Discrete transparent boundary conditions for the linearized Green - Naghdi system of equations

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Overview

- Linear Green - Naghdi model
- Boundary condition derivation
  - Exact continuous boundary conditions
    - Derivation : System (detailed) & Boussinesq
    - Stability
  - Discrete boundary conditions
    - Derivation : Boussinesq (detailed) & System
    - Consistency
- Test cases : outgoing and incoming wave
- Conclusions & Perspectives

Green - Naghdi equations : Linearization

We consider the classical Green - Naghdi system for the propagation of dispersive waves on shallow water:

$$\begin{cases} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + \frac{gh^2}{2} + \frac{1}{3}h^2\ddot{h})_x = 0, \end{cases} \qquad x \in \mathbb{R}, t > 0$$

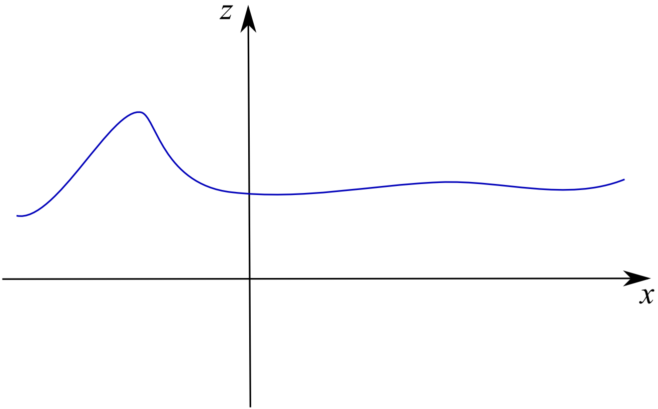
Linearized on the steady state $h = \mathrm{const}, u = 0$ nondimensionalized system takes form:

$$\begin{cases} \eta_t + w_x = 0, \\ w_t + \eta_x - \varepsilon w_{txx} = 0, \end{cases} \qquad x \in \mathbb{R}, t > 0$$

here ε is a parameter of dispersion. The coupled system describes the evolution of free surface elevation η and average fluid velocity w.

Motivation

- Restriction of the observation area.



Boundary conditions:

Hyperbolic system - *Riemann invariant*

Dispersive system - ?

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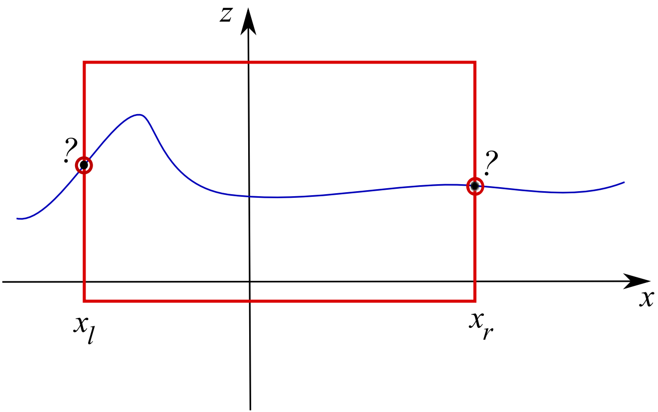

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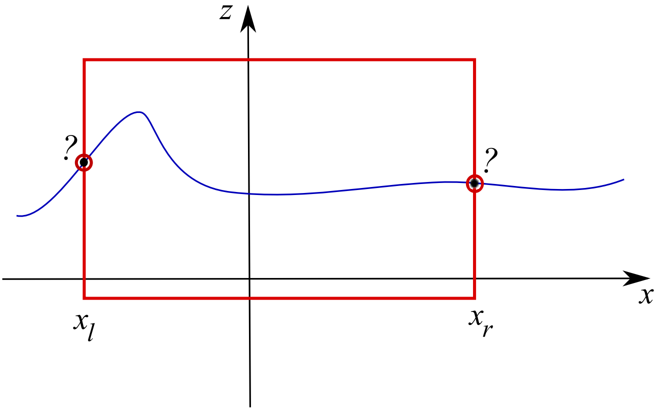


