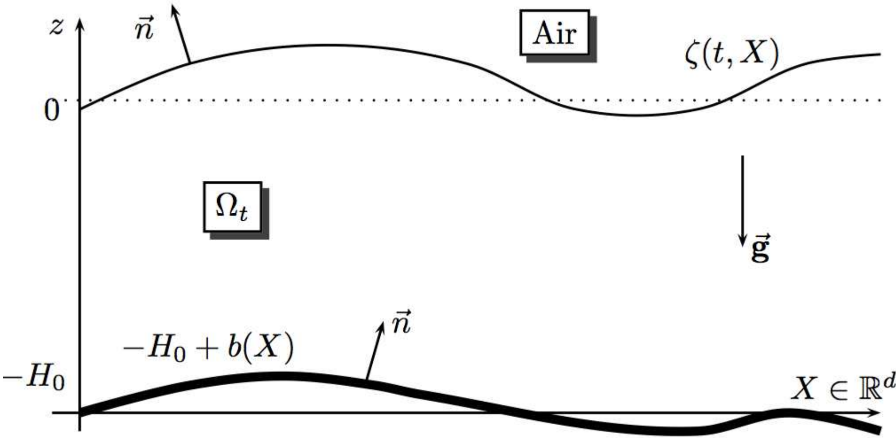
Notations



z
$\vec{n}$
Air
$\zeta(t, X)$
0
Ω_t
$\vec{\mathbf{g}}$
$\vec{n}$
$-H_0 + b(X)$
$-H_0$
$X \in \mathbb{R}^d$

Basic equations

In the fluid domain Ω_t

$$\begin{aligned}
\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} &= -\frac{1}{\rho} \nabla_{X,z} P - g\mathbf{e}_z \\
\text{div } \mathbf{U} &= 0, \\
\text{curl } \mathbf{U} &= 0
\end{aligned}$$

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At the surface

$$P = P_{\mathrm{atm}},$$

$$\partial_t \zeta - \underline{U} \cdot N = 0 \qquad \text{with } N = \begin{pmatrix} -\nabla\zeta \\ 1 \end{pmatrix},$$

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At the bottom

$$U_b \cdot N_b = 0 \qquad \text{with } N_b = \begin{pmatrix} -\nabla b \\ 1 \end{pmatrix}.$$

Dimension reduction

by harmonic analysis (Zakharov)

- $U = \nabla_{X,z}\Phi$ with $\Delta_{X,z}\Phi = 0$ in Ω_t
- U defined on Ω_t fully determine by $\psi = \Phi_{|_{z=\zeta}}$ defined on $\mathbb{R}^d$:

$$\begin{cases} \Delta_{X,z}\Phi = 0, \\ \Phi_{|_{z=\zeta}} = \psi, \qquad \partial_n \Phi_{|_{z=-h_0+b}} = 0. \end{cases}$$

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by vertical integration

- Remove the variable z by integrating Euler's equations vertically
- The equations reduce to a set of equations on ζ and Q where

$$Q(t,X) = \int_{-h_0}^{\zeta} V(t,X,z)dz$$

The water waves equations in (ζ, Q) variables

- Conservation of mass

$$\begin{cases} \int_{-h_0}^{\zeta} \left(\nabla \cdot V + \partial_z w\right) & = 0 \\ \partial_t \zeta - U \cdot N & = 0 \end{cases} \quad \rightsquigarrow \quad \boxed{\partial_t \zeta + \nabla \cdot Q = 0}$$

- Momentum equation
  - Pressure from vertical component of the Euler equation

$$\int_z^{\zeta} \left(\partial_t w + \mathbf{U} \cdot \nabla_{X,z} w + g + \frac{1}{\rho}\partial_z P\right) = 0$$

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⇝ The equations are exact Teshukov07

⇝ They are closed: one can reconstruct the full velocity field $\mathbf{U}$ in Ω from the knowledge of ζ and Q L. 17

How do they work?

