About fully-well-balanced schemes for shallow-water equations

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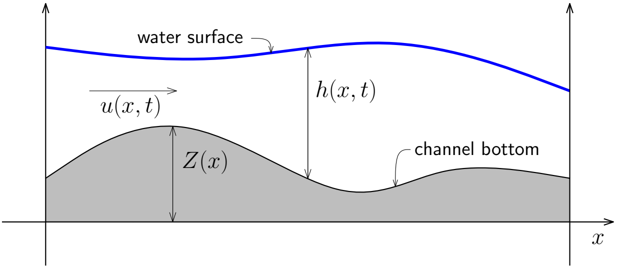
The shallow-water equations and their source terms

$$\begin{cases} \partial_t h + \partial_x(hu) = 0 \\ \partial_t(hu) + \partial_x\left(hu^2 + \dfrac{1}{2}gh^2\right) = -gh\partial_x Z \end{cases}$$

We can rewrite the equations as $\partial_t W + \partial_x F(W) = S(W)$, with $W = \begin{pmatrix} h \\ q \end{pmatrix}$.



Steady state solutions

Definition: Steady state solutions

W is a steady state solution iff $\partial_t W = 0$, i.e. $\partial_x F(W) = S(W)$.

In the shallow-water equations

$$\begin{cases} \partial_x q = 0 \\ \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z \end{cases}$$

The steady state solutions are given by

Bernoulli's equation

$$\begin{cases} q = \text{cst} \\ \dfrac{u^2}{2} + g(h+Z) = \text{cst} \end{cases}$$

Lake at rest

$$\begin{cases} u = 0 \\ h + Z = \text{cst} \end{cases}$$

Why Bernoulli instead of Lake at rest

Shallow-water with friction

$$\begin{cases} \partial_t h + \partial_x(hu) = 0 \\ \partial_t(hu) + \partial_x\left(hu^2 + \dfrac{1}{2}gh^2\right) = -gh\partial_x Z - \dfrac{kq|q|}{h^\eta} \end{cases}$$

Steady states

$$\begin{cases} \partial_x q = 0 \\ \partial_x\left(\dfrac{q^2}{h} + \dfrac{1}{2}gh^2\right) = -gh\partial_x Z - \dfrac{kq|q|}{h^\eta}. \end{cases}$$

Friction desapears as soon as the lake at rest is adopted

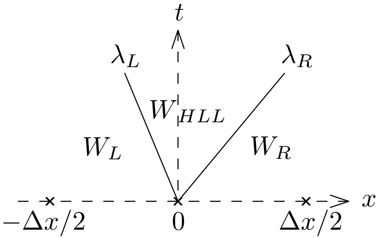
Contents

- Design a fully-well-balanced finite volume scheme
- Nonlinear scheme (robustness and entropy preserving)
- Linear scheme (robustness)
- High-order extensions

Non-exhaustive bibliography
Gosse (2000), *Castro* et al. (2007), *Fjordholm* et al. (2011), *Xing* et al. (2011)

The HLL scheme

To approximate solutions of
$\partial_t W + \partial_x F(W) = 0$
the HLL scheme
Harten, Lax, van Leer (1983)



Integral consistency condition (as per Harten and Lax)

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}(\Delta t, x; W_L, W_R)dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x; W_L, W_R)dx,$$

which gives $W_{HLL} = \dfrac{\lambda_R W_R - \lambda_L W_L}{\lambda_R - \lambda_L} - \dfrac{F(W_R) - F(W_L)}{\lambda_R - \lambda_L} = \begin{pmatrix} h_{HLL} \\ q_{HLL} \end{pmatrix}.$

We impose $h_L > 0$ and $h_R > 0$

Modification of the HLL scheme

$$\frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2}\left(\partial_t W_{\mathcal{R}} + \partial_x F(W_{\mathcal{R}}) \qquad\qquad \right)\,dx\,dt = 0$$