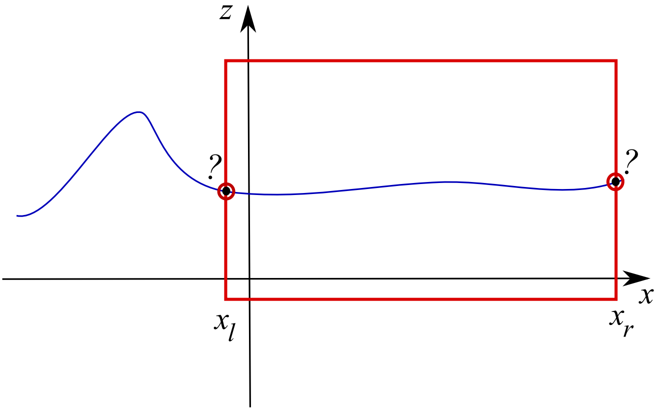
Boundary conditions:

Hyperbolic system - *Riemann invariant*

Dispersive system - *Schrödinger equation, non-reflecting ABC for WE*

Motivation

- Restriction of the observation area.



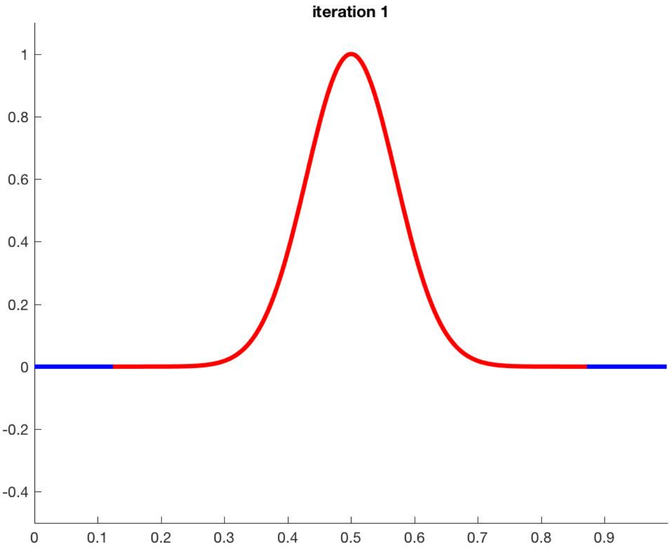
Domain decomposition for coupling different models

Generation of the waves *Schrödinger equation*

Objective Generation of the waves for the dispersive water waves problem

Numerical implementation

Results



Exact boundary conditions

We consider the initial value problem set on the whole space

$$\begin{aligned} \eta_t + w_x = 0, &\quad \forall x \in \mathbb{R}, \forall t > 0 \\ w_t + \eta_x - \varepsilon w_{txx} = 0, &\quad \forall x \in \mathbb{R}, \forall t > 0 \\ \eta(0,x) = \eta_0(x), \quad w(0,x) = w_0(x), &\quad \forall x \in \mathbb{R} \end{aligned}$$

$$\lim_{x\to\pm\infty} w(t,x) = \lim_{x\to\pm\infty} \eta(t,x) = 0,$$

where the initial data η_0, w_0 are compactly supported functions in a finite interval $[x_\ell, x_r]$. The problem set on the complement of $[x_\ell, x_r] \subset \mathbb{R}$:

$$\begin{aligned} \tilde{\eta}_t + \tilde{w}_x = 0, &\quad \forall x \in \mathbb{R} \setminus [x_\ell, x_r], \forall t > 0 \\ \tilde{w}_t + \tilde{\eta}_x - \varepsilon \tilde{w}_{txx} = 0, &\quad \forall x \in \mathbb{R} \setminus [x_\ell, x_r], \forall t > 0 \\ \tilde{\eta}(0,x) = 0, \quad \tilde{w}(0,x) = 0, &\quad \forall x \in \mathbb{R} \setminus [x_\ell, x_r] \end{aligned}$$

$$\tilde{\eta}(t,x_\ell) = \eta(t,x_\ell), \tilde{w}(t,x_\ell) = w(t,x_\ell), \tilde{\eta}(t,x_r) = \eta(t,x_r), \tilde{w}(t,x_r) = w(t,x_r),$$
$$\lim_{x\to\pm\infty} \tilde{w}(t,x) = \lim_{x\to\pm\infty} \tilde{\eta}(t,x) = 0.$$

Exact boundary conditions

Applying the Laplace transform :

$$
\begin{gathered}
s\mathcal{L}(\tilde{\eta}) + \mathcal{L}(\tilde{w}_x) = 0, \\
s\mathcal{L}(\tilde{w}) + \mathcal{L}(\tilde{\eta}_x) - \varepsilon s\mathcal{L}(\tilde{w}_{xx}) = 0.
\end{gathered}
$$

