of inequality holds for some length function (replacing the term $(d+1)$ in (2.1) by some power of $(d+1)$) are called groups with property RD [22] and have been extensively studied; they play for example a role in the proof of the Baum-Connes conjecture for discrete cocompact lattives of $SL_3(\mathbb{R})$ [31].

Another direction in which the Haagerup inequality was studied and extended is the theory of operator spaces. It concerns the same inequality when the function f is allowed to take operator values. This question was first studied by Haagerup and Pisier in [20], and the most complete inequality was then proved by Buchholz in [8]. One of its interests is that it gives an explanation of the $(d+1)$ term in the classical inequality. Indeed, in the operator valued case, the term $(d+1)\|f\|_2$ is replaced by a sum of $d+1$ different norms of f (which are all dominated by $\|f\|_2$ when f is scalar valued). More precisely if S denotes the canonical set of generators of F_r and their inverses, a function $f : F_r \to M_n(\mathbb{C})$ supported by the words of length d can be viewed as a family $(a_{(h_1,\dots,h_d)})_{(h_1,\dots,h_d) \in S^d}$ of matrices indexed by S^d in the following way: $a_{(h_1,\dots,h_d)} = f(h_1 h_2 \dots h_d)$ if $|h_1 \dots h_d| = d$ and $a_{(h_1,\dots,h_d)} = 0$ otherwise.

The family of matrices $a = (a_h)_{h\in S^d}$ can be seen in various natural ways as a bigger matrix, for any decomposition of $S^d \simeq S^l \times S^{d-l}$. If the a_h's are viewed as operators on a Hilbert space H ($H = \mathbb{C}^n$), then let us denote by M_l the operator from $H \otimes \ell^2(S)^{\otimes d-l}$ to $H \otimes \ell^2(S)^{\otimes l}$ having the following block-matrix decomposition:

$$M_l = \left(a_{(s,t)}\right)_{s\in S^l, t\in S^{d-l}}.$$

Then the generalization from [8] is

Theorem 2.1 ([8],Theorem 2.8)**.** *Let $f : F_r \to M_n(\mathbb{C})$ supported by the words of length d and define $(a_h)_{h\in S^d}$ and M_l for $0 \le l \le d$ as above. Then*

$$\left\| \sum_{g\in W_d} f(g) \otimes \lambda(g) \right\|_{M_n \otimes C^*_\lambda(F_r)} \le \sum_{l=0}^{d} \|M_l\|.$$

The same result has also been extended in [33] to the L^p-norms up to constants that are not bounded as $d \to \infty$. See also [48] and [24].

More recently and in the direction of free probability, Kemp and Speicher [27] discovered the striking fact that, whereas the constant $(d+1)$ is optimal in (2.1), when restricted to (scalar) functions supported by the set W_d^+ of words of length d in the generators $g_1, \dots, g_r$ but *not their inverses* (it is the holomorphic setting in the vocabulary of [26] and [27]), this constant $(d+1)$ can be replaced by a constant of order $\sqrt{d}$.

Theorem 2.2 ([27],Theorem 1.4)**.** *Let $f : F_r \to \mathbb{C}$ be a function supported on W_d^+. Then*

$$\left\| \sum_{g\in W_d^+} f(g)\lambda(g) \right\|_{C^*_\lambda(F_r)} \le \sqrt{e}\sqrt{d+1}\|f\|_2.$$

A similar result has been obtained when the operators $\lambda(g_1), \dots, \lambda(g_r)$ are replaced by free $\mathscr{R}$-diagonals elements: Theorem 1.3 in [27]. These results are proved using combinatorial methods: to get bounds on operator norms the authors first get bounds for the norms in the non-commutative L^p-spaces for p even integers, and make p tend to infinity. For an even integer, the L^p-norms are expressed in terms of moments and these moments are studied using the free cumulants.

In this paper we generalize and improve these results to the operator-valued case. As for the generalization of the usual Haagerup inequality, the operator valued inequality we get gives an explanation of the term $\sqrt{d+1}$: for operator coefficients this term has to be replaced by the ℓ^2 combination of the norms $\|M_l\|$ introduced above. A precise statement is the following. We state the result for the free group F_∞ on countably many generators $(g_i)_{i\in\mathbb{N}}$, but it of course applies for the free group with finitely many generators.

Theorem 2.3. *For $d \in \mathbb{N}$, denote by $W_d^+ \subset F_\infty$ the set of words of length d in the g_i's (but* not their inverses*). For $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ let $g_k = g_{k_1} \dots g_{k_d} \in W_d^+$.*

Let $a = (a_k)_{k\in\mathbb{N}^d}$ be a finitely supported family of matrices, and for $0 \le l \le d$ denote by $M_l = \left(a_{(k_1,\dots,k_l),(k_{l+1},\dots,k_d)}\right)$ the corresponding $\mathbb{N}^l \times \mathbb{N}^{d-l}$ block-matrix. Then

$$\left\| \sum_{k\in\mathbb{N}^d} a_k \otimes \lambda(g_k) \right\| \le 4^5\sqrt{e} \left(\sum_{l=0}^{d} \|M_l\|^2 \right)^{1/2}. \tag{2.2}$$

Note that even when $a_k \in \mathbb{C}$, this really is (up to the constant 4^5) an improvement of Theorem 2.2. Indeed it is always true that for any l, $\|M_l\|^2 \le Tr(M_l^* M_l) = \sum_k |a_k|^2$. There is equality when $l = 0$ or d but the inequality is in general strict when $0 < l < d$. Thus if the a_k's are scalars such that $\|(a_k)\|_2 = 1$ and $\|M_l\| \le 1/\sqrt{d}$ for $0 < l < d$, the inequality in Theorem 2.3 becomes $\left\|\sum_{k\in\mathbb{N}^d} a_k \lambda(g_k)\right\| \le 4^5\sqrt{3e}\|(a_k)\|_2$. Since the reverse inequality $\left\|\sum_{k\in\mathbb{N}^d} a_k\lambda(g_k)\right\| \ge \|(a_k)\|_2$ always holds, we thus get that $\left\|\sum_{k\in\mathbb{N}^d} a_k\lambda(g_k)\right\| \simeq \|(a_k)\|_2$ with constants independent of d. An example of such a family is given by the following construction: if p is a prime number and $a_{k_1,\dots,k_d} = \exp(2i\pi k_1 \dots k_d/p)/p^{d/2}$ for any $k_i \in \{1,\dots,p\}$ and $a_k = 0$ otherwise then obviously $\sum_k |a_k|^2 = 1$, whereas a computation (see Lemma 2.34) shows that $\|M_l\|^2 \le d/p$ if $0 < l < d$. It is thus enough to take $p \ge d^2$.

As in [27], the same arguments apply for the more general setting of $*$-free $\mathscr{R}$-diagonal elements ($*$-free means that the C^*-algebras generated are free). Moreover we get significant results already for the L^p-norms for p even integers. Recall that on a C^*-algebra $\mathcal{A}$ equipped with a trace τ, the p-norm of an element $x \in \mathcal{A}$ is defined by $\|x\|_p = \tau(|x|^p)^{1/p}$ for $1 \le p < \infty$, and that for $p = \infty$ the L^∞ norm is just the operator norm. In the following the algebra $M_n \otimes \mathcal{A}$ will be equipped with the trace $Tr \otimes \tau$. The most general statement we get is thus:

Theorem 2.4. *Let c be an $\mathscr{R}$-diagonal operator and $(c_k)_{k\in\mathbb{N}}$ a family of $*$-free copies of c on a tracial C^*-probability space $(\mathcal{A},\tau)$. Let $(a_k)_{k\in\mathbb{N}^d}$ be as above a finitely supported family of matrices and $M_l = \left(a_{(k_1,\dots,k_l),(k_{l+1},\dots,k_d)}\right)$ for $0 \le l \le d$ the corresponding $\mathbb{N}^l \times \mathbb{N}^{d-l}$ block-matrix.*

For $k = (k_1,\dots,k_d) \in \mathbb{N}^d$ denote $c_k = c_{k_1}\dots c_{k_d}$.

Then for any integer m,

$$\left\|\sum_{k\in\mathbb{N}^d} a_k \otimes c_k\right\|_{2m} \le 4^5\|c\|_2^{d-2}\|c\|_{2m}^2 e\sqrt{1+\frac{d}{m}}\left(\sum_{l=0}^{d}\|M_l\|_{2m}^2\right)^{1/2}. \tag{2.3}$$

For the operator norm,

$$\left\|\sum_{k\in\mathbb{N}^d} a_k \otimes c_k\right\| \le 4^5\|c\|_2^{d-2}\|c\|^2\sqrt{e}\left(\sum_{l=0}^{d}\|M_l\|^2\right)^{1/2}. \tag{2.4}$$

When the c_k's are circular these inequalities are valid without the factor $4^5\|c\|_2^{d-2}\|c\|^2$.

The outline of the proof of Theorem 2.4 is the same as the proof of Theorem 1.3 in [27]: we first prove the statement for the L^p-norms when $p = 2m$ is an even integer (letting $p \to \infty$ leads to the statement for the operator norm). This is done with the use of free cumulants that express moments in terms of non-crossing partitions (the definition of non-crossing partitions is recalled in part 2.1.2). More precisely to any integer n, any non-crossing partition π of the set $\{1,\dots,n\}$ and any family $b_1,\dots,b_n \in \mathcal{A}$ the free cumulant $\kappa_\pi[b_1,\dots,b_n] \in \mathbb{C}$ is defined (see [32] for a detailed introduction). When $\pi = 1_n$ is the partition into only one block, κ_π is denoted by κ_n. The free cumulants have the following properties:

- *Multiplicativity:* If $\pi = \{V_1,\dots,V_s\}$, $\kappa_\pi[b_1,\dots,b_n] = \prod_i \kappa_{|V_i|}[(b_k)_{k\in V_i}]$.
- *Moment-cumulant formula:* $\tau(b_1,\dots,b_n) = \sum_{\pi\in NC(n)} \kappa_\pi[b_1,\dots,b_n]$.

- *Characterization of freeness:* A family $(\mathcal{A}_i)_i$ of subalgebras is free iff all mixed cumulants vanish, *i.e.* for any n, any $b_k \in \mathcal{A}_{i_k}$ and any $\pi \in NC(n)$ then $\kappa_\pi[b_1, \dots, b_n] = 0$ unless $i_k = i_l$ for any k and l in a same block of π.

The first two properties characterize the free cumulants (and hence could be taken as a definition), whereas the third one motivates their use in free probability theory. Since the $*$-distribution of an operator $c \in (\mathcal{A}, \tau)$ is characterized by the trace of the polynomials in c and c^*, the cumulants involving only c and c^* (that is the cumulants $\kappa_\pi[(b_i)]$ with $b_i \in \{c, c^*\}$ for any i) depend only on the $*$-distribution of c.

In order to motivate the combinatorial study of certain non-crossing partitions in the first section, let us shortly sketch the proof of the main result. For details, see part 2.3.1. With the notation of Theorem 2.4 let $A = \sum a_k \otimes c_k$. For $k = (k(1), \dots, k(d)) \in \mathbb{N}^d$ set $\widetilde{a}_k = a_{(k(d),\dots,k(1)}$ and $\widetilde{c}_k = c_{k(d)} \dots c_{k(1)}$ so that $(\widetilde{c}_k)^* = c^*_{k(1)} \dots c^*_{k(d)}$. Then $A^* = \sum_k \widetilde{a}^*_k \otimes \widetilde{c}^*_k$, and for $p = 2m$ the p-th power of the p-norm of A is just the trace $Tr \otimes \tau$ of $(AA^*)^m$, which can be expressed by linearity as the sum of the terms of the form $Tr(a_{k_1}\widetilde{a}^*_{k_2} \dots a_{k_{2m-1}}\widetilde{a}^*_{k_{2m}}) \otimes \tau(c_{k_1}\widetilde{c}^*_{k_2} \dots c_{k_{2m-1}}\widetilde{c}^*_{k_{2m}})$. The expression $c_{k_1}\widetilde{c}^*_{k_2} \dots c_{k_{2m-1}}\widetilde{c}^*_{k_{2m}}$ is the product of $2dm$ terms of the form c_i or c^*_i (for $i \in \mathbb{N}$). Apply the moment-cumulant formula to its trace. Using the characterization of freeness with cumulants and then the multiplicativity of cumulants and the fact that cumulants only depend on the $*$-distribution we thus get

$$\left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi \in NC(2dm)} \kappa_\pi[c_{d,m}] \underbrace{\sum_{(k_1,\dots,k_{2m}) \prec \pi} Tr(a_{k_1}\widetilde{a}^*_{k_2} \dots \widetilde{a}^*_{k_{2m}})}_{\stackrel{\text{def}}{=} S(a,\pi,d,m)},$$

where for $k \in \mathbb{N}^{2dm}$ and $\pi \in NC(2dm)$ we write $k \prec \pi$ if $k_i = k_j$ whenever i and j belong to the same block of π and where

$$c_{d,m} = \overbrace{\underbrace{c, \dots, c}_{d}, \underbrace{c^*, \dots, c^*}_{d}, \dots, \underbrace{c, \dots, c}_{d}, \underbrace{c^*, \dots, c^*}_{d}}^{2m \text{ groups}}.$$

Up to this point we did not use the assumption that c is $\mathcal{R}$-diagonal. But as in [27], since the $\mathcal{R}$-diagonal operators are those operators for which the list of non-zero cumulants is very short (see part 2.3.1 for details), we get that the previous sum can be restricted to a sum over the partitions in the subset $NC^*(d, m) \subset NC(2dm)$, which is defined and extensively studied in part 2.1.2:

$$\left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi \in NC^*(d,m)} \kappa_\pi[c_{d,m}] S(a, \pi, d, m). \tag{2.5}$$

The term $\kappa_\pi[c_{d,m}]$ is easy to dominate (Lemma 2.30). When the a_k's are scalars the second term $S(a, \pi, d, m)$ can be dominated by $\|(a_k)\|^{2m}_{\ell^2}$ (by the usual Cauchy-Schwarz inequality). This is what is done in the proof of [27]. But here the fact that we are dealing with operators and not scalars forces to derive a more sophisticated Cauchy-Schwarz type inequality that may control explicitly the expressions $S(a, \pi, d, m)$ in terms of norms of the operators M_l. This is one of the main technical results in this paper, Corollary 2.29. This Corollary states that

$$|S(a, \pi, d, m)| \leq \prod_{l=0}^{d} \|M_l\|_{2m}^{2m\mu_l} \tag{2.6}$$

for some non-negative μ_l with $\sum_l \mu_l = 1$. Moreover the μ_l are explicitely described by some combinatorial properties of π. This inequality is proved through a process of "symmetrization" of partitions. The basic observation is that if one applies a simple Cauchy-Schwarz inequality to $S(a,\pi,d,m)$ (Lemma 2.25), this corresponds on the level of partitions to a certain combinatorial operation of symmetrization that is studied in the part 2.1.1. This observation was already used implicitely in [9], Lemma 2, in some special case: Buchholz indeed notices that for $d=1$ and if π is a pairing (*i.e.* has blocks of size 2), this Cauchy-Schwarz inequality corresponds to some transformation of pairings (for which he does not give a combinatorial description), and that iterating this inequality eventually leads to an domination of the form (2.6) (for $d=1$) but in which he does not compute the exponents μ_0 and μ_1.

In our more general setting it also appears that repeating this operation in an appropriate way turns every non-crossing partition $\pi \in NC^*(d,m)$ into one very simple and fully symmetric partition for which the expression $S(a,\pi,d,m)$ is exactly the ($2m$-power of the $2m$-) norm of one of the M_l's. This is stated and proved in Corollary 2.12 and Lemma 2.26. One important feature of our study of the symmetrization operation on $NC^*(d,m)$ is the fact that we are able to determine some combinatorial invariants of this operation (see part 2.1.3). This allows to keep track of the exponents of the $\|M_l\|_{2m}$ that progressively appear during the symmetrization process, and to compute the coefficients μ_l in (2.6).

The second technical result that we prove and use is a finer study of $NC^*(d,m)$. The main conclusion is Theorem 2.13 which expresses that partitions in $NC^*(d,m)$ have mainly blocks of size 2 and that $NC^*(d,m)$ is not very far from the set $NC(m)^{(d)}$ of non-decreasing chains of non-crossing partitions on m (in the sense that there is a natural surjection $NC^*(d,m) \to NC(m)^{(d)}$ such that the fiber of any point has a cardinality dominated by a term not depending on d). This combinatorial result is then generalized in Theorem 2.22 and Lemma 2.23, and then used to transform the sum in (2.5) into a sum over $NC(m)^{(d)}$ for which the combinatorics are well known by [11].

We prove also the following results, which are extensions to the non-holomorphic case of the previous results and their proofs. Let c be an $\mathscr{R}$-diagonal operator and $(c_k)_{k\in\mathbb{N}}$ a family of $*$-free copies of c on a tracial C^*-probability space $(\mathcal{A},\tau)$. For $\varepsilon = (\varepsilon_1,\dots,\varepsilon_d) \in \{1,*\}^d$ and $k=(k_1,\dots,k_d)\in\mathbb{N}^d$ denote $c_{k,\varepsilon} = c_{k_1}^{\varepsilon_1}\dots c_{k_d}^{\varepsilon_d}$. The result is an extension of Haagerup's inequality for the space generated by the $c_{k,\varepsilon}$ for the k,ε satisfying $k_i = k_{i+1} \Rightarrow \varepsilon_i = \varepsilon_{i+1}$, *i.e.* for which $\lambda(g)_{k,\varepsilon}$ has length d. Denote by I_d the set of such (k,ε).

Theorem 2.5. *Let $(a_{(k,\varepsilon)})_{(k,\varepsilon)\in(\mathbb{N}\times\{1,*\})^d}$ be a finitely spported family such that $a_{(k,\varepsilon)} = 0$ for $(k,\varepsilon)\notin I_d$. For $0\le l\le d$, let M_l be the matrix formed as above from $(a_{(k,\varepsilon)})$ for the decomposition $(\mathbb{N}\times\{1,*\})^d = (\mathbb{N}\times\{1,*\})^l \times (\mathbb{N}\times\{1,*\})^{d-l}$.*

Then for any $p\in 2\mathbb{N}\cup\{\infty\}$

$$\left\| \sum_{(k,\varepsilon)\in(\mathbb{N}\times\{1,*\})^d} a_{k,\varepsilon}\otimes c_{k,\varepsilon} \right\|_p \le 4^5 \|c\|_p^2 \|c\|_2^{d-2} (d+1) \max_{0\le l\le d} \|M_l\|_p.$$

Similarly for self-adjoint operators we have:

Theorem 2.6. *Let μ be a symmetric compactly supported probability measure on $\mathbb{R}$, and c a self-adjoint element of a tracial C^*-algebra distributed as μ.*

Let $(c_k)_{k\in\mathbb{N}}$ be self-adjoint free copies of c and $(a_{k_1,\dots,k_d})_{k_1,\dots,k_d\in\mathbb{N}}$ be a finitely supported family of matrices such that $a_{k_1,\dots,k_d} = 0$ if $k_i = k_{i+1}$ for some $1\le i<d$. Then for any

$p \in 2\mathbb{N} \cup \{\infty\}$

$$\left\| \sum_{(k_1,\ldots,k_d)\in\mathbb{N}^d} a_{k_1,\ldots,k_d} \otimes c_{k_1} \ldots c_{k_d} \right\|_p \leq 4^5 \|c\|_p^2 \|c\|_2^{d-2}(d+1) \max_{0\leq l\leq d} \|M_l\|_p. \qquad (2.7)$$

For the case of the semicircular law and scalar coefficient a_k, this result is not new. It is due to Bożejko [7], and was reproved using combinatorial methods by Biane and Speicher, Theorem 5.3.4 of [4]. Our proof is a generalization of their proof and uses it. Note also that the condition that $a_{k_1,\ldots,k_d} = 0$ if $k_i = k_{i+1}$ for some i is crucial to get (2.7): indeed if $a_{k_1,\ldots,k_d} = 0$ except for $a_{1,\ldots,1} = 1$ then we have the equality $\| \sum_{k\in\mathbb{N}^d} a_k \otimes c_{k_1} \ldots c_{k_d} \|_p = \|c_1^d\|_p = \|c\|_{dp}^d$, whereas $\max_l \|M_l\|_p = 1$ and if μ is not a Bernoulli measure $\|c\|_p^2 \|c\|_2^{d-2}(d+1) = o(\|c\|_{dp}^d)$ when $d \to \infty$. The inequality (2.7) thus does not hold for this choice of (a_k), even up to a constant.

These results are of some interest since they prove a new version of Haagerup's inequality in a broader setting, but they are still unsatisfactory since one would expect to be able to replace the term $(d+1)\max_{0\leq l\leq d} \|M_l\|$ by $\sum_{l=0}^d \|M_l\|$.

The paper is organized as follows: the first part only deals with combinatorics of non-crossing partitions. In the second part we use the results of the first part to get inequalities for the expressions $S(a, \pi, d, m)$. In the third and last part we finally prove the main results stated above.

Although some definitions are recalled, the reader will be assumed to be familiar with the basics of free probability theory and more precisely to its combinatorial aspect (non-crossing partitions, free cumulants, $\mathcal{R}$-diagonal operators...). For more on this see [32]. For the vocabulary of non-commutative L^p spaces nothing more than the definitions of the p-norm, the Cauchy-Schwarz inequality $|\tau(ab)| \leq \|a\|_2\|b\|_2$ and the fact that $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$ will be used.

2.1 Symmetrization of non-crossing partitions

For any integer n, we denote by $[n]$ the interval $\{1, 2, \ldots, n\}$, which we identify with $\mathbb{Z}/n\mathbb{Z}$ and which is endowed with the natural cyclic order: for $k_1, \ldots, k_p \in [n]$ we say that $k_1 < k_2 < \cdots < k_p$ for the cyclic order if there are integers $l_1, \ldots l_p$ such that $l_1 < l_2 < \cdots < l_p$, $k_i = l_i \mod n$ and $l_p - l_1 \leq n$. In other words, if the elements of $[n]$ are represented on the vertices of a regular polygon with n vertices labelled by elements of $[n]$ as in Figure 2.1, then we say that $k_1 < k_2 < \cdots < k_p$ if the sequence $k_1, \ldots k_p$ can be read on the vertices of the regular polygon by following the circle clockwise for at most one full circle.

If π is a partition of $[n]$, and $i \in [n]$, the block of π to which i belongs is denoted by $\pi(i)$. We also write $i \sim_\pi j$ if i and j belong to the same block of the partition π.

If the elements of $[n]$ are represented on the vertices of a regular polygon with n vertices, a partition π of $[n]$ is then represented on the regular polygon by drawing a path between i and j if $i \sim_\pi j$. See Figure 2.1 for an example.

2.1.1 Definitions and first observation

We introduce the operations P_k on the set of partitions of an even number $n = 2N$. This definition is motivated by Lemma 2.25.

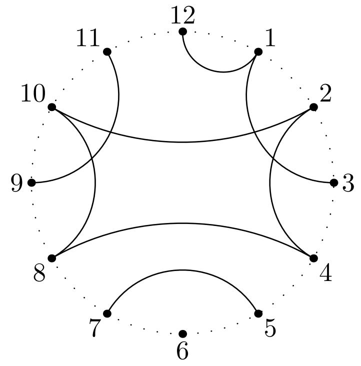


Figure 2.1: The partition $\{\{1,3,12\},\{2,4,8,10\},\{5,7\},\{6\},\{9,11\}\}$ represented graphically.

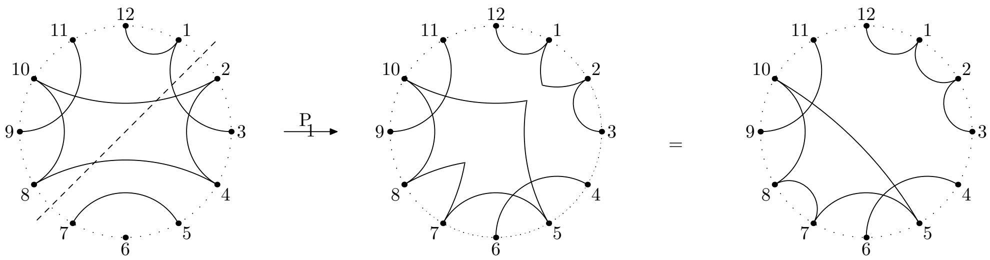


Figure 2.2: The operation P_1 on the partition $\{\{1,3,12\},\{2,4,8,10\},\{5,7\},\{6\},\{9,11\}\}$.

Definition 2.7. Let $k \in [2N]$, and I_k the subinterval of $[2N]$ of length N and ending with k, $I_k = \{k-N+1, k-N+2, \dots, k\}$ and $s_k^{(N)}$ (or simply s_k when no confusion is possible) the symmetry $s_k(i) = 2k+1-i$ (note that s_k is an involution of $[2N]$ that exchanges I_k and $[2N] \setminus I_k$). For a partition π of $[2N]$, $s_k(\pi)$ is the symmetric of π: $A \in s_k(\pi)$ if $s_k^{-1}(A) = s_k(A) \in \pi$. In other words $i \sim_{s_k(\pi)} j$ if and only if $s_k(i) \sim_\pi s_k(j)$.

For any partition π of $[2N]$, we denote by $P_k(\pi)$ the partition of $[2N]$ that we view as a symmetrization of π around k, and which is formally defined by the following: if one denotes $\pi' = P_k(\pi)$, then

$$\text{for } i,j \in I_k \qquad i \sim_{\pi'} j \text{ if and only if } i \sim_\pi j \tag{2.8}$$

$$\text{for } i,j \in [2N] \setminus I_k \qquad i \sim_{\pi'} j \text{ if and only if } s_k(i) \sim_\pi s_k(j) \tag{2.9}$$

$$\text{for } i \in I_k, j \notin I_k \quad i \sim_{\pi'} j \text{ if and only if } i \sim_\pi s_k(j) \text{and } \exists l \notin I_k, i \sim_\pi l. \tag{2.10}$$

It is straightforward to check that this indeed defines a partition of $[2N]$, and that it is symmetric with respect to k, that is $s_k(\pi') = \pi'$.

The operation P_k is perhaps more easily described graphically: represent π on a regular polygon as above, and draw the symmetry line going through the middle of the segment $[k, k+1]$. A graphical representation of $P_k(\pi)$ is then obtained by erasing all the half-polygon not containing k and replacing it by the mirror-image of the half-polygon containing k. See Figure 2.2 for an example.

The following lemma expresses the fact that applying sufficiently many times appropriate operators P_k, one can make a partition symmetric with respect to all the s_k's. See Figure 2.3 to see an example of this symmetrization process.

Lemma 2.8. *Let m be a positive integer.*

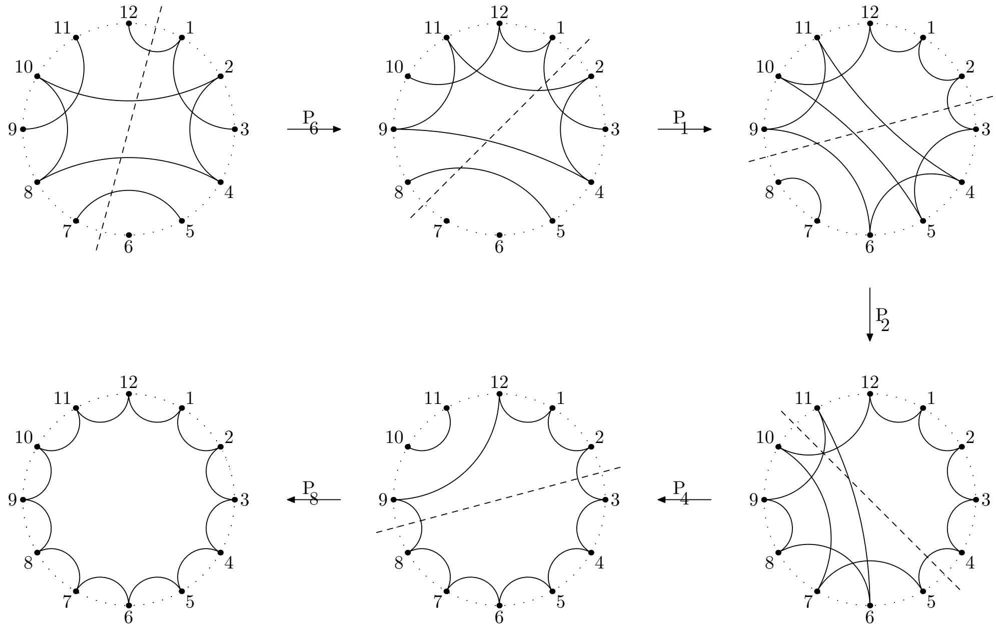


Figure 2.3: The symmetrization process starting from the partition $\{\{1, 3, 12\}, \{2, 4, 8, 10\}, \{5, 7\}, \{6\}, \{9, 11\}\}$.