Modification of the HLL scheme

$$\frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2}(\partial_t W_{\mathcal{R}} + \partial_x F(W_{\mathcal{R}}) \qquad\qquad\qquad)\, dx\, dt = 0$$

$$\begin{aligned}0 =& \frac{1}{\Delta t}\frac{1}{\Delta x}\left(\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x)dx - \int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(0, x)dx\right) + \\ & \frac{1}{\Delta t}\frac{1}{\Delta x}\left(\int_0^{\Delta t} F(W_{\mathcal{R}})\left(t, -\frac{\Delta x}{2}\right)dt - \int_0^{\Delta t} F(W_{\mathcal{R}})\left(t, \frac{\Delta x}{2}\right)dt\right)\end{aligned}$$

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x)dx = \frac{W_L + W_R}{2} - \frac{\Delta t}{\Delta x}(F(W_R) - F(W_L))$$

Modification of the HLL scheme

Harten-Lax consistency condition to $\partial_t W + \partial_x F(W) = S(W)$

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}(\Delta t, x; W_L, W_R)dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x; W_L, W_R)dx,$$

first step: compute $\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}(\Delta t, x)dx$ (straightforward)

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}(\Delta t, x)dx = \frac{W_L + W_R}{2} - \lambda_R \frac{\Delta t}{\Delta x}(W_R - W_R^*) + \lambda_L \frac{\Delta t}{\Delta x}(W_L - W_L^*)$$

second step: compute $\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x)dx$

Modification of the HLL scheme

$$\frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2}\left(\partial_t W_{\mathcal{R}} + \partial_x F(W_{\mathcal{R}}) - S(W_{\mathcal{R}})\right)\,dx\,dt = 0$$

Modification of the HLL scheme

$$\frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2}\left(\partial_t W_{\mathcal{R}} + \partial_x F(W_{\mathcal{R}}) - S(W_{\mathcal{R}})\right)\,dx\,dt = 0$$

$$\begin{aligned} 0 =& \frac{1}{\Delta t}\frac{1}{\Delta x}\left(\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x)dx - \int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(0, x)dx\right) + \\ & \frac{1}{\Delta t}\frac{1}{\Delta x}\left(\int_0^{\Delta t} F(W_{\mathcal{R}})\left(t, -\frac{\Delta x}{2}\right)dt - \int_0^{\Delta t} F(W_{\mathcal{R}})\left(t, \frac{\Delta x}{2}\right)dt\right) - \\ & \frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\Delta x/2}^{\Delta x/2} S(W_{\mathcal{R}})(t,x)\,dx\,dt \end{aligned}$$

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x)dx \simeq \frac{W_L + W_R}{2} - \frac{\Delta t}{\Delta x}\left(F(W_R) - F(W_L)\right) + \overline{S}\Delta t$$

Modification of the HLL scheme

4 unknowns to be determined: $W_L^* = \begin{pmatrix} h_L^* \\ q_L^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q_R^* \end{pmatrix}$

Relevant definition for $\overline{S}$ to approximate the source term $-gh\partial_x Z$

- Harten-Lax consistency gives us the following two relations:
  - $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$ (relation 1)
  - $q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}$ (relation 2)

Modification of the HLL scheme

Definition of $\overline{S}$ comes from relation 2

Steady state preserving: $\overline{S}$ is defined by enforcing

$$\begin{cases} h_L u_L = h_R u_R \\ \dfrac{(u_L)^2}{2} + g(h_L + Z_L) = \dfrac{(u_R)^2}{2} + g(h_R + Z_R) \end{cases} \quad \text{then} \quad \begin{cases} h_L^\star = h_L & u_L^\star = u_L \\ h_R^\star = h_R & u_R^\star = u_R \end{cases}$$

to get

$$-\Delta x \overline{S} = \left(h_R u_R^2 + g\frac{h_R^2}{2} \right) - \left(h_L u_L^2 + g\frac{h_L^2}{2} \right)$$

By involving the steady state assumptions

$$\Delta x \overline{S} = -g\frac{h_L h_R}{\bar{h}}(Z_R - Z_L) - g\frac{(h_R - h_L)^3}{4\bar{h}} \qquad \text{with} \qquad \bar{h} = \frac{h_L + h_R}{2}$$

$\hookrightarrow$ It is a necessary condition to recover all steady states

Modification of the HLL scheme

Steady state smoothness condition

$$\Delta x \overline{S} = -g \frac{h_L h_R}{\bar{h}} (Z_R - Z_L) - g \frac{(h_R - h_L)^3}{4\bar{h}}$$

