Analysis of Discontinuous Galerkin type methods for oceanic and climate circulations

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Introduction

- Climate, oceanic and atmospheric flows support the propagation of:
  - very fast acoustic waves.
  - the fast inertia-gravity modes (internal, external).
  - the slow mid-latitude Rossby modes.
  - equatorial waves (Kelvin, Rossby index 1 and 2).

  and many physical processes, equilibrium, conservation properties etc.

- Numerical methods badly represent the propagation of such waves and their properties at the discrete level.

- Many attemps to understand why over the past 50 years.

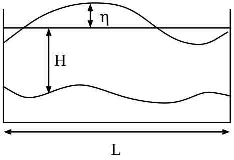
- To find out how to compute the waves accurately, error analyses are performed here mostly by studying the kernel of the discrete operators and by using the Fourier analysis.

- The aim of this talk is to present such Fourier results, and to propose a class of possible discretization schemes, that are not affected by the spurious solutions.

The shallow-water equations (2D)

They are derived by vertical integration of the momentum and continuity equations in the primitive system assuming:

- Horizontal displacements ($u_z = v_z = 0$)
- $H \ll L$
- The density ρ is constant
- The hydrostatic equilibrium ($p_z = -\rho g$)



Inviscid non linear SW system:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + f\mathbf{k}\times\mathbf{u} + g\nabla\eta &= 0, \\ \frac{\partial \eta}{\partial t} + \nabla\cdot\big((H+\eta)\mathbf{u}\big) &= 0, \end{aligned}$$

Inviscid linearized SW system:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + f\mathbf{k}\times\mathbf{u} + g\nabla\eta &= 0, \\ \frac{\partial \eta}{\partial t} + H\nabla\cdot\mathbf{u} &= 0, \end{aligned}$$

with appropriate initial and boundary conditions.

- $\mathbf{u} = (u, v)$ is the velocity field
- η is the surface elevation with respect to the reference level $z = 0$
- g is the gravitational acceleration and $\mathbf{k}$ is a unit vector in the vertical direction
- the mean depth H is assumed constant

Inertia-gravity and Rossby waves in 2–D

We consider the system

$$\begin{aligned} \mathbf{u}_t + f\mathbf{k}\times\mathbf{u} + g\nabla\eta &= 0, & (1)\\ \eta_t + H\nabla\cdot\mathbf{u} &= 0, & (2) \end{aligned}$$

Periodic solutions

$$\begin{aligned} \mathbf{u} &= \hat{\mathbf{u}}\, e^{i(kx+ly+\omega t)},\\ \eta &= \hat{\eta}\, e^{i(kx+ly+\omega t)}. \end{aligned}$$

After substitution in (1) and (2), the following system is obtained for the amplitudes

$$\begin{pmatrix} 0 & -f & igk \\ f & 0 & igl \\ iHk & iHl & 0 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\eta} \end{pmatrix} = -i\omega\, I_{3,3} \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\eta} \end{pmatrix}.$$

The dispersion relation, i.e. $\omega(k,l)$, is obtained by vanishing the 3×3 determinant of the matrix:

- geostrophic and inertia-gravity modes ($f=$ constant)
  - The (slow) solution $\omega = 0$ is the geostrophic mode.
  - The two other (fast) solutions correspond to the inertia-gravity modes $\omega_{AN} = \pm\sqrt{f^2 + gH(k^2+l^2)}$.

    Two limits: • $f=0$ (gravity wave), $\omega = \pm\sqrt{gH(k^2+l^2)}$.

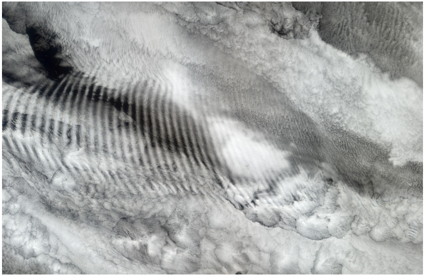
    • $gH(k^2+l^2) \ll f^2$ (inertial oscillations), $\omega = \pm f$.