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + gh\nabla\zeta + \nabla \cdot \Big(\int_{-h_0}^{\zeta} V \otimes V \Big) + h\mathbf{a}_{\mathrm{NH}}(\zeta, \mathbf{U}) = 0, \end{cases}$$

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- Let us decompose

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V(t,X,z) = \overline{V}(t,X) + V^*(t,X,z) \quad \text{with} \quad \overline{V} = \frac{1}{h}\int_{-h_0}^{\zeta} V(t,X,z)d
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- Therefore

$$\int_{-h_0}^{\zeta} V \otimes V = h\overline{V} \otimes \overline{V} + \int_{-h_0}^{\zeta} V^* \otimes V^* \quad (1)$$

$$= \frac{1}{h} Q \otimes Q + \mathbf{R} \quad (2)$$

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$$h\mathbf{a}_{\rm NH} = \frac{1}{\rho}\int_{-h_0}^{\zeta} \nabla P_{\rm NH} \quad \text{with} \quad P_{\rm NH} = \int_z^{\zeta} \big(\partial_t w + \mathbf{U} \cdot \nabla_{X,z} w\big)$$

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Goal

Show that this is approximately true for (almost) irrotational flows in shallow water.

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- We introduce three characteristic scales
  1. The characteristic water depth H_0
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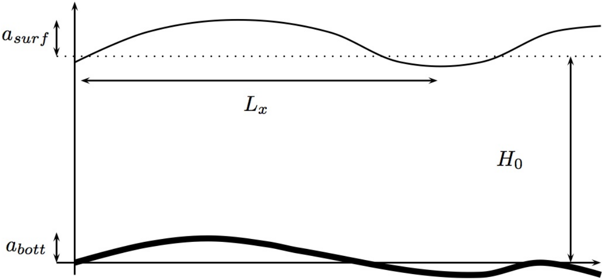

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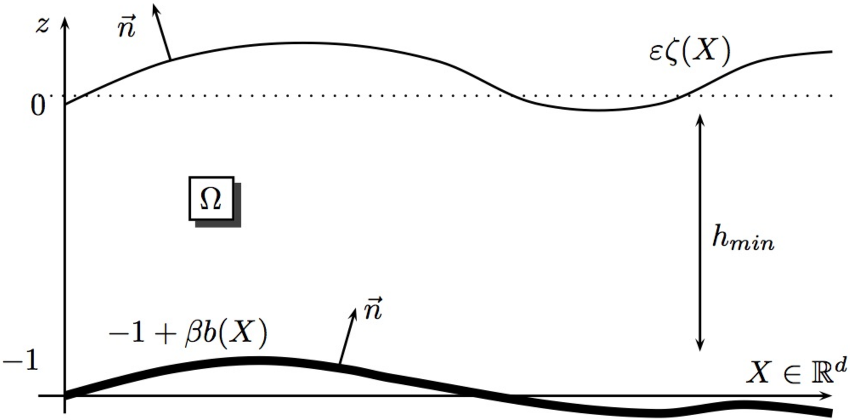
  1. The characteristic water depth H_0
  2. The characteristic horizontal scale L
  3. The order of the free surface amplitude a
- Two independent dimensionless parameters can be formed

$$\frac{a}{H_0} = \varepsilon \quad \text{(amplitude parameter),}$$

$$\frac{H_0^2}{L^2} = \mu \quad \text{(shallowness parameter).}$$

We proceed to the simple nondimensionalizations

$$X' = \frac{X}{L}, \qquad z' = \frac{z}{H_0}, \qquad \zeta' = \frac{\zeta}{a}, \quad \text{etc.}$$



Dimensionless equations

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + h\nabla\zeta + \varepsilon\nabla \cdot \left(\frac{1}{h} Q \otimes Q\right) + \varepsilon\nabla \cdot \mathbf{R} + h\mathbf{a}_{\mathrm{NH}}(\zeta, \mathbf{U}) = 0. \end{cases}$$

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and

$$\begin{pmatrix} V \\ w \end{pmatrix} = \nabla_{X,z}\Phi \quad \text{with} \quad \begin{cases} \partial_z^2 \Phi + \mu\Delta\Phi = 0, \\ \Phi_{|_{z=\varepsilon\zeta}} = \psi, \qquad \partial_z\Phi_{|_{z=-1}} = 0 \end{cases}$$

Structure of the velocity field

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- Non hydrostatic term

$$h\mathbf{a}_{\rm NH}(\zeta,\mathbf{U}) = \int_{-1}^{\varepsilon\zeta} \nabla \int_z^{\varepsilon\zeta} \big(\partial_t w + \varepsilon V\cdot\nabla w + \frac{\varepsilon}{\mu} w\partial_z w\big) = O(\mu)$$