To recover steady states but modifying the Rankine-Hugoniot relations

Smoothness correction

$$C_h \quad \text{such that} \quad \sup_{h \text{ smooth}} |\partial_x h| \leq C_h$$

$$\delta h = \begin{cases} h_R - h_L & \text{if } |h_R - h_L| \leq C_h \Delta x \\ C_h \Delta x & \text{otherwise} \end{cases}$$

To suggest the approximation

$$\Delta x \overline{S} = -g \frac{h_L h_R}{\bar{h}} (Z_R - Z_L) - g \frac{\delta h^3}{4\bar{h}}$$

$\hookrightarrow$ Only smooth steady states can be reached

Modification of the HLL scheme

Two relations are missing to solve the approximate Riemann Solver

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x \left(\frac{h}{u}^2 + \frac{1}{2} g h^2 \right) - g h \partial_x Z = 0 \\ \partial_t Z = 0 \end{cases}$$

Stationary wave: Riemann invariants

$$hu \qquad \text{and} \qquad \frac{u^2}{2} + g(h+Z)$$

To get the two missing relations

$$h_L^\star u_L^\star = h_R^\star u_R^\star$$

$$\frac{(u_L^\star)^2}{2} + g(h_L^\star + Z_L) = \frac{(u_R^\star)^2}{2} + g(h_R^\star + Z_R) \qquad \text{or a linearization}$$

Nonlinear scheme

Characterization of the intermediate states

$$\begin{cases} h_L u_L - h_R u_R = \lambda_L (h_L - h_L^\star) + \lambda_R (h_R^\star - h_R) \\ \left(h_L u_L^2 + g \dfrac{h_L^2}{2} \right) - \left(h_R u_R^2 + g \dfrac{h_R^2}{2} \right) - \Delta x \overline{S} = \\ \qquad\qquad \lambda_L (h_L u_L - h_L^\star u_L^\star) + \lambda_R (h_R^\star u_R^\star - h_R u_R) \\ h_L^\star u_L^\star = h_R^\star u_R^\star \\ \dfrac{(u_L^\star)^2}{2} + g(h_L^\star + Z_L) = \dfrac{(u_R^\star)^2}{2} + g(h_R^\star + Z_R) \end{cases}$$

To get

$$h_L^\star u_L^\star = h_R^\star u_R^\star = q^\star(w_L, w_R) \qquad \text{(explicit)}$$

and

$$\lambda_L (h_L - h_L^\star) + \lambda_R (h_R^\star - h_R) + q_R - q_L = 0$$

$$\frac{q^\star}{2} \left(\frac{1}{(h_L^\star)^2} - \frac{1}{(h_R^\star)^2} \right) + g(h_L^\star - h_R^\star) + g(Z_L - Z_R) = 0$$

Then $h_L^\star$ solution of a polynomial of degree 5 denoted $p_5(h_L^\star) = 0$

Nonlinear scheme

Theorem

Assume w_L and w_R in Ω

There exists $\lambda_L < 0$ and $\lambda_R > 0$ (large enough) such that p_5 admits 5 roots exactly. One of these roots, denotes $h_L^\star$, satisfies

$$0 < h_L^\star < \frac{\lambda_R - \lambda_L}{-\lambda_L}\left(\frac{\lambda_R h_R - \lambda_L h_L}{\lambda_R - \lambda_L} - \frac{q_R - q_L}{\lambda_R - \lambda_L}\right)$$

$$h_L^\star = h_L \qquad \text{if } w_L \text{ and } w_R \text{ define a steady state}$$

In addition $h_R^\star$ satisfies

$$0 < h_R^\star < \frac{\lambda_R - \lambda_L}{\lambda_R}\left(\frac{\lambda_R h_R - \lambda_L h_L}{\lambda_R - \lambda_L} - \frac{q_R - q_L}{\lambda_R - \lambda_L}\right)$$

$$h_R^\star = h_R \qquad \text{if } w_L \text{ and } w_R \text{ define a steady state}$$

