Cartan Subgroups and Generosity in $SL_2(\mathbb{Q}_p)$

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Abstract

We show that there exist a finite number of Cartan subgroups up to conjugacy in $SL_2(\mathbb{Q}_p)$ and we describe all of them. We show that the Cartan subgroup consisting of all diagonal matrices is generous and it is the only one up to conjugacy.

Keywords p-adic field ; Cartan subgroup ; generosity
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A subset X of a group G is *left-generic* if G can be covered by finitely many left-translates of X. We define similarly right-genericity. If X is G-invariant, then left-genericity is equivalent to right-genericity. This important notion in model theory was particulary developped by B. Poizat for groups in stable theories [**?**]. For a group of finite Morley-rank and X a definable subset, genericity is the same as being of maximal dimension [**?**, lemme 2.5]. The term generous was introduced in [**?**] to show some conjuguation theorem. A definable subset X of a group G is *generous* in G if the union of its G-conjugates, $X^G = \{x^g \mid (x,g) \in X \times G\}$, is generic in G.

In an arbitrary group G, we define a *Cartan subgroup* H as a maximal nilpotent subgroup such that every finite index normal subgroup $X \trianglelefteq H$ is of finite index in its normalizer $N_G(X)$. First we can remark that they are infinite because $N_G(1) = G$ and if H is finite then $\{1\}$ is of finite index in H. In connected reductive algebraic groups over an algebraically closed fields, the maximal torus is typically an example of a Cartan subgroup. Moreover it is the only one up to conjugation and it is generous. It has been remarked in [**?**] that, in the group $SL_2(\mathbb{R})$, also the Cartan subgroup consisting of diagonal matrices is generous. But it has also been remarked that in the case of $SL_2(\mathbb{R})$, there exists another Cartan subgroup, namely $SO_2(\mathbb{R})$, which is not generous.

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We will discuss here some apparently new remarks of the same kind in $SL_2(\mathbb{Q}_p)$. First we describe all Cartan subgroups of $SL_2(\mathbb{Q}_p)$. After we show that the Cartan subgroup consisting of diagonal matrices is generous and it is the only one up to conjugacy.

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Description of Cartan subgroups up to conjugacy

We note $v_p : \mathbb{Q}_p \longrightarrow \mathbb{Z}\bigcup\{+\infty\}$ the p-adic-valuation, and $ac : \mathbb{Q}_p^\times \longrightarrow \mathbb{F}_p$ the angular component defined by $ac(x) = res(p^{-v_p(x)}x)$ where $res : \mathbb{Q}_p \longrightarrow \mathbb{F}_p$ is the residue map.

With these notations, if $p \neq 2$, an element $x \in \mathbb{Q}_p^\times$ is a square if and only if $v_p(x)$ is even and $ac(x)$ is a square in $\mathbb{F}_p$. For $p = 2$, an element $x \in \mathbb{Q}_2$ can be written $x = 2^n u$ with $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_2^\times$, then x is a square if n is even and $u \equiv 1 (\mathrm{mod}\ 8)$ [**?**].

Fact 1 ([**?**]). *If $p \neq 2$, the group $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, it has for representatives $\{1, u, p, up\}$, where $u \in \mathbb{Z}_p^\times$ is such that $ac(u)$ is not a square in $\mathbb{F}_p$*

The group $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, it has for representatives $\{\pm1, \pm2, \pm5, \pm10\}$.

For any prime p, and any δ in $\mathbb{Q}_p^\times\backslash(\mathbb{Q}_p^\times)^2$, we put :

$$Q_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbb{Q}_p) \mid a \in \mathbb{Q}_p^\times \right\}$$
$$Q_\delta = \left\{ \begin{pmatrix} a & b \\ b\delta & a \end{pmatrix} \in SL_2(\mathbb{Q}_p) \mid a, b \in \mathbb{Q}_p \text{ and } a^2 - b^2\delta = 1 \right\}$$

Lemma 1.

$$\forall x \in Q_1\backslash\{I, -I\} \quad C_{SL_2(\mathbb{Q}_p)}(x) = Q_1$$
$$\forall x \in Q_\delta\backslash\{I, -I\} \quad C_{SL_2(\mathbb{Q}_p)}(x) = Q_\delta$$

The checking of these equalities is left to the reader.