The solutions of the system have the from

$$
\left(\begin{array}{c} \mathcal{L}(\tilde{w})(s,x) \\ \mathcal{L}(\tilde{\eta})(s,x) \end{array} \right) = \alpha_+^r V^+ e^{\lambda^+ x} + \alpha_-^r V^- e^{\lambda^- x}, \quad \forall x > x_r,
$$

$$
\left(\begin{array}{c} \mathcal{L}(\tilde{w})(s,x) \\ \mathcal{L}(\tilde{\eta})(s,x) \end{array} \right) = \alpha_+^\ell V^+ e^{\lambda^+ x} + \alpha_-^\ell V^- e^{\lambda^- x}, \quad \forall x < x_\ell
$$

where $\alpha_+^{r,\ell}$, $\alpha_-^{r,\ell}$ are constant coefficients, λ^+, λ^- are given by

$$
\lambda^+ = \sqrt[+]{\frac{s^2}{1+\varepsilon s^2}}, \quad \lambda^- = -\sqrt[+]{\frac{s^2}{1+\varepsilon s^2}},
$$

and V^+, V^- are the constant vectors:

$$
V^+ = (1, -\lambda^+/s)^T, \; V^- = (1, \lambda^+/s)^T.
$$

Exact boundary conditions

Laplace transform inversion

$$2\alpha_+^r = \mathcal{L}(\tilde{w})(s,x_r) - \sqrt[+]{1+\varepsilon s^2}\mathcal{L}(\tilde{\eta})(s,x_r) = 0,$$

$$2\alpha_-^\ell = \mathcal{L}(\tilde{w})(s,x_\ell) + \sqrt[+]{1+\varepsilon s^2}\mathcal{L}(\tilde{\eta})(s,x_\ell) = 0.$$

This leads to the conditions at the boundary points x_ℓ, x_r:

$$\mathcal{L}(w)(s,x_r) = \frac{1+\varepsilon s^2}{\sqrt[+]{1+\varepsilon s^2}}\mathcal{L}(\eta)(s,x_r), \quad \mathcal{L}(w)(s,x_\ell) = -\frac{1+\varepsilon s^2}{\sqrt[+]{1+\varepsilon s^2}}\mathcal{L}(\eta)(s,x_\ell)$$

The inversion of Laplace transform is explicit:

$$\varepsilon w_{tx}(t,x_r) - \eta(t,x_r) = -\int_0^t \frac{\mathcal{J}_2+\mathcal{J}_0}{2\sqrt{\varepsilon}}\left(\frac{s}{\sqrt{\varepsilon}}\right) w(t-s,x_r)ds - \sqrt{\varepsilon}w_t(t,x_r),$$

$$\varepsilon w_{tx}(t,x_\ell) - \eta(t,x_\ell) = \int_0^t \frac{\mathcal{J}_2+\mathcal{J}_0}{2\sqrt{\varepsilon}}\left(\frac{s}{\sqrt{\varepsilon}}\right) w(t-s,x_\ell)ds + \sqrt{\varepsilon}w_t(t,x_\ell),$$

where $\mathcal{J}_n$ is the Bessel function of the first kind:

$$\mathcal{J}_n(t) = \frac{1}{\pi}\int_0^\pi \cos(n\tau - t\sin\tau)d\tau$$

Proposition

The problem

$$\eta_t + w_x = 0, \quad w_t + \eta_x - \varepsilon w_{txx} = 0 \quad \forall x \in]x_\ell, x_r[, \forall t > 0$$

$$\eta(0,x) = \eta_0(x), \quad w(0,x) = w_0(x), \quad \forall x \in]x_\ell, x_r[$$

$$\varepsilon w_{tx}(t,x_r) - \eta(t,x_r) = -\int_0^t \frac{\mathcal{J}_2 + \mathcal{J}_0}{2\sqrt{\varepsilon}} \left(\frac{s}{\sqrt{\varepsilon}}\right) w(t-s,x_r)ds - \sqrt{\varepsilon} w_t(t,x_r),$$

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is $L^\infty(\mathbb{R}^+, H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ stable: for all $t > 0$ and for all smooth solution of (0.1)*, we have*

$$\int_{x_\ell}^{x_r} \frac{\eta^2(t,x)}{2} + \frac{w^2(t,x)}{2} + \varepsilon \frac{(w_x(t,x))^2}{2} dx \le \int_{x_\ell}^{x_r} \frac{\eta_0^2(x)}{2} + \frac{w_0^2(x)}{2} + \varepsilon \frac{(w_{0,x}(x))^2}{2} dx$$