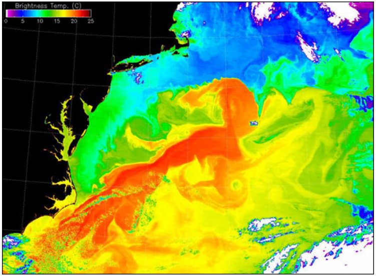
By using the quasi-geostrophic approximation we can also obtain a relation for the

- Rossby mode: $\omega = \dfrac{-\beta k}{\frac{1}{\lambda^2} + k^2 + l^2}$, with $f = f_0 + \beta y$ and $\lambda = \sqrt{gH}/f_0$.

High frequency inertia-Gravity

Low frequency Rossby waves





Matrix kernel analysis of the stationary SW model

A solution of the continuous stationary shallow water model

Continuous

$$\begin{aligned} f\mathbf{k}\times\mathbf{u}+g\nabla\eta &= 0, \quad (3)\\ H\nabla\cdot\mathbf{u} &= 0, \quad (4)\end{aligned}$$

Discrete

$$\begin{pmatrix} C & G \\ D & 0 \end{pmatrix}\begin{pmatrix}\mathbf{u}_h\\ \eta_h\end{pmatrix} := CDG\begin{pmatrix}\mathbf{u}_h\\ \eta_h\end{pmatrix} = 0,$$

with $CG := \begin{pmatrix} C & G\end{pmatrix}$ and $DO := \begin{pmatrix} D & 0\end{pmatrix}$,

should satisfy (Rostand and Le Roux, IJNMF 2007, SIAM 2008)

Property 1: $(3)\Rightarrow(4)$ $\quad \mathbf{u} = \dfrac{g}{f}k\times\nabla\eta = \dfrac{g}{f}\mathbf{rot}\,\eta = \dfrac{g}{f}\nabla^{\perp}\eta$

$\ker(CG)\subset\ker(DO)$

Property 2: $(4)\Rightarrow(3)$ $\quad \exists\zeta : \mathbf{u} = \nabla^{\perp}\zeta \Rightarrow \eta = \dfrac{f}{g}\zeta + c$

$\ker(D)\subset\{\mathbf{u}_h \mid \exists\eta_h : (\mathbf{u}_h,\eta_h)\in\ker(CG)\}$

Property 3: $\forall\eta, \exists\mathbf{u}$ verifying (3) and (4).

$\forall\eta_h, \exists\mathbf{u}_h : (\mathbf{u}_h,\eta_h)\in\ker(CDG)$

A necessary (but not sufficient) condition is that the ratio of momentum versus continuity equations remains the same for the continuous and discrete cases.

Property 4: $\dim\ker(C) = 0$ and $\dim\ker(G) = 1$

Mimetic discretization

$$\dim \ker(D) = \underbrace{\mathscr{D}_1}_{\textit{Geostrophic}} + \underbrace{\mathscr{D}_2}_{\textit{Spurious inertial}} + \underbrace{\mathscr{D}_3}_{\textit{Spurious pressure}}$$ (Le Roux, JCP 20012)

Existence of the discrete Helmholtz decomposition requires that the following diagram commutes:

$$\begin{array}{ccccc} H^1(\Omega) & \xrightarrow{\nabla^\perp} & H(div,\Omega) & \xrightarrow{\nabla\cdot} & L^2(\Omega) \\ \downarrow \pi_S & & \downarrow \pi_V & & \downarrow \pi_Q \\ S & \xrightarrow{\nabla^\perp} & V & \xrightarrow{\nabla\cdot} & Q \end{array}$$ (Arnold et al, 2002, 2006; Cotter, Shipton, 2012)

A necessary condition to avoid both spurious inertia-gravity and spurious Rossby waves, at least in the periodic plane, is to have $\dim(V) = 2\dim(Q)$.