Let $k \in \mathbb{N}$ such that $2^k \geq m$. Then for any partition π of $[2m]$, the partition $\pi_k = P_{2^k} P_{2^{k-1}} \dots P_2 P_1 P_m(\pi)$ is one of the four following partitions:

$$\begin{aligned} \pi_k = 0_{2m} &= \{\{j\}, j \in [2m]\} \\ \pi_k = c_m &= \{\{2j; 2j+1\}, j \in [m]\} \\ \pi_k = r_m &= \{\{2j-1; 2j\}, j \in [m]\} \\ \pi_k = 1_{2m} &= \{[2m]\} \end{aligned}$$

Proof. Let $A = I_m \cap \pi(1) \setminus \{1\}$ and $B = ([2m] \setminus I_m) \cap \pi(1)$. The four cases correspond respectively to the four following cases:

1. $A = B = \emptyset$.
2. $A = \emptyset$ and $B \neq \emptyset$.
3. $A \neq \emptyset$ and $B = \emptyset$.
4. $A \neq \emptyset$ and $B \neq \emptyset$.

In the first case, it is straightforward to prove by induction on k that π_k includes the blocks $\{i\}$ for any $i \in \{1, \dots, 2^{k+1}\}$.

If $A = \emptyset$ and $B \neq \emptyset$, then $P_m(\pi)$ includes the block $\{0, 1\}$ and this implies that $P_1 P_m(\pi)$ includes the blocks $\{0, 1\}$ and $\{2, 3\}$, which in turn implies that $P_2 P_1 P_m(\pi)$ includes the blocks $\{0, 1\}$,$\{2, 3\}$ and $\{4, 5\}$... More generally π_k includes the blocks $\{0, 1\}$, $\{2, 3\}$,...,$\{2^{k+1}, 2^{k+1} + 1\}$ (this can be proved by induction). For $2^{k+1} \geq 2m$ this is exactly $\pi_k = c_m$. We leave the details to the reader.

In the same way, in the third case it is easy to prove by induction on k that π_k includes the blocks $\{2l - 1, 2l\}$ for $l \in \{1, \dots, 2^k\}$.

The fourth case follows from a similar proof by induction that $\{0, 1, 2, \dots, 2^{k+1} + 1\}$ is contained in $\pi_k(1)$. The details are not provided. □

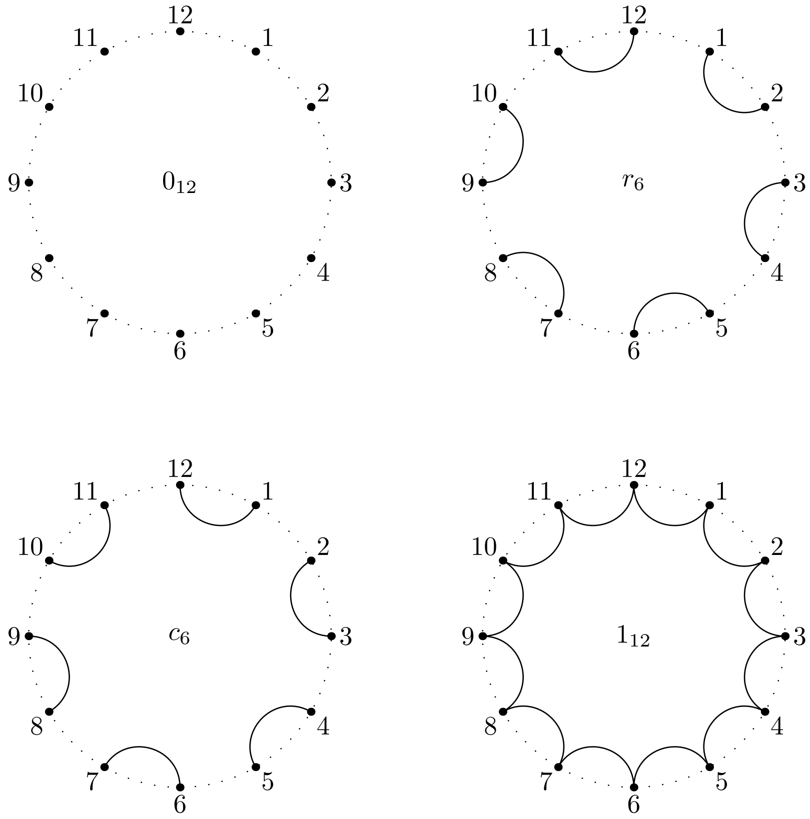


Figure 2.4: The partitions 0_{12}, r_6, c_6 and 1_{12}.

Although $P_k(\pi)$ is defined for any partition π, we will be mainly interested in the case when π is a non-crossing partition, and more precisely when $\pi \in NC^*(d, m)$.

2.1.2 Study of $NC^*(d, m)$

We first recall the definition of a non-crossing partition. A partition π of $[N]$ is called non-crossing if for any distinct $i < j < k < l \in [N]$, $i \sim_\pi k$ and $j \sim_\pi l$ implies $i \sim_\pi j$ (in this definition either take for $<$ the usual order on $\{1, \dots, N\}$ or the cyclic order since it gives to the same notion). More intuitively π is non-crossing if and only there is a graphical representation of π (on a regular polygon with n vertices as explained in the beginning of section 2.1) such that the paths lie inside the polygon and only intersect (possibly) at the vertices of the regular polygon. For example the partitions of Figures 2.1, 2.2 are crossing, whereas the partitions in Figures 2.4, 2.5, 2.6 are all non-crossing. The set of non-crossing partitions of $[N]$ is denoted by $NC(N)$. The cardinality of $NC(N)$ is known to be equal to the Catalan number $(2N)!/(N!(N+1)!)$ (see [29]), but we will only use that it is less that 4^{N-1}.

Following [27], we introduce the subset $NC^*(d, m)$ of $NC(2dm)$.

In the following, for a real number x one denotes by $\lfloor x \rfloor$ the biggest integer smaller than or equal to x.

Divide the set $[2dm]$ into $2m$ intervals $J_1 \dots J_{2m}$ of size d: the first one is $J_1 = \{1, 2 \dots, d\}$, and the k-th is $J_k = \{(k-1)d + 1, \dots, kd\}$.

To each element of $[2dm]$ we assign a label in $\{1, \dots, d\}$ in the following way: in any interval J_k of size d as above, the elements are labelled from 1 to d if k is odd and from d

to 1 if k is even. We shall denote by A_k the set of elements labelled by k.

Definition 2.9. A non-crossing partition π of $[2dm]$ belongs to $NC^*(d,m)$ if each block of the partition has an even cardinality, and if within each block, two consecutive elements i and j belong to intervals of size d of different parity. Formally, the last condition means that $\lfloor (i-1)/d \rfloor \neq \lfloor (j-1)/d \rfloor \mod 2$ or equivalently $k(i) \neq k(j) \mod 2$ when $i \in J_{k(i)}$ and $j \in J_{k(j)}$.

Here are some first elementary properties of $NC^*(d,m)$:

Lemma 2.10. *If $d = 1$, a non-crossing partition $\pi \in NC(2m)$ belongs to $NC^*(1,m)$ if and only if it has blocks of even cardinality.*

*A non-crossing partition of $[2dm]$ is in $NC^*(d,m)$ if and only if it has blocks of even cardinality and it connects only elements with the same labels (*i.e. *it is finer than the partition $\{A_1,\dots,A_d\}$).*

Proof. The first statement is a particular case of the second statement, which we now prove. For any $i \in [2dm]$ denote by $k(i)$ the integer such that $i \in J_{k(i)}$: $k(i) = 1 + \lfloor (i-1)/d \rfloor$. Let $\pi \in NC^*(d,m)$. Then by the definition of $NC^*(d,m)$ every block of π contains as many elements i such that $k(i)$ is odd than elements i such that $k(i)$ is even. We have to prove that if s and t are two consecutive elements of a block of π, then s and t have the same labellings. Assume for example that s belongs to an odd interval, *i.e.* $k(s)$ is odd, and denote by $l(s)$ the label of s. Then $s = (k(s)-1)d + l(s)$. In the same way, $k(t)$ is then even and if $l(t)$ is the label of t, we have that $t = k(t)d + 1 - l(t)$. Hence the number of elements $i \in \{s+1,\dots,t-1\}$ such that $k(i)\big(= 1 + \lfloor (i-1)/d \rfloor\big)$ is odd is equal to $d - l(s) + d \cdot (k(t) - k(s) - 1)/2$, and the number of elements i such that $k(i)$ is even is equal to $d - l(t) + d \cdot (k(t) - k(s) - 1)/2$. But since π is non-crossing, the interval $\{s+1,\dots,t-1\}$ is a union of blocks of π and therefore contains as many elements i such that $k(i)$ is odd than elements i such that $k(i)$ is even. This implies $l(s) = l(t)$. The proof is the same if $k(s)$ is even.

Now assume that $\pi \in NC(dm)$ has blocks of even cardinality and that π is finer than the partition $\{A_1,\dots,A_d\}$. Let s and t be two consecutive elements of a block of π. Then there is i such that $s,t \in A_i$. Since π is non-crossing and π is finer than $\{A_1,\dots,A_d\}$, the set $\{s+1,\dots,t-1\} \cap A_i$ is a union of blocks of π, and in particular it has an even cardinality. But $\{s+1,\dots,t-1\} \cap A_i$ is the set of elements labelled by i in the union of the intervals J_k for $k(s) < k < k(t)$ (for the cyclic order). Hence its cardinality is $k(t) - k(s) - 1$. Hence $k(t) - k(s)$ is odd. Since s and t are arbitrary, this proves that $\pi \in NC^*(d,m)$. □

Thus to any $\pi \in NC^*(d,m)$ we can assign d partitions $\pi|_{A_1},\dots,\pi|_{A_d}$, which are the restrictions of π to $A_1,\dots,A_d$ respectively. It is immediate that for any $i \in \{1,\dots,d\}$, $\pi|_{A_i} \in NC^*(1,m)$. See Figure 2.5 for an example. To study $NC^*(d,m)$, we thus begin with the study of $NC^*(1,m)$.

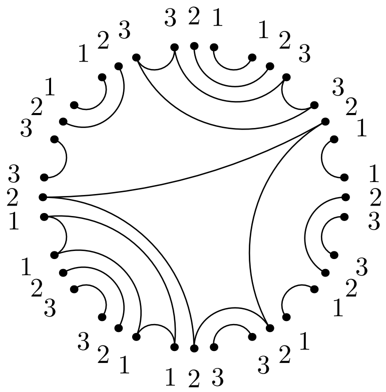
The first lemma shows that if k is a multiple of d, then P_k maps $NC^*(d,m)$ into itself:

Lemma 2.11. *If $k \in \mathbb{N}$ and $\pi \in NC(2N)$ then $P_k(\pi) \in NC(2N)$.*

If $k \in \mathbb{N}$ then for any $\pi \in NC^*(d,m)$, $P_{kd}(\pi) \in NC^*(d,m)$.

Moreover if $\pi \in NC^*(d,m)$, then for any $i \in \{1,\dots d\}$:

$$P_{kd}(\pi)|_{A_i} = P_k(\pi|_{A_i}).$$



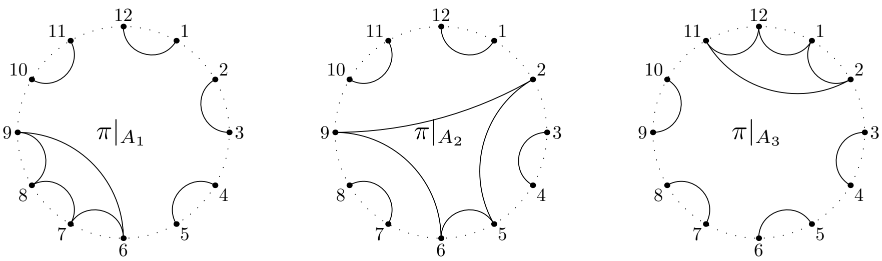


Figure 2.5: A graphical representation of a partition π in $NC^*(3,6)$ and the corresponding restrictions $\pi|_{A_1}$, $\pi|_{A_2}$ and $\pi|_{A_3}$.

Sketch of Proof. The first statement is obvious from the graphical point of view: if there are no crossing, the symmetrization map will not produce one.

The second statement follows from the characterization of Lemma 2.10: it is not hard to check that if π has blocks of even cardinality then $P_{kd}(\pi)$ also has. The fact that $P_{kd}(\pi)$ is finer that $\{A_1, \ldots, A_d\}$ if π is follows from the fact that $s_{kd}(A_i) = A_i$ for any k and $1 \leq i \leq d$.

The third statement follows from the fact that $s_{kd}^{(dm)}$ is characterized by the properties that for any $1 \leq j \leq 2m$, $s_{kd}^{(dm)}(J_i) = J_{s_k^{(m)}(i)}$ and $s_{kd}^{(dm)}(A_i) = A_i$ for $1 \leq i \leq d$. □

We have the following corollary of Lemma 2.8.

Corollary 2.12. *Let $\pi \in NC^*(d,m)$. Then for $2^k \geq m$, the partition*

$$\pi_k = P_{2^k d} P_{2^{k-1} d} \ldots P_{2d} P_d P_{md}(\pi)$$

is one of the $2d+1$ partitions $\sigma_l^{(d,m)}$ for $l = 0, 1, \ldots, d$ and $\widetilde{\sigma}_l^{(d,m)}$ for $l = 1, 2, \ldots, d$ defined by:

$$\sigma_l^{(d,m)}|_{A_i} = \begin{cases} c_m & \text{if } 1 \leq i \leq l \\ r_m & \text{if } l < i \leq d, \end{cases}$$

$$\widetilde{\sigma}_l^{(d,m)}|_{A_i} = \begin{cases} c_m & \text{if } 1 \leq i < l \\ 1_{2m} & \text{if } i = l \\ r_m & \text{if } l < i \leq d. \end{cases}$$

Moreover for any integer i, $P_{id}(\pi) = \pi$ when π is one of the partitions $\sigma_l^{(d,m)}$ for $l = 0, 1, \ldots, d$ and $\widetilde{\sigma}_l^{(d,m)}$ for $l = 1, 2, \ldots, d$.

Proof. Let k and π as above. By Lemma 2.11, $\pi_k|_{A_i} = P_{2^k}P_{2^{k-1}} \dots P_2 P_1 P_m(\pi|_{A_i})$, which is by Lemma 2.8 one of 0_{2m}, r_m, c_m and 1_{2m}. But since 0_{2m} does not have blocks of even sizes, only the three r_m, c_m and 1_{2m} are possible.

Let $1 \leq i < j \leq d$. If $\pi_k|_{A_i} = r_m$ or 1_{2m} then in particular $i \sim_{\pi_k} 2d+1-i$. Since π_k is non-crossing, $j \nsim_{\pi_k} 1-j$, which implies that $\pi_k\big|_{A_j} \neq c_m, 1_{2m}$. Thus $\pi_k\big|_{A_j} = r_m$. In the same way if $\pi_k\big|_{A_j} = c_m$ or 1_{2m} then $\pi_k|_{A_i} = c_m$. This concludes the proof.

Similarly, the second claims follows from the fact (easy to verify) that $P_i(\pi) = \pi$ for any $i \in [2m]$ when $\pi = 1_{2m}$, r_m or c_m. □

An important subset of $NC^*(d,m)$ is the subset $NC_2^*(d,m)$ of partitions in $NC^*(d,m)$ with blocks of size 2. As explained in part 3.1 of [27], $NC_2^*(d,m)$ is naturally in bijection with the non-decreasing chains (for the natural lattice structure on $NC(m)$) of length d of non-crossing partitions of $[m]$. Let us denote by $NC(m)^{(d)}$ this set of non-decreasing chains in $NC(m)$, for the order of refinement, given by $\pi \leq \pi'$ if π' is finer that π. The bijective map $NC_2^*(d,m) \to NC(m)^{(d)}$ extends naturally to a (of course non-bijective) map $NC^*(d,m) \to NC(m)^{(d)}$ which is of interest. We now describe the construction of this map.

Let $\pi \in NC^*(1,m)$, that is a non-crossing partition of $[2m]$ with blocks of even size. Then $\Phi(\pi)$ is the partition of $[m]$ defined by the fact that $\sim_{\Phi(\pi)}$ is the transitive closure of the relation that relates k and l if $2k \sim_\pi 2l$ or $2k-1 \sim_\pi 2l$ or $2k \sim_\pi 2l-1$ or $2k-1 \sim_\pi 2l-1$. That is $\Phi(\pi)$ is the partition obtained by identifying the $2k-1$ and $2k$ in $[2m]$ to get $k \in [m]$.

If $\pi \in NC^*(d,m)$, we define the map $\mathcal{P}$ by $\mathcal{P}(\pi) = (\Phi(\pi|_{A_1}), \dots, \Phi(\pi|_{A_d}))$. See Figure 2.6.

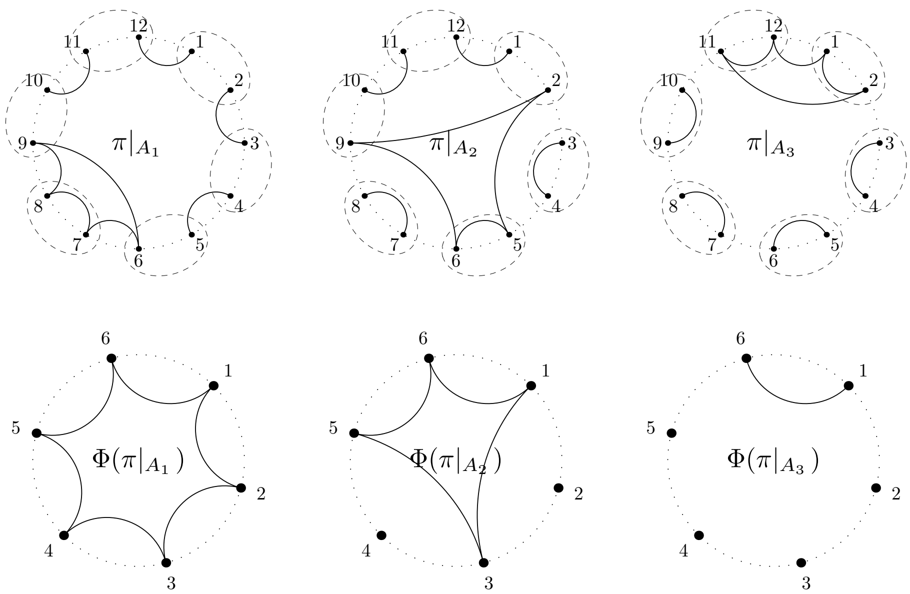


Figure 2.6: The map $\mathcal{P}$ for the partition $\pi \in NC^*(3,6)$ of Figure 2.5.

The map $\mathcal{P}$ is a good tool to make a finer study of $NC^*(d,m)$.

The main result in this section is that partitions in $NC^*(d,m)$ are not far from belonging to $NC_2^*(d,m)$:

Theorem 2.13. *For any $\sigma \in NC_2^*(d,m)$ there are less than 4^{2m} partitions $\pi \in NC^*(d,m)$ such that $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$.*

Moreover for such a π, the partition σ is finer than π and the number of blocks of π of size 2 is greater than $dm - 2m$, and every block has size at most $2m$.

Remark 2.14. The remarkable feature of $NC^*(d,m)$ illustrated in this Theorem is that the bounds we get on the number of $\pi \in NC^*(d,m)$ such that $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$ and on the elements of $[2dm]$ that do not belong to a block of size 2 of $\pi \in NC^*(d,m)$ *do not depend on d.*

In particular since the cardinality of $NC_2^*(d,m)$ is equal to the Fuss-Catalan number $1/m\binom{m(d+1)}{m-1}$ which is less that $e^m(d+1)^m$ (Corollary 3.2 in [27]) the first statement of the Theorem implies that the cardinality of $NC^*(d,m)$ is less that $\big(16e(d+1)\big)^m$.

This Theorem will follow from a series of lemmas. Here is the first one, which treats the case $d = 1$:

Lemma 2.15. *Let $\sigma \in NC_2^*(1,m)$ and $\pi \in NC^*(1,m)$ such that $\Phi(\pi) = \Phi(\sigma)$. Then σ is finer than π.*

More precisely if $\pi \in NC^*(1,m)$ and if $\{k_1 < k_2 \cdots < k_p\}$ is a block of $\Phi(\pi)$, then for any i, $2k_i \sim_\pi 2k_{i+1} - 1$ (with the convention $k_{p+1} = k_1$).

Proof. The first statement follows easily from the second one. We thus focus on the second statement. At least as far as partitions in $NC_2^*(1,m)$ are concerned, this is explained in the discussion preceding Corollary 3.2 in [27]. The proof is the same for a general $\pi \in NC^*(1,m)$, but for completeness we still provide a proof.

It is clear that $\Phi(\pi)(k) = \{k\}$ implies that $2k \sim_\pi 2k - 1$. Thus to prove the statement we have to prove that if k and l are consecutive and distinct elements of a block of $\Phi(\pi)$ then $2k \sim_\pi 2l - 1$.

The first element in $\pi(2k)$ after $2k$ is odd, that is of the form $2p-1$, because $2k$ is even and the parity alternates in blocks of π. We claim that $p = l$. Note that we necessarily have $k < l \leq p$ (again for the cyclic order) because $k \sim_{\Phi(\pi)} p$. Suppose that $k < l < p$. We get to a contradiction: indeed since $l \sim_{\Phi(\pi)} k$ and $\{2l-1, 2l\} \subset \{2k+1, 2k+2\dots, 2p-2\}$ there is at least one $j \in \{2k+1, 2k+2\dots, 2p-2\}$ and $i \in \{2p-1, 2p\dots 2k\}$ such that $i \sim_\pi j$. But by definition of p, $j \nsim_\pi 2k$ and $j \nsim_\pi 2p-1$. This contradicts the fact that π is non-crossing. □

We can now check that $\mathcal{P}$ is well-defined:

Lemma 2.16. *The map $\mathcal{P}$ from $NC^*(d,m)$ takes values in $NC(m)^{(d)}$.*

Proof. Let $\pi \in NC^*(d,m)$; we have to prove that if $1 \leq i < j \leq d$ then $\Phi(\pi|_{A_j})$ is finer than $\Phi(\pi|_{A_i})$.

Let $\{k_1 < k_2 \cdots < k_p\}$ be a block of $\Phi(\pi|_{A_i})$. Suppose that $\Phi(\pi|_{A_j})(k_1)$ is not contained in $\{k_1, k_2 \dots k_p\}$. Then there exist $1 \leq s \leq p$ and $l \notin \{k_1, k_2 \dots k_p\}$ such that k_s and l are consecutive elements of $\Phi(\pi|_{A_j})(k_1)$ (for the cyclic order). If $1 \leq t \leq p$ is such that $k_t < l < k_{t+1}$ (with again the convention $k_{p+1} = k_1$), we have by Lemma 2.15 that $2k_t \sim_{\pi|_{A_i}} 2k_{t+1} - 1$ and $2k_s \sim_{\pi|_{A_j}} 2l - 1$, which contradicts the fact that π is non-crossing. This shows that $\Phi(\pi|_{A_j})(k_1) \subseteq \{k_1, k_2 \dots k_p\} = \Phi(\pi|_{A_i})(k_1)$. Since k_1 was arbitrary, the proof is complete. □

Here is a last elementary lemma concerning general non-crossing partitions:

Lemma 2.17. *Let $N \in \mathbb{N}$ and $\pi \in NC(N)$ with α blocks. Then the number of $k \in [N]$ such that $k \sim_\pi k+1$ is greater or equal to $N - 2(\alpha - 1)$.*

Proof. For $\pi \in NC(N)$, let us denote by $c(\pi)$ the number of $k \in [N]$ such that $k \sim_\pi k+1$. We prove by induction on α that if $\pi \in NC(N)$ has α blocks, then $c(\pi) \geq N - 2(\alpha - 1)$. If $\alpha = 1$, this is clear since $c(\pi) = N$.

Assume that the statement of the lemma is true for all N and all $\pi \in NC(N)$ with α blocks. Take $\pi \in NC(N)$ with $\alpha + 1$ blocks. Since π is non-crossing there is a block of π, say A, which is an interval of size S. If $\pi\big|_{[N]\setminus A}$ is regarded as an element of $NC(N-S)$ then $c(\pi) \geq S - 1 + c(\pi\big|_{[N]\setminus A}) - 1$. By the induction hypothesis $c(\pi\big|_{[N]\setminus A}) \geq N - S - 2(\alpha - 1)$, which implies $c(\pi) \geq N - 2\alpha$ and thus concludes the proof. □

The next Lemma is the main result of this section, and Theorem 2.13 will easily follow from it:

Lemma 2.18. *Let $\sigma \in NC_2^*(d,m)$. Then there is a subset A of $[2dm]$ of size greater than $2dm - 4m$, which is a union of blocks of σ, and such that for any $\pi \in NC^*(d,m)$ with $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$ and any $k \in A$, $\pi(k) = \sigma(k)$.*

Proof. For any $1 \leq j \leq d$, denote by $\sigma_j = \Phi(\sigma\big|_{A_j})$. Denote by $\sigma_{d+1} = 0_m$. Fix now $1 \leq i \leq d$ and $\{k_1 < k_2 < \cdots < k_p\}$ a block of σ_i. As usual we take the convention that $k_{p+1} = k_1$. We claim that if $k_s \sim_{\sigma_{i+1}} k_{s+1}$ then for any $\pi \in NC^*(d,m)$ with $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$, $\pi(2dk_s - i + 1) = \{2dk_s - i + 1, 2dk_{s+1} - 2d + i\}$ $(= \sigma(2dk_s - i + 1)$ by Lemma 2.15).

Let us first check that this claim implies the Lemma. By Lemma 2.16, σ_{i+1} is finer than σ_i and in particular its restriction to $\{k_1, k_2, \ldots, k_p\}$ makes sense. By Lemma 2.17, the number of s's in $\{1, \ldots, p\}$ such that $k_s \sim_{\sigma_{i+1}} k_{s+1}$ is greater than $p - 2(|\sigma_{i+1}\big|_{\{k_1,k_2,\ldots,k_p\}}| - 1)$ where $|\sigma|$ is the number of blocks of σ. Thus summing over all blocks of σ_i we get at least $2m - 4(|\sigma_{i+1}| - |\sigma_i|)$ elements k in A_i such that $\pi(k) = \sigma(k)$ for any $\pi \in NC^*(d,m)$ with $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$. This allows to conclude the proof since

$$\sum_{i=1}^{d} (2m - 4(|\sigma_{i+1}| - |\sigma_i|)) = 2md - 4|\sigma_{d+1}| + 4|\sigma_1| > 2md - 4m.$$

Note that A is constructed as a union of blocks of σ.

We now only have to prove the claim. Assume that $k_s \sim_{\sigma_{i+1}} k_{s+1}$ and take $\pi \in NC^*(d,m)$ such that $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$. By Lemma 2.15 applied to $\Phi(\sigma|_{A_i}) = \sigma_i$, $2dk_s - i + 1 \sim_\pi 2dk_{s+1} - 2d + i$. Thus we only have to prove that if $k_s \sim_{\sigma_{i+1}} k_{s+1}$ there is no $k \in \{k_1, \ldots, k_p\} \setminus \{k_{s+1}\}$ such that $2dk_s - i + 1 \sim_\pi 2dk - 2d + i$.

But if $k_s \sim_{\sigma_{i+1}} k_{s+1}$ then $i \neq d$ (because $\sigma_{d+1} = 0_m$) and by Lemma 2.16, k_s and k_{s+1} are consecutive elements in $\sigma_{i+1}(k_s)$. Thus by Lemma 2.15, $2dk_s - i \sim_\pi 2dk_{s+1} - 2d + i + 1$. The condition that π is non-crossing implies the claim since for $k \in \{k_1, \ldots, k_p\} \setminus \{k_{s+1}\}$,

$$2dk_s - i < 2dk_s - i + 1 < 2dk_{s+1} - 2d + i + 1 < 2dk - 2d + i,$$

that is $(2dk_s - i + 1, 2dk - 2d + i)$ and $(2dk_s - i, 2dk_{s+1} - 2d + i + 1)$ are crossing.

□

We can now prove the Theorem.