Exact boundary conditions

Stability result

proof The time-derivation of the generalised kinetic energy is

$$\frac{d}{dt}\int_{x_\ell}^{x_r} \frac{\eta^2(t,x)}{2} + \frac{w^2(t,x)}{2} + \varepsilon\frac{(w_x(t,x))^2}{2}dx = \left[w(t,x)\left(\varepsilon w_{tx}(t,x) - \eta(t,x)\right)\right]_{x_\ell}^{x_r},$$

Integrating over the time interval $(0,t)$ gives:

$$\int_{x_\ell}^{x_r} \frac{\eta^2(t,x)}{2} + \frac{w^2(t,x)}{2} + \varepsilon\frac{(\partial_x w(t,x))^2}{2}dx - \int_{x_\ell}^{x_r} \frac{\eta_0^2(x)}{2} + \frac{w_0^2(x)}{2} + \varepsilon\frac{(\partial_x w_0(x))^2}{2}dx =$$

$$= \int_0^t w(s,x_r)(\varepsilon w_{tx} - \eta)(s,x_r)ds - \int_0^t w(s,x_\ell)(\varepsilon w_{tx} - \eta)(s,x_\ell)dt := J_r - J_\ell$$

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$$J_r \overset{?}{<} 0$$

Exact boundary conditions

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$$= \int_0^t w(s,x_r)(\varepsilon w_{tx} - \eta)(s,x_r)ds - \int_0^t w(s,x_\ell)(\varepsilon w_{tx} - \eta)(s,x_\ell)dt := J_r - J_\ell$$

$J_\ell \overset{?}{>} 0$

Exact boundary conditions

Stability result

$$J_r = -\int_0^t \left(\frac{\mathcal{J}_2 + \mathcal{J}_0}{2\sqrt{\varepsilon}} (\frac{\cdot}{\sqrt{\varepsilon}}) * w(s, x_r) + \sqrt{\varepsilon} w_t(s, x_r) \right) w(s, x_r) ds$$

Next, we fix $T > 0$ and denote $W(t) = w(t, x_r) 1_[0, T](t)$. One has $w'(t) = W'(t) + W(T)\delta_{t=T}$. By substituting into the formula for J_r, one has

$$J_r = -\int_0^\infty \left(\frac{\mathcal{J}_2 + \mathcal{J}_0}{2\sqrt{\varepsilon}} (\frac{\cdot}{\sqrt{\varepsilon}}) * W(s) + \sqrt{\varepsilon} W'(s) \right) W(s, x_r) ds - \sqrt{\varepsilon} W(T)^2.$$

By applying Plancherel's identity, one finds:

$$J_r = -\frac{1}{2\pi} \Re \int_{\mathbb{R}} \sqrt[+]{1 - \varepsilon\xi^2} |\widehat{W}|^2(\xi) d\xi - \sqrt{\varepsilon} |W(T)|^2 \leq 0.$$

$\square$

Linearized Boussinesq equation

The system is equivalent to the following equation:

$$(w - \varepsilon w_{xx})_{tt} - w_{xx} = 0, \quad \forall x \in \mathbb{R}, \forall t > 0.$$

Solution decreasing at infinity: this gives us one condition on the left boundary and one on the right one for the function $\mathcal{L}(w)$:

$$\mathcal{L}(w_x)(s, x_r) = -\frac{s}{\sqrt[+]{1 + \varepsilon s^2}} \mathcal{L}(w)(s, x_r), \quad \mathcal{L}(w_x)(s, x_\ell) = \frac{s}{\sqrt[+]{1 + \varepsilon s^2}} \mathcal{L}(w)(s, x_\ell$$

The inversion of Laplace transform can be found explicitly and finally we get

$$\begin{aligned} w_x(t, x_r) &= \frac{1}{\varepsilon} \int_0^t \mathcal{J}_1\left(\frac{s}{\sqrt{\varepsilon}}\right) w(t - s, x_r) ds - \frac{1}{\sqrt{\varepsilon}} w(t, x_r), \\ w_x(t, x_\ell) &= -\frac{1}{\varepsilon} \int_0^t \mathcal{J}_1\left(\frac{s}{\sqrt{\varepsilon}}\right) w(t - s, x_\ell) ds + \frac{1}{\sqrt{\varepsilon}} w(t, x_\ell), \end{aligned} \tag{1}$$