Proposition 1. *The groups Q_1 and Q_δ are Cartan subgroups of $SL_2(\mathbb{Q}_p)$*

Proof. One checks easily that Q_1 is abelian and the normalizer of Q_1 is :

$$N_{SL_2(\mathbb{Q}_p)}(Q_1) = Q_1 \cdot < \omega > \qquad \text{where} \qquad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For X a subgroup of Q_1, if $g \in N_{SL_2(\mathbb{Q}_p)}(X)$ and $x \in X$, then, using lemma **??**

$$Q_1 = C_{SL_2(\mathbb{Q}_p)}(x) = C_{SL_2(\mathbb{Q}_p)}(x^g) = C_{SL_2(\mathbb{Q}_p)}(x)^g = Q_1^g$$

It follows that $N_{SL_2(\mathbb{Q}_p)}(X) = N_{SL_2(\mathbb{Q}_p)}(Q_1) = Q_1 \cdot <\omega>$ and if X of finite index k in Q_1, then X is of index $2k$ in $N_{SL_2(\mathbb{Q}_p)}(X)$. We can see that for t in Q_1, $t^\omega = \omega^{-1} t \omega = t^{-1}$ and thus $[\omega, t] = t^2$.

If we note Γ_i the descending central series of $N_{SL_2(\mathbb{Q}_p)}(Q_1)$, we have $\Gamma_0 = N_{SL_2(\mathbb{Q}_p)}(Q_1)$, $\Gamma_1 = [N_{SL_2(\mathbb{Q}_p)}(Q_1), N_{SL_2(\mathbb{Q}_p)}(Q_1)] = Q_1^2$ and $\Gamma_i = [N_{SL_2(\mathbb{Q}_p)}(Q_1), \Gamma_{i-1}] = Q_1^{2^i}$. Observing that $Q_1 \cong \mathbb{Q}_p^\times$, we can conclude that the serie of Γ_i is infinite because Q_1 is infinite and $Q_1^{2^i}$ is of finite index in $Q_1^{2^{i-1}}$. Thus $N_{SL_2(\mathbb{Q}_p)}(Q_1)$ is not nilpotent. By the normalizer condition for nilpotent groups, if Q_1 is properly contained in a nilpotent group K, then $Q_1 < N_K(Q_1) \leq K$, here $N_K(Q_1) = Q_1 \cdot <\omega>$ which is not nilpotent, a contradiction. It finishes the proof that Q_1 is a Cartan subgroup.

For $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$, we check similarly that the group Q_δ is abelian. Since for all subgroups X of Q_δ, $C_{SL_2(\mathbb{Q}_p)}(X) = Q_\delta$, it follows that $N_{SL_2(\mathbb{Q}_p)}(X) = N_{SL_2(\mathbb{Q}_p)}(Q_\delta) = Q_\delta$, and if X is of finite index in Q_δ then X is of finite index in its normalizer. By the normalizer condition for nilpotent groups, Q_δ is nilpotent maximal. □

Proposition 2. *1.* $Q_1^{SL_2(\mathbb{Q}_p)} = \left\{ A \in SL_2(\mathbb{Q}_p) \mid tr(A)^2 - 4 \in (\mathbb{Q}_p^\times)^2 \right\} \bigcup \{I, -I\}$

2. For any $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$, there exist $\mu_1, ... \mu_n \in GL_2(\mathbb{Q}_p)$ such that

$$\bigcup_{i=1}^{n} Q_\delta^{\mu_i \cdot SL_2(\mathbb{Q}_p)} = \left\{ A \in SL_2(\mathbb{Q}_p) \mid tr(A)^2 - 4 \in \delta \cdot (\mathbb{Q}_p^\times)^2 \right\} \bigcup \{I, -I\}$$