Order $O(\mu)$ approx. : the NL shallow water equations

Regime

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Neglecting the $O(\mu)$ terms: Nonlinear shallow water equations

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Open problems

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  Just because we cannot prove that compressible flows with prescribed initial values exist doesnt mean

Order $O(\mu^2)$ approx. : the Serre-Green-Naghdi equations

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Neglect the $O(\mu^2)$ terms $\rightsquigarrow$ Serre-Green-Naghdi equations

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Order $O(\mu^2)$ approx. : the Serre-Green-Naghdi equations

Regime

$\varepsilon = O(1)$ (fully nonlinear), $\mu \ll 1$ (shallow water)

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Open problems

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Order $O(\mu^2)$ approx. : the Boussinesq equations

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$\mathbf{T} Q = -\frac{1}{3} \nabla \big(h^3 \nabla \cdot (\frac{Q}{h}) \big) = -\frac{1}{3} \nabla \nabla^T + O(\varepsilon\mu)$

Variants of the Boussinesq system

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Generalization to the the Fully nonlinear regime

$$\begin{cases} \partial_t\zeta + \nabla\cdot Q = 0, \\ (1 + \mu\mathbf{T}[\zeta, b]\frac{1}{h})\partial_t Q + h\nabla\zeta + \varepsilon\nabla\cdot\big(\frac{1}{h}Q\otimes Q\big) + \mu\varepsilon\mathcal{Q}(\zeta, Q) = 0. \end{cases}$$

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This is the constant diagonal SGN system used in Uhaina L.-MARCHE15

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⇝ How to generalize in the presence of vorticity?

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$\rightsquigarrow$ Problem: equation on ω is $d+1$-dimensional

Idea: Seek equation on $\mathbf{R}_\omega$ as for compressible gases

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\qquad , \\
\qquad .
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v^\sharp = \frac{12}{h^3} \int_{-1}^{\zeta} (z+1)^2 v_{\rm sh}^* \quad \text{and} \quad v_{\rm sh}^* = \int_z^\zeta \omega - \overline{\int_z^\zeta \omega}
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and

$$F = \int_{-1}^{\zeta} (v_{\rm sh}^*)^3 \quad \text{and} \quad E = \int_{-1}^{\zeta} (v_{\rm sh}^*)^2$$

$$\begin{cases} \partial_t \zeta + \partial_x (hv) = 0, \\ (1+\mu \mathbf{T})\partial_t v + \varepsilon v \partial_x v + \partial_x \zeta + \mathcal{Q} + \mu \frac{1}{h}\partial_x \mathbf{R}_\omega + \mu^{3/2}\mathcal{C}(v, v^\sharp) = O(\mu^2) \\ \partial_t \left(\frac{\mathbf{R}_\omega}{h^3}\right) + v\partial_x \left(\frac{\mathbf{R}_\omega}{h^3}\right) + \sqrt{\mu}\frac{1}{h^3}\partial_x F = O(\mu) \\ \partial_t \left(\frac{v^\sharp}{h}\right) + \overline{v}\partial_x \left(\frac{v^\sharp}{h}\right) = O(\sqrt{\mu}), \\ \partial_t \left(\frac{F}{h^4}\right) + \overline{v}\partial_x \left(\frac{F}{h^4}\right) = O(\sqrt{\mu}). \end{cases}$$

with

$$v^\sharp = \frac{12}{h^3}\int_{-1}^{\zeta} (z+1)^2 v_{\mathrm{sh}}^* \quad \text{and} \quad v_{\mathrm{sh}}^* = \int_z^{\zeta} \omega - \overline{\int_z^{\zeta} \omega}$$

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⇝ Valid for $2D$ with topography

⇝ Dynamics of the vorticity $d+1$ dimensional but d-dimensional equations

Properties and numerical simulations

- Local conservation of energy

$$\partial_t \mathfrak{e} + \nabla \cdot \mathfrak{F} = 0$$

with $\mathfrak{e} = \mathfrak{e}_p + \mathfrak{e}_k + \mathfrak{e}_{rot}$ and

$$\mathfrak{e}_p = \frac{1}{2} g \zeta^2, \qquad \mathfrak{e}_k = \frac{1}{2} h |\overline{V}|^2 + \frac{1}{6} h^3 |\nabla \cdot \overline{V}|^2, \qquad \mathfrak{e}_{rot} = \frac{1}{2} \mathrm{Tr}\mathbf{E}.$$