For example, to avoid spurious pressure (η) modes it is required that

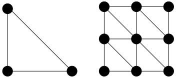
$$\pi_Q(\nabla\cdot\mathbf{u}) = \nabla\cdot\pi_V(\mathbf{u})$$

Fourier analysis (dispersion relations) at the discrete level

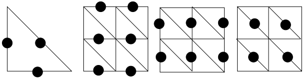
- Time is assumed to be continuous ($\frac{\partial}{\partial t} = i\omega$) and f is held constant.
- The discrete problem leads to a set of discrete equations in space (at node $j_1 = 1, 2, 3, \cdots$, for $\mathbf{u}$ and $j_2 = 1, 2, 3, \cdots$, for η) on a regular and uniform mesh (the meshlength parameter h is taken as a constant). In the following biased right triangles are used.
- Periodic solutions $\mathbf{u}_{j_1} = \hat{\mathbf{u}}_p \, e^{i(kx_{j_1} + ly_{j_1} + \omega t)}$ and $\eta_{j_2} = \hat{\eta}_q \, e^{i(kx_{j_2} + ly_{j_2} + \omega t)}$ are sought where $\hat{\mathbf{u}}_p$ and $\hat{\eta}_q$ are the Fourier amplitudes, with $p = 1, 2, 3 \cdots$, and $q = 1, 2, 3, \cdots$.
- When linear polynomials are employed to approximate $\mathbf{u}$ and η, the velocity and pressure unknowns are located at triangle vertices and we have $p = q = 1$ (for symmetry reasons).

  However, when mid-side, barycenter, internal, etc ..., nodes are used to locate velocity and surface-elevation nodal values we have $p > 1$ and $q > 1$. For example:

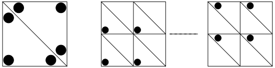
P_1 $(p, q = 1)$



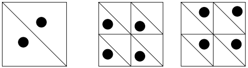
RT_0, P_1^{NC} $(p, q = 3)$



P_1^{DG} $(p, q = 6)$

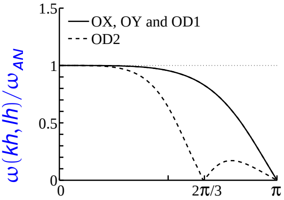


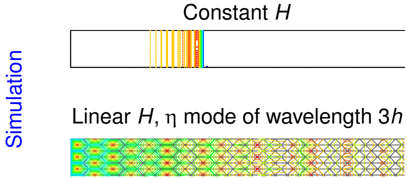
P_0 $(p, q = 2)$



A $n \times n$ system is obtained for the amplitudes and the dispersion relation is hence a polynomial of degree n in ω, leading to the existence of eventual spurious solutions.
We have $n = 2p + q$ or $n = p + q$ depending on the FE pairs.

Effects of spurious elevation modes: the $P_1 - P_1$ pair: $\dim \ker(G) \neq 1$



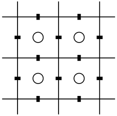


- The behavior of the smallest nonzero singular value σ_0 of the discrete divergence operator is related to the so-called discrete stability inf-sup (LBB) condition for mixed problems

$$\inf_{\vartheta_h \in Q_{j,h}} \sup_{\mathbf{u}_h \in \mathbf{V}_{j,h}} \frac{b(\mathbf{u}_h, \vartheta_h)}{\|\mathbf{u}_h\|_{\mathbf{V}_j} \ \|\vartheta_h\|_{Q_j/\ker B^T}} \geq \sigma_0, \tag{5}$$

  where ϑ_h denotes the test function, and B is the linear continuous operator defined as $< B\mathbf{u}, \vartheta >_{Q_j' \times Q_j} = b(\mathbf{u}, \vartheta) = \int_\Omega \nabla \cdot \mathbf{u} \, \vartheta \, d\mathbf{x}, \quad \forall \mathbf{u} \in \mathbf{V}_j, \forall \vartheta \in Q_j$.
- $\sigma_0 \neq 0$ is needed when $\dim(\mathbf{V}_{j,h})$ and $\dim(Q_{j,h})$ increase, to avoid a zero eigenvalue of the problem associated with a stationary spurious η mode $(\mathbf{u} = \mathbf{0}, \eta \in \ker(G_h), \eta \neq$ **constant**$)$.
- Stabilized FEM (Hughes et al., 1986): retrieving the information lost by the projection Π_{V_h}, i.e. $\mathbf{grad}\,\eta_h - \Pi_{V_h} \mathbf{grad}\,\eta_h$, for a bad choice of $V_{j,h}$ and $Q_{j,h}$ (when there are not enough $\mathbf{u}_h$ compared to η_h), as it is the case when $\mathbf{grad}$ is not injective, namely $\dim(\ker B_h^t) > 1$.