We put :

$$U = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{Q}_p \right\} \bigcup \left\{ \begin{pmatrix} -1 & u \\ 0 & -1 \end{pmatrix} \mid u \in \mathbb{Q}_p \right\} \text{ and } U^+ = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{Q}_p \right\}$$

If $A \in SL_2(\mathbb{Q}_p)$ satisfies $tr(A)^2 - 4 = 0$, then either $tr(A) = 2$ or $tr(A) = -2$, and A is a conjugate of an element of U. In this case, A is said *unipotent*. It follows, from Proposition **??** :

Corollary 3. *We have the following partition :*

$$SL_2(\mathbb{Q}_p) \backslash \{I, -I\} = (U \backslash \{I, -I\})^{SL_2(\mathbb{Q}_p)} \sqcup (Q_1 \backslash \{I, -I\})^{SL_2(\mathbb{Q}_p)} \sqcup \bigsqcup_{\delta \in \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2} \bigcup_{i=1}^{n} (Q_\delta^{\mu_i} \backslash \{I, -I\})^{SL_2(\mathbb{Q}_p)}$$

Remark. *If δ and δ' in $\mathbb{Q}_p^\times$ are in the same coset of $(\mathbb{Q}_p^\times)^2$, then, by Proposition* **??**, *if $x' \in Q_{\delta'}^{\mu'}$ with $\mu' \in GL_2(\mathbb{Q}_p)$, then there exists $x \in Q_\delta$, $\mu \in GL_2(\mathbb{Q}_p)$ and $g \in SL_2(\mathbb{Q}_p)$, such that $x' = x^{\mu \cdot g}$, thus, by lemma* **??**, *$Q_{\delta'} = C_{SL_2(\mathbb{Q}_p)}(x') = C_{SL_2(\mathbb{Q}_p)}(x)^{\mu \cdot g} = Q_\delta^{\mu \cdot g}$. Therefore the Corollary* **??** *makes sense.*

Proof of Proposition **??**. • If $A \in Q_1^{SL_2(\mathbb{Q}_p)}$, then there exists $P \in SL_2(\mathbb{Q}_p)$ such that

$$A = P \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} P^{-1}$$

with $a \in \mathbb{Q}_p^\times$. We have $tr(A) = a + a^{-1}$, so $tr(A)^2 - 4 = (a + a^{-1})^2 - 4 = (a - a^{-1})^2$ and $tr(A)^2 - 4 \in (\mathbb{Q}_p^\times)^2$.

Conversely , let A be in $SL_2(\mathbb{Q}_p)$ with $tr(A)^2 - 4$ a square. The caracteristic polynomial is $\chi_A(X) = X^2 - tr(A)X + 1$ and its discriminant is $\Delta = tr(A)^2 - 4 \in (\mathbb{Q}_p^\times)^2$, so χ_A has two distinct roots in $\mathbb{Q}_p$ and A is diagonalizable in $GL_2(\mathbb{Q}_p)$. There is $P \in GL_2(\mathbb{Q}_p)$, and $D \in SL_2(\mathbb{Q}_p)$ diagonal such that $A = PDP^{-1}$. If

$$P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

we put

$$\tilde{P} = \begin{pmatrix} \frac{\alpha}{det(P)} & \beta \\ \frac{\gamma}{det(P)} & \delta \end{pmatrix}$$

and we have $\tilde{P} \in SL_2(\mathbb{Q}_p)$ and $A = \tilde{P}D\tilde{P}^{-1} \in Q_1^{SL_2(\mathbb{Q}_p)}$.