Proof of Theorem 2.13. Let $\sigma \in NC_2^*(d,m)$. If $\pi \in NC^*(d,m)$ satisfies $\mathcal{P}(\pi) = \sigma$ then Lemma 2.15 applied to $\sigma|_{A_i}$ and $\pi|_{A_i}$ for $i = 1, \ldots, d$ proves that σ is finer than π, and 2.18 implies that π has at least $dm - 2m$ blocks of size 2. The fact that every block of π has size at most m just follows from the definition of $NC^*(d,m)$: π is indeed finer than $\{A_1, \ldots A_d\}$ with $|A_j| = 2m$.

We now prove the first statement of Theorem 2.13. Let A be the subset of $[2dm]$ given by Lemma 2.18. Then there is an injection:

$$\begin{array}{ccc} \{\pi \in NC^*(d,m), \mathcal{P}(\pi) = \mathcal{P}(\sigma)\} & \to & NC([2dm] \setminus A) \\ \pi & \mapsto & \pi\big|_{[2dm]\setminus A} \end{array}$$

In particular since there are less than 4^N non-crossing partitions on $[N]$, the first statement of the Theorem follows with 4^{2m} replaced by 4^{4m} because $[2dm] \setminus A$ has cardinality less than $4m$. To get the 4^{2m} just replace $[2dm] \setminus A$ by a set B that contains exactly one element of $\sigma(k)$ for any $k \in [2dm] \setminus A$. Then B has cardinality less than $2m$ because $[2dm] \setminus A$ is a union of blocks (=pairs) of σ, and the previous map is still an injection since $\pi \in NC^*(d,m)$ and $\mathcal{P}(\pi) = \mathcal{P}(\sigma)$ implies that σ is finer that π. □

2.1.3 Invariant of the P_k's

Motivated by Lemma 2.25 we are interested in invariants of the operations P_{kd} ($k \in [2m]$) on $NC^*(d,m)$. For $\pi \in NC^*(1,m)$ let $B(\pi)$ be the number of blocks in $\Phi(\pi)$. This is the fundamental observation:

Lemma 2.19. *For any $\pi \in NC^*(1,m)$,*

$$B(\pi) = \frac{1}{2}\left(B(P_k(\pi)) + B(P_{k+m}(\pi))\right).$$

This Lemma is a consequence of the following description, which proves that for any k, the set of blocks of $\Phi(\pi)$ but one is in bijection with the set of blocks of π that do not contain k and that begin with an odd element (after k for the cyclic order):

Lemma 2.20. *Let $k \in [2m]$ and $\pi \in NC^*(1,m)$. Then $B(\pi) - 1$ is equal to the number of $l \in [2m] \setminus \{k\}$ such that l is odd and such that for any $l' \sim_\pi l$, $l \le l' < k$ (for the cyclic order).*

Proof. Indeed the set of odd l's different from k such that $l' \sim_\pi l \Rightarrow l \le l' < k$ (for the cyclic order) is in bijection with the blocks of $\Phi(\pi)$ that do not contain $\lfloor (k+1)/2 \rfloor$.

The direct map consists in mapping to any such l the block $\Phi(\pi)(\lfloor (l+1)/2 \rfloor)$ and the reverse map gives to any block A of $\Phi(\pi)$ no containing $\lfloor (k+1)/2 \rfloor$ the smallest l greater than k (again for the cyclic order) such that $\lfloor (l+1)/2 \rfloor \in A$. The reader can check using Lemma 2.15 that these maps are indeed inverses of each other. □

Proof of Lemma 2.19. We use Lemma 2.20 with $k+1$ instead of k. For any $\pi \in NC^*(1,m)$ we denote by $F(\pi,k)$ the set of odd $l \in [2m] \setminus \{k+1\}$ such that $l' \sim_\pi l \Rightarrow l \le l' < k+1$. We know that $|F(\pi,k)| = B(\pi) - 1$. Moreover let us decompose $F(\pi,k)$ as the disjoint union of $F_1(\pi,k)$ and $F_2(\pi,k)$ defined by: $l \in F_1(\pi,k)$ if and only $l \in F(\pi,k)$ and $\pi(l) \subset I_{k+m}$; and $F_2(\pi,k)$ is the set of $l \in F(\pi,k)$ such that $\pi(l) \cap I_l \neq \emptyset$.

If $l \in I_{k+m}$ then $l \in F(P_{k+m}(\pi),k)$ if and only if $l \in F(\pi,k)$ because if $k+1 \le l' < l$, then $l' \sim_{P_{k+m}(\pi)} l$ if and only if $l' \sim_\pi l$.

Take now $l \notin I_{k+m}$. By definition of $F(\cdot,k)$, l is in $F(P_{k+m}(\pi),k)$ if and only if l is odd and l is the first element (after $k+1$ for the cyclic order) of a block of $P_{k+m}(\pi)$ contained in I_k, which is equivalent to the fact that $s_k(l) = 2k+1-l$ is even and is the last element of a block of π contained in I_{k+m}. Such a block then has first element odd, and thus belongs to $F_1(\pi,k)$ except if it is equal to $k+1$. To summarize, we have thus proved that

$$|F(P_{k+m}(\pi),k)| = |F(\pi,k) \cap I_{k+m}| + |F_1(\pi,k)| + 1 \tag{2.11}$$

if $k+1$ is odd and $\pi(k+1) \subset I_{k+m} = \{k+1, k+2, \ldots, k+m\}$, and

$$|F(P_{k+m}(\pi), k)| = |F(\pi, k) \cap I_{k+m}| + |F_1(\pi, k)| \tag{2.12}$$

otherwise.

We now compute $|F(P_k(\pi), k)|$. If $l \in I_k$ then as above $l \in F(P_k(\pi), k)$ if and only if $l \in F(\pi, k)$. If $l \notin I_k$ then $l \in F(P_k(\pi), k)$ if and only if l is odd and l is the first element strictly after $k+1$ (in the cyclic order) of a block of $P_k(\pi)$ not containing $k+1$. By construction of $P_k(\pi)$ this is equivalent to the fact that $s_k(l) = 2k+1-l$ is even, belongs to I_k, is different from k and is the last element before k in a block of π. The first element (strictly after k in the cyclic order) of such a block is then in $F_2(\pi, k)$ except if it is equal to $k+1$. Reciprocally, if l' is the last element of a block containing an element of $F_2(\pi, k)$ then $l = s_k(l') \in F(P_k(\pi), k)$ except if $l' = k$. The same is true if $\pi(k+1) \not\subset I_{k+m}$, $k+1$ is odd and if l' denotes the last element in $\pi(k+1)$. Thus

$$\begin{aligned} |F(P_k(\pi), k)| &= |F(\pi, k) \cap I_k| + |F_2(\pi, k)| - 1_{k \text{ is even}} + 1_{k \text{ is even and } \pi(k+1) \not\subset I_{k+m}} \\ &= |F(\pi, k) \cap I_k| + |F_2(\pi, k)| - 1_{k \text{ is even and } \pi(k+1) \subset I_{k+m}}. \end{aligned}$$

Summing this last equality with (2.11) or (2.12) yields

$$\begin{aligned} &|F(P_k(\pi), k)| + |F(P_{k+m}(\pi), k)| = \\ &\qquad |F(\pi, k) \cap I_k| + |F_2(\pi, k)| + |F(\pi, k) \cap I_{k+m}| + |F_1(\pi, k)| = 2|F(\pi, k)|. \end{aligned}$$

This concludes the proof since by Lemma 2.20 for any $\sigma \in NC^*(1, m)$, $|F(\sigma, k)| = B(\sigma) - 1$. □

2.1.4 Study of $NC(d, m)$

Another relevant subset of $NC(2dm)$ is the set $NC(d, m)$ of partitions π with blocks of even cardinality and that connect only elements of different intervals J_k. In other words for all $i, j \in [2dm]$, $i \nsim_\pi j$ if $i, j \in J_k$.

The following observation is very simple but, in view of Theorem 2.5 or 2.6, it is the motivation for the introduction of $NC(d, m)$:

Lemma 2.21. *Let $\pi \in NC(2dm)$ with blocks of even cardinality. Then $\pi \in NC(d, m)$ if and only if π does not connect two consecutive elements of a same subinterval J_i. In other words, $i \sim_\pi i+1$ only if i is a multiple of d.*

Proof. The *only if* part of the proof is obvious. The converse follows from the fact that a non-crossing partition always contains an interval (if π is non-crossing with blocks of even size, and $s < t \in J_i$ with $s \sim_\pi t$ and $t \neq s+1$, apply this fact to $\pi\big|_{\{s, s+1, \ldots, t-1\}}$). □

The purpose of this section is to generalize Theorem 2.13. Namely we prove

Theorem 2.22. *The cardinality of $NC(d, m)$ is less than $(4d+4)^{2m}$.*

Moreover for any $\pi \in NC(d, m)$ the number of blocks of π of size 2 is greater than $(d-2)m$.

The idea of the proof is similar to the proof of Theorem 2.13: we try to reduce to the subset of $NC(d, m)$ consisting of partitions into pairs. For this we introduce the map $Q = Q^{(N)}$ from the set of non-crossing partitions of $[2N]$ into blocks of even sizes to the set of non-crossing partitions of $[2N]$ into pairs. The map Q has the property that if $\pi \in NC(2N)$ has blocks of even sizes, then $Q(\pi)$ is finer than π and any block $\{k_1, \dots, k_{2p}\}$ of π with $1 \leq k_1 < \dots < k_{2p} \leq 2N$ becomes p blocks of $Q(\pi)$, namely $\{k_1, k_2\}, \dots, \{k_{2p-1}, k_{2p}\}$. It is straightforward to check that this indeed defines a non-crossing partition of $[2N]$ into pairs. Note that unlike in the rest of the paper here the element $1 \in [2N]$ plays a specific role in the definition of Q and we abandon the cyclic symmetry of $[2N]$. But this has the advantage to allow to define an order relation on the set of pairs of elements of $[2N]$: we will say that a pair (i, j) covers a pair (k, l) if $1 \leq i < k < l < j \leq 2N$.

A noteworthy property of Q is that if $\sigma = Q(\pi)$ then two blocks (=pairs) of σ cannot be contained in the same block of π if one covers the other. In other words if $1 \leq i < k < l < j \leq 2N$ with $i \sim_\sigma j$ and $k \sim_\sigma l$ then $i \nsim_\pi k$.

Following the notation of section 3.1 in [27], the image $Q(NC(d, m))$ is denoted by $\mathscr{I}(d, m)$; it is the set of partitions of π into pairs that do not connect elements of a same subinterval J_k for $k = 1, \dots, 2m$. We are not aware of any nice combinatorial description of $\mathscr{I}(d, m)$ as for $NC_2^*(d, m)$, but a precise bound for its cardinality is known: by the proof of Theorem 5.3.4 in [4], the cardinality of $\mathscr{I}(d, m)$ is equal to $\tau(T_d(s)^{2m})$ where T_d is the d-th Tchebycheff polynomial and s is a semicircular element of variance 1 in a tracial C^*-algebra (A, τ). In particular since $\|T_d(s)\| = d+1$ we have that $|\mathscr{I}(d, m)| \leq (d+1)^{2m}$. Theorem 2.22 will thus follow from the following more general statement:

Lemma 2.23. *Suppose that $[2N]$ is divided into k non-empty intervals $S_1, \dots, S_k$ and let σ be a non-crossing partition of $[2N]$ into pairs that do not connect elements of a same subinterval S_i. Then there are at most 4^{k-2} non-crossing partitions π of $[2N]$ that do not connect elements of a same subinterval S_i and such that $Q(\pi) = \sigma$. Moreover for such a π there are at most $2k-4$ elements $i \in [2N]$ for which $\pi(i)$ is not a pair.*

Proof. We prove this statement by induction on N. For simplicity of notation we will assume that the intervals $S_1, \dots, S_k$ are ordered, *i.e.* that if $i \in S_s$ and $j \in S_t$ with $s < t$ then $i < j$.

If $N = 1$ and σ is as above then $\sigma = 1_2$, $k = 2$, and there is only one $\pi \in NC(2)$ with $Q(\pi) = \sigma$. This proves the assertion for $N = 1$.

Assume that the above statement holds for $1, 2, \dots, N-1$ and take σ as above. Consider the set $\{\{s_i, t_i\}, i = 1 \dots p\}$ of outermost blocks (=pairs) of σ, *i.e* the set of pairs of σ that are not being covered by another block of σ. If we order the s_i's and t_i's so that $s_i < t_i$ and $s_i < s_{i+1}$ then we have that $s_1 = 1$, $s_{i+1} = t_i + 1$ and $t_p = 2N$.

By the property of Q mentioned above, a partition $\pi \in NC(2N)$ that does not connect elements of the same interval S_j (for $j = 1, \dots, k$) satisfies $Q(\pi) = \sigma$ if and only if the following properties are satisfied:

- For any $1 \leq i \leq p$, $\{s_i + 1, \dots, t_i - 1\}$ is a union of blocks of π, the non-crossing partition $\pi|_{\{s_i+1,\dots t_i-1\}}$ does not connect elements of the same subinterval $S_j \cap \{s_i + 1, \dots t_i - 1\}$ for $j = 1, \dots, k$, and $Q(\pi|_{\{s_i+1,\dots t_i-1\}}) = \sigma|_{\{s_i+1,\dots t_i-1\}}$.
- Any block of $\pi|_{\{s_1,t_1,s_2,t_2,\dots,s_p,t_p\}}$ is a union of pairs $\{s_i, t_i\}$ and does not contain 2 elements of a same interval S_j.

Define $k_+(i)$ and $k_-(i)$ for $1 \leq i \leq p$ by $s_i \in S_{k_-(i)}$ and $t_i \in S_{k_+(i)}$. Then for any $1 \leq i \leq p$, $k_-(i) < k_+(i)$ and for $i < p$, $k_+(i) \leq k_-(i+1)$.

Since $\{s_i+1, \dots t_i-1\}$ intersects at most $k_+(i)-k_-(i)+1$ different intervals S_j, we have by the induction hypothesis that the number of non-crossing partitions of $\{s_i+1, \dots, t_i-1\}$

that satisfy the first point above is at most $4^{k_+(i)-k_-(i)-1}$, and for such a partition at most $2(k_+(i)-k_-(i)-1)$ elements of $\{s_i+1,\dots t_i-1\}$ do not belong to a pair.

Moreover the set of non-crossing partitions of $\{s_1,t_1,s_2,t_2,\dots,s_p,t_p\}$ that satisfy the second point is in bijection with the set of non-crossing partitions of $\{s_i, i=1\dots p\}$ such that $s_i \nsim s_{i+1}$ if $k_+(i)=k_-(i+1)$. Its cardinality is in particular less than (or equals) the number of non-crossing partitions of $[p]$, which is less than 4^{p-1}. Therefore the total number of non-crossing partitions π of $[2N]$ that do not connect elements of a same subinterval S_j and such that $Q(\pi)=\sigma$ is less than

$$4^{p-1}\prod_{i=1}^{p} 4^{k_+(i)-k_-(i)-1} \leq 4^{k-2}.$$

We used the inequality $\sum_{i=1}^p k_+(i)-k_-(i)-1 \leq k-1-p$.

To prove that for such a π at most $2k-4$ elements of $[2N]$ do not belong to a pair of π, note that for an element $j\in[2N]$ the block $\pi(j)$ is not a pair either if $j\in\{s_1,t_1,\dots,s_p,t_p\}$ or if j belongs to a block of $\pi\big|_{\{s_i+1,\dots t_i-1\}}$ which is not a pair for some $1\leq i\leq p$. If $k_+(i)<k_-(i+1)$ for some i then we are done since $2p+\sum_{i=1}^p 2k_+(i)-2k_-(i)-2 \leq 2k-4$. To conclude the proof we thus have to check that if $k_+(i)=k_-(i+1)$ for any $1\leq i<p$ then there are at least 2 elements of $\{s_1,t_1,\dots,s_p,t_p\}$ that belong to a pair of π. But this amounts to showing that a non-crossing partition of $[p]$ such that $i\nsim i+1$ for any $1\leq i<p$ contains at least one singleton, which is clear. □

The following Lemma is also an easy extention of Lemma 2.8. Remember that the partitions $\sigma_l^{(d,m)}$ and $\widetilde{\sigma}_l^{(d,m)}$ are defined in Corollary 2.12:

Lemma 2.24. *Fix integers d and m.*

For any $k\in[2m]$ and $\pi\in NC(d,m)$ the partition $P_{kd}(\pi)$ also belongs to $NC(d,m)$.

Let $k\in\mathbb{N}$ such that $2^k\geq m$. Then for any partition $\pi\in NC(d,m)$, the partition $\pi_k = P_{2^k}P_{2^{k-1}}\dots P_2P_1P_m(\pi)$ is one of the $2d+1$ partitions $\sigma_l^{(d,m)}$ for $0\leq l\leq d$ or $\widetilde{\sigma}_l^{(d,m)}$ for $1\leq l\leq d$.

Proof. The first point is straightforward.

The proof of the second point is the same as Lemma 2.8: depending on the fact that $\{1,2,\dots,dm\}\cap\pi(i)\setminus\{i\}$ and $\{dm+1,\dots,2dm\}\cap\pi(i)$ are empty or not for $i=1,\dots,d$, we prove by induction on k that π_k has the right properties. The details are left to the reader. □

2.2 Inequalities

For any partition π of $[2N]$, and any $k=(k_1,\dots,k_{2N})\in\mathbb{N}^{2N}$, we write $k\prec\pi$ if for any $i,j\in[2N]$ such that $i\sim_\pi j$, $k_i=k_j$.

Let $a=(a_k)_{k\in\mathbb{N}^N}$ be a finitely supported family of matrices. For any $k=(k_1,\dots,k_N)\in\mathbb{N}^N$ let $\widetilde{a}_k = a_{(k_N,k_{N-1},\dots,k_1)}$.

For such a and for a partition π of $[2N]$, we denote by $S(a,\pi,N,1)$ the following quantity:

$$S(a,\pi,N,1) = \sum_{k,l\in\mathbb{N}^N,(k,l)\prec\pi} Tr(a_k\widetilde{a}_l^*). \tag{2.13}$$

More generally for integers m, d, for a finitely supported family of matrices $a = (a_k)_{k\in\mathbb{N}^d}$ and a partition π of $[2dm]$, we define

$$S(a, \pi, d, m) = \sum_{k_1,\dots,k_{2m}\in\mathbb{N}^d,(k_1,\dots,k_{2m})\prec\pi} Tr(a_{k_1}\widetilde{a}^*_{k_2}a_{k_3}\dots a_{k_{2m-1}}\widetilde{a}^*_{k_{2m}}). \tag{2.14}$$

In this equation and in the rest of the paper an element $k = (k_1, \dots, k_{2m}) \in (\mathbb{N}^d)^{2m}$ is identified with an element of $\mathbb{N}^{2dm}$. Therefore the expression $k \prec \pi$ has a meaning for $\pi \in NC(2dm)$.

The following application of the Cauchy-Schwarz inequality is what motivates the introduction of the operations P_k on the partitions of $[2N]$. The same use of the Cauchy-Schwarz inequality has been made in the second part of [9].

Lemma 2.25. *For a partition π of $[2N]$ and a finitely supported family of matrices $a = (a_k)_{k\in\mathbb{N}^N}$,*

$$|S(a, \pi, N, 1)| \le (S(a, P_0(\pi), N, 1))^{1/2} (S(a, P_N(\pi), N, 1))^{1/2}.$$

More generally for a partition π of $[2dm]$, for a finitely supported family of matrices $a = (a_k)_{k\in\mathbb{N}^d}$ and any integer i

$$|S(a, \pi, d, m)| \le (S(a, P_{di}(\pi), d, m))^{1/2} \left(S(a, P_{(m+i)d}(\pi), d, m)\right)^{1/2}. \tag{2.15}$$

Proof. The second statement for $i = 0$ follows from the first one by replacing N by dm. Indeed for any and $k = (k_1, \dots, k_m) \in (\mathbb{N}^d)^m \simeq \mathbb{N}^{dm}$, denote $\beta_k = a_{k_1}\widetilde{a}^*_{k_2}a_{k_3}\dots a_{k_m}$ if m is odd and $\beta_k = a_{k_1}\widetilde{a}^*_{k_2}a_{k_3}\dots\widetilde{a}^*_{k_m}$ if m is even. We claim that $S(a, \pi, d, m) = S(\beta, \pi, dm, 1)$. We give a proof when m is odd, the case when m is even is similar. It is enough to prove that if $k = (k_1, \dots, k_m) \in (\mathbb{N}^d)^m$ then $\widetilde{\beta}^*_k = \widetilde{a}^*_{k_1}a_{k_2}\dots\widetilde{a}^*_{k_m}$. But if $r : \mathbb{N}^d \to \mathbb{N}^d$ denotes the map $r(s_1, \dots, s_d) = (s_d, \dots, s_1)$ we have that

$$\begin{aligned}
\widetilde{\beta}^*_k &= \beta^*_{r(k_m),\dots,r(k_1)} = \left(a_{r(k_m)}\dots\widetilde{a}^*_{r(k_2)}a_{r(k_1)}\right)^* \\
&= a^*_{r(k_1)}\widetilde{a}_{r(k_2)}\dots a^*_{r(k_m)} \\
&= \widetilde{a}^*_{k_1}a_{k_2}\dots\widetilde{a}^*_{k_m}.
\end{aligned}$$

For a general i the following argument based on the trace property allow to reduce to the case $i = 0$: for a partition π of $[2dm]$ and any $n \in [2dm]$ denote $\tau_n(\pi)$ the partition such that $s \sim_{\tau_n(\pi)} t$ if and only if $s + n \sim_\pi t + n$, so that $P_{n+k}(\pi) = (\tau_n^{-1} \circ P_k \circ \tau_n)(\pi)$ for any integer k. Moreover by the trace property $S(a, \pi, d, m) = S(a, \tau_{di}(\pi), d, m)$ if n is even and $S(a, \pi, d, m) = S(\widetilde{a}^*, \tau_{di}(\pi), d, m)$ if i is even (here $\widetilde{a}^*$ denotes the family $(\widetilde{a}^*_k)_{k\in\mathbb{N}^d}$). Therefore if one assumes that the inequality (2.15) is satisfied for any π and any a but only for $i = 0$, then we can deduce it for a general i in the following way. Denote $b = (a_k)_{k\in\mathbb{N}^d}$ if i is even and $b = (\widetilde{a}^*_k)_{k\in\mathbb{N}^d}$ if i is odd and :

$$\begin{aligned}
|S(a, \pi, d, m)|^2 &= |S(b, \tau_{di}(\pi), d, m)|^2 \\
&\le S(b, P_0(\tau_{di}(\pi)), d, m)S(b, P_{dm}(\tau_{di}(\pi)), d, m) \\
&= S(b, \tau_{di}(P_{di}(\pi)), d, m)S(b, \tau_{di}(P_{dm+di}(\pi)), d, m) \\
&= S(a, P_{di}(\pi), d, m)S(a, P_{(m+i)d}(\pi), d, m)
\end{aligned}$$

We now prove the first statement. We take the same notation as in Definition 2.7.

Let us clarify the notation for the rest of the proof. In the whole proof, for a set X we see a $k \in \mathbb{N}^X$ as a function from X to $\mathbb{N}$, and for an integer N we will identify $\mathbb{N}^N$ with

$\mathbb{N}^{[N]}$. In particular, if X and Y are disjoint subsets of a set Z, and if $k \in \mathbb{N}^X$ and $l \in \mathbb{N}^Y$, $[k,l]$ will denote the element of $\mathbb{N}^{X \cup Y}$ corresponding to the function on $X \cup Y$ that has k as restriction to X and l as restriction to Y.

Let us denote by A the union of the blocks of π that are contained in $I_N = \{1, \dots N\}$, by B the union of the blocks of π that are contained in $[2N] \setminus I_N = \{N+1, \dots, 2N\} = I_{2N}$ and by C the rest of $[2N]$. In the following equations, s will vary in $\mathbb{N}^A$, t in $\mathbb{N}^{I_N \setminus A}$, u in $\mathbb{N}^B$ and v in $\mathbb{N}^{I_{2N} \setminus B}$. For such s,t,u and v and with the previous notation, $[s,t,u,v] \prec \pi$ if and only if $s \prec \pi|_A$, $[t,v] \prec \pi|_C$ and $u \prec \pi|_B$. For $k \in \mathbb{N}^{I_{2N}}$ (*i.e.* k is a function $k : I_{2N} \to \mathbb{N}$), we will also abusively denote $\widetilde{a}_k \stackrel{\text{def}}{=} \widetilde{a}_{(k(N+1),\dots,k(2N))}$. With this notation the definition in (2.13) becomes

$$\begin{aligned} S(a,\pi,N,1) &= \sum_{\substack{s \in \mathbb{N}^A, t \in \mathbb{N}^{I_N \setminus A}, u \in \mathbb{N}^B, v \in \mathbb{N}^{I_{2N} \setminus B} \\ [s,t,u,v] \prec \pi}} Tr(a_{[s,t]} \widetilde{a}^*_{[u,v]}) \\ &= \sum_{\substack{t, v \\ [t,v] \prec \pi|_C}} Tr\left(\big(\sum_{s \prec \pi|_A} a_{[s,t]} \big) \big(\sum_{u \prec \pi|_B} \widetilde{a}_{[u,v]} \big)^* \right). \end{aligned}$$

Thus

$$|S(a,\pi,N,1)| \leq \sum_{[t,v] \prec \pi|_C} \left| Tr\left(\big(\sum_{s \prec \pi|_A} a_{[s,t]} \big) \big(\sum_{u \prec \pi|_B} \widetilde{a}_{[u,v]} \big)^* \right) \right|.$$

Applying the Cauchy-Schwarz inequality for the trace, we get

$$|S(a,\pi,N,1)| \leq \sum_{[t,v] \prec \pi|_C} \left\| \sum_{s \prec \pi|_A} a_{[s,t]} \right\|_2 \left\| \sum_{u \prec \pi|_B} \widetilde{a}_{[u,v]} \right\|_2.$$

The classical Cauchy-Schwarz inequality yields

$$|S(a,\pi,N,1)| \leq (1)^{1/2} (2)^{1/2}$$

where

$$\begin{aligned} (1) &= \sum_{[t,v] \prec \pi|_C} \left\| \sum_{s \prec \pi|_A} a_{[s,t]} \right\|_2^2 \\ (2) &= \sum_{[t,v] \prec \pi|_C} \left\| \sum_{u \prec \pi|_B} \widetilde{a}_{[u,v]} \right\|_2^2. \end{aligned}$$

We claim that $(1) = S(a, P_N(\pi), N, 1)$ and $(2) = S(a, P_0(\pi), N, 1)$. We only prove the first equality, the second is proved similarly (or follows from the first). But

$$\begin{aligned}
(1) &= \sum_{[t,v]\prec\pi|_C} \Big\| \sum_{s\prec\pi|_A} a_{[s,t]} \Big\|_2^2 \\
&= \sum_{[t,v]\prec\pi|_C} Tr\left(\big(\sum_{s\prec\pi|_A} a_{[s,t]}\big)\cdot\big(\sum_{s\prec\pi|_A} a_{[s,t]}\big)^* \right) \\
&= Tr\left(\sum_{[t,v]\prec\pi|_C} \sum_{s\prec\pi|_A} \sum_{s'\prec\pi|_A} a_{[s,t]} a^*_{[s',t]} \right) \\
&= Tr\left(\sum_{[t,v]\prec\pi|_C} \sum_{s\prec\pi|_A} \sum_{s'\prec\pi|_A} a_{[s,t]} \widetilde{a}^*_{r([s',t])} \right),
\end{aligned}$$

where on the last line for any $k=(k_1,\dots,k_N)\in\mathbb{N}^{I_N}$, $r(k)\in\mathbb{N}^{I_N}$ is defined by $r(k)=(k_N,k_{N-1},\dots,k_1)$.

By definition of B, for any $j\in I_{2N}\setminus B$ there is $i\in I_N\setminus A$ such that $i\sim_\pi j$. Thus for any $t\in\mathbb{N}^{I_N\setminus A}$ there is exactly one or zero $v\in\mathbb{N}^{I_{[2N]}\setminus B}$ such that $[t,v]\prec\pi_C$, depending whether $t\prec\pi_{I_N\setminus A}$ or not.

