A survey of Cartan Subgroups in Model Theory

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Plan

1. Cartan Subgroups, Albebraicaly Closed Fields and Groups of finite Morley rank
2. Cartan Subgroups and *o*-minimal Structures
3. Cartan Subgroups and *p*-adic Fields

Cartan subgroup

Definition

If G is an arbitrary group, a subgroup Q of G is called a **Cartan subgroup** *if it satisfies the two following conditions :*

1. *Q is nilpotent and maximal with this property ;*
2. *If $X \leq Q$ is a subgroup which is normal in Q and of finite index in Q, then X is of finite index in its normalizer $N_G(X)$.*

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Remark : If G is an infinite group then all its Cartan subgroup are infinite.
If not, $X = 1$ is subgroup of finite index in Q but X is not of finite index in its normalizer $N_G(X) = G$.

Generosity

Definition

Le G be a group and X a subset of G.

1. *We say that X is* **generic** *in G if G can be covered by finitely many translates of X.*

$$G = \bigcup_{i=1}^{n} g_i \cdot X \quad \text{with } g_1, ..., g_n \in G$$

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2. *We say that X is* **generous** *if the union of its conjugates $X^G = \bigcup_{g \in G} X^g$ is generic.*

Plan

Theorem

Let G be connected affine algebraic group over an algebraically closed field, and Q a subgroup of G.
The followings are equivalent :

1. *Q is a Cartan subgroup.*
2. *Q is the centralizer of a maximal torus.*

Cartan subgroups in algebraic groups

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Let G be connected affine algebraic group over an algebraically closed field, and Q a subgroup of G.
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Corollary

- *Every Cartan subgroup of G is algebraic and connected.*
- *Any two Cartan subgroups are conjugated.*

Jordan decomposition and groups of finite Morley rank

Theorem (Jordan decomposition)

Let G be commutative linear algebraic group.

1. *If $a \in G$, then it can be written as $a = a_s \cdot a_u$, where :*
   - *a_s is semi-simple (ie. diagonalizable in some extension);*
   - *a_u is unipotent (ie. $(a_u - I)$ is nilpotent).*
2. *$G = G_s \times G_u$*
   - *G_s is the set of semi-simple elements of G;*
   - *G_u is the set of unipotent elements of G.*

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Question: How do we get a similar result for groups of finite Morley rank?

In groups of finite Morley rank, we can define:

- x is semi-simple if x is contained in a Cartan subgroup.
- x is unipotent if $d(x)$ intersects no Cartan subgroup.

Carter subgroups

Definition

Let G be a group of finite Morley rank.
A subgroup of G is called a **Carter subgroup** *if it is definable, definably connected, nilpotent and of finite index in its normalizer in G.*

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Theorem

Let G be a group of finite Morley rank, and C a subgroup of G. The followings are equivalent:

1. *C is a Carter subgroup;*
2. *C is a connected Cartan subgroup.*

Connected solvable groups of finite Morley rank - I

Theorem (Frécon, 2006)

Let G be a connected solvabe group of finite Morlay rank and C be a subgroup of G.
C is a Carter subgroup iff C is nilpotent and selfnormalizing.

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Definition

Let G be a group and H be a subgroup.
H is called **anormal** *if for all $g \in G$, $g \in < H, H^g >$.*

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Theorem

Let G be a connected solvable group of finite Morley rank, and C a subgroup.
C is Carter subgroup iff C is a minimal anormal subgroup.

Connected solvable groups of finite Morley rank - II

Theorem (Frécon - 2006)

In a connected solvable group of finite Morley rank, there exist Carter subgroups and any two of them are conjugated.

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In a definable subgroup of a connected solvable group of finite Morley rank, there exist Carter subgroups and any two of them are conjugated.

Groups of finite Morley rank

Theorem (Frécon, Jaligot - 2005)

Let G be a group of finite Morley rank, then G contains a Carter subgroup. Futhermore any abelian divisible torsion subgroup of G is contained in a Carter subgroup of G.

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In any group of finite Morley rank, all generous Carter subgroups are conjugated.

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Conjecture

Carter subgroups of a group of finite Morley rank are conjugated.

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In any group of finite Morley rank, Carter subgroups are generous.

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Theorem (Otero, Baro, Jaligot - 2011)

In a definable group in an o-minimal structure, any Carter subgroup is the connected component of some Cartan subgroup.

Consequence: Cartan subgroups are definable.

Main result for o-minimal structure

Theorem (Otero, Baro, Jaligot - 2011)

Let G be a definable group in an o-minmal structure.

1. *G has Cartan subgroups;*
2. *G has finitely many conjugacy classes of Cartan subgroups;*
3. *only one of them is generous.*

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Example : $SL_2(\mathbb{R})$ has 2 conjugacy classes of Cartan subgroups:

- $Q_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}^\times \right\}$ is generous;
- $Q_{-1} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R} \text{ and } a^2 + b^2 = 1 \right\}$ is not generous.

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Cartan Subgroups in $SL_2(\mathbb{Q}_p)$

$$Q_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbb{Q}_p) \mid a \in \mathbb{Q}_p^\times \right\}$$

$$Q_\delta = \left\{ \begin{pmatrix} a & b \\ b\delta & a \end{pmatrix} \in SL_2(\mathbb{Q}_p) \mid a, b \in \mathbb{Q}_p \text{ and } a^2 - b^2\delta = 1 \right\}$$

where δ is a non-square in $\mathbb{Q}_p$.