• If A is in $Q_\delta^{\mu \cdot SL_2(\mathbb{Q}_p)} \backslash \{I, -I\}$ with $\mu \in GL_2(\mathbb{Q}_p)$, then $tr(A) = 2a$ and there exists $b \neq 0$ such that $a^2 - b^2\delta = 1$. So $tr(A)^2 - 4 = 4a^2 - 4 = 4(b^2\delta + 1) - 4 = (2b)^2\delta \in \delta \cdot (\mathbb{Q}_p^\times)^2$

Conversely we proceed as in the real case and the root $i \in \mathbb{C}$. The discriminant of χ_A, $\Delta = tr(A)^2 - 4$ is a square in $\mathbb{Q}_p(\sqrt{\delta})$, and the caracteristic polynomial χ_A has two roots in $\mathbb{Q}_p(\sqrt{\delta})$: $\lambda_1 = \alpha + \beta\sqrt{\delta}$ and $\lambda_2 = \alpha - \beta\sqrt{\delta}$ (with $\alpha, \beta \in \mathbb{Q}_p$). For the two eigen values λ_1 and λ_2, A has eigen vectors :

$$v_1 = \begin{pmatrix} x + y\sqrt{\delta} \\ x' + y'\sqrt{\delta} \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} x - y\sqrt{\delta} \\ x' - y'\sqrt{\delta} \end{pmatrix}$$

In the basis $\{(x, x'), (y, y')\}$, the matrix A can be written :

$$\begin{pmatrix} a & b \\ b\delta & a \end{pmatrix}$$

with $a, b \in \mathbb{Q}_p$. We can conclude that there exists $P \in GL_2(\mathbb{Q}_p)$ such that :

$$A = P \begin{pmatrix} a & b \\ b\delta & a \end{pmatrix} P^{-1}$$

We proved that $Q_\delta^{GL_2(\mathbb{Q}_p)} = \left\{A \in SL_2(\mathbb{Q}_p) \mid tr(A)^2 - 4 \in \delta \cdot (\mathbb{Q}_p^\times)^2\right\} \cup \{I, -I\}$.

Let us now study the conjugation in $GL_2(\mathbb{Q}_p)$ and in $SL_2(\mathbb{Q}_p)$. For the demonstration, we note : $S = SL_2(\mathbb{Q}_p)$, $G = GL_2(\mathbb{Q}_p)$ and $Ext(S) = \{f \in Aut(S) \mid f(M) = M^P \text{ for } M \in S, P \in G\}$, $Int(S) = \{f \in Aut(S) \mid f(M) = M^P \text{ for } M \in S, P \in S\}$. Let $P, P' \in G$ and $M \in S$ then :

$$M^P = M^{P'} \Leftrightarrow P^{-1}MP = P'^{-1}MP' \Leftrightarrow P'P^{-1}M = MP'P^{-1} \Leftrightarrow PP' \in C_G(M)$$

So P and P' define the same automorphism if and only if $P'P^{-1} \in C_G(S) = Z(G) = \mathbb{Q}_p \cdot I_2$, then $Ext(S) \cong GL_2(\mathbb{Q}_p)/Z(G) \cong PGL_2(\mathbb{Q}_p)$, and similarly $Int(S) \cong SL_2(\mathbb{Q}_p)/Z(S) \cong$

$PSL_2(\mathbb{Q}_p)$. It is known that $PGL_2(\mathbb{Q}_p)/PSL_2(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$. Finally $Int(S)$ is a normal subgroup of finite index in $Ext(S)$, and there exist $\mu_1, ..., \mu_n \in GL_2(\mathbb{Q}_p)$ such that :

$$Q_\delta^{GL_2(\mathbb{Q}_p)} = Q_\delta^{\mu_1 \cdot SL_2(\mathbb{Q}_p)} \cup ... \cup Q_\delta^{\mu_n \cdot SL_2(\mathbb{Q}_p)}$$

□

Theorem 4. *The subgroups Q_1, Q_δ (for $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$) and the externally conjugate $Q_\delta^{\mu_i}$ (for $\mu_1, ..., \mu_n \in GL_2(\mathbb{Q}_p)$) are the only Cartan subgroups up to conjugacy of $SL_2(\mathbb{Q}_p)$*

Proof. It is clear that the image of a Cartan subgroup by an automorphism is also a Cartan subgroup. For the demonstration we note $S = SL_2(\mathbb{Q}_p)$ and B the following subgroup of $SL_2(\mathbb{Q}_p)$:

$$B = \left\{ \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{Q}_p^\times, u \in \mathbb{Q}_p \right\}$$

With these notations, we can easily check for $g \in U \backslash \{I, -I\}$ that $C_S(g) = U$ and $N_S(U) = B$. Moreover it is known that every $q \in B$ can be written as $q = tu$ where $t \in Q_1$ and $u \in U$.