The claim that $(1)=S(a,P_N(\pi),N,1)$ thus follows from the observation that for $k,l\in\mathbb{N}^N$, $(k,l)\prec P_N(\pi)$ if and only there are $s,s'\in\mathbb{N}^A$ and $t\in\mathbb{N}^{I_N\setminus A}$ such that $k=[s,t]$, $l=r([s',t])$ and $s\prec\pi|_A$, $s'\prec\pi|_A$ and $t\prec\pi\big|_{I_N\setminus A}$. □

We now have to observe that the quantities $S(a,\sigma_l^{(d,m)},d,m)$ for $l=0,\dots,d$ and $S(a,\widetilde{\sigma}_l^{(d,m)},d,m)$ for $l=0,\dots,d$ have simple expressions.

A (finitely supported) family of matrices $a=(a_k)_{k\in\mathbb{N}^d}$ can be made in various natural ways into a bigger matrix, for any decomposition of $\mathbb{N}^d\simeq\mathbb{N}^l\times\mathbb{N}^{d-l}$. If the a_k's are viewed as operators on a Hilbert space H ($H=\mathbb{C}^\alpha$ if the a_k's are in $M_\alpha(\mathbb{C})$), then let us denote by M_l the operator from $H\otimes\ell^2(\mathbb{N})^{\otimes d-l}$ to $H\otimes\ell^2(\mathbb{N})^{\otimes l}$ having the following block-matrix decomposition:

$$\left(a_{[s,t]}\right)_{s\in\mathbb{N}^{\{1,\dots,l\}},t\in\mathbb{N}^{\{l+1,\dots,d\}}}.$$

Note that since (a_k) has finite support, the above matrix has only finitely many nonzero entries, and hence corresponds to a finite rank operator. In particular, it belongs to $S_p\left(H\otimes\ell^2(\mathbb{N})^{\otimes d-l};H\otimes\ell^2(\mathbb{N})^{\otimes l}\right)$ for any $p\in(0,\infty]$.

Lemma 2.26. *Let d, m, $a=(a_k)_{k\in\mathbb{N}^d}$ and M_l as above, and σ_l and $\widetilde{\sigma}_l$ defined in Corollary 2.12. Then for $l\in\{0,1,\dots,d\}$:*

$$S(a,\sigma_l^{(d,m)},d,m)=\|M_l\|^{2m}_{S_{2m}\left(H\otimes\ell^2(\mathbb{N})^{\otimes d-l};H\otimes\ell^2(\mathbb{N})^{\otimes l}\right)}.$$

Moreover for $l\in\{1,\dots,d\}$

$$S(a,\widetilde{\sigma}_l^{(d,m)},d,m)\le\|M_l\|^{2m}_{S_{2m}\left(H\otimes\ell^2(\mathbb{N})^{\otimes d-l};H\otimes\ell^2(\mathbb{N})^{\otimes l}\right)}.$$

Remark 2.27. It is also true that

$$S(a,\widetilde{\sigma}_l^{(d,m)},d,m)\le\|M_{l-1}\|^{2m}_{S_{2m}\left(H\otimes\ell^2(\mathbb{N})^{\otimes d-l};H\otimes\ell^2(\mathbb{N})^{\otimes l}\right)},$$

but we will only use the inequality stated in the lemma. This inequality follows from the one stated by conjugating by the rotation $k\in[2dm]\mapsto k+d$.

Proof. We fix $l \in \{0, \ldots, d\}$. For any $s = (s_1, \ldots, s_l) \in \mathbb{N}^l$ we denote by $A_s = (a_{s,t})_{t \in \mathbb{N}^{d-l}}$ viewed as a row matrix. As an operator, A_s thus acts from $H \otimes \ell^2(\mathbb{N})^{\otimes d-l}$ to H. For $s, s' \in \mathbb{N}^l$, if $r(k_1, \ldots, k_d) = (k_d, \ldots, k_1)$

$$A_s A_{s'}^* = \sum_{t \in \mathbb{N}^{d-l}} a_{s,t} a_{s',t}^* = \sum_{t \in \mathbb{N}^{d-l}} a_{s,t} \widetilde{a}_{r(s',t)}^*.$$

Hence for $s^{(1)}, s^{(2)}, \ldots, s^{(m)} \in \mathbb{N}^l$, if $s^{(m+1)} = s^{(1)}$,

$$\prod_{i=1}^m A_{s^{(i)}} A_{s^{(i+1)}}^* = \sum_{t^{(1)}, \ldots, t^{(m)} \in \mathbb{N}^{d-l}} a_{s^{(1)},t^{(1)}} \widetilde{a}_{r(s^{(2)},t^{(1)})}^* a_{s^{(2)},t^{(2)}} \widetilde{a}_{r(s^{(3)},t^{(2)})}^* \ldots \widetilde{a}_{r(s^{(1)},t^{(m)})}^*.$$

But for $k \in \mathbb{N}^{[2dm]}$, $k \prec \sigma_l^{(d,m)}$ if and only if there exist $s^{(1)}, s^{(2)}, \ldots, s^{(m)} \in \mathbb{N}^l$ and $t^{(1)}, t^{(2)}, \ldots, t^{(m)} \in \mathbb{N}^{d-l}$ such that for all i, $(k_{2di+1}, k_{2di+2}, \ldots, k_{2di+d}) = (s^{(i)}, t^{(i)})$ and $(k_{2di+2d}, k_{2di+2d-1}, \ldots, k_{2di+d+1}) = (s^{(i+1)}, t^{(i)})$. Thus summing over $s^{(1)}, s^{(2)}, \ldots, s^{(m)} \in \mathbb{N}^l$ in the preceding equation leads to

$$\sum_{s^{(1)}, s^{(2)}, \ldots, s^{(m)} \in \mathbb{N}^l} \prod_{i=1}^m A_{s^{(i)}} A_{s^{(i+1)}}^* = \sum_{(k_1, \ldots, k_{2m}) \prec \sigma_l^{(d,m)}} a_{k_1} \widetilde{a}_{k_2}^* a_{k_3} \ldots a_{k_{2m-1}} \widetilde{a}_{k_{2m}}^*.$$

Taking the trace and using the trace property we get

$$\begin{aligned} S(a, \sigma_l^{(d,m)}, d, m) &= \sum_{s^{(1)}, s^{(2)}, \ldots, s^{(m)} \in \mathbb{N}^l} Tr\left(\prod_{i=1}^m A_{s^{(i)}}^* A_{s^{(i)}}\right) \\ &= Tr\left[\left(\sum_{s \in \mathbb{N}^l} A_s^* A_s\right)^m\right] \\ &= Tr\left[(M_l^* M_l)^m\right] \end{aligned}$$

where the last identity follows from the fact that $M_l = \sum A_s \otimes e_{s1}$. This concludes the proof for $\sigma_l^{(d,m)}$. For $\widetilde{\sigma}_l^{(d,m)}$ with $1 \le l \le d$, the same kind of computations yield to

$$S(a, \widetilde{\sigma}_l^{(d,m)}, d, m) = \sum_{s_l \in \mathbb{N}} Tr\left[\left(\sum_{s \in \mathbb{N}^{l-1}} A_{(s,s_l)}^* A_{(s,s_l)}\right)^m\right].$$

To conclude we only have to use Lemma 2.28 below. □

Lemma 2.28. *Let $X_1, X_2 \ldots X_N$ be matrices. Then for any integer $m \ge 1$*

$$\sum_{i=1}^N Tr((X_i^* X_i)^m) \le Tr((\sum_{i=1}^N X_i^* X_i)^m).$$

Proof. This is a general inequality for the non-commutative L^p-norms. Indeed, for any $\alpha, N \in \mathbb{N}$, and $p \in [2, \infty]$, the map

$$\begin{array}{rccc} T: & M_{N,1}(M_\alpha(\mathbb{C})) & \to & M_N(M_\alpha(\mathbb{C}) \\ & \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} & \mapsto & \begin{pmatrix} X_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & X_N \end{pmatrix} \end{array}$$

is a contraction for all p-norms. For $p = 2$, this is easy because T is an isometry. For $p = \infty$ this is also obvious. For a general $p \in (2, \infty)$ the claim follows by interpolation.

Applied for $p = 2m$, this concludes the proof since for an integer m,

$$\left\| \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} \right\|_{2m}^{2m} = Tr((\sum_{i=1}^{N} X_i^* X_i)^m)$$

and

$$\left\| \begin{pmatrix} X_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & X_N \end{pmatrix} \right\|_{2m}^{2m} = \sum_{i=1}^{N} Tr((X_i^* X_i)^m).$$

□

We are now able to state and prove the main result of this section. Recall that for a partition π of $NC^*(1, m)$, $B(\pi)$ was defined in part 2.1.3 as the number of blocks of the partition $\Phi(\pi)$ (the map Φ was defined after Corollary 2.12).

Corollary 2.29. *Let $\pi \in NC^*(d, m)$. Then if a and M_l are as in Lemma 2.26,*

$$|S(a, \pi, d, m)| \leq \prod_{l=0}^{d} \|M_l\|_{S_{2m}\left(H \otimes \ell^2(\mathbb{N})^{\otimes d-l}; H \otimes \ell^2(\mathbb{N})^{\otimes l}\right)}^{2m\mu_l}$$

where $\mu_l = \left(B(\pi\big|_{A_{l+1}}) - B(\pi|_{A_l})\right)/(m-1)$ where we take the convention that $B(\pi|_{A_0}) = 1$ and $B(\pi\big|_{A_{d+1}}) = m$.

Proof. The idea is, as in Lemma 2 and Corollary 3 of [9], to iterate the inequality of Lemma 2.25, except that here the combinatorial invariants (Lemma 2.19) of the map $\pi \mapsto (P_{kd}(\pi), P_{kd+md}(\pi))$ allow us to precisely determine the exponents of each $\|M_l\|_{2m}$. In the rest of the proof since no confusion is possible, we will simply denote $\sigma_l = \sigma_l^{(d,m)}$ and $\widetilde{\sigma}_l = \widetilde{\sigma}_l^{(d,m)}$, and S will denote the set $\{\sigma_l, 0 \leq l \leq d\} \cup \{\widetilde{\sigma}_l, 0 \leq l \leq d\}$. Fix $\pi \in NC^*(d, m)$.

Maybe the clearest way to write out a proof is using the basic vocabulary of probability theory (for a reference see for example [18]). Let us consider the (homogeneous) Markov chain $(\pi_n)_{n \geq 0}$ on (the finite state space) $NC^*(d, m)$ given by $\pi_0 = \pi$ and $\pi_{n+1} = P_{id}(\pi_n)$ where i is uniformly distributed in $[2m]$ and independent from $(\pi_k)_{0 \leq k \leq n}$ (note that $\pi_{n+1} \in NC^*(d, m)$ if $\pi_n \in NC^*(d, m)$ by Lemma 2.11). Corollary 2.12 implies that the sequence $(\pi_n)_n$ is almost surely eventually equal to one of the σ_l or $\widetilde{\sigma}_l$. Its second statement indeed expresses that if $\pi_n \in S$ then $\pi_N = \pi_n$ for all $N \geq n$; it suffices therefore to prove that $p_n \stackrel{\text{def}}{=} \mathbb{P}(\pi_n \notin S) \to 0$ as $n \to \infty$. But if k is fixed with $2^{k-2} \geq m$, its first statement implies that $p_k \leq 1 - (1/2m)^k = c < 1$ for any starting state π_0. From the equality $p_{n+k} = p_n \mathbb{P}(\pi_{n+k} \notin S \big| \pi_n \notin S)$ and the Markov property we get that $p_{n+k} \leq c p_n$ for any integer $n \in \mathbb{N}$, from which we deduce that $p_n \leq c^{\lfloor n/k \rfloor} \to 0$ as $n \to \infty$.

Let us denote $\lambda_l(\pi) = \mathbb{P}(\lim_n \pi_n = \sigma_l)$ and $\widetilde{\lambda}_l(\pi) = \mathbb{P}(\lim_n \pi_n = \widetilde{\sigma}_l)$ for $0 \leq l \leq d$ (take $\widetilde{\lambda}_0(\pi) = 0$); note that $\sum_l \lambda_l(\pi) + \widetilde{\lambda}_l(\pi) = 1$.

Lemma 2.19 and the last statement of Lemma 2.11 show that for any $i \in \{1, \ldots, d\}$ the sequence $B(\pi_n|_{A_i})$ is a martingale. In particular since $\pi_0 = \pi$, $B(\pi|_{A_i}) = \mathbb{E}\left[B(\pi_n|_{A_i})\right]$

for any $n \geq 0$. Letting $n \to \infty$ we get

$$\begin{aligned} B(\pi|_{A_i}) &= \sum_{l=0}^{d} \lambda_l(\pi) B(\sigma_l|_{A_i}) + \sum_{l=1}^{d} \widetilde{\lambda}_l(\pi) B(\widetilde{\sigma}_l|_{A_i}) \\ &= \sum_{l=0}^{d} \left(\lambda_l(\pi) + \widetilde{\lambda}_l(\pi)\right)(1 + (m-1)1_{l<i}) \\ &= 1 + (m-1) \sum_{0 \leq l < i} \lambda_l(\pi) + \widetilde{\lambda}_l(\pi). \end{aligned}$$

We used the fact that $B(\sigma_l|_{A_i}) = B(\widetilde{\sigma}_l|_{A_i}) = 1 + (m-1)1_{l<i}$. This follows from the observations that since $\Phi(c_m) = \Phi(1_{2m}) = 1_m$, $B(c_m) = |1_m| = 1$ and that since $\Phi(r_m) = 0_m$, $B(r_m) = m$. Subtracting the equalities above for i and $i+1$ gives

$$(\lambda_i(\pi) + \widetilde{\lambda}_i(\pi))(m-1) = B(\pi\big|_{A_{i+1}}) - B(\pi|_{A_i}) \tag{2.16}$$

with the convention that $B(\pi|_{A_0}) = 1$ and $B(\pi\big|_{A_{d+1}}) = m$.

On the other hand Lemma 2.25 implies that the sequence $M_n = \log|S(a, \pi_n, d, m)|$ is a submartingale. As above letting $n \to \infty$ in the inequality $M_0 \leq \mathbb{E}[M_n]$ yields

$$\log|S(a, \pi, d, m)| \leq \sum_{l=0}^{d} \lambda_l(\pi) \log|S(a, \sigma_l, d, m)| + \sum_{l=1}^{d} \widetilde{\lambda}_l(\pi) \log|S(a, \widetilde{\sigma}_l, d, m)|.$$

If we denote simply by $\|M_l\|_{2m}$ the quantity $\|M_l\|_{S_{2m}\left(H\otimes\ell^2(\mathbb{N})^{\otimes d-l}; H\otimes\ell^2(\mathbb{N})^{\otimes l}\right)}$, then by Lemma 2.26 this inequality becomes

$$|S(a, \pi, d, m)| \leq \prod_{l=0}^{d} \|M_l\|_{2m}^{2m(\lambda_l(\pi)+\widetilde{\lambda}_l(\pi))}.$$

This inequality, combined with (2.16), concludes the proof.

□

2.3 Main result

We are now able to prove the main results of this paper. We first treat the "holomorphic" setting (Theorems 2.3 and 2.4) for which the results we get are completely satisfactory.

2.3.1 Holomorphic setting

It is a generalization to operator coefficients of the main result of [27]. When the coefficients a_k are taken to be scalars, the techniques of our Theorem 2.4 give a new proof and an improvement of the theorem 1.3 of [27]. In [27], Kemp and Speicher introduce free Poisson variables to get an upper bound, whereas our proof is more combinatorial and lies is the study of $NC^*(d, m)$ that is done is part 2.1.2. We refer to [32] or to the paper [27] for definitions and facts on free cumulants and $\mathscr{R}$-diagonal operators. We just recall that the $*$-distribution of a variable c in a C^*-probability space is characterized by its free cumulants, which are the family of complex numbers $\kappa_n[c^{\varepsilon_1}, \dots, c^{\varepsilon_n}]$, for $n \in \mathbb{N}$ and $\varepsilon_i \in \{1, *\}$. Moreover the $\mathcal{R}$-diagonal operators are exactly the operators c for which the cumulants $\kappa_n[c^{\varepsilon_1}, \dots, c^{\varepsilon_n}]$ vanish except if n is even and if 1's and $*$'s alternate in the sequence $\varepsilon_1, \dots, \varepsilon_n$. Since the family $\lambda(g_1), \dots, \lambda(g_r)$ (where $g_1, \dots, g_r$ are the generators of the free group F_r) form an example of $*$-free $\mathcal{R}$-diagonal operators, Theorem 2.3 is a particular case of Theorem 2.4, that is why do not include a proof.

Proof of Theorem 2.4. The start of the proof is the same as in the proof of Theorem 1.3 of [27], and was sketched in the Introduction. Fix $p = 2m \in 2\mathbb{N}$.

As in (2.14), if $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ denote by $\widetilde{a}_k = a_{(k_d,\dots,k_1)}$ and $\widetilde{c}_k = c_{(k_d,\dots,k_1)} = c_{k_d} \dots c_{k_1}$. First develop the norms:

$$\begin{aligned}\left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} &= \sum_{k_1,\dots,k_{2m} \in \mathbb{N}^d} Tr(a_{k_1} a_{k_2}^* \dots a_{k_{2m}}^*) \tau(c_{k_1} c_{k_2}^* \dots c_{k_{2m}}^*) \\ &= \sum_{k_1,\dots,k_{2m} \in \mathbb{N}^d} Tr(a_{k_1} \widetilde{a}_{k_2}^* \dots \widetilde{a}_{k_{2m}}^*) \tau(c_{k_1} \widetilde{c}_{k_2}^* \dots \widetilde{c}_{k_{2m}}^*).\end{aligned}$$

Take $k_1, \dots, k_{2m} \in \mathbb{N}^d$; if $k_l = (k_l(1), k_l(2), \dots, k_l(d))$ then

$$c_{k_1} \widetilde{c}_{k_2}^* \dots \widetilde{c}_{k_{2m}}^* = c_{k_1(1)} c_{k_1(2)} \dots c_{k_1(d)} c_{k_2(1)}^* \dots c_{k_2(d)}^* \dots c_{k_{2m}(d)}^*$$

and by the fundamental property of cumulants:

$$\tau(c_{k_1} \widetilde{c}_{k_2}^* \dots \widetilde{c}_{k_{2m}}^*) = \sum_{\pi \in NC(2dm)} \kappa_\pi[c_{k_1(1)}, \dots, c_{k_1(d)}, c_{k_2(1)}^*, \dots, c_{k_2(d)}^*, \dots, c_{k_{2m}(d)}^*].$$

Denote $k = (k_1, \dots, k_{2m}) \in (\mathbb{N}^d)^{2m} \simeq \mathbb{N}^{2dm}$. Since freeness is characterized by the vanishing of mixed cumulants (Theorem 11.16 in [32]), $\kappa_\pi[c_{k_1(1)}, \dots, c_{k_{2m}(d)}^*]$ is non-zero only if $k \prec \pi$, and in this case we claim that it is equal to $\kappa_\pi[c_{d,m}]$ where

$$c_{d,m} = \overbrace{\underbrace{c, \dots, c}_{d}, \underbrace{c^*, \dots, c^*}_{d}, \dots, \underbrace{c, \dots, c}_{d}, \underbrace{c^*, \dots, c^*}_{d}}^{2m \text{ groups}}. \tag{2.17}$$

Relabel the sequence $k_1(1), \dots, k_{2m}(d)$ by $k_1, \dots, k_{2dm}$, and denote also by $\varepsilon_1, \dots, \varepsilon_{2dm}$ the corresponding sequence of 1's and $*$'s, in such a way that $\kappa_\pi[c_{k_1(1)}, \dots, c_{k_{2m}(d)}^*] = \kappa_\pi[\big(c_{k_i}^{\varepsilon_i}\big)_{1 \le i \le 2dm}]$ and $\kappa_\pi[c_{d,m}] = \kappa_\pi[\big(c^{\varepsilon_i}\big)_{1 \le i \le 2dm}]$. By the definition of κ_π, we have

$$\kappa_\pi[\big(c_{k_i}^{\varepsilon_i}\big)_{1 \le i \le 2dm}] = \prod_{V \in \pi} \kappa_{|V|}[(c_{k_i}^{\varepsilon_i})_{i \in V}]$$

where the products runs over by the blocks of π. Similarly

$$\kappa_\pi[c_{d,m}] = \prod_{V \in \pi} \kappa_{|V|}[(c^{\varepsilon_i})_{i \in V}].$$

Our claim thus follows from the observation that if $k \prec \pi$ then for any block V of π there is an index s such that $k_i = s$ for all $i \in V$, and the equality $\kappa_{|V|}[(c_s^{\varepsilon_i})_{i \in V}] = \kappa_{|V|}[(c^{\varepsilon_i})_{i \in V}]$ expresses just the fact that c and c_s have the same $*$-distribution and therefore the same cumulants.

The next claim is that since c is $\mathscr{R}$-diagonal, $\kappa_\pi[c_{d,m}]$ is non-zero only if $\pi \in NC^*(d,m)$. Since with the previous notation $\kappa_\pi[c_{d,m}] = \prod_{V \in \pi} \kappa_{|V|}[(c^{\varepsilon_i})_{i \in V}]$, this amounts to showing that if there is a block V of π which is not of even cardinality or for which 1's and $*$'s do not alternate in the sequence $(\varepsilon_i)_{i\in V}$, then $\kappa_{|V|}[(c^{\varepsilon_i})_{i\in V}] = 0$. But this is exactly the definition of $\mathcal{R}$-diagonal operators. Thus we get

$$\left\| \sum_{k\in\mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi\in NC^*(d,m)} \kappa_\pi[c_{d,m}] \sum_{(k_1,\dots,k_{2m})\prec\pi} Tr(a_{k_1}\widetilde{a}^*_{k_2}\dots\widetilde{a}^*_{k_{2m}}),$$

or with the notation introduced in (2.14)

$$\left\| \sum_{k\in\mathbb{N}^d} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi\in NC^*(d,m)} \kappa_\pi[c_{d,m}] S(a,\pi,d,m). \tag{2.18}$$

Up to this point we have mainly reproduced the beginning of the proof of Theorem 1.3 of [27] (the authors of [27] only deal with scalar a_k's but there is no other difference).

We can now use the study of $NC^*(d,m)$ that we did in part 2.1.2. Recall in particular that there is a map $\mathcal{P} : NC^*(d,m) \to NC(m)^{(d)}$ the properties of which are summarized in Theorem 2.13.

Take $(\sigma_1,\dots,\sigma_d) \in NC(m)^{(d)}$ and denote $\mu_l = (|\sigma_{l+1}| - |\sigma_l|)/(m-1)$ where $|\sigma|$ denotes the number of blocks of σ with the convention $|\sigma_0| = 1$ and $|\sigma_{d+1}| = m$. If $\pi \in NC^*(d,m)$ and $\mathcal{P}(\pi) = (\sigma_1,\dots,\sigma_d)$ then by Corollary 2.29, $|S(a,\pi,d,m)| \leq \prod_{l=0}^d \|M_l\|_{2m}^{2m\mu_l}$.

Thus by the first part of Theorem 2.13, we have that

$$\left| \sum_{\pi\in NC^*(d,m), \mathcal{P}(\pi)=(\sigma_1,\dots,\sigma_d)} \kappa_\pi[c_{d,m}] S(a,\pi,d,m) \right| \leq 4^{2m} \prod_{l=0}^d \|M_l\|_{2m}^{2m\mu_l} \max_{\mathcal{P}(\pi)=(\sigma_1,\dots,\sigma_d)} |\kappa_\pi[c_{d,m}]|.$$

But by the second statement of Theorem 2.13 and Lemma 2.30 below (recall that for $\tau(c) = \kappa_1[c] = 0$ since c is $\mathcal{R}$-diagonal)

$$|\kappa_\pi[c_{d,m}]| \leq \|c\|_2^{2dm} \left(\frac{16\|c\|_{2m}}{\|c\|_2} \right)^{4m},$$

which implies

$$\left| \sum_{\pi\in NC^*(d,m), \mathcal{P}(\pi)=(\sigma_1,\dots,\sigma_d)} \kappa_\pi[c_{d,m}] S(a,\pi,d,m) \right| \leq 4^{10m} \prod_{l=0}^d \|M_l\|_{2m}^{2m\mu_l} \|c\|_2^{2dm} \left(\frac{\|c\|_{2m}}{\|c\|_2} \right)^{4m}. \tag{2.19}$$

But by Theorem 3.2 in [11], for any non-negative integers $s_0,\dots,s_d$ such that $\sum_i s_i = m-1$, the number of $(\sigma_1,\dots,\sigma_d) \in NC(m)^{(d)}$ such that $|\sigma_{l+1}| - |\sigma_l| = s_l$ for any $0 \leq l \leq d$ (with the conventions $|\sigma_0| = 1$ and $|\sigma_{d+1}| = m$) is equal to $(1/m)\binom{m}{s_0}\binom{m}{s_1}\dots\binom{m}{s_d}$. Thus from (2.18) we deduce

$$\left\|\sum_{k\in\mathbb{N}^d} a_k\otimes c_k\right\|_{2m}^{2m} \le 4^{10m}\|c\|_2^{2dm}\left(\frac{\|c\|_{2m}}{\|c\|_2}\right)^{4m}$$
$$\sum_{s_0+\cdots+s_d=m-1}(1/m)\binom{m}{s_0}\binom{m}{s_1}\dots\binom{m}{s_d}\prod_{l=0}^{d}\|M_l\|_{2m}^{2ms_l/(m-1)}.$$

Denote for simplicity $\gamma_l=\|M_l\|_{2m}^{2m/(m-1)}$. Since the number of $s_0,\dots,s_d\in\mathbb{N}$ such that $s_0+\cdots+s_d=m-1$ is equal to $\binom{m+d-1}{d}$, this inequality becomes

$$\left\|\sum_{k\in\mathbb{N}^d} a_k\otimes c_k\right\|_{2m}^{2m} \le 4^{10m}\|c\|_2^{2dm}\left(\frac{\|c\|_{2m}}{\|c\|_2}\right)^{4m}\binom{m+d-1}{d}$$
$$\sup_{s_0+\cdots+s_d=m-1}(1/m)\binom{m}{s_0}\binom{m}{s_1}\dots\binom{m}{s_d}\prod_{l=0}^{d}\gamma_l^{s_l}.$$

Now use the fact that for any integers N and n, $\binom{N}{n}\le (N/n)^n(N/(N-n))^{N-n}$ with the convention $(N/0)^0=1$. For a fixed N, this can be proved by induction on $n\le N/2$ using the fact that $x\in\mathbb{R}^+\mapsto x\log(1+1/x)$ is increasing. Thus

$$\binom{m+d-1}{d}\le\binom{m+d}{d}\le\left(1+\frac{m}{d}\right)^d\left(1+\frac{d}{m}\right)^m.$$

But since log is concave, if $s_0+\cdots+s_d=m-1$,

$$\begin{aligned}\prod_{l=0}^{d}\left(\frac{m}{m-s_l}\right)^{m-s_l} &= \exp\left((md+1)\sum_0^d\frac{m-s_l}{md+1}\log\big(m/(m-s_l)\big)\right)\\ &\le \exp\left((md+1)\log\big(\sum_0^d m/(md+1)\big)\right)\\ &= \exp\big((md+1)\log\big(1+(m-1)/(md+1)\big)\big)\le\exp(m)\end{aligned}$$

and

$$\begin{aligned}\prod_{l=0}^{d}\left(\frac{m\gamma_l}{s_l}\right)^{s_l} &= \exp\left((m-1)\sum_0^d\frac{s_l}{m-1}\log\big(m\gamma_l/s_l\big)\right)\\ &\le \exp\left((m-1)\log\big(m/(m-1)\sum_0^l\gamma_l\big)\right)\\ &= (\gamma_0+\dots\gamma_l)^{m-1}\left(\frac{m}{m-1}\right)^{m-1}\end{aligned}$$

But $(m/(m-1))^{m-1}\le m$ for any $m\ge 1$. This leads to

$$\left\|\sum_{k\in\mathbb{N}^d} a_k\otimes c_k\right\|_{2m}^{2m} \le 4^{10m}\|c\|_2^{2dm}\left(\frac{\|c\|_{2m}}{\|c\|_2}\right)^{4m}$$
$$\left(1+\frac{m}{d}\right)^d\left(1+\frac{d}{m}\right)^m\exp(m)\,(\gamma_0+\dots\gamma_l)^{m-1}. \quad (2.20)$$

Noting that since $2m/(m-1) \geq 2$,

$$(\gamma_0 + \dots \gamma_l)^{m-1} = \|(\|M_l\|_{2m})_l\|^{2m}_{\ell^{2m/(m-1)}(\{0,\dots,d\})} \leq \|(\|M_l\|_{2m})_l\|^{2m}_{\ell^2(\{0,\dots,d\})}$$

and taking the $2m$-th root in (2.20) one finally gets

$$\left\| \sum_{k \in \mathbb{N}^d} a_k \otimes c_k \right\|_{2m} \leq 4^5 \sqrt{e(1+d/m)} \left(1 + \frac{m}{d}\right)^{d/2m} \|c\|_2^d \left(\frac{\|c\|_{2m}}{\|c\|_2}\right)^2 \|(\|M_l\|_{2m})_l\|_{\ell^2(\{0,\dots,d\})} .$$

To conclude for the case $m < \infty$, just note that $\left(1 + \frac{m}{d}\right)^{d/m} \leq e$.