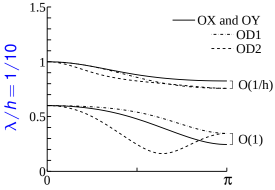
Coriolis $f-$modes: C-grid and RT, BDM and $BDFM$ elements: dim ker$(C) \neq 0$

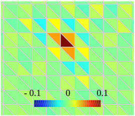


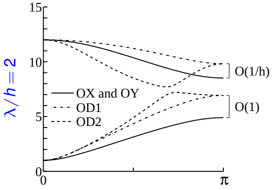
		Inertia part		Gravity part
$\left(\frac{\omega}{f}\right)^2_{AN}$	=	1	+	$\left(\frac{\lambda}{h}\right)^2 \left([kh]^2 + [lh]^2\right)$
$\left(\frac{\omega}{f}\right)^2_{C}$	=	$\cos^2 \frac{kh}{2} \cos^2 \frac{lh}{2}$	+	$4\left(\frac{\lambda}{h}\right)^2 \left(\sin^2 \frac{kh}{2} + \sin^2 \frac{lh}{2}\right)$

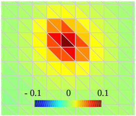
- The numerical example with the $RT_0 - P_0$ pair considers the geostrophic balance
  - The fluid is initially at rest and a point mass source and sink are prescribed
  - The Coriolis parameter is held constant
  - the RT0 scheme exhibits a checkerboard-like pattern of noise in the elevation field around the mass source and sink points when $\lambda = \sqrt{gH}/f$ is not resolved

    (Le Roux et al., SIAM 2008)









- Use a fine mesh to well resolve the Rossby radius.

Existence of spurious inertial oscillations: $\omega = \pm f$

For all FE pairs having two velocity components per node, with $n = 2p + q$ and $p \geq q$, the selected discrete equations lead to the following $(2p+q) \times (2p+q)$ matrix system for the amplitudes:

$$S\,X = 0, \quad \text{with} \quad S = \begin{pmatrix} M_{2p,2p} & -gD^{*}_{2p,q} \\ HD_{q,2p} & i\omega\, N_{q,q} \end{pmatrix}, \quad X = \left(\widehat{\mathbf{u}}_1, \cdots, \widehat{\mathbf{u}}_p, \widehat{\eta}_1, \cdots, \widehat{\eta}_q\right), \qquad (6)$$

where $\widehat{\mathbf{u}}_1, \cdots, \widehat{\mathbf{u}}_p$, $\widehat{\eta}_1, \cdots, \widehat{\eta}_q$ are the Fourier amplitudes, $D_{q,2p}$ and $N_{q,q}$ are the divergence and surface-elevation mass matrices, respectively, $M_{2p,2p}$ is the velocity mass / Coriolis matrix.

Theorem (Le Roux, JCP 2012)

For all FE pairs having two velocity components per node, with $n = 2p + q$ and $p \geq q$, the general dispersion relation obtained from (6) is such that

$$\det S = 0 = \omega^{q} \left(\omega^2 - f^2\right)^{p-q} \mathrm{P}_{2q}(\omega),$$

where $\mathrm{P}_{2q}(\omega)$ is a polynomial of degree $2q$ in ω (inertia-gravity solutions).

Consequences

The FE pairs having 2 velocity components per node are subject to

- Physical geostrophic modes $\omega = 0$ of multiplicity $q - 1$ if $q > 1$.
- Non physical solutions $\omega = \pm f$ if $p > q$, namely spurious inertial modes of multiplicity $p - q$.

