The length of the graph of an increasing function with an almost everywhere zero derivative.

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1 A general result.

We recall Lebesgue's theorem about the derivation of monotonic functions, whose proof is elementary and uses no more than the definition of derivative and the definition of a null set.

Theorem 1 *Let f be an increasing function defined on $[a, b]$. Then f has almost everywhere a finite derivative.*

Assuming Theorem (1), prove the following :

Problem 1 *Let f be a continuous, strictly increasing function, defined on $[a, b]$, with a zero derivative at almost every point of $[a, b]$. Then f is of bounded variation. Its graph has a length. Prove that this length is $b - a + f(b) - f(a)$.*

Remark : The conclusion is still true assuming only f non decreasing.

2 An example.

Let $0 < u < 1$, and $u = 1 - v$. Riesz and Nagy, in their book, Leçons d'Analyse Fonctionelle, give the construction of a function f, satisfying, when $u \neq 1/2$, the hypothesis of problem (1), with $[a, b] = [0, 1]$ and $f(0) = 0, f(1) = 1$. f is constructed by the following way:

Let be the sequence (f_n) of polygonal functions defined on $[0, 1]$ by

1. $f_1(t) = t$.
2. f_n is affine on each $[k2^{-n}, (k+1)2^{-n}]$ with $0 \leq k \leq 2^n - 1$.
   (a) $f_n\left(\frac{2k}{2^n}\right) = f_{n-1}\left(\frac{k}{2^{n-1}}\right)$
   (b) $f_n\left(\frac{2k+1}{2^n}\right) = uf_{n-1}\left(\frac{k}{2^{n-1}}\right) + vf_{n-1}\left(\frac{k+1}{2^{n-1}}\right)$

It can be easily shown that the sequence (f_n) is convergent. The function f is the limit of (f_n). The length of f_n's graph is

$$L_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \sqrt{1 + 4^n u^{2k} v^{2n-2k}}.$$

and the length of the graph of f is $\lim_{n\to\infty} L_n$. Using the result of problem (1) we know that (for $u \neq 1/2$), $\lim L_n = 2$. It is interesting to find a "direct" proof of this, ignoring the interpretation of L_n, as the length of f_n's graph.

Problem 2 *Discuss the existence and value of the following limit*

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \sqrt{1 + 4^n u^{2k} v^{2n-2k}}.$$

with u, v positive parameters.

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