Letting $m \to \infty$ and noting that $\left(1 + \frac{m}{d}\right)^{d/m} \to 1$ concludes the proof for the operator norm.

When the c_k's are circular, since $\kappa_\pi[c_{d,m}] = 1$ if $\pi \in NC_2^*(d,m)$ and $\kappa_\pi[c_{d,m}] = 0$ otherwise, we can replace (2.19) by

$$\left| \sum_{\pi \in NC^*(d,m), \mathcal{P}(\pi) = (\sigma_1, \dots, \sigma_d)} \kappa_\pi[c_{d,m}] S(a, \pi, d, m) \right| \leq \prod_{l=0}^{d} \|M_l\|_{2m}^{2m\mu_l} .$$

Following the rest of the arguments we get the claimed results. □

We still have to prove this Lemma that was used in the above proof.

Lemma 2.30. *Let $\pi \in NC(n)$ a non-crossing partition that has at least K blocks of size 2 and in which all blocks have a size at most N.*

Let $c_1, \dots, c_n$ be elements of a tracial C^*-probability space $(\mathcal{A}, \tau)$ that are centered: $\tau(c_k) = 0$ for all k. Let $m_p = \max_k \|c_k\|_p$ for $p = 2, N$. Then

$$|\kappa_\pi[c_1, \dots, c_n]| \leq m_2^{2K} (16 m_N)^{n-2K} . \tag{2.21}$$

Proof. Since both $\pi \mapsto \kappa_\pi$ and the right-hand side of (2.21) are multiplicative, we only have to prove (2.21) when $\pi = 1_n$ with $n \leq N$. Then as usual κ_π is denoted by κ_n. If $n = 1$ it is obvious since $\kappa_1(c_1) = \phi(c_1) = 0$.

If $n = 2$, then $K = 1$ and $\kappa_2(c_k, c_l) = \tau(c_k c_l) - \tau(c_k)\tau(c_l) = \tau(c_k c_l)$. By the Cauchy-Schwarz inequality we get $|\kappa_2(c_k, c_l)| \leq m_2^2$.

We now focus on the case $n > 2$, and then $K = 0$. This is essentially done in the proof of Lemma 4.3 in [27] but we have to replace the inequality $|\tau(c_{k_1} \dots c_{k_l})| \leq m_\infty^l$ by Hölder's inequality $|\tau(c_{k_1} \dots c_{k_l})| \leq m_N^l$ for any $l \leq n \leq N$. Following the proof of Lemma 4.3 in [27], we thus get that

$$\kappa_n[c_1, \dots, c_n] \leq 4^{n-1} \sum_{\sigma \in NC(n)} {m_n}^n \leq 4^{2n} {m_N}^n.$$

□

2.3.2 Non-holomorphic setting

Here we consider Theorems 2.5 and 2.6. We only sketch their proofs. The idea is the same as in the holomorphic setting, except that here the relevant subset of non-crossing partitions is the set $NC(d,m)$ introduced and studied in part 2.1.4.

Sketch of proof of Theorem 2.6. We will use that if c has a symmetric distribution, then c has vanishing odd cumulants. This means that $\kappa_\pi[c,\dots,c]=0$ unless π has only blocks of even cardinality. To check this, by the multiplicativity of free cumulants, we have to prove that $\kappa_n[c,\dots,c]=\kappa_{1_n}[c,\dots,c]=0$ if n is odd. But this is clear: since $-c$ and c have the same distribution, $\kappa_n[c,\dots,c]=\kappa_n[-c,\dots,-c]$. On the other hand since κ_n is n-linear, $\kappa_n[-c,\dots,-c]=(-1)^n\kappa_n[c,\dots,c]$.

Take $(c_k)_{k\in\mathbb{N}}$ and $(a_k)_{k\in\mathbb{N}^d}$ as in Theorem 2.6 and define $\widetilde{a}_k$ and $c_{k_1,\dots,k_d}$ as in the proof of Theorem 2.4. Assume for simplicity that c_k is normalized by $\|c_k\|_2=1$. Denote by I the set of $k=(k_1,\dots,k_d)\in\mathbb{N}^d$ such that for any $1\leq i<d$ $k_i\neq k_{i+1}$. Then for $p=2m$ we have that

$$\left\|\sum_{k\in I} a_k\otimes c_k\right\|_{2m}^{2m}=\sum_{k_1,\dots,k_{2m}\in I} Tr(a_{k_1}\widetilde{a}_{k_2}^*\dots\widetilde{a}_{k_{2m}}^*)\tau(c_{k_1}c_{k_2}\dots c_{k_{2m}}).$$

Expanding the moment $\tau(c_{k_1}\dots c_{k_{2m}})$ using cumulants we get

$$\tau(c_{k_1}c_{k_2}\dots c_{k_{2m}})=\sum_{\pi\in NC(2dm)}\kappa_\pi[c_{k_1(1)},\dots,c_{k_1(d)},c_{k_2(1)},\dots,c_{k_2(d)},\dots,c_{k_{2m}(d)}].$$

By freeness of the family $(c_k)_{k\in\mathbb{N}}$, by the assumption on the vanishing of odd moments and by Lemma 2.21 such a cumulant is equal to 0 except if $\pi\in NC(d,m)$ and $(k_1,\dots,k_{2m})\prec\pi$, in which case it is equal to $\kappa_\pi[c,c\dots,c]$. We get

$$\left\|\sum_{k\in I} a_k\otimes c_k\right\|_{2m}^{2m}=\sum_{\pi\in NC(d,m)}\kappa_\pi[c,\dots,c]S(a,\pi,d,m).$$

But by Lemma 2.24, Lemma 2.26 and an iteration of Lemma 2.25 we get that for any $\pi\in NC(d,m)$

$$S(a,\pi,d,m)\leq\max_{0\leq l\leq d}\|M_l\|_{2m}^{2m}.$$

On the other hand (remembering that $\|c\|_2=1$), Theorem 2.22 and Lemma 2.30 imply that for $\pi\in NC(d,m)$,

$$|\kappa_\pi[c,\dots,c]|\leq\left(16\|c\|_{2m}\right)^{4m}.$$

This yields

$$\left\|\sum_{k\in I} a_k\otimes c_k\right\|_{2m}^{2m}\leq\sum_{\pi\in NC(d,m)}\left(16\|c\|_{2m}\right)^{4m}\max_{0\leq l\leq d}\|M_l\|_{2m}^{2m}.$$

But by Theorem 2.22 $NC(d,m)$ has cardinality less than $4^{2m}(d+1)^{2m}$. Taking the $2m$-th root in the preceding equation we thus get

$$\left\|\sum_{k\in I} a_k\otimes c_k\right\|_{2m}\leq 4^5(d+1)\|c\|_{2m}^2\max_{0\leq l\leq d}\|M_l\|_{2m}.$$

This proves Theorem 2.6 for the case when $p\in 2\mathbb{N}$. For $p=\infty$ just make $p\to\infty$. □

For Theorem 2.5 the proof is the same except that we have to be slightly more careful in the beginning. Recall that I_d is the set of $(k_1, \varepsilon_1, \dots, k_d, \varepsilon_d) \in (\mathbb{N} \times \{1, *\})^d$ such that $\lambda(g_{k_1})^{\varepsilon_1} \dots \lambda(g_{k_d})^{\varepsilon_d}$ corresponds to an element of length d in the free group F_∞. For a family of matrices $(a_{k,\varepsilon})_{(k,\varepsilon)\in I_d}$ denote by

$$\breve{a}_{k,\varepsilon} = a_{(k_d,\dots,k_1),(\overline{\varepsilon}_d,\dots\overline{\varepsilon}_1)}$$

where $\overline{*} = 1$ and $\overline{1} = *$. The motivation for this notation is the following: for $(k, \varepsilon) \in I_d$ denote by $c_{k,\varepsilon} = c_{k_1}^{\varepsilon_1} \dots c_{k_d}^{\varepsilon_d}$, so that if $\breve{c}_{k,\varepsilon}$ is defined as $\breve{a}_{k,\varepsilon}$, we have that $\breve{c}_{k,\varepsilon}^* = c_{k,\varepsilon}$.

For $k = (k_1, \dots, k_{2m}) \in (\mathbb{N}^d)^{2m}$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{2m}) \in (\{1, *\})^{2m}$ and $\pi \in NC(2dm)$ with blocks of even cardinality we will also write $(k, \varepsilon) \prec \pi$ if $k_i = k_j$ for all $i \sim_\pi j$ and if in addition for each block $\{i_1 < \dots < i_{2p}\}$ of π, 1's and $*$'s alternate in the sequence $\varepsilon_{i_1}, \varepsilon_{i_2}, \dots, \varepsilon_{i_{2p}}$.

Last we denote, for $\pi \in NC(d, m)$

$$\widetilde{S}(a, \pi, d, m) = \sum_{(k,\varepsilon)\prec\pi} Tr(a_{k_1,\varepsilon_1} \breve{a}_{k_2,\varepsilon_2}^* a_{k_3,\varepsilon_3} \dots \breve{a}_{k_{2m},\varepsilon_{2m}}^*).$$

The proofs of Lemma 2.25 and Lemma 2.26 still apply with this notation:

Lemma 2.31. *Let $\pi \in NC(d, m)$, and take a finitely supported family of matrices $a = (a_{k,\varepsilon})_{(k,\varepsilon)\in I_d}$ as above. For any integer i*

$$\left|\widetilde{S}(a, \pi, d, m)\right| \le \left(\widetilde{S}(a, P_{di}(\pi), d, m)\right)^{1/2} \left(\widetilde{S}(a, P_{(m+i)d}(\pi), d, m)\right)^{1/2}.$$

Lemma 2.32. *Let d, m, $a = (a_{k,\varepsilon})_{(k,\varepsilon)\in I_d}$ and M_l be as in Theorem 2.5, and σ_l and $\widetilde{\sigma}_l$ as defined in Corollary 2.12. Then for $l \in \{0, 1, \dots, d\}$:*

$$\widetilde{S}(a, \sigma_l^{(d,m)}, d, m) = \|M_l\|_{S_{2m}\left(H\otimes\ell^2(\mathbb{N})^{\otimes d-l}; H\otimes\ell^2(\mathbb{N})^{\otimes l}\right)}^{2m}.$$

Moreover for $l \in \{1, \dots, d\}$

$$\widetilde{S}(a, \widetilde{\sigma}_l^{(d,m)}, d, m) \le \|M_l\|_{S_{2m}\left(H\otimes\ell^2(\mathbb{N})^{\otimes d-l}; H\otimes\ell^2(\mathbb{N})^{\otimes l}\right)}^{2m}.$$

We leave the proofs to the reader.

Sketch of the proof of Theorem 2.5. Use the same notation as above. Take $m \in \mathbb{N}$. Then as for the self-adjoint case we expand the $2m$-norm as follows:

$$\left\| \sum_{(k,\varepsilon)\in I_d} a_{k,\varepsilon} \otimes c_{k,\varepsilon} \right\|_{2m}^{2m} = \sum_{(k_1,\varepsilon_1),\dots,(k_{2m},\varepsilon_{2m})\in I_d} Tr(a_{k_1,\varepsilon_1} \breve{a}_{k_2,\varepsilon_2}^* \dots \breve{a}_{k_{2m},\varepsilon_{2m}}^*) \tau(c_{k_1,\varepsilon_1} c_{k_2,\varepsilon_2} \dots c_{k_{2m},\varepsilon_{2m}}).$$

By the freeness, the definition of I_d, Lemma 2.21 and the fact that the c_k's are $\mathscr{R}$-diagonal, the expression of the moment $\tau(c_{k_1,\varepsilon_1} \dots c_{k_{2m},\varepsilon_{2m}})$ becomes simply

$$\tau(c_{k_1,\varepsilon_1} \dots c_{k_{2m},\varepsilon_{2m}}) = \sum_{\pi\in NC(d,m)} 1_{(k,\varepsilon)\prec\pi} \kappa_\pi[c_{k_1(1)}^{\varepsilon_1(1)}, \dots, c_{k_{2m}(d)}^{\varepsilon_{2m}(d)}].$$

Where if $(k,\varepsilon) \prec \pi$ and $\alpha_n(c) = \kappa_{2n}[c, c^*, c, c^*, \dots, c, c^*] = \kappa_{2n}[c^*, c, c^*, c, \dots, c^*, c]$ we have that

$$\kappa_\pi[c_{k_1(1)}^{\varepsilon_1(1)} \dots c_{k_{2m}(d)}^{\varepsilon_{2m}(d)}] = \prod_{V \text{ block of } \pi} \alpha_{|V|/2}(c).$$

In particular this quantity (which we will abusively denote by $\kappa_\pi(c)$) does not depend on (k,ε). We therefore get

$$\left\| \sum_{k\in I} a_k \otimes c_k \right\|_{2m}^{2m} = \sum_{\pi \in NC(d,m)} \kappa_\pi[c] \widetilde{S}(a,\pi,d,m).$$

From this point the proof of Theorem 2.6 applies except that we use Lemma 2.32 and an iteration of Lemma 2.31 instead of Lemma 2.26 and an iteration of Lemma 2.25. □

2.3.3 Lower bounds

Here we get some lower bounds on the norms we investigated before. For example the following minoration is classical:

Lemma 2.33. *Let $(c_k)_{k\in\mathbb{N}}$ be circular $*$-free elements with $\|c\|_1 = 1$. Then for any finitely supported family of matrices $(a_{k_1,\dots,k_d})_{k_1,\dots,k_d\in\mathbb{N}}$ the following inequality holds:*

$$\| \sum_{k_1,\dots,k_d \in \mathbb{N}} a_{k_1,\dots,k_d} \otimes c_{k_1} \dots c_{k_d} \| \geq \max_{0\leq l\leq d} \|M_l\|.$$

Proof. We use the following (classical) realization of free circular elements on a Fock space. Let $H = H_1 \oplus_2 H_2$ be a Hilbert space with an orthonormal basis given by $(e_k)_{k\in\mathbb{N}} \cup (f_k)_{k\in\mathbb{N}}$ ((e_k) is a basis of H_1 and (f_k) of H_2). Let $\mathcal{F}(H) = \mathbb{C}\Omega \oplus \oplus_{n\geq 1} H^{\otimes n}$ be the full Fock space constructed on H and for $k \in \mathbb{N}$ $s(k)$ (resp. $\widetilde{s}(k)$) the operator of creation by e_k (resp. f_k). Define finally $c_k = s_k + \widetilde{s}_k^*$. It is well-known that $(c_k)_{k\in\mathbb{N}}$ form of $*$-free family of circular variables for the state $\langle \cdot \Omega, \Omega\rangle$ which is tracial on the C^*-algebra generated by the c_k's.

Let K be the Hilbert space on which the a_k's act ($K = \mathbb{C}^\alpha$ if $a_k \in M_\alpha(\mathbb{C})$). Then if P_k denotes the orthogonal projection from $\mathcal{F}(H) \to H_2^{\otimes k}$, for $0 \leq l \leq d$ the operator $(\mathrm{id} \otimes P_l) \circ \sum_{k_1,\dots,k_d\in\mathbb{N}} a_{k_1,\dots,k_d} \otimes c_{k_1} \dots c_{k_d} \Big|_{K\otimes H_1^{\otimes d-l}}$ corresponds to M_l if it is viewed as an operator from $K \otimes H_1^{\otimes d-l} \simeq K \otimes \ell^2(\mathbb{N})^{\otimes d-l}$ to $K \otimes H_2^{\otimes l} \simeq K \otimes \ell^2(\mathbb{N})^{\otimes l}$ for the identification $H_1 \simeq \ell^2$ and $H_2 \simeq \ell^2$ with the orthonormal bases (e_k) and (f_k).

This proves the Lemma. □

We also prove the following Lemma which was stated in the introduction.

Lemma 2.34. *Let p be a prime number and define $a_{k_1,\dots,k_d} = \exp(2i\pi k_1 \dots k_d/p)$ for any $k_i \in \{1,\dots,p\}$.*

Then $\|(a_k)\|_2 = p^{d/2}$ and for any $1 \leq l \leq d-1$ the matrix M_l defined by $M_l = \left(a_{(k_1,\dots,k_l),(k_{l+1},\dots,k_d)}\right) \in M_{p^l,p^{d-l}}(\mathbb{C})$ satisfies $\|M_l\| \leq p^{d/2}\sqrt{(d-1)/p}$.

Proof. Since $\|M_l\|^2 = \|M_l M_l^*\|$ we compute the matrix $M_l M_l^* \in M_{p^l,p^l}(\mathbb{C})$.

For any $s = (s_1, \dots s_l)$ and $t = (t_1,\dots,t_l) \in \{1,\dots,p\}^l$ the s,t-th entry of $M_l M_l^*$ is equal to

$$\sum_{(k_{l+1},\dots,k_d)\in\{1,\dots,p\}^{d-l}} \exp\left(2i\pi(s_1 \dots s_l - t_1 \dots t_l) k_{l+1} \dots k_d/p\right).$$

If $s_1 \dots s_l = t_1 \dots t_l \mod p$ then this quantity is equal to p^{d-l} whereas otherwise, $\omega = \exp\left(2i\pi(s_1 \dots s_l - t_1 \dots t_l)/p\right)$ is a primitive p-th root of 1, and it is straightforward to check that for such an ω,

$$\begin{aligned}\sum_{(k_{l+1},\dots,k_d)\in\{1,\dots,p\}^{d-l}} \omega^{k_{l+1}\dots k_d} &= \sum_{k_{l+1},\dots,k_{d-1}} \sum_{k_d=1}^{p} \left(\omega^{k_{l+1}\dots k_{d-1}}\right)^{k_d} \\ &= \sum_{k_{l+1},\dots,k_{d-1}} p 1_{k_{l+1}\dots k_{d-1}=0 \mod p} \\ &= p(p^{d-l-1} - (p-1)^{d-l-1}).\end{aligned}$$

We therefore have that

$$M_l M_l^* = (p^{d-l} - p(p-1)^{d-l-1})\Big(1\Big)_{s,t\in[p]^l} + p(p-1)^{d-l-1}\Big(1_{s_1\dots s_l = t_1\dots t_l}\Big)_{s,t\in[p]^l}.$$

The norm of an $N \times N$ matrix with entries all equal to 1 is N.

Moreover if $[p]^l = \{(s_1,\dots,s_l)\}$ is decomposed depending on the value of $s_1 \dots s_l$ modulo p, the matrix $\Big(1_{s_1\dots s_l=t_1\dots t_l}\Big)_{s,t\in[p]^l}$ is a block-diagonal matrix with blocks having all entries equal to 1. Its norm is therefore equal to

$$\begin{aligned}&\max_{i\in[p]} \left|\left\{(s_1,\dots,s_l) \in [p]^l, s_1\dots s_l = i \mod p\right\}\right| \\ &\qquad = \left|\left\{(s_1,\dots,s_l) \in [p]^l, s_1\dots s_l = 0\right\}\right| = p^l - (p-1)^l.\end{aligned}$$

By the triangle inequality the norm of $M_l M_l^*$ is thus less than

$$\begin{aligned}&p^{l+1}(p^{d-l-1} - (p-1)^{d-l-1}) + p(p-1)^{d-l-1}(p^l - (p-1)^l) \\ &\qquad = p^d - p(p-1)^{d-1} \le (d-1)p^{d-1}\end{aligned}$$

□

Chapter 3

Operator space valued Hankel matrices

Introduction

In this chapter I study Hankel matrices in the vector-valued non-commutative L^p-space $S^p[E]$. The main result is a characterization of the norm of such matrices in terms of vector-valued Besov spaces $B_p^s(E)_+$ defined in the second section. The surprising fact is that these norms only depend on the Banach-space structure of E. The main result is the following.

If $\varphi = \sum_{n\in\mathbb{N}} a_n z^n$ is a formal series with a_n belonging to an operator space E, we denote $a_n = \widehat{\varphi}(n)$ ($\widehat{\varphi}(n)$ coïncides with the Fourier coefficient of φ when $\varphi \in L^1(\mathbb{T}; E)$), the Hankel matrix Γ_φ is defined by its matrix representation

$$\Gamma_\varphi = (\widehat{\varphi}(j+k))_{j,k\geq 0}.$$

Theorem 3.1. *Let $1 \leq p < \infty$. A Hankel matrix $(a_{j+k})_{j,k\geq 0}$ belongs to $S^p[E]$ if and only if the formal series $\sum_{n\geq 0} a_n z^n$ belongs to $B_p^{1/p}(E)_+$.*

More precisely there is a constant $C > 0$ such that for any operator space E and any formal series $\varphi = \sum_{n\geq 0} a_k z^k$

$$C^{-1} \|\varphi\|_{B_p^{1/p}(E)_+} \leq \|\Gamma_\varphi\|_{S_p[E]} \leq Cp \|\varphi\|_{B_p^{1/p}(E)_+}.$$

Remark 3.2. In the first version of this thesis the inequality that was proved was

$$C^{-1} \|\varphi\|_{B_p^{1/p}(E)_+} \leq \|\Gamma_\varphi\|_{S_p[E]} \leq Cp^2 \|\varphi\|_{B_p^{1/p}(E)_+}.$$

I am grateful to Quanhua Xu for pointing out to me the improvement in page 94 allowing to replace p^2 by p in the previous equation. I should also mention that after the final version of this thesis was written, I managed to slightly change the proof and get a term $\sqrt{p}$ instead of p, and this $\sqrt{p}$ is optimal. The reader is refered to a forthcoming paper on the subject for more on this.

As often for results on non-commutative L^p spaces this result is proved using the complex interpolation method. For $p = 1$ the above theorem can be proved directly. A first natural attempt to derive the Theorem for any p would be to get something for $p = \infty$. Bounded Hankel operators are well-known with Nehari's theorem and its operator valued version, which states that for $E \subset B(\ell^2)$ and $p = \infty$, Γ_φ belongs to $B(\ell^2) \otimes E$ if and only if there is a function $\psi \in L^\infty(\mathbb{T}; B(\ell^2))$ such that $\widehat{\psi}(k) = \widehat{\varphi}(k)$ for $k > 0$. But for non-injective operator spaces, this seems very complicated (at least to me) to relate this function ψ to properties of E. Another natural attempt would be to interpolate between $p = 2$ and $p = 1$ since often for $p = 2$ results are obvious. But I would like to point out that here the Theorem is non trivial for $p = 2$ as well. We are thus led to pass from a problem with only one parameter p to a problem with more (3 here) parameters to "get room" in order to be able to use the interpolation method. This is done with the so-called generalized Hankel matrices.

For real (or complex) numbers α, β the generalized Hankel matrix with symbol φ is defined by

$$\Gamma_\varphi^{\alpha,\beta} = \Big((1+j)^\alpha(1+k)^\beta\widehat{\varphi}(j+k)\Big)_{j,k\geq 0}.$$

Our main theorem characterizes, for an operator space E and a $1 \leq p \leq \infty$, the generalized Hankel matrices that belong to $S^p[E]$ under the conditions that $\alpha + 1/2p > 0, \beta + 1/2p > 0$.

Theorem 3.3. *Let $1 \leq p \leq \infty$ and $\alpha, \beta > -1/2p$. Then for a formal series $\varphi = \sum_{n\geq 0}\widehat{\varphi}(n)z^n$ with $\widehat{\varphi}(n) \in E$, $\Gamma_\varphi^{\alpha,\beta} \in S_p[E]$ if and only if $\varphi \in B_p^{1/p+\alpha+\beta}(E)_+$.*

More precisely, for all $M > 0$, there is a constant $C = C_M$ (depending only on M, not on p, E) such that for all such φ, all $1 \leq p \leq \infty$ and all $\alpha, \beta \in \mathbb{R}$ such that $-1/2p < \alpha, \beta < M$,

$$C^{-1}\|\varphi\|_{B_p^{1/p+\alpha+\beta}(E)_+} \leq \left\|\Gamma_\varphi^{\alpha,\beta}\right\|_{S_p[E]} \leq \frac{C}{\sqrt{(\alpha+1/2p)(\beta+1/2p)}}\|\varphi\|_{B_p^{1/p+\alpha+\beta}(E)_+}. \tag{3.1}$$

Note that surprisingly, this theorem shows that the condition $\Gamma_\varphi^{\alpha,\beta} \in S_p[E]$ only depends on the Banach space structure of E (whereas the Banach space structure of $S_p[E]$ depends on the operator space structure of E).

These results extend results of Peller in the scalar case or in the case when $E = S^p$ ([34],[38],[35], [40]). In the scalar case Peller's theorem indeed shows that the space of Hankel matrices in S^p is isomorphic to a Besov space $B_{p+}^{1/p}$. The case when $E = S^p$ shows that this isomorphism is in fact a complete isomorphism. The results stated above show that this isomorphism has the stronger property of being *regular* as well as its inverse in the sense of [41]. In this chapter I made the choice to use the vocabulary of regular operators, but one could easily avoid this notion.

These results should be considered as remarks on Peller's proof rather than new theorems, since the steps presented here are all close to one of Peller's proofs ([40], sections 8 and 9 of Chapter 6). There are still some adaptations to make since for example the result for $p = 2$ is non-trivial here whereas it is obvious in Peller's case. For completeness we still provide a detailed proof.

Peller's classical results also have an extension to the case $0 < p < 1$. Here there are some obstructions: we should first of all clarify the notion vector-valued non-commutative L^p spaces when for $p < 1$. But even then, since the proof given here really lies on the duality and interpolation, some new ideas would be needed.

This chapter is organized as follows: in the first section we recall definitions and facts on regular operators. In the second section we give definitions and classical results on Besov spaces of analytic functions $B_{p,q+}^s$ that will be used later. All results are proved. In the third and last section we prove the main result.

3.1 Background on regular operators

3.1.1 Commutative case

We start by recalling the definition of regular operators in the commutative setting.

Definition 3.4. A linear operator $u : \Lambda_1 \to \Lambda_2$ between Banach lattices is said to be regular if for any Banach space X, $u \otimes id_X : \Lambda_1(X) \to \Lambda_1(X)$ is bounded. Equivalently (taking for $X = \ell_n^\infty$), if there is a constant C such that for any n and $f_1, \dots, f_n \in \Lambda_1$,

$$\left\| \sup_k |u(f_k)| \right\|_{\Lambda_2} \leq C \left\| \sup_k |f_k| \right\|_{\Lambda_1}.$$

The smallest such C is denoted by $\|u\|_r$.

This theory applies in particular if Λ_1 and Λ_1 are (commutative) L^p spaces: when $p = 1$ or $p = \infty$ a map is regular if and only if it is bounded. Similarly, a map that is simultaneously bounded $L^1 \to L^1$ and $L^\infty \to L^\infty$ is regular on L^p. This is not far from being a characterization since it is known that the set of regular operators: $L^p \to L^p$ coincides with the interpolation space (for the second complex interpolation method) between $B(L^\infty, L^\infty)$ and $B(L^1, L^1)$.

We refer to [2] for facts on the complex interpolation method.

3.1.2 Non-commutative case

Let S be a subspace of a non-commutative L^p space constructed on a hyperfinite von Neumann algebra. In the sequel for an operator space E we will denote by $S[E]$ the (closure of) the subspace $S \otimes E$ of the vector valued non-commutative L^p-space $L^p(\tau; E)$ defined in [42].

Definition 3.5. A linear map $u : S \to T$ between subspaces of non-commutative L^p spaces as above is said to be regular if for any operator space E, $u \otimes id_E : S[E] \to T[E]$ is bounded. As in the commutative case $\|u\|_r$ will denote the best constant C such that $\|u \otimes id_E\|_{S[E] \to T[E]} \leq C$ for all E.

The set of regular operators equipped with this norm will be denoted by $B_r(S, T)$.

