Linear groups definable in p-minimal structures

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THEOREM

Let $\mathcal{L}$ be a language containing the language of fields and the exponential function, such that $\mathbb{Q}_p$ is a p-minimal structure in this language. Let $\mathcal{Q}_p$ be an $\mathcal{L}$-elementary extension of $\mathbb{Q}_p$.
If G is a commutative linear p-connected group $\mathcal{L}$-definable in $\mathcal{Q}_p$ then G is $\mathcal{L}$-definably isomorphic to a semi-algebraic linear group.

p-MINIMALITY AND DIMENSION

The p-minimality was introduced in [1], by Haskell and Macpherson in 1997, on the model of o-minimality for p-adics.

Definition 1. *Let $\mathcal{L}$ be a language extending $\mathcal{L}_d$ and let $\mathcal{K}$ be a $\mathcal{L}$-structure ($\mathcal{K}$ is a p-valued field whose value group is a $\mathbb{Z}$-group). We say that $\mathcal{K}$ is p-minimal if, for every $\mathcal{K}'$ elementarily equivalent to $\mathcal{K}$, every definable subset of K' is quantifier-free definable in $\mathcal{L}_d$.*

Example. Let $\mathcal{L}_{an}$ be the language of fields extended with all analytic restricted functions, $\mathbb{Q}_p$ in $\mathcal{L}_{an}$ is a p-minimal structure.

Definition 2. *Let X be a definable subset of K^n, $\dim X$ is the greatest integer r for which there is a projection $\pi : K^n \longrightarrow K^r$ such that $\pi(X)$ has non-empty interior in K^r.*

Fact 3. *Let X and Y be definable subsets of K^m :*

Additivity *If f is a definable function from X to Y, whose fibers have constant dimension $m \in \mathbb{N}$, then $\dim X = \dim(Im\ f) + m$;*

Finite sets *X is finite iff $\dim X = 0$;*

Monotonicity $\dim(X \cup Y) = \max\{\dim X, \dim Y\}$.

p-CONNEXITY

Definition 7. *Let G be group.*

- *We say that G is* p-connected *if it does not contain any subgroup of index coprime to p.*
- *We say that G is* p'-divisible *if for every n coprime to p and for all $x \in G$ there is $y \in G$ such that $x = y^n$.*

Proposition 8. *If G is a p'-divisible group then G is p-connected.*

Example. $\mathbb{Q}_p^+$ and $\mathbb{Z}_p^+$ are p-connected.

Proposition/Definition 9. *Let G be a group, and $G^\square$ a subgroups. TFAE:*

1. *$G^\square$ is the biggest p-connected normal subgroup of G ;*
2. *$G^\square$ is the intersection of all normal subgroups of G of index coprime to p.*

We call $G^\square$ the p-connected componant.

Example. If $G = \mathbb{Q}_p^\times$, then $G^\square = 1 + p\mathbb{Z}_p$.

p-ADIC EXPONENTIAL AND LOGARITHM

Proposition/Definition 4. *Let K a finite extension of $\mathbb{Q}_p$, we define :*

- *the* exponential *by* $\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}$, *its convergence domain is* $E_p = \{x \in K \mid v_p(x) > \frac{1}{p-1}\}$;
- *the* logarithm *by* $\log(1+x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} x^n$, *its convergence domain contains* $1 + E_p$.

Fact 5. *The fonction* exp *is an isomorphism from E_p^+ to $(1+E_p)^\times$, and* log *is its inverse.*

Application 6. *If K is a finite extension of $\mathbb{Q}_p$ then :*

$$K^\times = \mathbb{Z} \times k^\times \times (1 + \pi\mathcal{O})^\times$$

$(1+\pi\mathcal{O})^\times$ contains a subgroup of finite index isomorphic to ${\mathbb{Z}_p^+}^m$.

JORDAN DECOMPOSITION AND TORI

Let G be a algebraic subgroup of $GL_n(K)$, and $g \in G$, we say that:

- g is *semi-simple* if g is diagonalizable over some finite extension of K, we denote $G_s = \{g \in G \mid g \text{ is semi-simple }\}$;
- g is *unipotent*, if there exist m such that $(g - I)^m = 0$, we denote $G_u = \{g \in G \mid g \text{ is unipotent }\}$.

Fact 10 ([2]). *Let G a commutative algebraic linear group, then :*

$$G = G_s \times G_u$$

Fact 11. *We have $G_s = G_a \cdot G_d$ and $G_a \cap G_d$ is finite, where :*

- *G_d is a* split torus, *(elements of G_d are diaonalizable over K) ;*
- *G_a is an* anisotropic torus, *(elements of G_a are not digonalizable over K).*

SKETCH OF THE PROOF

Jordan decomposition for p-connected linear definable groups in $\mathcal{Q}_p$:

Lemma 12. *For $p \neq 2$, let H be an algebraic linear commutative group defined over $\mathcal{Q}_p$. We denote G the p-conected component of $H(\mathcal{Q}_p)$, then G is semi-algebraically isomorphic to :*

$$T \times (1 + p\mathcal{Z}_p)^{\times m} \times {\mathcal{Q}_p^+}^l$$

where T is an anisotropic torus over $\mathcal{Q}_p$.

Definable subgroups of $\mathcal{Z}_p^+$ and $\mathcal{Q}_p^+$:

Lemma 13. *The $\mathcal{L}$-definable subgroups of ${\mathcal{Z}_p^+}^m$ are semi-algebraic and semi-algebraically isomorphic to ${\mathcal{Z}_p}^{m'}$.*

Lemma 14. *The $\mathcal{L}$-definable subgroups of ${\mathcal{Q}_p^+}^l$ are semi-algebraic and semi-algebraically isomorphic to $\mathcal{Q}_p^{l_1} \times \mathcal{Z}_p^{l_2}$.*

What does anisotropic torus over $\mathcal{Q}_p$ look like?

Lemma 15. *If T is an anisotropic torus over $\mathcal{Q}_p$ of dimension n, then:*

$$T = \widetilde{T} \times T^\square$$

where $\widetilde{T} = res(T)$ and $T^\square$ contains a subgroup of finite index definably isomorphic (by exponential) to $\mathcal{Z}_p^n$.

WORK IN PROGRESS ...

The next step is to study nilpotent groups. We expect a similar result to be true for these groups...
What about language without exponential ? We should describe semi-algebraic subgroups of anisotropic torus ...

REFERENCES

[1] Dierdre Haskell and Dugald Macpherson. A versin of o-minimality for the p-adics. *Journal of Symbolic Logic*, 62(4):1075–1092, 1997.

[2] Armand Borel. *Linear Algebraic Groups*. Graduate Texts in Mathematics. Springer, second enlarged edition edition, 1991.