Consider K a Cartan subgroup of $SL_2(\mathbb{Q}_p)$. We will show that K is a conjugate of Q_1 or of one of the Q_δ^μ (for $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$ and $\mu \in GL_2(\mathbb{Q}_p)$). First we prove K cannot contain a unipotent element other than I or $-I$. Since a conjugate of a Cartan subgroup is still a Cartan subgroup, it suffices to show that $K \cap U = \{I, -I\}$.

In order to find a contradiction, let $u \in K$ be a element of U different from I or $-I$, u is in $K \cap B$. If $\alpha \in N_S(K \cap B)$, then we have that $u^\alpha \in K \cap B$, and since $tr(u^\alpha) = tr(u) = \pm 2$, u^α is still in U. Therefore $U = C_S(u) = C_S(u^\alpha) = C_S(u)^\alpha = U^\alpha$ and so α is in $N_S(U) = B$. It follows $N_S(K \cap B) \leq B$ and finally $N_K(K \cap B) = K \cap B$. By the normalizer condition $K \cap B$ cannot be proper in K, then $K \leq B$.

It is known (see for example [?, Lemma 0.1.10]) that if K is a nilpotent group and $H \trianglelefteq K$ a non trivial normal subgroup, then $H \cap Z(K)$ is not trivial. If we assume that $K \nleq U^+$, since $K \leq B = N_S(U^+)$, $K \cap U^+$ is normal in K, and so $K \cap U^+$ contains a non trivial element x of the center $Z(K)$. For $q \in K \backslash U^+$, there are $t \in Q_1 \backslash \{I\}$ and $u \in U$ such that $q = tu$. We have $[x, q] = I$ so $[x, t] = I$, so $t = -I$ because $C_S(x) = U$. Therefore $K \leq U$. Since K is maximal nilpotent and U abelian, $K = U$. But U is not a Cartan subgroup, because it is of infinite index in its normalizer B. A contradiction.

Since K does not contain a unipotent element, K intersects a conjugate of Q_1 or of one of the Q_δ^μ (for $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$ and $\mu \in GL_2(\mathbb{Q}_p)$) by Corollary **??**, we note Q this subgroup. Let us show that $K = Q$. Let be x in $K \cap Q$, and $\alpha \in N_K(K \cap Q)$, then $x^\alpha \in Q$, and, by lemma **??**, $Q = C_S(x^\alpha) = C_S(x)^\alpha = Q^\alpha$. Thus $\alpha \in N_S(Q)$, and $N_K(K \cap Q) \leq N_S(Q)$.

1rst case Q is a conjugate of Q_1, then $N_S(Q) = Q \cdot < \omega' >$ where $\omega' = \omega^g$ if $Q = Q_1^g$. We have also $\omega'^2 \in Q$ and $t^{\omega'} = t^{-1}$ for $t \in Q$. One can check that $N_S(Q \cdot < \omega' >) = Q \cdot < \omega' >$, if $\omega' \in K$ then $N_K(Q \cdot < \omega' > \cap K) = Q \cdot < \omega' > \cap K$, by normalizer condition $K \leq Q \cdot < \omega' >$. If we note n the nilpotency classe of K, and $t \in K \cap Q$ then $[\omega', \omega', ..., \omega', t] = t^{2^n} = 1$, so t is an n^{th} root of unity, so $K \cap Q$ and $K = (K \cap Q) \cdot < \omega >$ are finite. A contradiction, so $\omega' \notin K$. Then $N_K(Q \cap K) \leq Q \cap K$, it follows by normalizer condition that $K \leq Q$, and by maximality of K, $K = Q$.