Since classical L^p spaces are special cases of non-commutative L^p spaces, this notion applies also for commutative L^p spaces (but fortunately the two notions coincide). This notion was defined and studied in [41]. In particular the following result was proved:

Theorem 3.6 (Pisier)**.** *Let $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \widetilde{\tau})$ be hyperfinite von Neumann algebras with normal faithful traces. Then a map $u : L^p(\tau) \to L^p(\widetilde{\tau})$ is regular is and only if it is a linear combination of bounded completely positive operators. Moreover isomorphically*

$$B_r(L^p, L^p) \simeq \left[CB(L^\infty, L^\infty), CB(L^1, L^1)\right]^\theta \text{ for } \theta = 1/p.$$

The following result was also proved:

Theorem 3.7. *Let $1 \leq p < \infty$. Then $u : L^p(\tau) \to L^p(\widetilde{\tau})$ is regular if and only if $u^* : L^{p'}(\tau) \to L^{p'}(\tau)$ is regular, and $\|u\|_r = \|u^*\|_r$.*

3.2 Vector valued Besov spaces

In this section we introduce the Besov spaces of analytic functions $B^s_{p,q+}$. Before that we need some facts on Fourier multipliers. Everything in this section is classical (the results are stated in [40], and they are proved for the real line instead of the unit circle in [2]), but we give precise proofs in order to get quantitative bounds on the norms of the different isomorphisms.

3.2.1 Fourier Multipliers on the circle

Here $\mathbb{T}$ will denote the unit circle: $\mathbb{T} = \{z \in \mathbb{C}, |z| = 1\}$ and will be equipped with its Haar probability measure.

The Fourier multiplier with symbol $(\lambda_k)_{k\in\mathbb{Z}}$ ($\lambda_k \in \mathbb{C}$) is the linear map on the polynomials in z and $\overline{z}$ denoted by $M_{(\lambda_k)_k}$ and mapping $\sum_{k\in\mathbb{Z}} a_k z^k$ to $\sum_{k\in\mathbb{Z}} \lambda_k a_k z^k$. For $1 \leq p \leq \infty$ we say that the Fourier multiplier is bounded on L^p if the map $M_{(\lambda_k)_k}$ can be extended to a bounded operator on $L^p(\mathbb{T})$ such that for $f \in L^p(\mathbb{T})$, $g = M_{(\lambda_k)_k}(f)$ satisfies $\widehat{g}(k) = \lambda_k \widehat{f}(k)$.

Similarly if X is a Banach space the multiplier $M_{(\lambda_k)_k}$ is said to be bounded on $L^p(\mathbb{T};X)$ if $M_{(\lambda_k)_k} \otimes id_X$ extends to a continuous map on $L^p(\mathbb{T};X)$ (which we still denote by $M_{(\lambda)_k)}$), such that for $f \in L^p(\mathbb{T};X)$, $g = (M_{(\lambda_k)_k} \otimes id_X)(f)$ satisfies $\widehat{g}(k) = \lambda_k \widehat{f}(k)$.

In the vocabulary of part 3.1 a multiplier $M_{(\lambda_k)_k}$ is said to be regular on L^p if it is bounded on $L^p(\mathbb{T};X)$ for any Banach space X.

For example if $\lambda_k = \widehat{\mu}(k)$ for some complex Borel measure μ on $\mathbb{T}$ then $M_{(\lambda_k)_k}$ is bounded on $L^p(\mathbb{T};X)$ ($1 \leq p \leq \infty$) for any Banach space X since it corresponds to the convolution map $f \mapsto \mu \star f$. Its regular norm on L^p is therefore equal to the total variation of μ.

The following Lemma will be essential.

Lemma 3.8. *Let $\lambda = (\lambda_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ satisfying $\|\lambda\|_2 < \infty$. Then the Fourier multiplier with symbol λ is bounded on every L^p and*

$$\left\|M_{(\lambda_k)_k}\right\|_{L^p \to L^p} \leq \frac{2}{\sqrt{\pi}} \sqrt{\|\lambda\|_2 \|(\lambda_{k+1} - \lambda_k)_k\|_2}.$$

It is even regular and its regular norm on L^p is less than $2/\sqrt{\pi}\sqrt{\|\lambda\|_2\|(\lambda_{k+1} - \lambda_k)_k\|_2}$.

Proof. Since $\|(\lambda_k)\|_2 < \infty$, the function $f : z \mapsto \sum_{k\in\mathbb{Z}} \lambda_k z^k$ is in L^2 and $\|f\|_2 = \|(\lambda_k)\|_2$. Similarly, the function $g : z \mapsto (1-z)f(z)$ satisfies $\|g\|_2 = \|(\lambda_k - \lambda_{k+1})_{k\in\mathbb{Z}}\|_2$.

Since the multiplier with symbol (λ_k) corresponds to the convolution by f, by the remark preceding the Lemma we only have to prove that $\|f\|_1^2 \lesssim \|f\|_2 \|g\|_2$. But for any $0 < s < 1/2$:

$$\begin{aligned}
\|f\|_1 &= \int_0^1 |f(e^{2i\pi t})| dt \\
&= \int_{-s}^{s} |f(e^{2i\pi t})| dt + \int_s^{1-s} \frac{1}{|1 - e^{2i\pi t}|} |(1 - e^{2i\pi t}) f(e^{2i\pi t})| dt \\
&\leq \sqrt{2s} \|f\|_2 + \sqrt{\int_s^{1-s} \frac{1}{|1 - e^{2i\pi t}|^2} dt} \|g\|_2
\end{aligned}$$

by the Cauchy-Schwarz inequality. The remaining integral can be computed:

$$\begin{aligned}\int_s^{1-s} \frac{1}{|1-e^{2i\pi t}|^2}dt &= 2\int_s^{1/2} \frac{1}{4\sin^2(\pi t)}dt \\ &= \frac{1}{2}\left[\frac{-\cos(\pi t)}{\pi\sin(\pi t)}\right]_s^{1/2} = \frac{1}{2\pi\tan(\pi s)} \leq \frac{1}{2\pi^2 s}\end{aligned}$$

where we used that $\tan x \geq x$ for all $0 \leq x \leq \pi/2$. Taking $s = \|g\|_2/2\pi\|f\|_2 \leq 1/2$ we get the desired inequality. □

The following consequence will be also used a lot:

Lemma 3.9. *Let $I = [a,b] \subset \mathbb{Z}$ be an interval of size N and take $(\lambda_k)_{k\in\mathbb{C}} \in \mathbb{C}^{\mathbb{Z}}$.*

Then for any $1 \leq p \leq \infty$, any Banach space X and any $f \in L^p(\mathbb{T};X)$ such that $\widehat{f}$ is supported in I,

$$\left\|M_{(\lambda_k)_k} f\right\|_{L^p(\mathbb{T};X)} \leq 2\|f\|_p \max\left(\sup_{k\in I}|\lambda_k|, \sqrt{N\sup_{k\in I}|\lambda_k| \sup_{a\leq k<b}|\lambda_k-\lambda_{k+1}|}\right). \tag{3.2}$$

In other words, the restriction of the multiplier M_λ to the subspace of $L^p(\mathbb{T})$ of functions with Fourier transform vanishing outside of I has a regular norm less than the right-hand side of this inequality.

Proof. Consider the multiplier M_μ with symbol $(\mu_k)_{k\in\mathbb{Z}}$ where $\mu_k = \lambda_k$ if $k \in I$, $\mu_k = 0$ if $k \leq a-N$ or if $k \geq b+N$, and μ_k is affine on the intervals $[a-N,a]$ and $[b,b+N]$.

Since M_μ and M_λ coincide on the space of functions such that $\widehat{f}(k) = 0$ for $k \notin I$, the claim will follow from the fact that the regular norm of M_μ is less that the right-hand side of (3.2). For this we use Lemma 3.8, so we have to dominate $\|(\mu_k)\|_2$ and $\|(\mu_{k+1}-\mu_k)\|_2$. Since both sequences $(\mu_k)_k$ and $(\mu_{k+1}-\mu_k)_k$ are supported in $]a-N,b+N]$ which is of size less than $3N$, their ℓ^2-norm is less than $\sqrt{3N}$ times their ℓ^∞ norm. The inequality $\sup_k|\mu_k| \leq \sup_{k\in I}|\lambda_k|$ is obvious by definition of μ_k. On the other hand we have $|\mu_{k+1}-\mu_k| = |\lambda_{k+1}-\lambda_k|$ if $k \in [a,b[$, and $|\mu_{k+1}-\mu_k| \leq \sup_{k\in I}|\lambda_k|/N$ otherwise since μ_k is affine on the intervals of size $N+1$ $[a-N,a]$ and $[b,b+N]$.

Thus by Lemma 3.8,

$$\|M_\mu\|_{L^p(\mathbb{T};X)\to L^p(\mathbb{T};X)} \leq \frac{2\sqrt{3}}{\sqrt{\pi}}\max\left(\sup_{k\in I}|\lambda_k|, \sqrt{N\sup_{k\in[a,b[}|\lambda_k)|\sup_{k\in I}|\lambda_k-\lambda_{k+1}|}\right).$$

This concludes the proof since $3 \leq \pi$. □

For all $n \in \mathbb{N}$, $n > 0$ we define the function W_n on $\mathbb{T}$ by

$$\widehat{W_n}(k) = \begin{cases} 2^{-n+1}(k-2^{n-1}) & \text{if } 2^{n-1} \leq k \leq 2^n \\ 2^{-n}(2^{n+1}-k) & \text{if } 2^n \leq k \leq 2^{n+1} \\ 0 & \text{otherwise.}\end{cases}$$

We also define $W_0(z) = z+1$.

Note that for all $k \in \mathbb{N}$, $\sum_{n\in\mathbb{N}}\widehat{W_n}(k) = 1$ (finite sum).

Since for $n > 0$, $\|(\widehat{W}_n(k))_k\|_2 \leq \sqrt{2^n}$ and $\|(\widehat{W}_n(k)-\widehat{W}_n(k+1))_k\|_2 = \sqrt{3/2^n}$, Lemma 3.8 implies the multiplier $f \mapsto W_n \star f$ has regular norm less than $2\sqrt{3/\pi} \leq 2$ on $L^p(\mathbb{T})$ any $1 \leq p \leq \infty$. The same is obvious for W_0.

3.2.2 Besov spaces of vector-valued analytic functions

We define the weighted ℓ_p spaces $\ell_p^s(\mathbb{N};X)$ for $p>0$, $s\in\mathbb{R}$ and a Banach space X as the space of sequences $(x_n)_{n\in\mathbb{N}}\in X^{\mathbb{N}}$ such that $\|(x_n)_n\|_{\ell_p^s(\mathbb{N};X)}=\|(2^{ns}\|x_n\|_X)_{n\in\mathbb{N}}\|_p<\infty$.

We will deal in this paper with Besov spaces of "analytic functions", which are defined in the following way. First note that the reader should take the term "analytic" with care. Elements of the Besov spaces are indeed defined as formal series $\sum_{k\geq 0}x_k z^k$ with $z\in\mathbb{T}$. The term analytic means that the formal series are indexed by $\mathbb{N}$ and not $\mathbb{Z}$ (in particular this has nothing to do with analytic maps defined on the real analytic manifold $\mathbb{T}$).

Let X be a Banach space; $p,q>0$ and s real numbers. The Besov space $B_{p,q}^s(X)_+$ is defined as the space of formal series $f(z)=\sum_{k\in\mathbb{N}}x_k z^k$ with $x_k\in X$ such that $(2^{ns}\|W_n\star f\|_p)_{n\in\mathbb{N}}\in\ell_q$, with the norm $\|(2^{ns}\|W_n\star f\|_p)_{n\in\mathbb{N}}\|_q$. Here by $W_n\star f$ we mean the (finite sum) $\sum_{k\geq 0}\widehat{W_n}(k)x_k z^k$, and this coincides with the obvious notion when $f\in L^1(\mathbb{T};X)$.

Remark 3.10 (Elements of $B_{p,q}^s(X)_+$ as functions)**.** It is easy to see that when $s>0$, any $f\in B_{p,q}^s(X)_+$ corresponds to a function belonging to $L^p(\mathbb{T};X)$ (and therefore also to $L^1(\mathbb{T};X)$). In this case the series $\sum_{n\geq 0}W_n\star f$ indeed converges in $L^p(\mathbb{T};X)$ (because $\sum_{n\geq 0}\|W_n\star f\|_p<\infty$). It is also immediate to see that for any s, $\|x_k\|_X\leq C\|f\|_{B_{p,q}^s(X)_+}k^{-s}$ for some constant $C>0$, and thus that for any $f\in B_{p,q}^s(X)_+$, $\sum_{k\geq 0}x_k z^k$ converges for all z in the unit ball $\mathbb{D}$ of $\mathbb{C}$.

On the opposite when $s<0$ there are elements $f=\sum_{k\geq 0}x_k z^k\in B_{p,q}^s(X)_+$ such that the sequence x_k is not even bounded (and thus cannot represent a function in $L^1(\mathbb{T};X)$).

The space can be equivalently defined as a subspace of $\ell_q^s(\mathbb{N};L^p(\mathbb{T};X))$ with the isometric injection

$$\begin{array}{rcl} B_{p,q}^s(X)_+ & \longrightarrow & \ell_q^s(\mathbb{N};L^p(\mathbb{T};X)) \\ f & \mapsto & (W_n\star f)_{n\in\mathbb{N}} \end{array}$$

Moreover the image of $B_{p,q}^s(X)_+$ in the isometric injection is a complemented subspace. The complementation map is given by

$$\begin{array}{rcl} P:\ell_q^s(\mathbb{N};L^p(\mathbb{T};X)) & \longrightarrow & B_{p,q}^s(X)_+ \\ (a_n) & \mapsto & (W_0+W_1)\star a_0+\sum_{n\geq 1}(W_{n-1}+W_n+W_{n+1})\star a_n \end{array}$$

and has norm less than $C2^{2|s|}$ for some constant $C\leq 20$. Indeed, if $V_n=W_{n-1}+W_n+W_{n+1}$ if $n\geq 1$ and $V_0=W_0+W_1$, then $W_m\star V_n=0$ if $|n-m|>2$, and moreover if $|n-m|\leq 2$, $\|(W_m\star V_n)\star a_n\|_p\leq 4\|a_n\|_p$ by Lemma 3.8. This implies that

$$\begin{array}{rcl} \left\|\sum_{n\geq 0}V_n\star a_n\right\|_{B_{p,q}^s(X)_+} & \leq & \sum_{-2\leq\epsilon\leq 2}4\left\|(2^{ns}\|a_{n+\epsilon}\|_p)_{n\in\mathbb{N}}\right\|_q \\ & \leq & 4\left(2^{-2s}+2^{-s}+1+2^s+2^{2s}\right)\left\|(2^{ns}\|a_{n+\epsilon}\|_p)_{n\in\mathbb{N}}\right\|_q. \end{array}$$

When $p=q$, the Besov space $B_{p,q}^s(X)_+$ is also denoted by $B_p^s(X)_+$. In this case B_{p+}^s is a subspace of $\ell_p^s(\mathbb{N};L^p(\mathbb{T}))$ which is just the L^p space of $\mathbb{N}\times\mathbb{T}$ with respect to the product measure of the Lebesgue measure on $\mathbb{T}$ and the measure on $\mathbb{N}$ giving mass 2^{nsp} to $\{n\}$. Moreover (at least for $p<\infty$) $B_p^s(X)_+$ is the closure of $B_{p+}^s\otimes X$ in the vector-valued L^p space $L^p(\mathbb{N}\times\mathbb{T};X)$. This will allow to speak of regular operators between B_{p+}^s and an other (subspace of a) non-commutative L^p space. Note in particular that the above remark shows that B^s_{p+} is a complemented subspace of $L^p(\mathbb{N}\times\mathbb{T})$ and that the complementation map P (which does not depend on p) is regular.

As a consequence of the complementation, we have the following property of Besov spaces:

Theorem 3.11. *The properties of the Besov spaces with respect to duality are: if $p,q<\infty$*

$$B^s_{p,q}(X)^*_+ = B^{-s}_{p',q'}(X^*)_+ .$$

For the natural duality $\langle f,g\rangle = \sum_{n\geq 0}\langle \widehat{f}(n),\widehat{g}(n)\rangle$.

For a real (or complex) number α and an integer n, we define the number D^α_n by $D^\alpha_0 = 1$ and for $n\geq 1$,

$$D^\alpha_n = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} = \prod_{j=1}^{n}\left(1+\frac{\alpha}{j}\right).$$

For any $t\in\mathbb{R}$, we define the maps I_t and $\widetilde{I}_t$ by

$$I_t(\sum_{k\geq 0} a_k z^k) = \sum_{k\geq 0}(1+k)^t a_k z^k.$$

$$\widetilde{I}_t(\sum_{k\geq 0} a_k z^k) = \sum_{k\geq 0} D^t_k a_k z^k.$$

The boundedness properties of the maps I_t and $\widetilde{I}_t$ are described by the following result:

Theorem 3.12. *Let $M>0$ be a real number. There is a constant $C=C_M$ (depending only on M) such that for any $1\leq p,q\leq\infty$, any $|t|\leq M$, any $s\in\mathbb{R}$, and any Banach space X,*

$$\|I_t : B^s_{p,q}(X)_+ \to B^{s-t}_{p,q}(X)_+\|, \|I_t^{-1} : B^{s-t}_{p,q}(X)_+ \to B^s_{p,q}(X)_+\| \leq C.$$

Moreover if $-1/2\leq t\leq M$,

$$\|\widetilde{I}_t : B^s_{p,q}(X)_+ \to B^{s-t}_{p,q}(X)_+\|, \|\widetilde{I}_t^{-1} : B^{s-t}_{p,q}(X)_+ \to B^s_{p,q}(X)_+\| \leq C.$$

Proof. Fix $M>0$ (and even $M\geq 1$) and take $|t|\leq M$. Let us treat the case of I_t. Let $f=\sum_{k\geq 0} a_k z^k \in B^s_{p,q}(X)_+$. Since the maps $f\mapsto W_n\star f$ and $f\mapsto I_t f$ are both multipliers, they commute, and we have that

$$\|I_t f\|_{B^{s-t}_{p,q}(X)_+} = \left\|(2^{|n|s}\|I_t/2^{nt}(W_n\star f)\|_p)_{n\in\mathbb{N}}\right\|_q .$$

To show that $\|I_t\|\leq C$, it is therefore enough to show that the multiplier $I_t/2^{nt}$ (the symbol of which is $((1+k)/2^n)^t$) is bounded by some constant C on the subspace of $L^p(\mathbb{T},X)$ consisting of functions whose Fourier transform is supported in $]2^{n-1},2^{n+1}[$. This follows from Lemma 3.9. We indeed have $((1+k)/2^n)^t\leq 2^{|t|}$ for $k\in]2^{n-1},2^{n+1}[$. To dominate the difference $|((2+k)/2^n)^t-((1+k)/2^n)^t|$ for $2^{n-1}<k<2^{n+1}-1$, just dominate the derivative of $x\mapsto(x/2^n)^t$ on the interval $[2^{n-1},2^{n+1}]$ by $|t|2^{|t-1|}/2^n\leq M2^{M+1}/2^n$. The multiplier $I_t/2^{nt}$ is thus bounded by $4\sqrt{M}2^M$.

This shows that

$$\|I_t : B^s_{p,q}(X)_+ \to B^{s-t}_{p,q}(X)_+\| \le 4\sqrt{M}2^M$$

Since $I_{-t} = {I_t}^{-1}$, the inequality for I_{-t} follows.

By the same argument, to dominate the norms of $\widetilde{I}_t$ (resp. its inverse), we have to get a uniform bound on $\sup_k |\lambda_k|$ and $2^n \sup_k |\lambda_{k+1} - \lambda_k|$ where $\lambda_k = D^t_k/2^{nt}$ (resp. $\lambda_k = 2^{nt}/D^t_k$). This amounts to showing that there is a constant $C(M)$ (depending on M only) such that $1/C(M) \le |D^t_k/2^{nt}| \le C(M)$ and $|D^t_{k+1}/2^{nt} - D^t_k/2^{nt}| \le C(M)/2^n$ for $2^{n-1} \le k < 2^{n+1}$ (the inequality $|2^{nt}/D^t_{k+1} - 2^{nt}/D^t_k| \le C(M)^3/2^n$ will follow from the formula $|1/x - 1/y| = |y-x|/|xy|$). The first inequality can be proved by taking the logarithm, noting that $\log(1+t/j) = t/j + O(1/j^2)$ up to constants depending only on M if $-1/2 \le t \le M$, and remembering that $\sum_1^N 1/j = \log N + O(1)$. The second inequality follows easily since $D^t_{k+1} - D^t_k = t/(k+1)D^t_k$. □

We also use the following characterization of Besov spaces of analytic vector-valued functions:

Theorem 3.13. *Let $M > 0$. Then there is a constant $C = C_M$ (depending only on M) such that for all $0 < s < M$, for all Banach spaces X, all $1 \le p \le \infty$ and all $f : \mathbb{T} \to X$,*

$$C^{-1}\|f\|_{B^{-s}_{p,p}(X)_+} \le \left\|(1-|z|)^{s-1/p} f\right\|_{L^p(\mathbb{D},dz;X)} \le \frac{C}{s}\|f\|_{B^{-s}_{p,p}(X)_+}.$$

Proof. The left-hand side inequality is easier. For any $0 < r < 1$, let f_r denote the function $f_r(\theta) = f(re^{i\theta})$. Then

$$\left\|(1-|z|)^{s-1/p} f\right\|_{L^p(\mathbb{D},dz;X)} = \left(\int_0^1 (1-r)^{ps-1}\|f_r\|_p^p r dr\right)^{1/p}.$$

Let $1 - 2^{-n} \le r \le 1 - 2^{-n-1}$ with $n \ge 1$. Then $\|f_r\|_p \ge \|W_n \star f_r\|_p/2$. But f is the image of f_r by the multiplier with symbol $(r^{-k})_{k\in\mathbb{Z}}$. Note that for $2^{n-1} \le k \le 2^{n+1}$, $r^{-k} \le 2^4$, and for $2^{n-1} \le k < 2^{n+1}$, $r^{-k-1} - r^{-k} = (1-r)r^{-k-1} \le 2^{-n+1}2^4 = 2^{-n+5}$. Thus since multipliers commute and since the Fourier transform of $W_n \star f$ vanishes outside of $]2^{n-1}, 2^{n+1}[$, Lemma 3.9 implies

$$\|W_n \star f\|_p \le 2\|W_n \star f_r\|_p 2^5 \le 2^6\|f_r\|_p.$$

Moreover $(1-r)^{ps-1} \ge 2^{-ps}2^{-nsp+n}$. Integrating over r, we thus get that for $n \ge 1$:

$$2^{-nsp}\|W_n \star f\|_p^p \le C^p \int_{1-2^{-n}}^{1-2^{-n-1}} (1-r)^{ps-1}\|f_r\|_p^p r dr$$

where C depends only on M. For $n = 0$ the same inequality is very easy. Summing over p and taking the p-th root, we get the first inequality

$$\|f\|_{B^{-s}_{p,p}(X)_+} \le C\left\|(1-|z|)^{s-1/p} f\right\|_{L^p(\mathbb{D},dz;X)}.$$

For the right-hand side inequality, note that since $\sum_n \widehat{W_n}(k) = 1$ for all $k \ge 0$, we have that for any $r > 0$

$$\|f_r\|_p \le \sum_{n\ge 0} \|W_n \star f_r\|_p.$$

Then as above since $W_n \star f_r$ is the image of $W_n \star f$ by the Fourier multiplier of symbol r^k, Lemma 3.9 again implies than

$$\|W_n \star f_r\|_p \leq 2r^{2^{n-1}} \max(1, \sqrt{2^{n+1}(1-r)}) \|W_n \star f\|_p.$$

If m is such that $1 - 2^{-m} \leq r \leq 1 - 2^{-m-1}$ then

$$r^{2^{n-1}} = \left((1-2^{-m-1})^{2^{m+1}}\right)^{2^{n-m-2}} \leq e^{-2^{n-m-2}}$$

and

$$\max(1, \sqrt{2^{n+1}(1-r)}) \leq \max(1, \sqrt{2}^{n+1-m}).$$

If for $k \in \mathbb{Z}$ one denotes $b_k = 2e^{-2^{k-2}} \max(1, \sqrt{2}^{k+1}) 2^{ks}$ one thus has

$$\|W_n \star f_r\|_p \leq 2^{ms} b_{n-m} 2^{-ns} \|W_n \star f_r\|_p.$$

If $a_n = 2^{-ns}\|W_n \star f_r\|_p$ for $n \geq 0$ and $a_n = 0$ if $n < 0$, summing the previous inequality over n we thus get

$$\|f_r\|_p \leq 2^{ms} \sum_{n \geq 0} b_{n-m} a_n = 2^{ms} (a \star b)_m.$$

Let us raise this inequality to the power p, multiply by $r(1-r)^{ps-1} \leq 2^{-mps} 2^{m+1}$ and integrate on $[1-2^{-m}, 1-2^{-m-1}]$. One gets

$$\int_{1-2^{-m}}^{1-2^{-m-1}} (1-r)^{ps-1} \|f_r\|_p^p r dr \leq (a \star b)_m^p.$$

Summing over m this leads to

$$\left\| (1-|z|)^{s-1/p} f \right\|_{L^p(\mathbb{D}, dz; X)} \leq \left(\sum_{m \geq 0} (a \star b)_m^p \right)^{1/p} \leq \|a \star b\|_{\ell^p(\mathbb{Z})}.$$

Now note that $\|a \star b\|_{\ell^p(\mathbb{Z})} \leq \|a\|_p \|b\|_1 = \|f\|_{B^{-s}_{p,p}(X)_+} \|b\|_1$. We are just left to prove that $b \in \ell^1(\mathbb{Z})$ and $\|b\|_1 \leq C/s$ with some constant C depending only on M. If $k \geq 0$, we have $|b_k| \leq 2\sqrt{2} e^{-2^{k-2}} 2^{k(M+1/2)}$ which proves that $\sum_{k \geq 0} b_k \leq C_1$ for some constant depending only on M. If $k < 0$, $|b_k| \leq 2^{ks+1}$, which proves that $\sum_{k<0} |b_k| \leq 2/(2^s - 1) \leq C_2/s$ for some universal constant. This concludes the proof. □

3.3 Operator space valued Hankel matrices

In this section we finally prove the main result stated in the Introduction, Theorem 3.3. We prove the two sides of (3.1) separately. We first show how we can derive the left-hand side inequality from the right-hand side for $\alpha = \beta = 1$ by duality.

For the right-hand side, we first recall a proof for the cases when $p = 1$ or $p = \infty$ (this was contained in Peller's proof since for non-commutative L^1 or L^∞ spaces, regularity and complete boundedness coincide). Then we derive the case of a general p by an interpolation argument.

3.3.1 Left-hand side of (3.1)

In this section we assume that the right-hand side of (3.1) holds for $\alpha = \beta = 1$, that is to say the operator

$$\begin{array}{rcl} B_{p+}^{1/p+2} & \to & S^p \\ \varphi & \mapsto & \Gamma_\varphi^{1,1} \end{array}$$

is regular for every $1 \leq p \leq \infty$.

Fix now $1 \leq p \leq \infty$ and $\alpha, \beta > -1/2p$. We prove that the map $\Gamma_\varphi^{\alpha,\beta} \mapsto \varphi$ is regular from the subspace of S^p formed of all the matrices of the form $\Gamma_\varphi^{\alpha,\beta}$ to $B_{p+}^{1/p+\alpha+\beta}$.

For $\psi \in B_{p'+}^{1/p'+2}$ define the matrix

$$\begin{aligned} \widetilde{\Gamma}_\psi^{1,1} &= \left(\frac{D_j^{\alpha+1}}{(1+j)^\alpha} \frac{D_k^{\beta+1}}{(1+k)^\beta} \widehat{\psi}(j+k) \right)_{j,k\geq 0} \\ &= diag\left(\frac{D_j^{\alpha+1}}{(1+j)^{\alpha+1}} \right) \cdot \Gamma_\psi^{1,1} \cdot diag\left(\frac{D_k^{\beta+1}}{(1+k)^{\beta+1}} \right). \end{aligned}$$

First note that since $\sup_{-1/2\leq\alpha\leq M} \sup_{j\geq 0} D_j^{\alpha+1}/(1+j)^{\alpha+1} < \infty$ the assumption with p' implies that the operator $T : \psi \mapsto \widetilde{\Gamma}_\psi^{1,1}$ is also regular from $B_{p'+}^{1/p'+2}$ to $S^{p'}$ with regular norm bounded by some constant depending only on M.