Stability result

Proposition

Any smooth solution of the problem

$$(w-\varepsilon w_{xx})_{tt}-w_{xx}=0,\quad \forall x\in[x_l,x_r],\forall t>0$$

$$w(0,x)=w_0(x),\quad w_t(0,x)=v_0(x),\quad \forall x\in]x_\ell,x_r[$$

$$\left(1+\varepsilon\frac{\partial^2}{\partial t^2}\right)w_x(t,x_r)=-\int_0^t\frac{\mathcal{J}_2+\mathcal{J}_0}{2\sqrt{\varepsilon}}\left(\frac{t-s}{\sqrt{\varepsilon}}\right)w_t(s,x_r)ds-\sqrt{\varepsilon}w_{tt}(t,x_r),$$

$$\left(1+\varepsilon\frac{\partial^2}{\partial t^2}\right)w_x(t,x_\ell)=\int_0^t\frac{\mathcal{J}_2+\mathcal{J}_0}{2\sqrt{\varepsilon}}\left(\frac{t-s}{\sqrt{\varepsilon}}\right)w_t(s,x_\ell)ds+\sqrt{\varepsilon}w_{tt}(t,x_\ell),$$

satisfies for all $t>0$ *the following estimate:*

$$\int_{x_\ell}^{x_r}\left(\frac{(w_t)^2}{2}+(w_x)^2+\varepsilon\,(w_{tx})^2\right)(t,x)dx\leq \int_{x_\ell}^{x_r}\left(\frac{v_0^2}{2}+(w_{0,x})^2+\varepsilon\,(v_{0,x})^2\right)dx.$$

Discrete transparent boundary conditions

We write down the centred-Crank Nicholson discretization :

$$\frac{\eta_{j+1/2}^{n+1}-\eta_{j+1/2}^{n}}{\delta t}+\frac{1}{2}\left(\frac{w_{j+1}^{n+1}-w_{j}^{n+1}}{\delta x}+\frac{w_{j+1}^{n}-w_{j}^{n}}{\delta x}\right)=0,$$

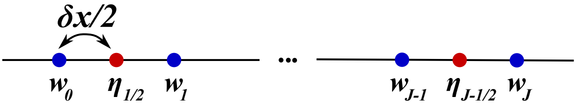
$$\frac{w_{j}^{n+1}-w_{j}^{n}}{\delta t}-\varepsilon\left(\frac{w_{j+1}^{n+1}-2w_{j}^{n+1}+w_{j-1}^{n+1}}{\delta x^2}-\frac{w_{j+1}^{n}-2w_{j}^{n}+w_{j-1}^{n}}{\delta x^2}\right)$$

$$+\frac{1}{2}\left(\frac{\eta_{j+1/2}^{n+1}-\eta_{j-1/2}^{n+1}}{2\delta x}+\frac{\eta_{j+1/2}^{n}-\eta_{j-1/2}^{n}}{2\delta x}\right)=0,$$

$$1\le j\le J, n\in\mathbb{N}$$

where $\delta t>0$, $\delta x>0$ the time and space step, respectively, $w_j^n\approx w(n\delta t, x_l+j\delta x)$ and number of space cells J is calculated as

$$J=\frac{x_r-x_l}{\delta x}$$



Discrete transparent boundary conditions

We will apply the discrete $\mathcal{Z}-$transform with respect to time index n, which is defined as follows

$$\hat{u}(z) = \mathcal{Z}\{(u)_n\}(z) = \sum_{n\geq 0} u_n z^{-n}, \quad |z| > R > 0,$$

C. Besse, M. Ehrhardt, I. Lacroix-Violet, *Discrete artificial boundary conditions for the linearised Korteweg-de Vries equation** Num.Meth. for PDE, V. 32, Issue 5, (2016) 1455-1484.
C. Besse, Mesognon B., Noble P., *Discrete Artificial Boundary Condition for the Benjamin- Bona-Mahoney equation**
C. Besse, P. Noble, D. Sanchez, *Discrete transparent boundary conditions for the mixed KDV-BBM equation**.