2nd case Q is a conjuguate of Q_δ (for $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$), then $N_S(Q) = Q$. It follows similarly that $K = Q$.

□

Generosity of the Cartan subgroups

Our purpose is now to show the generosity of the Cartan subgroup Q_1. It follows from the next more general proposition :

Proposition 5. *1. The set $W = \{A \in SL_2(\mathbb{Q}_p) \mid v_p(tr(A)) < 0\}$ is generic in $SL_2(\mathbb{Q}_p)$.*

2. The set $W' = \{A \in SL_2(\mathbb{Q}_p) \mid v_p(tr(A)) \geq 0\}$ is not generic in $SL_2(\mathbb{Q}_p)$.

Proof. 1. We consider the matrices :

$$A_1 = I, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} 0 & -b^{-1} \\ b & 0 \end{pmatrix}$$

with $v_p(a) > 0$ and $v_p(b) > 0$.

We show that $SL_2(\mathbb{Q}_p) = \bigcup_{i=1}^4 A_i W$. Suppose there exists

$$M = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \in SL_2(\mathbb{Q}_p)$$

such that $M \notin \bigcup_{i=1}^4 A_i W$.

Since $M \notin A_1 W \bigcup A_2 W$, we have $x + v = \varepsilon$ and $y - u = \delta$ with $v_p(\varepsilon) \geq 0$ and $v_p(\delta) \geq 0$. Since $M \notin A_3 W$, we have $ax + a^{-1}v = \eta$ with $v_p(\eta) \geq 0$. We deduce $a(\varepsilon - v) + a^{-1}v = \eta$ and $v = \frac{\eta - a\varepsilon}{a^{-1} - a}$. Similarly, it follows from $M \notin A_4 W$ that $u = \frac{\theta - b\delta}{b^{-1} - b}$ with some θ such that $v_p(\theta) \geq 0$.

Since $v_p(a) > 0$, we have $v_p(a + a^{-1}) < 0$. From $v_p(\eta - a\varepsilon) \geq min\{v_p(\eta); v_p(a\varepsilon)\} \geq 0$, we deduce that $v_p(v) = v_p(\frac{\eta - a\varepsilon}{a + a^{-1}}) = v_p(\eta - a\varepsilon) - v_p(a + a^{-1}) > 0$. Similarly $v_p(u) > 0$. It follows that $v_p(x) = v_p(\varepsilon - v) \geq 0$ and $v_p(y) \geq 0$.

Therefore $v_p(det(M)) = v_p(xv - uy) \geq min\{v_p(xv), v_p(uy)\} > 0$ and thus $det(M) \neq 1$, a contradiction .

2. We show that the family of matrices $(M_x)_{x \in \mathbb{Q}_p^\times}$ cannot be covered by finitely many $SL_2(\mathbb{Q}_p)$-translates of W', where :

$$M_x = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Q}_p)$. Then $tr(A^{-1}M_x) = dx + ax^{-1}$. If $v_p(x) > max\{|v_p(a)|, |v_p(d)|\}$ then $v_p(tr(A^{-1}M_x)) < 0$ and $M_x \notin AW'$.

Therefore for every finite family $\{A_j\}_{i \le n}$, there exist $x \in \mathbb{Q}_p$ such that $M_x \notin \bigcup_{j=1}^{n} A_j W'$. □

Remark. *We remark that the sets W and W' form a partition of $SL_2(\mathbb{Q}_p)$. They are both definable in the field language because the valuation v_p is definable in $\mathbb{Q}_p$.*

Lemma 2. $W \subseteq Q_1^{SL_2(\mathbb{Q}_p)}$ *and for* $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$ *and* $\mu \in GL_2(\mathbb{Q}_p)$, $Q_\delta^{\mu \cdot SL_2(\mathbb{Q}_p)} \subseteq W'$