Remark

The steady state property is not satisfied by the other roots

Corollary

The resulting Godunov type scheme is positive and full well-balanced

Nonlinear Scheme: Discrete entropy inequality

General principle (Gallice 03, Chalons et al 10)
Conservation laws with source term $\partial_t w + \partial_x f(w) = S(w)$
Entropy inequalities $\partial_t \eta(w) + \partial_x G(w) \leq \sigma(w)$
Approximate Riemann solver (with constant intermediate states w_ℓ)
consistent with the entropy inqualities if

$$\sum_{k=1}^{\ell} \lambda_\ell(\eta(w_{\ell+1}) - \eta(w_\ell)) \geq G(w_R) - G(w_L) - \Delta x \tilde{\sigma}(\Delta x, \Delta t, w_L, w_R)$$

Objective: Establish

$$\lambda_L(\eta(w_L^\star) - \eta(w_L)) + \lambda_R(\eta(w_R) - \eta(w_R^\star)) \leq G(w_R) - G(w_L) - \Delta x \tilde{\sigma}(\Delta x, \Delta t, w_L, w_R)$$

where

$$\eta(w) = h\frac{u^2}{2} + g\frac{h^2}{2} \quad G(w) = \left(h\frac{u^2}{2} + gh^2\right)u \qquad \lim_{\substack{\Delta x,\, \Delta t \to 0 \\ w_L,\, w_R \to w}} \tilde{\sigma} = -ghu\Delta x Z$$

Nonlinear Scheme: Discrete entropy inequality

Behaviors of the intermediate states

$$h_L^\star = h^{HLL} - \alpha(Z_R - Z_L)\frac{(h^{HLL})^3}{\tilde{q}^2/2 - (h^{HLL})^3} + (Z_R - Z_L)\varepsilon(Z_R - Z_L)$$

$$h_R^\star = h^{HLL} + (1-\alpha)(Z_R - Z_L)\frac{(h^{HLL})^3}{\tilde{q}^2/2 - (h^{HLL})^3} + (Z_R - Z_L)\varepsilon(Z_R - Z_L)$$

$$q^\star = \tilde{q} - \frac{g}{\lambda_R - \lambda_L}\frac{h_L h_R}{\bar{h}}(Z_R - Z_L) \qquad \tilde{q} = q^{HLL} + \frac{g}{\lambda_R - \lambda_L}\frac{\delta h^3}{4\bar{h}}$$

$w^{HLL} = (h^{HLL}, q^{HLL})$ constant intermediate state coming from HLL scheme

$$(\lambda_R \eta(w_R) - \lambda_L \eta(w_L)) - (\lambda_R - \lambda_L)\eta(w^{HLL}) + g\frac{h_L h_R}{\bar{h} h^{HLL}} q^{HLL}(Z_R - Z_L)$$
$$+ (Z_R - Z_L)\varepsilon(Z_R - Z_L) + \mathcal{O}(\Delta x^3)$$

But we have $\eta(w^{HLL}) \leq \dfrac{\lambda_R \eta(w_R) - \lambda_L \eta(w_L)}{\lambda_R - \lambda_L} - \dfrac{1}{\lambda_R - \lambda_L}(G(w_R) - G(w_L))$

To obtain the required entropy inequality up to

$(Z_R - Z_L)\varepsilon(Z_R - Z_L) = \Delta x \varepsilon(\Delta x)$

The nonlinear scheme turns out to be impossible to be coded !

Linearization of the Riemann Invariant preservation relations

Example

$$h_L^* u_L^* = h_R^* u_R^*$$
$$h_L^* \frac{u_L^2}{2h_L} + g(h_L^* + z_L) = h_R^* \frac{u_R^2}{2h_R} + g(h_R^* + z_R)$$

to get

$$h_L^* u_L^* = h_R^* u_R^* = q^{HLL} - \frac{\Delta x}{\lambda_R - \lambda_L}\, \bar{S}$$
$$h_L^* = \frac{(\lambda_R - \lambda_L)\left(g + \frac{u_R^2}{2h_R}\right) h^{HLL} + g\lambda_R(z_R - z_L)}{\lambda_R\left(g + \frac{u_L^2}{2h_L}\right) - \lambda_L\left(g + \frac{u_R^2}{2h_R}\right)} \quad > 0$$
$$h_R^* = \frac{(\lambda_R - \lambda_L)\left(g + \frac{u_L^2}{2h_L}\right) h^{HLL} - g\lambda_L(z_L - z_R)}{\lambda_R\left(g + \frac{u_L^2}{2h_L}\right) - \lambda_L\left(g + \frac{u_R^2}{2h_R}\right)} \quad > 0$$