Effects of spurious inertial oscillations: $\omega = \pm f$

- η plays the role of a constant and $\mathbf{u}$ only depends on time, i.e. $\mathbf{u} = \mathbf{u}(t)$.
- $\mathbf{u}(t)$ is the discrete counterpart of the solution of the continuous equation $\mathbf{u}_t + f\mathbf{k} \times \mathbf{u} = 0$:

$$u(t) = V \sin(ft + \Phi) \quad \text{and} \quad v(t) = V \cos(ft + \Phi),$$

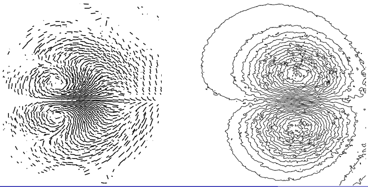
where $\mathbf{u}(t)$ rotates with frequency f. This is observed when f is a constant depending on

The ratio of the spurious inertial modes to the total number of discrete modes:

	$P_1^{DG} - P_1$	$P_2 - P_1$	$P_1^{NC} - P_1$	$P_1^{DG} - P_2$
$\frac{2(p-q)}{n}$	10/13 77 %	6/9 67 %	4/7 57 %	4/16 25 %

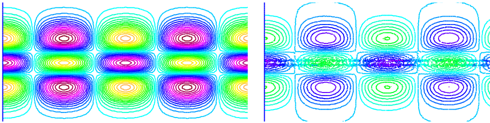
Rossby mode $f = f_0 + \beta y$, u (9 periods)

$P_1^{DG} - P_1$ $\qquad$ $P_2 - P_1$



Rossby wave of index 2, $f = \beta y$, v (5 periods)

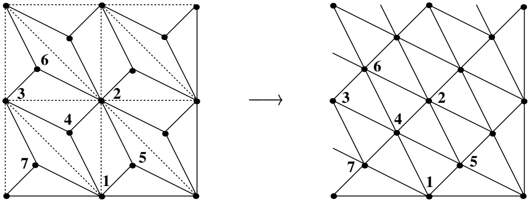
Continuous: -1.14, 1.14 $\qquad$ $P_1^{DG} - P_2$: -2.50, 2.36



Finite Element Exterior Calculus: a few candidates

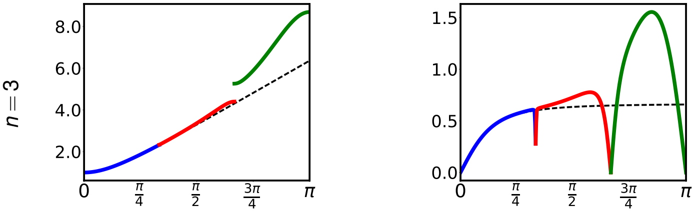
- The $P_1^{NC} - P_{1-C}$ pair (2D) (Le Roux, SIAM 2001)

The P_{1-C} element:



- The $BDFM_1 - P_1^{DG}$ pair (2D) (Cotter and Shipton, JCP 2012) $V = \{\mathbf{u} \in H(div) : \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, with $\mathbf{u}_1 |_K \in (P_1(K))^2$ and $\mathbf{u}_2 |_K \in \{\mathbf{u} \in (P_2(K))^2 : \mathbf{u} \cdot \mathbf{n} = 0$ on $\partial K\}\}$

- The $P_n^C - P_{n-1}^{DG}$ pair (1D) (Eldred and Le Roux)



DG discretization of the 2D linear shallow-water system

Let $\{\tau_h\}_{h>0}$ denote a partition of the domain Ω into a finite number of disjoint open elements Ω_K:

$$\overline{\Omega} = \cup_K \overline{\Omega}_K \quad \text{and} \quad \Omega_{K^+} \cap \Omega_{K^-} = \emptyset \ \text{ for } \ K^+ \neq K^-,$$

where the meshlength parameter h is assumed to be constant. Further, let Γ be the finite ensemble of interelement boundaries $\Gamma_e = \overline{\partial\Omega_{K^+}} \cap \overline{\partial\Omega_{K^-}}$ with $K^+ > K^-$ inside the domain, and

$$\overline{\Gamma} = \cup_e \overline{\Gamma}_e \quad \text{and} \quad \Gamma_e \cap \Gamma_f = \emptyset \ \text{ for } \ e \neq f.$$

Each $\Gamma_e \in \Gamma$ is associated with a unique fixed unit normal $\mathbf{n}_e = (n_e^x, n_e^y)$. Finally, for any function $\iota \in V_K = H^1(\Omega_K)$ (ι may represent u, v or η), the trace of ι on Γ_e is denoted by $\iota^{\pm}$, with

$$\iota^-(\mathbf{x}) = \lim_{\epsilon \to 0^-} \iota(\mathbf{x} + \epsilon \mathbf{n}_e), \quad \iota^+(\mathbf{x}) = \lim_{\epsilon \to 0^+} \iota(\mathbf{x} + \epsilon \mathbf{n}_e),$$

where $\mathbf{x} \in \Gamma_e$.