Proof. Let be $A \in SL_2(\mathbb{Q}_p)$ with $v_p(tr(A)) < 0$.
For $p \neq 2$, since $v_p(tr(A)) < 0$, $v_p(tr(A)^2 - 4) = 2v_p(tr(A))$ and $ac(tr(A)^2 - 4) = ac(tr(A)^2)$, so $tr(A)^2 - 4$ is a square in $\mathbb{Q}_p$.
For $p = 2$, we can write $tr(A) = 2^n u$ with $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^\times$. Then $tr(A)^2 - 4 = 2^{2n}(u^2 - 4 \cdot 2^{-2n})$. Since $n \le -1$, $u^2 - 4 \cdot 2^{-2n} \equiv u^2 \equiv 1 (\bmod 8)$, so $tr(A)^2 - A \in (\mathbb{Q}_2^\times)^2$.

In all cases, by the proposition **??**, $W \subseteq Q_1^{SL_2(\mathbb{Q}_p)}$ and ,by complementarity, $Q_\delta^{\mu \cdot SL_2(\mathbb{Q}_p)} \subseteq W'$. □

We can now conclude with the following corollary, similar to [**?**, Remark 9.8] :

Corollary 6. *1. The Cartan subgroup Q_1 is generous in $SL_2(\mathbb{Q}_p)$.*

2. The Cartan subgroups Q_δ^μ (for $\delta \in \mathbb{Q}_p^\times \backslash (\mathbb{Q}_p^\times)^2$ and $\mu \in GL_2(\mathbb{Q}_p)$) are not generous in $SL_2(\mathbb{Q}_p)$.

Structure of groups Q_δ

In this section, we will take some specific value for δ. δ will be one of the representative elements $\{\alpha, p, \alpha p\}$ for the non suare in $\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$. Everything we can do here stay true up to conjugacy for every δ.

With these notations, we can remark that $Q_\delta \subseteq SL_2(\mathbb{Z}_p)$. We put :

$$Z_{n,\delta} := \left\{ \begin{pmatrix} a & b \\ b\delta & a \end{pmatrix} \in SL_2(\mathbb{Q}_p) \mid b \in p^n \mathbb{Z}_p, a \in 1 + p^{2n}\delta\mathbb{Z}_p \text{ and } a^2 - b^2\delta = 1 \right\}$$

Remark. *For $\begin{pmatrix} a & b \\ b\delta & a \end{pmatrix} \in Z_{0,\delta}$, then : $b \in p^n \mathbb{Z}_p^\times$ iff $a \in 1 + p^{2n}\delta\mathbb{Z}_p^\times$.*

The groups $Z_{n,\delta}$ form an infinite descending chain of definable subgroups. Before to show the man theorem of this section, let us etablish some technical lemmas.

Lemma 3. *For $\delta \in \{p, \alpha p\}$ and $n \ge 0$ (or for $\delta = \alpha$ and $n \ge 1$).*

1. $Z_{n,\delta} / Z_{n+1,\delta} \cong \mathbb{Z}/p\mathbb{Z}$

2. If $x \in Z_{n,\delta} \backslash Z_{n+1,\delta}$ then $x^{p^k} \in Z_{n+k} \backslash Z_{n+k+1}$

References

[1] Elias Baro, Eric Jaligot, and Margarita Otero. Cartan subgroups of groups definable in o-minimal structures. arXiv:1109.4349v2 [math.GR], 2011.

[2] Eric Jaligot. Generix never gives up. *J. Symbolic Logic*, 71(2):599–610, 2006.

[3] Bruno Poizat. *Groupes stables*. Nur al-Mantiq wal-Ma'rifah [Light of Logic and Knowledge], 2. Bruno Poizat, Lyon, 1987. Une tentative de conciliation entre la géométrie algébrique et la logique mathématique. [An attempt at reconciling algebraic geometry and mathematical logic].

[4] Jean-Pierre Serre. *Cours d'arithmétique*. puf, 1970.

[5] Frank Wagner. *Stable groups*, volume 240. London Mathematical Society Lecure Note Series, 1997.