A linearized scheme

Objective: Get another *easy* fully-well-balanced scheme

Assume that W_L and W_R define a steady state

$$\begin{cases} \partial_x q = 0 \\ \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -g h \partial_x Z \end{cases}$$

with $[X] = X_R - X_L$, we set

$$\frac{1}{\Delta x} \left(q_0^2 \left[\frac{1}{h} \right] + \frac{g}{2} \left[h^2 \right] \right) = \overline{S}.$$

which rewrittes

$$q_0^2 \left(\frac{1}{h_R} - \frac{1}{h_L} \right) + \frac{g}{2} \left((h_R)^2 - (h_L)^2 \right) = \overline{S} \Delta x.$$

Determination of h_L^* and h_R^*

The intermediate water heights satisfy the following relation:

$$-(q^\star)^2\left(\frac{h_R^* - h_L^*}{h_L^* h_R^*}\right) + \frac{g}{2}(h_L^* + h_R^*)(h_R^* - h_L^*) = \overline{S}\Delta x.$$

Instead of the above relation, we choose the following linearization:

$$\frac{-(q^*)^2}{h_L h_R}(h_R^* - h_L^*) + \frac{g}{2}(h_L + h_R)(h_R^* - h_L^*) = \overline{S}\Delta x,$$

which can be rewritten as follows:

$$\underbrace{\left(\frac{-(q^*)^2}{h_L h_R} + \frac{g}{2}(h_L + h_R)\right)}_{\alpha}(h_R^* - h_L^*) = \overline{S}\Delta x.$$

Determination of h_L^* and h_R^*

With the consistency relation between h_L^* and h_R^*, the intermediate water heights satisfy the following linear system:

$$\begin{cases} \alpha(h_R^* - h_L^*) = \overline{S}\Delta x, \\ \lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}. \end{cases}$$

Using both relations linking h_L^* and h_R^*, we obtain

$$\begin{cases} h_L^* = h_{HLL} - \dfrac{\lambda_R \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}, \\ h_R^* = h_{HLL} - \dfrac{\lambda_L \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}, \end{cases}$$

where $\alpha = \left(\dfrac{-(q^*)^2}{h_L h_R} + \dfrac{g}{2}(h_L + h_R) \right)$ with $q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}$.

Correction to ensure non-negative h_L^* and h_R^*

However, these expressions of h_L^* and h_R^* do not guarantee that the intermediate heights are non-negative: instead, we use the following cutoff (see Audusse, Chalons, Ung (2014)):

$$\begin{cases} h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \bar{S} \Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right), \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \bar{S} \Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right). \end{cases}$$

Note that this cutoff does not interfere with:

- the consistency condition $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$;
- the well-balance property, since it is not activated when W_L and W_R define a steady state.

Summary

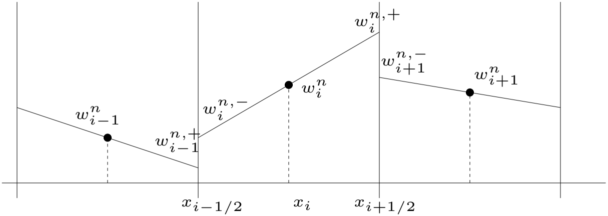
The two-state approximate Riemann solver with intermediate states $W_L^* = \begin{pmatrix} h_L^* \\ q^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q^* \end{pmatrix}$ given by

$$\begin{cases} q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}, \\ h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right), \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right), \end{cases}$$

is consistent, non-negativity-preserving and well-balanced

Second-order MUSCL schemes derivation

- Piecewise constant approximations



- MUSCL : van Leer(79)

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}(F^{\pm}_{i+1/2} - F^{\pm}_{i-1/2}) \qquad F^{\pm}_{i+1/2} = F(w_{i+1}^{n,-}, w_i^{n,+})$$

Inner approximations $w_i^{n,\pm} = w^h(x_{i\pm 1/2}, t^n) = w_i^n + \Delta w_i^{n,\pm}$

$\Delta w_i^{n,\pm}$ given by a limitation procedure

Well-balance high-order fully well-balanced scheme: MUSCL

Avoid to solve Bernoulli's equation within the reconstruction step
reconstruction procedure $\rightsquigarrow$ scheme no longer well-balanced