Discrete transparent boundary conditions

Recurrence relation

$$\hat{\eta}_{j+1/2} = -\frac{1}{s(z)\delta x}(\hat{w}_{j+1} - \hat{w}_j),$$

$$-\frac{\varepsilon s(z)}{\delta x^2}\hat{w}_{j-1} + s(z)\left(1 + \frac{2\varepsilon}{\delta x^2}\right)\hat{w}_j - \frac{\varepsilon s(z)}{\delta x^2}\hat{w}_{j+1} + \frac{\hat{\eta}_{j+1/2} - \hat{\eta}_{j-1/2}}{\delta x} = 0 \quad ,$$

where,

$$s(z) = \frac{2}{\delta t}\frac{1 - z^{-1}}{1 + z^{-1}}.$$

<u>Note</u> $\boxed{|z| > 1 \implies \Re\left(s(z)\right) > 0}$ $\boxed{\mathcal{Z}(u_{n+1}) = z\mathcal{Z}(u_n)}$

$$\boxed{(1 + \varepsilon s^2(z))\hat{w}_{j-1} - 2\left(1 + s(z)\left(\varepsilon + \frac{\delta x^2}{2}\right)\right)\hat{w}_j + (1 + \varepsilon s^2(z))\hat{w}_{j+1} = 0, \quad 1 \le j \le J, n \in \mathbb{N}}$$

Discrete transparent boundary conditions

Recurrence relation, solution

This solution of the linear recurrence written in the form:

$$\hat{w}_j = A_+ r_+(z)^j + A_- r_-(z)^j$$

where $r_\pm$ are the roots of characteristic polynomial associated with the recurrence:

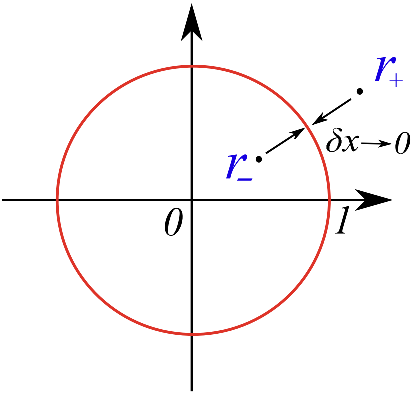
$$P(r) = (1 + \varepsilon s^2(z))r^2 - 2\left(1 + s^2(z)\left(\varepsilon + \frac{\delta x^2}{2}\right)\right) r + (1 + \varepsilon s^2(z)).$$

The explicit formulae for the roots reads

$$r_\pm(z) = 1 + \frac{s^2(z)\delta x^2}{2(1 + \varepsilon s^2(z))} \pm \frac{s(z)\delta x\sqrt{\delta x^2 + 4(1 + \varepsilon s^2(z))}}{2(1 + \varepsilon s^2(z))}.$$

Discrete transparent boundary conditions

Roots separation, assymptotics



Proposition 2 The roots of characteristic polynomial associated with the linear recurrence relation have the following property

$$P(r) = (1+\varepsilon s^2(z))r^2 - 2\left(1 + s^2(z)\left(\varepsilon + \frac{\delta x^2}{2}\right)\right) r + (1+\varepsilon s^2(z)),$$

$$|r_+(z)| > 1, \quad |r_-(z)| < 1.$$

Remark 1 The further strategy of discrete boundary conditions construction will be based on the fact proved in Proposition 2.

Solutions space decomposition : $E^s(z) \cup E^u(z)$

$E^s(z)$: solutions decrease to 0 with $j \to \infty$

$E^u(z)$: solutions decrease to 0 with $j \to -\infty$

Choice of the space step δx in order to separate the roots r_+, r_- well. Take into account the relation between δt and ε for the dispersive effects.

$$1 + \varepsilon s^2 = 1 + \frac{4\varepsilon}{\delta t^2} \frac{z-1}{z+1}$$

Discrete boundary conditions

Inversion of Z transform

$$\hat{w}_1 = \left(1 + \frac{2\delta x^2 (z-1)^2}{\Lambda z^2 - 2\mu z + \Lambda} + \frac{2\delta x (z-1)\sqrt{\Gamma z^2 - 2\nu z + \Gamma}}{\Lambda z^2 - 2\mu z + \Lambda}\right)\hat{w}_0,$$

$$\hat{w}_{J+1} = \left(1 + \frac{2\delta x^2 (z-1)^2}{\Lambda z^2 - 2\mu z + \Lambda} - \frac{2\delta x (z-1)\sqrt{\Gamma z^2 - 2\nu z + \Gamma}}{\Lambda z^2 - 2\mu z + \Lambda}\right)\hat{w}_J,$$

here $\Lambda = 4\varepsilon + \delta t^2$, $\mu = 4\varepsilon - \delta t^2$, $\Gamma = \Lambda + \delta x^2$, $\nu = \mu + \delta x^2$.