Recall that by Theorem 3.11 $B_{p+}^{-1/p'-2} \simeq (B_{p'+}^{1/p'+2})^*$ if $p > 1$ (and $(B_{p+}^{-1/p'-2})^* \simeq B_{p'+}^{1/p'+2}$ if $p < \infty$). Since $B_{p+}^{1/p-3}$ is complemented in $\ell_p^{1/p-3}(\mathbb{N}; L^p)$ with a regular complementation map, Theorem 3.7 implies that the dual map $T^* : S^p \to B_{p+}^{-1/p'-2} = B_{p+}^{1/p-3}$ is also regular.

It is now enough to compute explicitly the restriction of T^* to the set of matrices of the form $\Gamma_\varphi^{\alpha,\beta}$ to conclude. Indeed for any analytic $\varphi : \mathbb{T} \to \mathbb{C}$ such that $\Gamma_\varphi^{\alpha,\beta} \in S^p$, and any $\psi \in B_{p'+}^{1/p'+2}$ we have

$$\begin{aligned} \left\langle T^* \Gamma_\varphi^{\alpha,\beta}, \psi \right\rangle &= \left\langle \Gamma_\varphi^{\alpha,\beta}, T\psi \right\rangle \\ &= \sum_{j,k\geq 0} D_j^{\alpha+1} D_k^{\beta+1} \widehat{\varphi}(j+k) \widehat{\psi}(j+k) \\ &= \sum_{n\geq 0} D_n^{\alpha+\beta+3} \widehat{\varphi}(n) \widehat{\psi}(n) \\ &= \langle \widetilde{I}_{\alpha+\beta+3}\varphi, \psi \rangle. \end{aligned}$$

We used that for all $\alpha, \beta \in \mathbb{R}$, and all $n \in \mathbb{N}$

$$\sum_{j+k=n} D_j^\alpha D_k^\beta = D_n^{\alpha+\beta+1},$$

which follows from the equality $\sum_{n\geq 0} D_n^\alpha x^n = (1+x)^{-\alpha-1}$ for $|x| < 1$.

Thus we have that $T^*\Gamma_\varphi^{\alpha,\beta} = \widetilde{I}_{\alpha+\beta+3}\varphi$. By Theorem 3.12 the map $\left(\widetilde{I}_{\alpha+\beta+3}\right)^{-1}$ is regular as a map from $B_{p+}^{1/p-3}$ to $B_{p+}^{1/p+\alpha+\beta}$. Hence the map $\Gamma_\varphi^{\alpha,\beta} \mapsto \varphi$ is regular from the subspace of S^p formed of all the matrices of the form $\Gamma_\varphi^{\alpha,\beta}$ to $B_{p+}^{1/p+\alpha+\beta}$. This concludes the proof (it is immediate from the proof that the regular norm of this map only depends on M).

3.3.2 Right hand side of (3.1) for $p = 1$ and $p = \infty$

Now we prove that for a formal series $\varphi = \sum_{k\geq 0} \widehat{\varphi}(k) z^k$ with $\widehat{\varphi}(k) \in E$, it is sufficient that φ belongs to $B_{p+}^{1/p+\alpha+\beta}$ to ensure that $\Gamma_\varphi^{\alpha,\beta} \in S_p[E]$. Since for $p = 1$ or $p = \infty$, regularity and complete boundedness coincide, the case $p = 1$ and $p = \infty$ are contained in Peller's result (see [40]). We will still provide a proof which is more precise as far as constants are concerned.

Proof of sufficiency for p $= 1$. Let E be an arbitrary operator space. Since (formally) $\varphi = \sum_0^\infty W_n \star \varphi$, and $\|\varphi\|_{B_1^{1+\alpha+\beta}(E)_+} = \sum_{n\geq 0} 2^{n(1+\alpha+\beta)} \|W_n \star \varphi\|_1$, by the triangle inequality replacing φ by $W_n \star \varphi$ it is enough to prove that, if $\varphi = \sum_{k=0}^m a_k z^k$ with $a_k \in E$,

$$\left\|\Gamma_\varphi^{\alpha,\beta}\right\|_{S_1[E]} \leq C \frac{(1+m)^{1+\alpha+\beta}}{\sqrt{(\alpha+1/2)(\beta+1/2)}} \|\varphi\|_{L^1(\mathbb{T};E)}.$$

But we can write

$$\Gamma_\varphi^{\alpha,\beta} = \int_{\mathbb{T}} \left(\varphi(z)(1+j)^\alpha (1+k)^\beta \overline{z}^{j+k}\right)_{0\leq j,k\leq m} dz$$

and compute, for $z \in \mathbb{T}$,

$$\left\|\left(\varphi(z)(1+j)^\alpha (1+k)^\beta \overline{z}^{j+k}\right)_{0\leq j,k\leq m}\right\|_{S_1[E]} = \|\varphi(z)\|_E \left\|\left((1+j)^\alpha (1+k)^\beta \overline{z}^{j+k}\right)_{0\leq j,k\leq m}\right\|_{S_1},$$

with

$$\left\|\left((1+j)^\alpha (1+k)^\beta \overline{z}^{j+k}\right)_{0\leq j,k\leq m}\right\|_{S_1} = \left\|\left((1+j)^\alpha\right)_{j=0\ldots m}\right\|_{\ell^2} \left\|\left((1+k)^\beta\right)_{k=0\ldots m}\right\|_{\ell^2}.$$

Thus the lemma follows from the fact that

$$\left\|\left((1+j)^\alpha\right)_{j=0\ldots m}\right\|_{\ell^2}^2 \leq C \frac{(1+m)^{2\alpha+1}}{2\alpha+1}$$

for a constant C which depends only on $M = \max\{\alpha, \beta\}$. □

Proof of the sufficiency for p $= \infty$. Note that in this case $\alpha, \beta > 0$. Assume that $E \subset B(H)$ for a Hilbert space H. Then $S_\infty(E)$ is isometrically contained in $B(\ell^2(H))$. In this proof we use the fact that $H \hat{\otimes} \overline{H} \simeq B(H)_*$ isometrically through the duality $\langle T, \xi \otimes \overline{\eta}\rangle = \langle T\xi, \eta\rangle$.

Let $\varphi \in B_\infty^{\alpha+\beta}(E)_+$. We have that

$$\|\Gamma_\varphi^{\alpha,\beta}\|_{S_\infty[E]} = \sup_{\substack{\|(a_k)_{k\in\mathbb{N}}\|_{\ell^2(H)} \leq 1 \\ \|(b_k)_{k\in\mathbb{N}}\|_{\ell^2(H)} \leq 1}} \left\langle \Gamma_\varphi^{\alpha,\beta} (a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \right\rangle_{\ell^2(H)}$$

where the sup can be restricted to finitely supported families $(a_k)_{k\in\mathbb{N}} \in \ell^2(H)$ and $(b_k)_{k\in\mathbb{N}} \in \ell^2(H)$.

$$\left\langle \Gamma_\varphi^{\alpha,\beta}(a_k), (b_k) \right\rangle = \sum_{j,k\geq 0} \left\langle (1+j)^\alpha (1+k)^\beta \widehat{\varphi}(j+k) a_k, b_j \right\rangle_H.$$

Define the functions $f : \mathbb{T} \to H$ and $g : \mathbb{T} \to \overline{H}$ by

$$f(z) = \sum_{k\geq 0} (1+k)^{\beta} a_k z^k.$$

$$g(z) = \sum_{j\geq 0} (1+j)^{\alpha} \overline{b_j} z^j.$$

Then for the duality $\langle T, \xi \otimes \overline{\eta} \rangle_{B(H), H\otimes \overline{H}} = \langle T\xi, \eta\rangle$ between $B(H)$ and $H \otimes \overline{H}$, we have that

$$\left\langle \Gamma_{\varphi}^{\alpha,\beta}(a_k), (b_k) \right\rangle = \sum_{m\geq 0} \left\langle \widehat{\varphi}(m), \widehat{f\otimes g}(m) \right\rangle.$$

We thus have to show that $f \otimes g \in B_{\infty}^{\alpha+\beta}(B(H))_{+}^{*}$. But by Theorem 3.11, $B_{\infty}^{\alpha+\beta}(B(H))_{+}$ is the dual space of $B_1^{-\alpha-\beta}(B(H)_*)_+$, so that from $B(H)_* \simeq H \hat{\otimes} \overline{H}$, we have to prove that

$$\|f\otimes g\|_{B_1^{-\alpha-\beta}(H\hat{\otimes}\overline{H})_+} \leq \frac{C}{\sqrt{\alpha\beta}} \|(a_k)\|_{\ell^2(H)} \|(b_k)\|_{\ell^2(H)}. \tag{3.3}$$

From Theorem 3.13,

$$\|f\otimes g\|_{B_1^{-\alpha-\beta}(H\hat{\otimes}\overline{H})_+} \leq C \left\| (1-|z|)^{\alpha-1/2+\beta-1/2} f\otimes g \right\|_{L^1(\mathbb{D},dz;H\hat{\otimes}\overline{H})}.$$

By the Cauchy-Schwarz inequality, we get that

$$\|f\otimes g\|_{B_1^{-\alpha-\beta}(H\hat{\otimes}\overline{H})_+} \leq C \left\| (1-|z|)^{\beta-1/2} f \right\|_{L^2(\mathbb{D},dz;H)} \left\| (1-|z|)^{\alpha-1/2} g \right\|_{L^2(\mathbb{D},dz;\overline{H})}. \tag{3.4}$$

To dominate the terms $\left\|(1-|z|)^{\beta-1/2} f\right\|_{L^2(\mathbb{D},dz;H)}$ and $\left\|(1-|z|)^{\alpha-1/2} g\right\|_{L^2(\mathbb{D},dz;\overline{H})}$ one could use again Theorem 3.13 to get an inequality of the form

$$\|f\otimes g\|_{B_1^{-\alpha-\beta}(H\hat{\otimes}\overline{H})_+} \lesssim \frac{1}{\alpha\beta} \|(a_k)\|_{\ell^2(H)} \|(b_k)\|_{\ell^2(H)}.$$

But these terms can be computed directly and this leads to a better bound (this was kindly explained to me by Quanhua Xu). Indeed,

$$\begin{aligned} \left\| (1-|z|)^{\beta-1/2} f \right\|_{L^2(\mathbb{D},dz;H)}^2 &= \int_0^1 (1-r)^{2\beta-1} \sum_{k\geq 0} (1+k)^{2\beta} \|a_k\|^2 r^{2k+1} dr \\ &= \sum_{k\geq 0} \|a_k\|^2 (1+k)^{2\beta} \int_0^1 (1-r)^{2\beta-1} r^{2k+1} dr. \end{aligned}$$

Integrating by parts $2k+1$ times, one gets

$$\int_0^1 (1-r)^{2\beta-1} r^{2k+1} dr = \frac{(2k+1)2k(2k-1)\dots 1}{2\beta(2\beta+1)\dots(2\beta+2k+1)} = \frac{1}{2\beta} \prod_{i=1}^{2k+1} \frac{i}{2\beta+i}.$$

It is then easy to show that $(1+k)^{2\beta} \prod_{i=1}^{2k+1} i/(2\beta+i)$ is bounded above (and below) by some constant depending only on M as long as $\beta \leq M$. This implies that

$$\left\| (1-|z|)^{\beta-1/2} f \right\|_{L^2(\mathbb{D},dz;H)}^2 \lesssim \frac{1}{\beta} \sum_k \|a_k\|^2.$$

The same computation for α together with (3.4) leads finally to the desired inequality (3.3). This concludes the proof.

□

3.3.3 Right hand side of (3.1) for a general p.

Let us first reformulate the right-hand side of (3.1).

Denote by D the infinite diagonal matrix $D_{j,j} = 1/(1+j)$ and $D_{j,k} = 0$ if $j \neq k$. Let p, α and β as in Theorem 3.3. Define $\widetilde{\alpha} = \alpha + 1/2p$ and $\widetilde{\beta} = \beta + 1/2p$. Then for any φ

$$\Gamma_\varphi^{\alpha,\beta} = D^{1/2p} \Gamma_\varphi^{\widetilde{\alpha},\widetilde{\beta}} D^{1/2p}.$$

Moreover Theorem 3.12 implies that the map $I_{\widetilde{\alpha}+\widetilde{\beta}} : B_{p+}^{\widetilde{\alpha}+\widetilde{\beta}} \to B_{p+}^0$ is a regular isomorphism (with regular norms of the map and its inverse depending only on M as far as $\alpha, \beta < M$).

The main result of this section is thus

Lemma 3.14. *Let $M > 0$. Take $0 < \alpha, \beta < M$ and $1 \leq p \leq \infty$. The map*

$$\begin{array}{rcl} T_p : B_{p+}^0 & \to & S^p \\ \varphi & \mapsto & D^{1/2p} \left(\widehat{\varphi}(j+k) \dfrac{(1+j)^\alpha (1+k)^\beta}{(1+j+k)^{\alpha+\beta}} \right)_{j,k \geq 0} D^{1/2p} \end{array}$$

is regular, with regular norm less that $C/\sqrt{\alpha\beta}$ for some constant C depending only on M.

As explained above, this result is equivalent to the right-hand side inequality in (3.1). More precisely the above Theorem for some $\alpha, \beta > 0$ and $1 \leq p \leq \infty$ is equivalent to the right-hand side inequality in (3.1) for the same p but with α and β replaced by $\alpha - 1/2p$, $\beta - 1/2p$. In the proof below Pisier's Theorem 3.6 on interpolation of regular operators is used, but the reader unfamiliar with regular operators can as well directly use complex interpolation of vector-valued Besov spaces and Schatten classes.

Proof of Lemma 3.14. We have already seen that the map T_p is regular(=completely bounded) when $p = 1$ or $p = \infty$. Therefore up to the change of density given by D, T_p is simultaneously completely bounded on B_{1+}^0 and $B_{\infty+}^0$, which should imply that T_p is regular.

To check this more rigorously, we use Pisier's Theorem 3.6. Since the Besov space B_{p+}^0 is a complemented subspace of $L^p(\mathbb{N} \times \mathbb{T})$ (where $\mathbb{N} \times \mathbb{T}$ is equipped with the product of the counting measure on $\mathbb{N}$ and the Lebesgue measure on $\mathbb{T}$), and since the complementation map P is regular and is the same for every p, T_p naturally extends to a map $T_p \circ P : L^p(\mathbb{N} \times \mathbb{T}) \to S^p$ which is still completely bounded for $p = 1, \infty$.

To show that T_p is regular, we show that $T_p \circ P \in \left[CB(L^\infty, B(\ell^2)), CB(L^1, S^1)\right]_\theta$ (where the first L^∞ and L^1 spaces are $L^\infty(\mathbb{N} \times \mathbb{T})$ and $L^1(\mathbb{N} \times \mathbb{T})$). Since by the equivalence theorem for complex interpolation $[A_0, A_1]_\theta \subset [A_0, A_1]^\theta$ with constant 1 for any compatible Banach spaces A_0, A_1 (Theorem 4.3.1 in [2]), Theorem 3.6 will imply that $T_p \circ P$ is regular and hence its restriction to B_{p+}^0 T_p too.

Consider the analytic map $f(z)$ with values in $CB(L^1, S^1) + CB(L^\infty, B(\ell^2))$ given by $f(z) = D^{z/2} T_\infty \circ P D^{z/2}$ (f takes in fact values in $CB(L^\infty, B(\ell^2))$). Then $f(1/p) = T_p \circ P$. The conjugation by a unitary is a complete isometry on $B(\ell^2)$ and on S^1. Therefore if $Re(z) = 0$, $\|f(z)\|_{CB(L^\infty, B(\ell^2))} = \|T_\infty \circ P\|_{CB(L^\infty, B(\ell^2))} \leq C/\sqrt{\alpha\beta}$ and if $Re(z) = 1$, $\|f(z)\|_{CB(L^1, S^1)} = \|T_1 \circ P\| CB(L^1, S^1) \leq C/\sqrt{\alpha\beta}$.

This proves that

$$\|T_p\|_{B_r(L^p, S^p)} \leq C/\sqrt{\alpha\beta}.$$

□

Appendix A

Some remarks and questions on operator Lipschitz functions

Here we present some remarks and questions concerning operator Lipschitz functions. The type of questions we are interested in is: given a norm on a von Neumann algebra and a function $f : \mathbb{R} \to \mathbb{R}$, is the function $A \mapsto f(A)$ (defined for any self-adjoint A by the continuous functional calculus) Lipschitz for the norm we are considering? (We will be mainly interested in the operator norm or the L^p norms). The reader is also refered to the recent survey preprint [36].

This question can take two slightly different forms. The first one is: for a given norm, give a necessary and sufficient condition for a function f to be Lipschitz for this norm. The second one is: given a certain class of functions, for which spaces are these functions operator Lipschitz? Of course if one can answer completely one of these questions the answer to the other question will follow, but the fact is that there are still many open problems. I would like to address a few in this Appendix. Except perhaps for Lemma A.16, the material here is not new and contains mainly results that either are classical or for which a reference is given.

For the operator norm, much has been done for the first question by Peller in [39]: he indeed found some necessary condition and some sufficient condition (which do not coincide!) for a function f to be operator Lipschitz on $B(H)$, but there are still open questions.

Here we will focus on the second question for the class of all Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ (the case of functions of the Hölder class was recently treated by Peller and Aleksandrov, see [1]) . For this already much is known, and it has become quite clear that this is related to the UMD property. For the operator norm, the following is classical and seems to go back to Farforovskaya [13] (see Lemma A.13 below for a simple proof):

Theorem A.1. *There are some Lipschitz functions $\mathbb{R} \to \mathbb{R}$ that are not operator Lipschitz on $B(\ell^2)$, for example the map $f(t) = |t|$.*

The following Theorem was proved very recently by Potapov and Sukochev, answering a long-standing open question:

Theorem A.2 (Potapov and Sukochev). *For any $1 < p < \infty$ there exists a constant C_p such that for any 1-Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ and any two selfadjoint operators A, B on ℓ^2 with $A - B$ belonging to p-Schatten class S^p (*i.e.* such that $\|A - B\|_p \overset{def}{=} Tr(|A - B|^p)^{1/p} < \infty$), $f(A) - f(B) \in S^p$ and moreover*

$$\|f(A) - f(B)\|_p \leq C_p \|A - B\|_p.$$

The proof presented in the first preprint version of this result was based on an extrapolation argument and a factorization result in non-commutative H^p spaces. Studying their proof I noticed that these two tools were not necessary and I found a shortcut to their proofs. This same shortcut was also noticed independently by Potapov and Sukochev. I present it here in section A.1.

These two results seem to answer completely the question. But I would like to raise some questions related to the constants appearing.

Notation A.3. Let us denote by ${S^p}_{sa}$ (resp. $S^p_{n,sa}$) the set of self-adjoint elements of S^p (resp. S^p_n).

For a function $f : \mathbb{R} \to \mathbb{R}$ we will denote by $\|f\|_{Lip,S^p}$ (resp. $\|f\|_{Lip,S^p_n}$) the best constant C such that the inequality $\|f(A) - f(B)\|_p \leq C\|A - B\|_p$ holds for any A, B self-adjoint in $B(\ell^2)$ such that $A - B \in S^p$ (resp. for any $A, B \in S^p_{n,sa}$ self-adjoint). Take $\|f\|_{Lip,S^p} = \infty$ (resp. $\|f\|_{Lip,S^p_n} = \infty$) if no such C exists.

The key observation is that these quantities $\|f\|_{Lip,S^p}$ and $\|f\|_{Lip,S^p_n}$ are closely related to the norms of Schur multipliers on S^p and S^p_n. Recall that the Schur multiplier with symbol $(a_{i,j})_{1\leq i,j\leq n}$ is the linear map on M_n mapping $(x_{i,j})$ to $(a_{i,j}x_{i,j})$, and denote by $\|(a_{i,j})\|_{Schur(S^p_n)}$ its norm. For an infinite matrix we use the similar notation $\|(a_{i,j})\|_{Schur(S^p)}$. In the same way we denote by $\|(a_{i,j})\|_{Schur(S^p_{n,sa})}$ and $\|(a_{i,j})\|_{Schur({S^p}_{sa})}$ their restrictions to the self-adjoint part of S^p_n and S^p. One way to relate operator Lipschitz functions with Schur multipliers is with the use of double operator integrals as defined by Birman and Solomyak (see the survey [5]). If one restricts to finite matrices, this is very simple. Take indeed self-adjoint $A, B \in M_n$ and a function $f : \mathbb{R} \to \mathbb{R}$, and decompose $A = UD_\lambda U^*$ and $B = V^*D_\mu V$ where U and V are unitary and D_λ (resp D_μ) is diagonal with diagonal entries $\lambda_1, \ldots, \lambda_n$ (resp. $\mu_1, \ldots, \mu_n$). If M is the Schur multiplier with symbol $(f(\lambda_i) - f(\mu_j))/(\lambda_i - \mu_j)$ if $\lambda_i \neq \mu_j$ and 0 otherwise (such a multiplier is called multiplier of divided differences), then

$$f(A) - f(B) = U \cdot M\,(U^*(A - B)V) \cdot V^*.$$

This immediatly implies that $\|f\|_{Lip,S^p_n}$ is less than the supremum over all λ_i and μ_j of the norm on S^p_n of the preceding Schur multiplier. In the following, in order to also get easily the reverse inequality, we also present an alternative approach by differentiation to relate operator-Lipschitz functions with Schur multipliers.

Let us first state some properties of operator-Lipschitz functions:

Proposition A.4. *For any $f : \mathbb{R} \to \mathbb{R}$, the following hold :*

- *for any $1 \leq p \leq \infty$, $\|f\|_{Lip,S^p_1} = \|f\|_{Lip}$.*
- *For any integer n and any $1 \leq p \leq \infty$,*

$$\|f\|_{Lip,S^p_n} \leq \|f\|_{Lip,S^p_{n+1}}.$$

$$\|f\|_{Lip,S^p} = \lim_{n\to\infty} \|f\|_{Lip,S^p_n}.$$

- *$\|f\|_{Lip,S^p_n} < \infty$ if f is Lipschitz, $n \in \mathbb{N}$ and $1 \leq p \leq \infty$*
- *$\|f\|_{Lip,S^2_n} = \|f\|_{Lip}$ and if $2 \leq p \leq q \leq \infty$,*

$$\|f\|_{Lip,S^p_n} \leq \|f\|_{Lip,S^q_n}.$$

- *if* $1/p + 1/p' = 1$*,* f *is operator Lipschitz on* S_n^p *(resp.* S^p*) if and only if* f *is operator Lipschitz on* $S_n^{p'}$ *(resp.* $S^{p'}$*) :*

$$\|f\|_{Lip,S_n^p} = \|f\|_{Lip,S_n^{p'}},$$

$$\|f\|_{Lip,S^p} = \|f\|_{Lip,S^{p'}}.$$

Proof. The first equality is obvious since the self-adjoint part of S_1^p coincides isometrically with $\mathbb{R}$. The second inequality $\|f\|_{Lip,S_n^p} \le \|f\|_{Lip,S_{n+1}^p}$ follows from the embedding of S_n^p into S_{n+1}^p for example in the upper-left corner of S_{n+1}^p. The third inequality is just an approximation argument and is left to the reader.

It is also easy to deduce from Lemma A.5 below that for any n, the quantities $\|f\|_{Lip,S_n^p}$ and $\|f\|_{Lip}$ are equivalent for a C^1 function f that is Lipschitz (the case f arbitrary Lipschitz follows by approximation). This will also be made more precise in Lemma A.14.

To prove the next inequality as well as the equality $\|f\|_{Lip,S_n^2} = \|f\|_{Lip}$ we can also restrict ourselves to the case when f is a C^1 function, and in this case the inequalities follow also from interpolation and Lemma A.5, since we even get, if $1/p = \theta/2 + (1-\theta)/q$, that

$$\|f\|_{Lip,S_n^p} \le \|f\|_{Lip,S_n^q}^{\theta} \|f\|_{Lip}^{1-\theta}.$$

For the last equalities, we can restrict to the finite-dimensional case, and by a density argument we can restrict to the case when f is of class C^1. Since the Schur-multipliers are self-dual (and more precisely since for a Schur multiplier with self-adjoint symbol $\phi_{i,j}$, $\|(\phi_{i,j})\|_{Schur(S_{n,sa}^p)} = \|(\phi_{i,j})\|_{Schur(S_{n,sa}^{p'})}$), the claim is a consequence of Lemma A.5 below. ☐

Lemma A.5. *If* f *is a* C^1 *function, then* $A \mapsto f(A)$ *is also a* C^1 *function on the set of self-adjoint* $n \times n$ *matrices, and its differential at a matrix* $A = diag(\lambda_1, \dots, \lambda_n)$ *(with* $\lambda_i \in \mathbb{R}$*) is equal to the Schur multiplier with symbol* $(f(\lambda_i) - f(\lambda_j))/(\lambda_i - \lambda_j)$ *if* $\lambda_i \neq \lambda_j$ *and* $f'(\lambda_i)$ *if* $\lambda_i = \lambda_j$*.*

For a general self-adjoint A*, if* $A = UA'U^*$ *for a unitary* U *and a diagonal* A'*, the differential* Df_A *at* A *is equal to* $AdU \circ Df_{A'} \circ AdU^*$*.*

Here we denote by AdU the conjugation by the unitary U: $AdU(B) = UBU^*$.

Proof. We first treat the case when f is a polynomial; by linearity it is enough to do it if $f(t) = t^m$ for $m \in \mathbb{N}$. Then for any self-adjoint $A, B \in M_n(\mathbb{C})$, we have

$$f(A+B) - f(A) = \sum_{k=0}^{m-1} A^k B A^{m-1-k} + o(B).$$

If $A = diag(\lambda_1, \dots, \lambda_n)$ is diagonal this equation becomes

$$f(A+B) - f(A) = \left(\sum_{k=0}^{m-1} \lambda_i^k \lambda_j^{m-1-k} b_{i,j} \right)_{i,j \le n} + o(B).$$

If $\lambda_i \neq \lambda_j$, we recognize $\sum_{k=0}^{m-1} \lambda_i^k \lambda_j^{m-1-k} = (\lambda_i^m - \lambda_j^m)/(\lambda_i - \lambda_j)$, whereas for $\lambda_i = \lambda_j$, $\sum_{k=0}^{m-1} \lambda_i^k \lambda_j^{m-1-k} = m\lambda_i^{m-1}$. The equation $Df_{AdU(A)} = AdU \circ Df_A AdU^*$ is obvious.

For a general f, we use an approximation argument: fix some interval I of the real line and take a sequence of polynomials P_k such that P_k and P_k' converge to f and f' uniformly on a neighbourhoud of I. From the first part for any k the differential of $A \mapsto P_k(A)$ at A is up to a unitary conjugation a Schur multiplier of divided differences. Since the derivative of P_k is uniformly bounded on I, we get a uniform bound on the differential of $A \mapsto P_k(A)$ at every A the spectrum of which lie in I. In praticular, if the spectrum of A lie in I, we get that

$$P_k(A+B) - P_k(A) = DP_{kA}(B) + o(B)$$

with a $o(B)$ uniform in k. If we decompose $A = UA'U^*$ with $A' = diag(\lambda_1, \dots, \lambda_n)$ diagonal and U unitary, then the differential DP_{kA} is equal to $AdU \circ S_k \circ AdU^*$ where S_k is the Schur multiplier with symbol $(P_k(\lambda_i) - P_k(\lambda_j))/(\lambda_i - \lambda_j)$ if $\lambda_i \neq \lambda_j$ and $P_k'(\lambda_i)$ if $\lambda_i = \lambda_j$. As $k \to \infty$, we have that $S_k \to S$ the Schur multiplier with symbol $(f(\lambda_i) - f(\lambda_j))/(\lambda_i - \lambda_j)$ if $\lambda_i \neq \lambda_j$ and $f(\lambda_i)$ if $\lambda_i = \lambda_j$. We therefore get

$$f(A+B) - f(A) = (AdU \circ S \circ AdU^*)(B) + o(B).$$

This concludes the proof. □

Theorem A.6. *Assume that* $\|f\|_{Lip} = 1$. *Then*

$$\left| \|f\|_{Lip, S_n^p} - \sup_{\lambda_1 < \dots < \lambda_n} \left\| \left(\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} 1_{i \neq j} \right) \right\|_{Schur(S^p_{n,sa})} \right| \leq 1.$$

Proof. When f is a C^1 function Lemma A.5 implies that

$$\|f\|_{Lip, S_n^p} = \sup_{\lambda_1 < \dots < \lambda_n} \left\| \left(\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} 1_{i \neq j} + f'(\lambda_i) 1_{i=j} \right) \right\|_{Schur(S^p_{n,sa})}.$$

The claim follows from the fact that the norm on S_n^p of the diagonal Schur multiplier with symbol $f'(\lambda_i)1_{i=j}$ is $\max_i |f'(\lambda(i)| \leq 1$.