Well-balance recovery

We suggest a convex combination between the high-order scheme W_{HO} and the well-balanced scheme W_{WB}:

$$W_i^{n+1} = \theta_i^n (W_{HO})_i^{n+1} + (1 - \theta_i^n)(W_{WB})_i^{n+1},$$

with θ_i^n the parameter of the convex combination, such that:

- if $\theta_i^n = 0$, then the (infinity-order) well-balanced scheme is used;
- if $\theta_i^n = 1$, then the second-order scheme is used.

next step: derive a suitable expression for θ_i^n

Choice of θ_i^n

Introduce the steady states error evaluations

$$\hat{e}_i^n = \max\big(|q_i^n - q_{i-1}^n|, |q_{i+1}^n - q_i^n|\big)$$

$$\check{e}_i^n = \max\big(|\Phi_i^n - \Phi_{i-1}^n|, |\Phi_{i+1}^n - \Phi_i^n|\big) \qquad \text{with} \qquad \Phi = \frac{q^2}{2h^2} + g(h+Z)$$

Fix $\varepsilon_m = 10^{-12}$ a measure of the machine precision

Theorem

Introduce the following two conditions:

(C_1) $\quad \hat{e}_i^n < \varepsilon_m$ and $\check{e}_i^n < \varepsilon_m$

(C_2) $\quad |h_i^{n+1,MUSCL} - h_i^n| \leq (e_h)_i^n$ and $|q_i^{n+1,MUSCL} - q_i^n| \leq (e_q)_i^n$

$$\text{Define } \theta_i^n = \begin{cases} 0 & \text{if } (\mathcal{C}_1) \text{ or } (\mathcal{C}_2) \text{ holds,} \\ 1 & \text{otherwise.} \end{cases}$$

Then the scheme $W_i^{n+1} = \theta_i^n (W_{MUSCL})_i^{n+1} + (1-\theta_i^n)(W_{WB})_i^{n+1}$ is fully well-balanced and second-order accurate.

Choice of θ_i^n

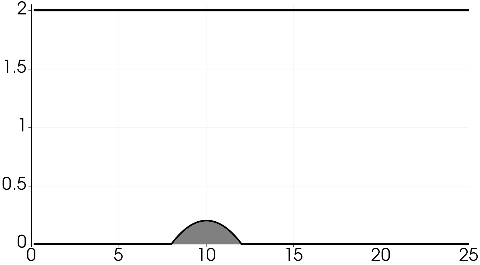
We have fixed

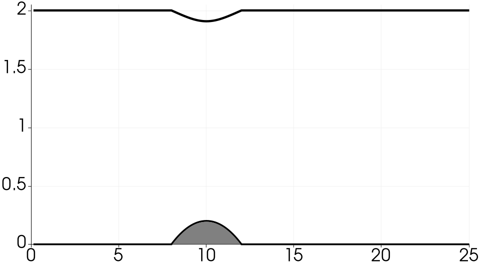
$$(e_h)_i^n = \hat{e}_i^n \Delta t \Delta x \frac{\Delta t}{\Delta x} \frac{q_i^n}{(h_i^n)^3} + \check{e}_i^n \Delta t \Delta x \frac{\Delta t^2}{\Delta x^2} \frac{q_i^n}{(h_i^n)^2} + \frac{\Delta x^3}{(h_i^n)^2}$$

$$(e_q)_i^n = \hat{e}_i^n \Delta t \Delta x \frac{q_i^n}{(h_i^n)^3} + \check{e}_i^n \Delta t \Delta x \frac{\Delta t}{\Delta x} \frac{q_i^n}{(h_i^n)^2} + \Delta x^3 \frac{q_i^n}{(h_i^n)^3}$$

Verification of the well-balance: topography

subcritical flow test case (see Goutal, Maurel (1997))





left panel: initial free surface at rest; water is injected from the left boundary

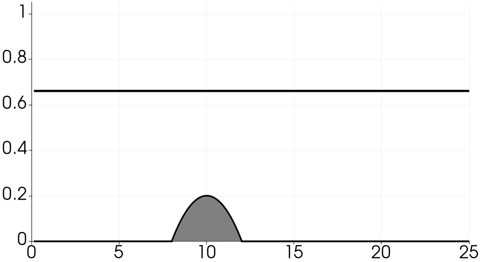
right panel: free surface for the steady state solution, after a transient state