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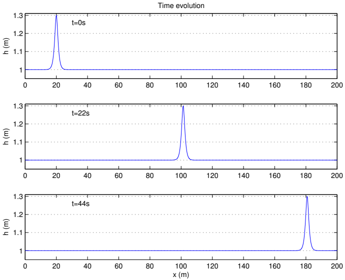
- Influence of vorticity on solitary wave



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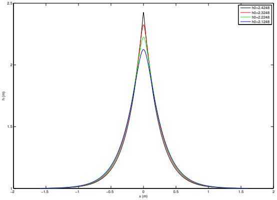
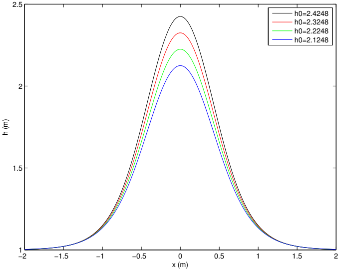
- Influence of vorticity on solitary wave
- Left and right going solitary waves of different shape





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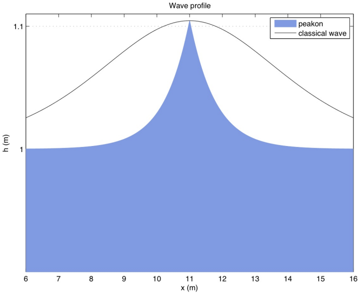
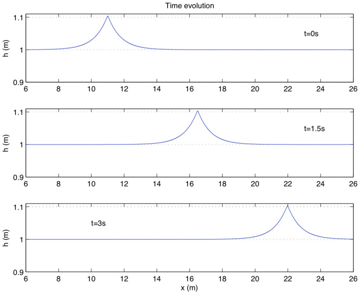
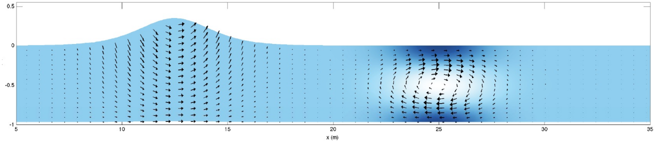
- Influence of vorticity on solitary wave
- Left and right going solitary waves of different shape
- Existence of peakons!





- Dynamics is *d* dimensional and wave current interaction is nonlinear!



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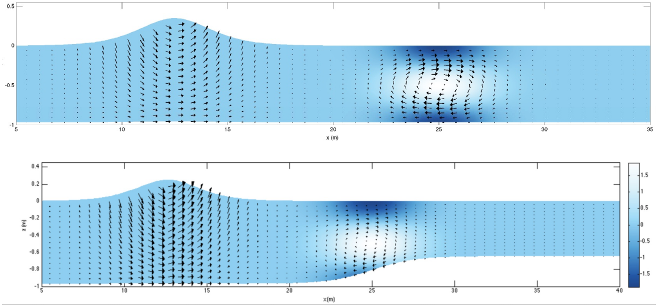
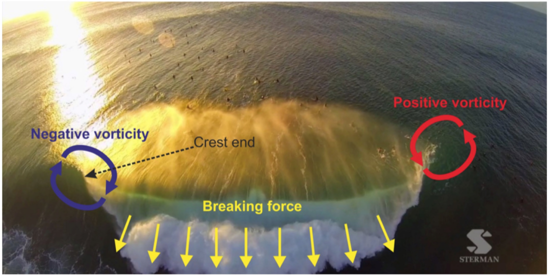


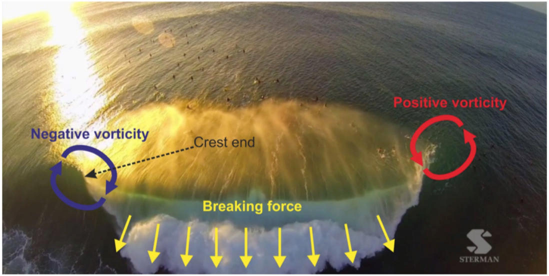
(F. Marche, D. L. 2015)

To do:

- Two dimensional computations
- Exhibit the transfer mechanisms between horizontal and vertical vorticity
- Study the vorticity generation FOR NSW [GAVRILYUK-RICHARD]



Adapté de Clark et al. (2012, GRL)



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Thank you for your attention!