A general 1-Lipschitz function f can be approximated uniformly by C^1 1-Lipschitz functions. Therefore by Lemma A.5 for any A that has n different eigenvalues $\lambda_1, \dots, \lambda_n$,

$$\limsup_{B \to A} \frac{\|f(A) - f(B)\|_p}{\|A - B\|_p} \leq 1 + \left\| \left(\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} 1_{i \neq j} \right) \right\|_{Schur(S^p_{n,sa})}.$$

We therefore get that

$$\frac{\|f(A) - f(B)\|_p}{\|A - B\|_p} \leq 1 + \sup_{\lambda_1 < \dots < \lambda_n} \left\| \left(\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} 1_{i \neq j} \right) \right\|_{Schur(S^p_{n,sa})}$$

if there are only finitely many matrices C on the segment $[A, B]$ with an eigenvalue with multiplicity greater than (or equal to 2). By a density argument this inequality is always true.

For the reverse inequality, by the same approximation argument as above we get that if $A = diag(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 < \dots < \lambda_n$

$$\limsup_{B \to A} \frac{\|f(A) - f(B)\|_p}{\|A - B\|_p} \geq \left\| \left(\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} 1_{i \neq j} \right) \right\|_{Schur(S^p_{n,sa})} - 1.$$

□

But note that for any linear map $u : S_n^p \to S_n^p$ we have that $\|u\| \le 2\|u\big|_{S_{n,sa}^p}\|$. It is therefore natural to introduce the following constants:

$$C_p(n) = \sup_{\|f\|_{Lip\le 1}} \sup_{\lambda_1<\cdots<\lambda_n} \left\|\left(\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}1_{i\neq j}\right)\right\|_{Schur(S_n^p)}$$

and similarly

$$C_p(\infty) = \sup_n C_p(n) = \lim_n C_p(n),$$

so that for all $n \ge 1$ and all $1 \le p \le \infty$

$$C_p(n)/2 - 1 \le \sup_{\|f\|_{Lip}\le 1} \|f\|_{Lip,S_n^p} \le 1 + C_p(n). \tag{A.1}$$

We also introduce

$$\widetilde{C}_p(n) = \sup_{\|f\|_{Lip\le 1}} \left\|\left(\frac{f(i)-f(j)}{i-j}1_{i\neq j}\right)\right\|_{Schur(S_n^p)}$$

and

$$\widetilde{C}_p(\infty) = \sup_n \widetilde{C}_p(n) = \lim_n \widetilde{C}_p(n).$$

The following properties are very easy:

Proposition A.7. *For any n and any $1 \le p \le \infty$, if $1/p' + 1/p \le 1$*

$$C_p(n) = C_{p'}(n).$$

Moreover for any $1 \le p \le \infty$,

$$\widetilde{C}_p(\infty) = C_p(\infty).$$

Proof. The first equality is obvious by duality; for the second one the inequality $\widetilde{C}_p(\infty) \le C_p(\infty)$ is also immediate. For the other direction note that

$$\widetilde{C}_p(N) = \sup_{\|f\|_{Lip\le 1}} \sup_{q\in\mathbb{N}} \left\|\left(\frac{f(i/q)-f(j/q)}{i/q-j/q}1_{i\neq j}\right)\right\|_{Schur(S_N^p)}$$

because the map $f \mapsto qf(\cdot/q)$ induces a bijection of the set of 1-Lipschitz real functions. On the other hand since $\mathbb{Q}$ is dense in $\mathbb{R}$ and by translation

$$C_p(n) = \sup_{\|f\|_{Lip\le 1}} \sup_{0<\lambda_1<\cdots<\lambda_n\in\mathbb{Q}} \left\|\left(\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}1_{i\neq j}\right)\right\|_{Schur(S_n^p)}.$$

The claim thus follows from the fact that if $(\phi_{i,j})_{i,j\le n}$ is a submatrix of $(\psi_{i,j})_{i,j\le N}$ then the norm $\|(\phi_{i,j})\|_{Schur(S_n^p)}$ is less than the norm $\|(\psi_{i,j})\|_{Schur(S_N^p)}$. ☐

The following result restates and precises Theorem A.2 and A.1 with the previous notation.

Theorem A.8. *If $1 < p < \infty$,*

$$C_p(\infty) < \infty.$$

For $p = 1$ or $p = \infty$ there is a constant C such that for $n \ge 2$,

$$C^{-1}\log n \le C_p(n) \le C\log n.$$

A.1 Operator Lipschitz functions on S^p

Here we prove Sukochev's and Potapov's Theorem A.2, or equivalently that with the notation introduced above, $C_p(n) \leq C_p$ for some constant C_p depending only on p. The proof is strongly related to the UMD property of S^p, and from the proof it is also easy to show that if E is a rearragement-invariant sequence space such that the corresponding space S_E is UMD, then all Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ remain Lipschitz on the self-adjoint part of S_E

Here are the two main ingredients from [47] we will use. The first one is a Fourier-transform trick (note that this kind of trick was already used in Lemma 1.7 of [23]):

Lemma A.9 ([47], Lemma 6). *There exists a function $g : \mathbb{R} \to \mathbb{C}$ such that:*

- $\int_{\mathbb{R}} |s|^m |g(s)| ds < \infty$ *for any* $m \in \mathbb{N}$.
- *for any* $0 < \lambda < \mu$ *we have*

$$\frac{\lambda}{\mu} = \int_{\mathbb{R}} g(s) \lambda^{is} \mu^{-is} ds.$$

The second ingredient is the following, which is a consequence of the vector-valued Marcinkiewicz multiplier theorem, due to Bourgain:

Lemma A.10 ([47], Lemma 5). *Let $1 < p < \infty$. There exists $K_p > 0$ such that for $s \in \mathbb{R}$, $n \in \mathbb{N}$ and if $M(s)$ the Schur multiplier on M_n with symbol $|k-l|^{is}$ (with the convention $0^{is} = 0$). Then*

$$\|M(s)\|_{S_n^p \to S_n^p} \leq K_p(1 + |s|).$$

Remark A.11. In [47] this lemma is stated for the Schur multiplier $|k-l|^{is} 1_{k>l}$, but since $|k-l|^{is} = |k-l|^{is} 1_{k<l} + |k-l|^{is} 1_{k>l}$ the above version follows from it.

Let us now prove the theorem:

Proof of Theorem A.2. As explained above it is enough to show that for any n, any $\lambda_1 < \cdots < \lambda_n$ and any function $f : \mathbb{R} \to \mathbb{R}$ with $\|f\|_{Lip} = 1$, if $\phi_{k,l} = (f(\lambda_k) - f(\lambda_l))/(\lambda_k - \lambda_l) 1_{k \neq l}$ for $\lambda_k, \lambda_l \leq n$, then $\|(\phi_{k,l})\|_{Schur(S_n^p)} \leq C_p$. In fact the proof gives that the completely bounded norm of this Schur multiplier is less than C_p. Since any 1-Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is the difference of two non-decreasing 1-Lipschitz functions, it is also enough to treat the case when f is non-decreasing (and even strictly increasing).

Then for $k \neq l$ since f is increasing and 1-Lipschitz we have $0 \leq \phi_{k,l} \leq 1$, and hence Lemma A.9 implies that

$$\phi_{k,l} = \int_{\mathbb{R}} g(s) |f(\lambda_k) - f(\lambda_l)|^{is} |\lambda_k - \lambda_l|^{-is} ds.$$

For any increasing sequence $\mu_1 < \cdots < \mu_n$ denote by $M(s, (\mu_k)_k)$ the Schur multiplier with symbol $|\mu_k - \mu_l|^{is}$, so that

$$M_\phi = \int_{\mathbb{R}} g(s) M(s, (f(\lambda_k))_k) M(-s, (\lambda_k)_k) ds.$$

Since $\int_{\mathbb{R}} |g(s)|(1+|s|)^2 ds < \infty$ if we prove that $M(s, (\mu_k)_k)$ is bounded on S_n^p with norm less than $K_p(1+|s|)$ (with K_p given by Lemma A.10) then we will be done since it would imply that

$$\|M_\phi\|_{S_n^p \to S_n^p} \leq K_p^2 \int_{\mathbb{R}} |g(s)|(1+|s|)^2 ds.$$

By a density argument it is enough to prove the bound on the norm of $M(s,(\mu_k)_k)$ when μ_k are all rational numbers. But then if N is an integer such that $N\mu_k \in \mathbb{Z}$, the equality $|\mu_k - \mu_l|^{is} = N^{-is}|N\mu_k - N\mu_l|^{is}$ implies that we can assume that $\mu_k \in \mathbb{Z}$ for all k (end even that $\mu_k \in \mathbb{N}$ by adding to the μ_k's a large enough number).

If $1 \leq \mu_1 < \cdots < \mu_n$ are integers then the matrix $(|\mu_k - \mu_l|^{is})_{1\leq k,l\leq n}$ is a submatrix of the matrix $(|k-l|^{is})_{1\leq k,l\leq \mu_n}$. The Schur multiplier $M(s,(\mu_k)_{1\leq k\leq n})$ is thus a restriction of the Schur multiplier $M(s,(k)_{1\leq k\leq \mu_n})$, which is just the multiplier $M(s)$ of Lemma A.10 with n replaced by μ_n and which is therefore bounded by $K_p(1+|s|)$. □

In fact, the above proof also applies to any semi-finite von Neumann algebra with a trace:

Proposition A.12. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a normal faithful trace τ. For $1 < p < \infty$ there is a constant $C_p < \infty$ such that for $f : \mathbb{R} \to \mathbb{R}$*

$$\|f\|_{Lip,L^p(\mathcal{M},\tau)} \leq C_p \|f\|_{Lip}$$

where $\|f\|_{Lip,L^p(\mathcal{M},\tau)}$ is defined as $\|f\|_{Lip,S^p}$ (replacing S^p by $L^p(\mathcal{M},\tau)$).

Proof. Assume $\|f\|_{Lip} \leq 1$. Since self-adjoint operators in $\mathcal{M}$ can be approximated in $L^p(\mathcal{M})$ by self-adjoint operators with finite spectrum, it is enough to prove that

$$\|f(A) - f(B)\|_p \leq C_p \|A - B\|_p$$

for any self-adjoint operators $A, B \in \mathcal{M}$ with finite spectrum (say $\lambda_1 < \cdots < \lambda_n$ and $\mu_1 < \cdots < \mu_m$). Let $A = \sum_i \lambda_i E_i$ and $B = \sum_i \mu_j F_j$ be its spectral decomposition, *i.e.* the E_i's (resp. the F_j's) are orthogonal projections with $\sum_i E_i = 1$ (resp. $\sum_j F_j = 1$). Then as explained in the introduction (and as is easy to check),

$$f(A) - f(B) = \sum_{i,j} \frac{f(\lambda_i) - f(\mu_j)}{\lambda_i - \mu_j} 1_{\lambda_i \neq \mu_j} E_i (A-B) F_j.$$

In other word if the elements of $X \in \mathcal{M}$ are viewed as block matrices $(E_i X F_j)_{i\leq n, j\leq m}$, then $f(A) - f(B)$ is the image of $A - B$ by the Schur multiplier with symbol $\frac{f(\lambda_i)-f(\mu_j)}{\lambda_i - \mu_j} 1_{\lambda_i \neq \mu_j}$, and the same proof as above shows that this Schur multiplier is completely bounded for the p-norm by some constant C_p. This yields the desired inequality. □

A.2 Operator Lipschitz functions on M_n

Here we present the known results on the order of growth of $C_\infty(n) = C_1(n)$.

As was observed at least in [13], this constant $C_\infty(n)$ is not bounded. Perhaps the easiest proof is the one that relates this constant $C_\infty(n)$ to the norm of the triangular projection on $M_n(\mathbb{C})$ (it is well-known that the norm of the triangular projection on $M_n(\mathbb{C})$ is of the order of $\log n$);

Lemma A.13. *Denote by T the triangular projection on $M_n(\mathbb{C})$: $T(a_{i,j}) = (a_{i,j} 1_{i\leq j})$. For any integer n*

$$C_\infty(2n) \geq 2\,\|T\|_{M_n(\mathbb{C})\to M_n(\mathbb{C})} - 1.$$

Note that the proof below also applies to the constant $C_p(2n)$, and is implies that $1 + C_p(\infty)$ is greater than 2 times the norm of the triangular projection of S^p.

Proof. Take $f(t) = |t|$ and $\lambda_k = (-A)^k$ for $k = 1, \dots 2n$, and consider the Schur multiplier with symbol $(f(\lambda_i) - f(\lambda_j))/(\lambda_i - \lambda_j)1_{i\neq j}$ which converges as $A \to \infty$ to $(-1)^{\max(i,j)}1_{i\neq j}$. We thus only have to show that the Schur multiplier with symbol $(-1)^{\max(i,j)}1_{i\neq j}$ has norm on $M_{2n}(\mathbb{C})$ greater than $2\|T\|_{M_n(\mathbb{C})\to M_n(\mathbb{C})} - 1$.

If $E_n = \{2, 4, \dots, 2n\}$ (resp. $O_n = \{1, 3, \dots, 2n-1\}$) denotes the set of even (resp. odd) integers less than $2n$, consider the submatrix $\left((-1)^{\max(i,j)}1_{i\neq j}\right)_{i\in O_n, j\in E_n}$. It is an $n \times n$ matrix, and if we take the indices in $\{1, \dots, n\}$, its (k, l) entry is 1 if $k \le l$ and -1 otherwise. In other words the Schur multiplier corresponding to this matrix is $-Id + 2T$. This concludes the proof. □

For the other direction, Farforovskaya proved the following inequality. This is outdated by Theorem A.18 below, but the proof is an example that questions on operator-Lipschitz functions are related to the triangular projection.

Lemma A.14 (Farforovskaya, [14])**.** *There is a constant $C > 0$ such that*

$$C_\infty(n) \le C(1 + \log n)^2.$$

Proof. It is enough to prove that $C_\infty(2^n) \le Cn^2$ for $n \ge 1$; in fact we prove by induction on $n \ge 1$ that

$$C_\infty(2^n) \le n\|T\|_{M_{2^n}(\mathbb{C})\to M_{2^n}(\mathbb{C})}. \tag{A.2}$$

For $n = 1$ this is follows from the observation that the multiplier with symbol $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has norm less than 1.

Assume that (A.2) for n and take $f : \mathbb{R} \to \mathbb{R}$ with $\|f\|_{Lip} = 1$ and $\lambda_1 < \dots < \lambda_{2^{n+1}}$. Decompose the matrix $(f(\lambda_i) - f(\lambda_j))/(\lambda_i - \lambda_j)$ into 4 $2^n \times 2^n$ block-matrices:

$$\begin{aligned} \left(\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}\right)_{i,j\le 2^{n+1}} &= \begin{pmatrix} \left(\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}\right)_{i,j\le 2^n} & \left(\frac{f(\lambda_i)-f(\lambda_{2^n+j})}{\lambda_i-\lambda_{2^n+j}}\right)_{i,j\le 2^n} \\ \left(\frac{f(\lambda_{2^n+i})-f(\lambda_j)}{\lambda_{2^n+i}-\lambda_j}\right)_{i,j\le 2^n} & \left(\frac{f(\lambda_{2^n+i})-f(\lambda_{2^n+j})}{\lambda_{2^n+i}-\lambda_i}\right)_{i,j\le 2^n} \end{pmatrix} \\ &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{aligned}$$

The norm of the Schur multiplier with symbol $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ is the maximum of the norms of the Schur multiplier with symbol A and the one with symbol D, which is less than $C_\infty(2^n)$.

The norm of the Schur multiplier with symbol $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ is the maximum of the norms of the Schur multiplier with symbol B and the one with symbol C. Let us for instance treat the case of C. For this write

$$\frac{f(\lambda_{2^n+i}) - f(\lambda_j)}{\lambda_{2^n+i} - \lambda_j} = \frac{f(\lambda_{2^n+i}) - f(\lambda_{2^n})}{\lambda_{2^n+i} - \lambda_j} + \frac{f(\lambda_{2^n}) - f(\lambda_j)}{\lambda_{2^n+i} - \lambda_j}.$$

For a fixed i, the total variation norm of the sequence indexed by j, $\frac{f(\lambda_{2^n+i})-f(\lambda_{2^n})}{\lambda_{2^n+i}-\lambda_j}$ is less than 1. In the same way, for any j the total variation norm of the sequence indexed by i, $\frac{f(\lambda_{2^n})-f(\lambda_j)}{\lambda_{2^{n+1}+1-i}-\lambda_j}$ is less than 1. Lemma A.15 below thus implies that

$$\left\|\left(\frac{f(\lambda_{2^n+i}) - f(\lambda_j)}{\lambda_{2^n+i} - \lambda_j}\right)\right\|_{Schur(M_{2^n})} \le 2\|T\|_{M_{2^n}}.$$

Putting the two inequalities together we get

$$C_\infty(2^{n+1}) \leq C_\infty(2^n) + 2\|T\|_{M_{2^n}},$$

which concludes the proof. □

Lemma A.15. *Take an integer n. For a (finite) sequence $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ denote by $\|a\|_{bv}$ its bounded variation norm,* i.e. $\|a\|_{bv} = |a_1| + \sum_{k=1}^{n-1} |a_{k+1} - a_k|$. *Then for any matrix of real number $(\phi_{i,j})_{1 \leq i,j \leq n}$*

$$\|(\phi_{i,j})\|_{Schur(M_n(\mathbb{C}))} \leq \|T\|_{M_{2^n}} \max_i \|(\phi_{i,j})_j\|_{bv}. \tag{A.3}$$

Moreover, the term $\max_i \|(\phi_{i,j})_i\|_{bv}$ *in* (A.3) *can be replaced by* $\max_i \|(\phi_{i,n+1-j})_j\|_{bv}$, $\max_j \|(\phi_{n+1-i,j})_i\|_{bv}$, *or* $\max_j \|(\phi_{i,j})_i\|_{bv}$.

Proof. We only prove the inequality with $\max_i \|(\phi_{i,j})_i\|_{bv}$, the others are proved similarly (or follow from the unitary invariance of the operator norm on $M_n(\mathbb{C})$).

Consider the norm on $\mathbb{C}^{n^2}$ given by $\max_i \|(\phi_{i,j})_j\|_{bv}$. It is enough to prove (A.3) when $(\phi_{i,j})$ is an extrem point of the corresponding norm, that is if for all i there is a $k_i \leq n$ such that $\phi_{i,j} = 1_{j \leq k_i}$. Since reordering the rows on the matrix $\phi_{i,j}$ does not change the norm $\|(\phi_{i,j})\|_{Schur(M_n(\mathbb{C}))}$, we can assume that the sequence k_i is non-decreasing. Then the matrix $(1_{j \leq k_i})$ is a submatrix of the matrix $(1_{j \leq i})_{i,j} \otimes (1)_{i,j \leq n}$. The norm of the corresponding Schur multiplier is thus less than $\|T \otimes \mathrm{id}\|_{M_n \otimes M_n} = \|T\|_{CB(M_n, M_n)} = \|T\|_{M_n \to M_n}$. □

The following is also true. This result is of course outdated by Theorem A.18 (but was proved earlier). We still include it in this thesis; the idea is similar to the proof of Theorem A.18.

Lemma A.16. *There is a constant C such that for all n*

$$\widetilde{C}_\infty(n) \leq C \log n.$$

In the proof, we will need the following classical result on Schur multipliers on $M_n(\mathbb{C})$ (see [43] for a proof).

Theorem A.17. *Let M_φ be the Schur multiplier on M_n with symbol $\varphi_{i,j} \in \mathbb{C}$ for $1 \leq i,j \leq n$. Then*

$$\|M_\varphi\|_{M_n(\mathbb{C}) \to M_n(\mathbb{C})} = \|M_\varphi\|_{CB(M_n(\mathbb{C}), M_n(\mathbb{C}))} = \inf_{x_i, y_j \in \ell^2, \langle x_i, y_j \rangle = \varphi_{i,j}} \sup_i \|x_i\| \sup_j \|y_j\|.$$

In particular, if $\sup_j \|(\varphi_{i,j})_i\|_{\ell^2} < 1$, then $\|M_\varphi\| < 1$.

Proof of Lemma A.16. We prove by induction on n that $\widetilde{C}_\infty(2^n) \leq 4n$. The inequality $\widetilde{C}_\infty(2) \leq C_\infty(2) \leq 1$ has already been proved in the proof of Lemma A.14.

Take $f : \mathbb{R} \to \mathbb{R}$ with $\|f\|_{Lip} = 1$, and denote by $\varphi_{i,j} = (f(i) - f(j))/(i-j) 1_{i \neq j}$ for all $i, j \leq 2^{n+1}$. Let $\varphi'_{i,j} = \varphi_{s(i),s(j)}$ where $s(i)$ is the smallest even integer greater than or equal to i: $s(2k) = 2k$ and $s(2k-1) = 2k$ for any integer k. Let $a_{i,j} = \varphi_{i,j} - \varphi'_{i,j}$. Then the matrix $(\varphi'_{i,j})_{i,j \leq 2^{n+1}}$ is constant equal to $\varphi_{2k,2l}$ in the 2×2 blocks $\{2k-1, 2k\} \times \{2l-1, 2l\}$, it therefore corresponds to the matrix $(\varphi_{2i,2j})_{i,j \leq 2^n} \otimes (1)_{i,j \leq 2}$. The Schur multiplier on $M_{2^{n+1}}(\mathbb{C})$ with symbol $\varphi'_{i,j}$ has thus a norm less that the completely bounded norm of the

Schur multiplier on $M_{2^n}(\mathbb{C})$ with symbol $\varphi_{2i,2j}$. Since the completely bounded norm of a Schur multiplier is equal to its norm we thus get that

$$\left\|(\varphi'_{i,j})_{i,j\leq 2^{n+1}}\right\|_{Schur(M_{2^{n+1}}(\mathbb{C}))} \leq \widetilde{C}_\infty(2^n).$$

We now dominate the norm $\left\|(a_{i,j})_{i,j\leq 2^{n+1}}\right\|_{Schur(M_{2^{n+1}}(\mathbb{C}))}$ using the criterion noted before the proof, We have have to bound $\|(a_{i,j})_i\|_{\ell^2}$ for any j. Note $|a_{i,j}| \leq 1$ if $i = j$ and $a_{i,j} \leq 2/|i-j|$ otherwise; this implies that $\|(a_{i,j})_i\|_{\ell^2} \leq \sqrt{1 + 2\sum_{n\geq 1} 4/n^2} \leq 4$. We therefore get that $\|(a_{i,j})\|_{Schur(M_{2^{n+1}})} \leq 4$.

These two inequalities together imply that

$$\left\|(\varphi_{i,j})_{i,j\leq 2^{n+1}}\right\|_{Schur(M_{2^{n+1}}(\mathbb{C}))} \leq 4 + \widetilde{C}_\infty(2^n).$$

Taking the supremum over f yields

$$\widetilde{C}_\infty(2^{n+1}) \leq 4 + \widetilde{C}_\infty(2^n).$$

This completes the proof. □

In fact the following was proved very recently in [37].

Theorem A.18. *There is a constant C such that for all $n \geq 2$.*

$$C_\infty(n) \leq C \log n.$$

Sketch of the proof. Since $C_\infty(n) = C_1(n)$ and since S^1_n is the convex hull of operators of rank one, it is enough to show that if T is a rank one and norm one operator, and if M is a Schur multiplier of divided differences, the sequence $s_1 \geq \cdots \geq s_n$ of singular values of $M(T)$ satisfy $s_k \leq C\|T\|_1/k$ for any k; *i.e.* $M(T)$ belongs to the weak S^1 space $S^{1,\infty}$ with $\|M\|_{1,\infty} \leq C\|T\|_1$.

A rank one operator in the unit ball of S^1_n has the form $(\xi_i\eta_j)_{i,j}$ with $\xi = (\xi_1, \ldots, \xi_n)$ and $\eta = (\eta_1, \ldots, \eta_n)$ in the unit ball of ℓ^2_n. We are thus left to study a matrix of the form

$$S = \left(\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}\xi_i\eta_j 1_{i\neq j}\right)_{i,j\leq n}$$

for a 1-Lipschitz function f and $\lambda_1 < \cdots < \lambda_n$, and to show that its restriction to a subspace of ℓ^2_n of codimension less than Ck has norm less than C/k for some constant C (of course not depending on n). For this it is enough to prove that there is a subspace of codimension at most Ck on which the restriction of S has Hilbert-Schmidt norm less than $C/\sqrt{k}$.

The first remark is that we can assume that $|\xi_i|, |\eta_j| \leq 1/\sqrt{k}$ for all i and j (because the number of i's (resp. j's) such that this does not hold is less than k).

Then the idea is, as in Lemma A.16, to use the fact that the coefficients $f(\lambda_i)-f(\lambda_j)/\lambda_i-\lambda_j$ "do not change much" locally, so that S is close to a matrix of small rank. More precisely it is possible to get a partition of $\{1, \ldots, n\}$ into at most k intervals $I_1, \ldots, I_k$ such that the following holds: for any $i \in \{1, \ldots, n\}$, if $i \in I_l = \{a, a+1, \ldots, b\}$ then denote $c(i) = (\lambda_a + \lambda_b)/2$ and $s(i) = \lambda_b - \lambda_a$. Then the matrix S is close (*i.e.* has distance less than $C/\sqrt{k}$ for the Hilbert-Schmidt norm) to the matrix

$$\left(\xi_i\eta_j\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - c(j)} 1_{s(i)>s(j)}\right) + \left(\xi_i\eta_j\frac{f(\lambda_i) - f(\lambda_j)}{c(i) - \lambda_j} 1_{s(j)>s(i)}\right).$$

It remains to check that these two matrices have rank less than $2k$. For example for the first one

$$\left(\xi_i\eta_j\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-c(j)}1_{s(i)>s(j)}\right)=\left(\frac{\xi_i f(\lambda_i)\eta_j}{\lambda_i-c(j)}1_{s(i)>s(j)}\right)-\left(\frac{\xi_i\eta_j f(\lambda_j)}{\lambda_i-c(j)}1_{s(i)>s(j)}\right),$$

and both these matrices have rank less than k because for any i, both $c(j)$ and $1_{s(i)>s(j)}$ remain constant when $j\in I_l$ for any l, and because multiplication on the right can only reduce the rank. □

A.3 Comment on the constants

Recall that the constants $C_\infty(n)$ and $C_p(\infty)$ satisfy the property in (A.1). We are interested in the following questions:

Question A.19. What is the order of growth of $C_p(\infty)$ as $p\to\infty$?

Question A.20. Does the condition $A-B\in S^1$ imply that $f(A)-f(B)\in S^{1,\infty}$ for a Lipschitz f?

This second question was raised in [37]. Nazarov and Peller's result (Theorem A.18 above) is a step in this direction since it proves that if $A-B$ is of rank one then $\|A-B\|_{S^{1,\infty}}\le C\|T\|_1$. Note that this does not answer the question beacause $\|\cdot\|_{S^{1,\infty}}$ is not a norm (does not satisfy the triangle inequality).

From the proof of Theorem A.2, the bound we get on $C_p(\infty)$ of the order K_p^2, where K_p is the constant appearing in Bourgain's vector valued Marcinkiewicz multiplier theorem for S^p.

In the proofs of section A.2, we saw that the behaviour of $C_n(\infty)$ is related to the norm of the triangular projection. On the other hand since the triangular projection has (completely bounded) norm of order $\log n$ on M_n, Theorem A.18 can be (artificially) stated as $C_p(n)\simeq\|T\|_{CB(S_n^p,S_n^p)}$ if $p=1$ or $p=\infty$. It is therefore natural to wonder whether this still holds for $1<p<\infty$ (for $p=2$ this is obvious, and as noted after Lemma A.13, the inequality $C_p(n)\gtrsim\|T\|_{S_n^p\to S_n^p}$ is known to hold in full generality).

Since the (completely bounded) norm of the triangular projection on S^p is of order p for $p>2$, this would imply that $C_p(\infty)$ is of order p, and hence it would reprove that $C_p(n)$ is of order $\log n$. This follows from Hölder's inequality, which implies that for $p=\log n$ and $A\in M_n(\mathbb{C})$, $\|A\|_\infty\le\|A\|_p\le\|A\|_\infty n^{1/p}=e\|A\|_\infty$.

On the other hand, as Pisier pointed out to me, if one can prove that $C_p(\infty)$ grows faster than p as $p\to\infty$, this would prove (by the real interpolation method) that the condition $A-B\in S^1$ does not imply that $A-B\in S^{1,\infty}$.

Thus these two questions are strongly related.

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