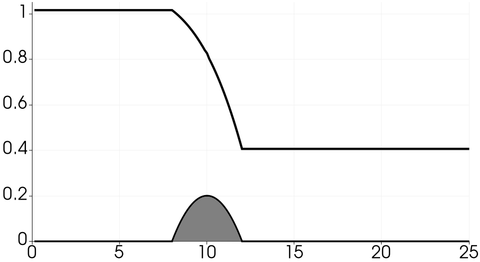
$$\mathcal{E} = \frac{u^2}{2} + g(h + Z)$$

	L^1	L^2	L^∞
errors on q	6.65e-14	6.99e-14	8.26e-14
errors on $\mathcal{E}$	1.18e-13	1.25e-13	1.53e-13

Verification of the well-balance: topography

transcritical flow test case (see Goutal, Maurel (1997))





left panel: initial free surface at rest; water is injected from the left boundary

right panel: free surface for the steady state solution, after a transient state

$$\mathcal{E} = \frac{u^2}{2} + g(h+Z)$$

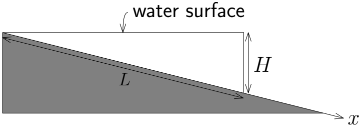
	L^1	L^2	L^∞
errors on q	1.47e-14	1.58e-14	2.04e-14
errors on $\mathcal{E}$	1.67e-14	2.13e-14	4.26e-14

Order of accuracy verification

N	WB		MUSCL		θ-WB	
25	5.46e-01	—	2.89e-01	—	2.94e-01	—
50	2.84e-01	0.94	2.84e-02	3.34	2.41e-02	3.61
100	1.55e-01	0.87	7.36e-03	1.95	5.99e-03	2.01
200	8.11e-02	0.94	1.90e-03	1.95	1.51e-03	1.99
400	4.10e-02	0.98	5.15e-04	1.88	4.41e-04	1.78

Table : L^2-error on Φ for the approximation of a smooth solution.

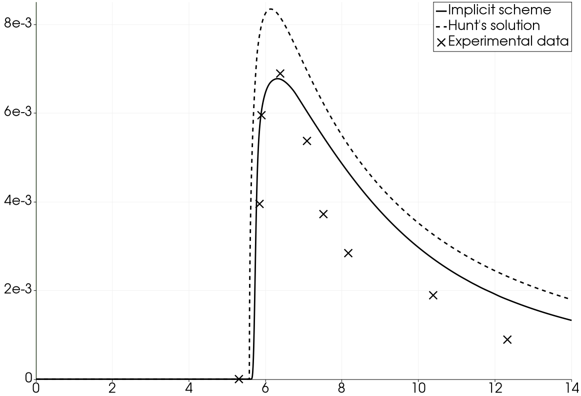
Dry dam-break: Hunt's asymptotic solution



↑

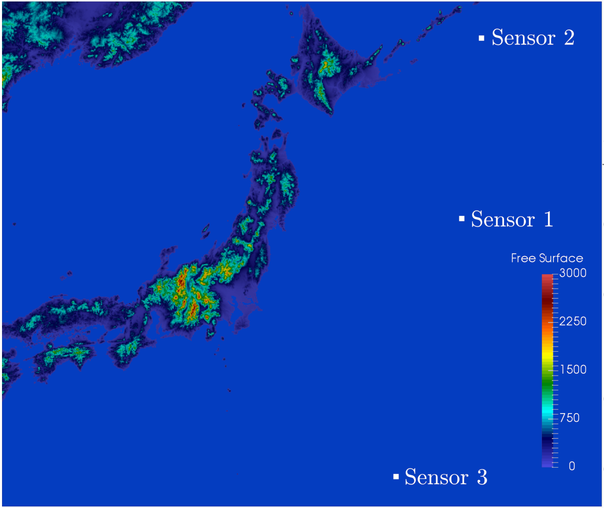
initial condition for the dry dam-break on a sloping channel