Discrete boundary conditions

Consistency result

The Crank-Nicolson approximation $O(\delta t^2 + \delta x^2)$.
The derived discrete boundary conditions provide the same order of approximation:

Theorem

Let $w(t,x)$ be a smooth solution, then one has for all compact $K \in \mathcal{C}^+$, all $s \in K$:

$$\hat{w}(e^{s\delta t}, \delta x) - r_+(e^{s\delta t})\hat{w}(e^{s\delta t}, 0) = O(\delta t^2 + \delta x^2)$$

$$\hat{w}(e^{s\delta t}, 1) - r_+(e^{s\delta t})\hat{w}(e^{s\delta t}, Jdx) = O(\delta t^2 + \delta x^2)$$

where $r_\pm(z)$ the roots of recurrence relation and $\mathcal{Z}-$transform of $w(\cdot, x)$ for all $x \in [0,1]$ defined as

Numerical implementation

Scheme

$$
\begin{aligned}
-a_+ w_{j+1}^{n+1} + (1+2a_+) w_j^{n+1} - a_+ w_{j-1}^{n+1} &= 2(-a_- w_{j+1}^{n} + (1+2a_-) w_j^{n} - a_- w_{j-1}^{n}) \\
&- (-a_+ w_{j+1}^{n-1} + (1+2a_+) w_j^{n-1} - a_+ w_{j-1}^{n-1}), \quad 1 \le j \le J, n \in \mathbb{N},
\end{aligned}
$$

where

$$
a_- = \frac{\varepsilon - \delta t/4}{\delta x^2}, \quad a_+ = \frac{\varepsilon + \delta t/4}{\delta x^2}.
$$

The linear system to solve numerically is:

$$
M_{n+1} W^{n+1} = 2 M_n W^n - M_{n+1} W^{n-1} + V, n \in \mathbb{N},
$$

Collocated grid

The numerical scheme reads as follow:

$$
\begin{aligned}
&\frac{\eta_j^{n+1}-\eta_j^n}{\delta t}+\frac{1}{2}\left(\frac{w_{j+1}^{n+1}-w_{j-1}^{n+1}}{2\delta x}+\frac{w_{j+1}^{n}-w_{j-1}^{n}}{2\delta x}\right)=0,\\
&\frac{w_j^{n+1}-w_j^n}{\delta t}-\varepsilon\left(\frac{w_{j+1}^{n+1}-2w_j^{n+1}+w_{j-1}^{n+1}}{\delta x^2}-\frac{w_{j+1}^{n}-2w_j^{n}+w_{j-1}^{n}}{\delta x^2}\right)\\
&\qquad\qquad+\frac{1}{2}\left(\frac{\eta_{j+1}^{n+1}-\eta_{j-1}^{n+1}}{2\delta x}+\frac{\eta_{j+1}^{n}-\eta_{j-1}^{n}}{2\delta x}\right)=0,
\end{aligned}
$$

Collocated grid

Left boundary condition:

$$
\begin{aligned}
(1+P^u)\widehat{w}_1 &= S^u\widehat{w}_0 - 2\delta x P^u s(z)\widehat{\eta}_0, \\
2\delta x\, s(z)\widehat{\eta}_1 + S^u\widehat{w}_1 &= (1+P^u)\widehat{w}_0.
\end{aligned}
$$

Right boundary condition:

$$
\begin{aligned}
(1+P^s)\widehat{w}_{J+1} &= S^s\widehat{w}_J - 2\delta x\, P^s s(z)\widehat{\eta}_J, \\
2\delta x\, s(z)\widehat{\eta}_{J+1} + S^s\widehat{w}_{J+1} &= (1+P^s)\widehat{w}_J.
\end{aligned}
$$