Remark : For $p \neq 2$ (resp. $p = 2$), there are 4 cosets (resp. 8 cosets) of $(\mathbb{Q}_p^\times)^2$ in $\mathbb{Q}_p^\times$.

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Proposition

The group Q_1 and Q_δ are Cartan subgroups of $SL_2(\mathbb{Q}_p)$.

Cartan subgroups and conjugacy

Lemma

1. $Q_1^{SL_2(\mathbb{Q}_p)} = \{A \in SL_2(\mathbb{Q}_p) \mid tr(A)^2 - 4 \in (\mathbb{Q}_p^\times)^2\} \cup \{I, -I\}$

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2. *For any* $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$,

$$Q_\delta^{\ SL_2(\mathbb{Q}_p)} = \{A \in SL_2(\mathbb{Q}_p) \mid tr(A)^2 - 4 \in \delta \cdot (\mathbb{Q}_p^\times)^2\} \cup \{I, -I\}$$

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2. *For any* $\delta \in \mathbb{Q}_p^\times \setminus (\mathbb{Q}_p^\times)^2$*, there exist* $\mu_1, ..., \mu_n \in GL_2(\mathbb{Q}_p)$ *such that*

$$\bigcup_{i=1}^{n} Q_\delta^{\mu_i \cdot SL_2(\mathbb{Q}_p)} = \{A \in SL_2(\mathbb{Q}_p) \mid tr(A)^2 - 4 \in \delta \cdot (\mathbb{Q}_p^\times)^2\} \cup \{I, -I\}$$

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Consequence : If $\delta \equiv \delta' \mod (\mathbb{Q}_p^\times)^2$ then Q_δ and $Q_{\delta'}$ are conjugated in $GL_2(\mathbb{Q}_p)$. In particular they are isomorphic.

Complete description of $SL_2(\mathbb{Q}_p)$

$$U = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{Q}_p \right\} \cup \left\{ \begin{pmatrix} -1 & u \\ 0 & -1 \end{pmatrix} \mid u \in \mathbb{Q}_p \right\}$$

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Corollary

$$SL_2(\mathbb{Q}_p) = U^{SL_2(\mathbb{Q}_p)} \cup Q_1^{SL_2(\mathbb{Q}_p)} \cup \bigcup_{\delta \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2} \bigcup_{i=1}^{n} Q_\delta^{\mu_i \cdot SL_2(\mathbb{Q}_p)}$$

Remark : This is a disjoint union if we exclude $\{I, -I\}$.

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Corollary

$$SL_2(\mathbb{Q}_p) = U^{SL_2(\mathbb{Q}_p)} \cup Q_1^{SL_2(\mathbb{Q}_p)} \cup \bigcup_{\delta \in \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2} \bigcup_{i=1}^{n} Q_\delta^{\mu_i \cdot SL_2(\mathbb{Q}_p)}$$

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Theorem

The subgroups Q_1, Q_δ (for $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$) and the externally conjugates $Q_\delta^{\mu_i}$ (for $\mu_1, \ldots, \mu_n \in GL_2(\mathbb{Q}_p)$) are the only Cartan subgroups up to conjugacy of $SL_2(\mathbb{Q}_p)$.

Generosity of Cartan Subgroups

Lemma

1. *The set* $W = \{A \in SL_2(\mathbb{Q}_p) \mid v_p(tr(A)) < 0\}$ *is generic in* $SL_2(\mathbb{Q}_p)$.
2. *The set* $W' = \{A \in SL_2(\mathbb{Q}_p) \mid v_p(tr(A)) \geq 0\}$ *is not generic in* $SL_2(\mathbb{Q}_p)$.

Lemma

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Corollary

1. *The Cartan subgroup* Q_1 *is generous in* $SL_2(\mathbb{Q}_p)$.
2. *The Cartan subgroup* Q_δ *(for* $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$ *and* $\mu \in GL_2(\mathbb{Q}_p)$*) are not generous in* $SL_2(\mathbb{Q}_p)$.

Model-theoretic consequences

Corollary

Every infinite commutative definable subgroup of $SL_2(\mathbb{Q}_p)$ *has dimension 1.*

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Every infinite commutative definable subgroup of $SL_2(\mathbb{Q}_p)$ has dimension 1.

$$Z_\delta^n = \left\{ \begin{pmatrix} a & b \\ b\delta & a \end{pmatrix} \mid b \in p^n \delta \mathbb{Z}_p,\ a \in 1 + p^{2n} \delta \mathbb{Z}_p \text{ and } a^2 - b^2\delta = 1 \right\} \subseteq Q_\delta$$

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Corollary

If G is an infinite commutative definable subgroup of $SL_2(\mathbb{Q}_p)$ then it is one of the following subgroups: Q_1, Q_δ, U, Z_δ^n, U^n, a definable subgroup of Q_1 or one of their conjugates.

Model-theoretic consequences

Corollary

Every infinite nilpotent definable subgroup of $SL_2(\mathbb{Q}_p)$ has dimension 1.

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