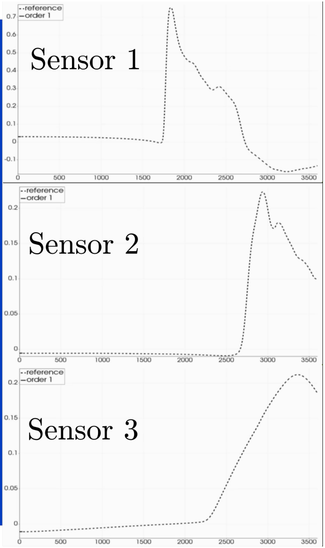
water height with respect to the time at a fixed position →



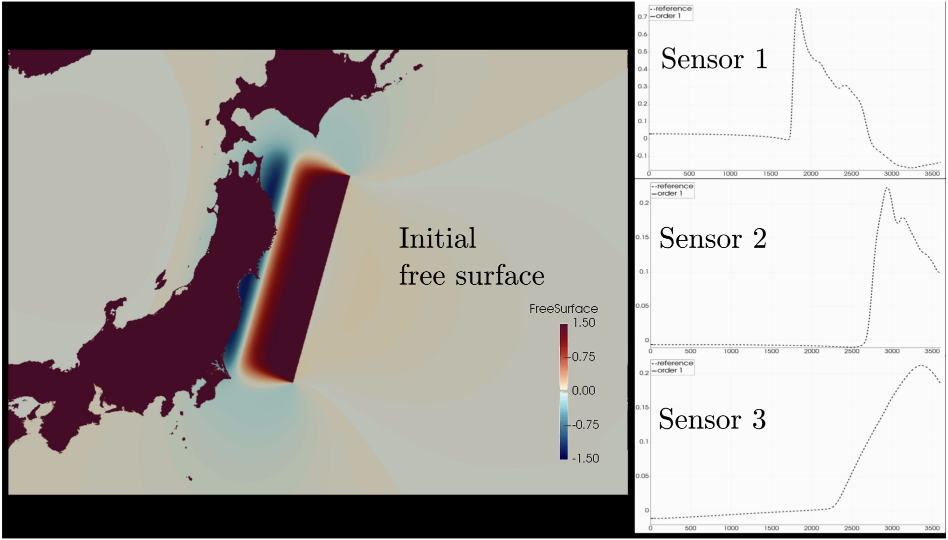
See Hunt (1984) for the experimental points and the solution, valid far enough away from the initial dam.

Simulation of the 2011 Tōhoku tsunami

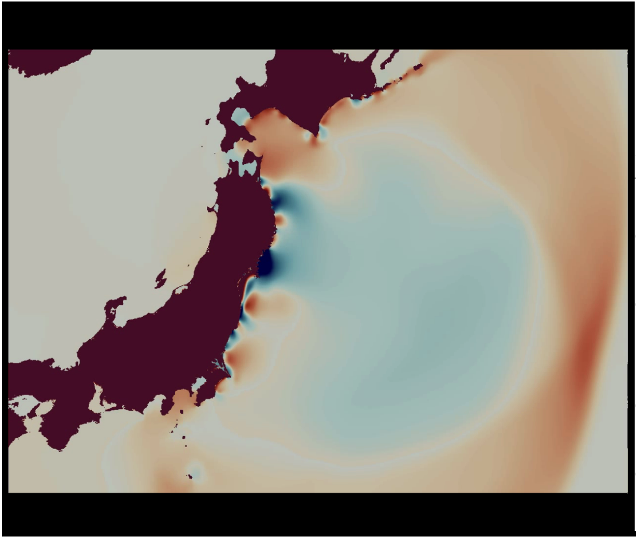


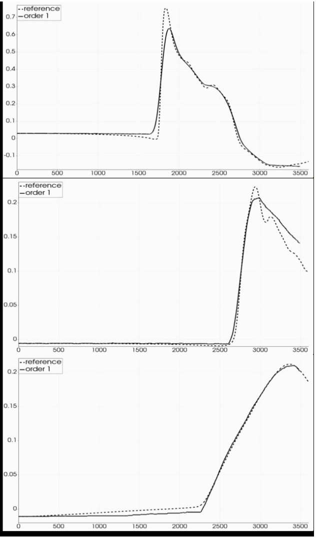


Simulation of the 2011 Tōhoku tsunami



Simulation of the 2011 Tōhoku tsunami





Thank you for your attention!