<u>Explicit inversion is not possible!</u>

Numerical implementation

Dispersive effects. Gaussian initial data

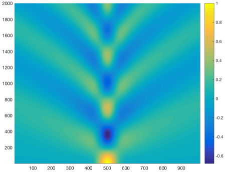


Figure 1: $\varepsilon = 10^{-2}$

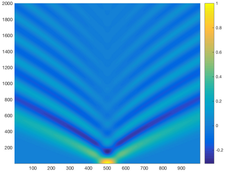


Figure 2: $\varepsilon = 10^{-3}$

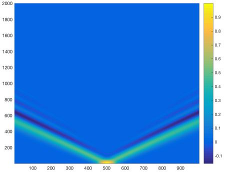


Figure 3: $\varepsilon = 10^{-4}$

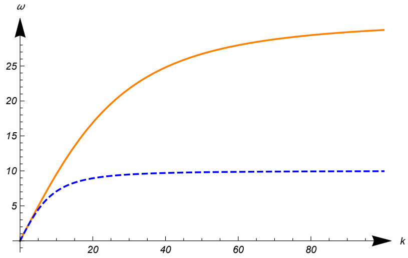
Numerical implementation

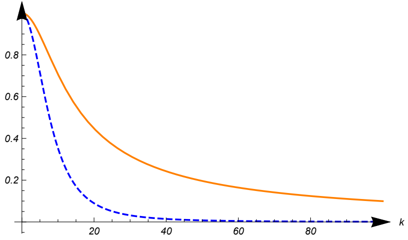
Dispersive effects. Wave package

$$\omega^2(k) = \frac{k^2}{1+\varepsilon k^2},$$

$$v_\varphi(k) = \frac{\omega(k)}{k} = \frac{1}{\sqrt{1+\varepsilon k^2}}, \quad v_g(k) = \frac{d\omega(k)}{dk} = \frac{1}{(1+\varepsilon k^2)^{3/2}}.$$

Group velocity is always less than phase velocity (see right Figure **??**).





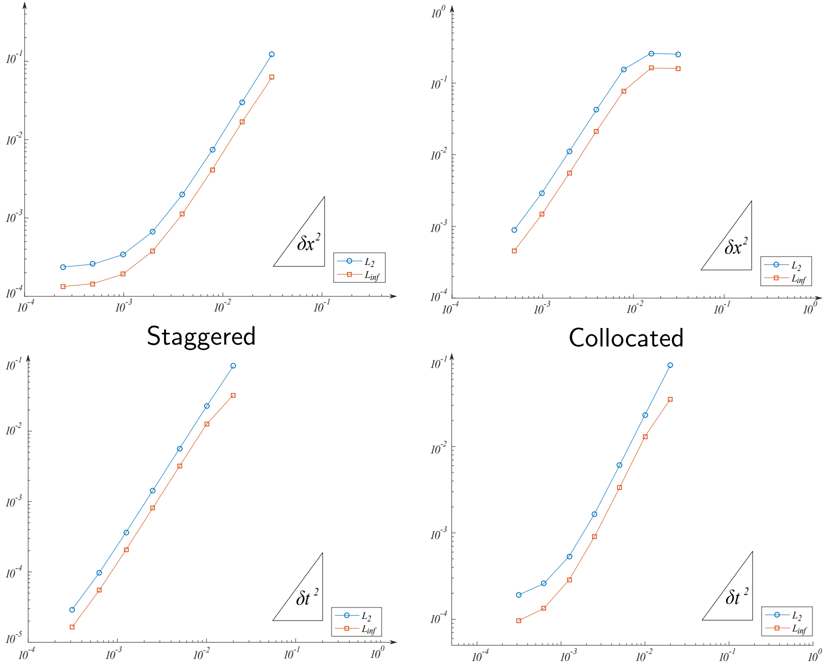
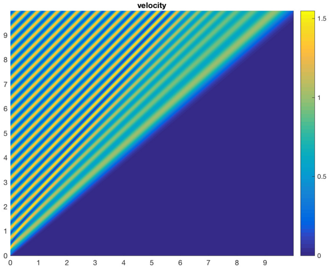


Figure 4: Evolution of the error functions.

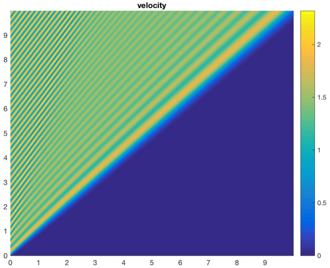
Numerical implementation

Incoming wave

$w^{in}(x,t) = \beta\cos(kx - \omega(k)t)$ with wave number $k = 2\pi p$, $p \in N$



$p = 4, \varepsilon = 10^{-3}$



$p = 8, \varepsilon = 10^{-3}$

Figure 5: Evolution of incoming wave solution for different wave number.

Conclusions and Perspectives

- Theoretical results
  - Exact transparent boundary condition are constructed (energy dissipation is shown)
  - Discrete conditions are derived
  - Consistency of the discrete conditions is proofed
- Numerics
  - The applicability of the technique is shown numerically as well
    - Outgoing wave
    - Incoming wave

Purpose for the further investigation:

Non-linear cases
2D case
Two-layer models Green-Naghdi type