Let $\mathbf{w} = (u, v, \eta)$. The linear shallow-water equations yield

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{w}) + \mathbf{S}(\mathbf{w}) = 0, \quad \text{with} \quad \mathbf{F}(\mathbf{w}) = \begin{pmatrix} g\eta & 0 \\ 0 & g\eta \\ Hu & Hv \end{pmatrix} \quad \text{and} \quad \mathbf{S}(\mathbf{w}) = \begin{pmatrix} fv \\ -fu \\ 0 \end{pmatrix}, \tag{7}$$

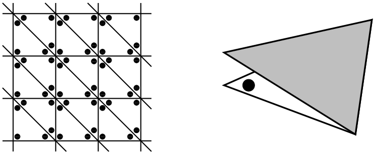
where $\mathbf{F}(\mathbf{w})$ and $\mathbf{S}$ are the flux and source terms, respectively.

The weak formulation is obtained by multiplying (7) by an arbitrary test function $\psi(\mathbf{x}) \in V_K$ and integrate the flux term by parts, using Green's theorem, over each element Ω_K:

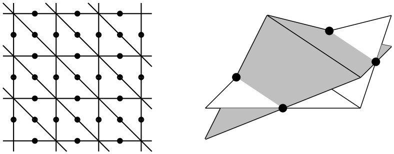
$$\int_{\Omega_K} \frac{\partial \mathbf{w}}{\partial t} \cdot \psi \, d\mathbf{x} - \int_{\Omega_K} \mathbf{F}(\mathbf{w}) \cdot \nabla\psi \, d\mathbf{x} + \int_{\partial\Omega_K} \mathbf{F}(\mathbf{w}) \cdot \mathbf{n}_e \cdot \psi \, ds + \int_{\Omega_K} \mathbf{S}(\mathbf{w}) \cdot \psi \, d\mathbf{x} = 0. \tag{8}$$

The discrete schemes

P_1^{DG}



P_1^{NC}



Let $\mathscr{P}^m(\Omega_K)$ the space of polynomials of degree $\leq m$ and $V_{h,K} = \{\iota : \iota|_{\Omega_K} \in \mathscr{P}^m(\Omega_K)\} \subset V_K$.
u, v, η are approximated in $V_{h,K}$ and these are denoted by $\mathbf{w}_h = (u_h, v_h, \eta_h)$. Equation (8) leads to

$$\int_{\Omega_K} \frac{\partial \mathbf{w}_h}{\partial t} \cdot \psi \, d\mathbf{x} - \int_{\Omega_K} \mathbf{F}(\mathbf{w}_h) \cdot \nabla \psi \, d\mathbf{x} + \int_{\partial \Omega_K} \mathbf{F}(\mathbf{w}_h^*) \cdot \mathbf{n}_e \cdot \psi \, ds + \int_{\Omega_K} \mathbf{S}(\mathbf{w}_h) \cdot \psi \, d\mathbf{x} = 0, \qquad (9)$$

where $\mathbf{w}_h^* = (u_h^*, v_h^*, \eta_h^*)$ denotes the numerical trace of $\mathbf{w}$ on the boundary element $\partial \Omega_K$.

To complete the definition of the approximate solution $\mathbf{w}_h$, it only remains to choose a unique numerical flux $\mathbf{w}^*$ at the cell interface as to render the method consistent and stable.

The Polynomial viscosity matrix approach (PVM)

Equation (7) is projected in the normal direction $\mathbf{n}_e = (n_e^x, n_e^y)$, with $\|\mathbf{n}_e\| = 1$, assuming the source term $\mathbf{S}(\mathbf{w})$ is disregarded. This yields

$$\frac{\partial \mathbf{w}}{\partial t} + A\frac{\partial \mathbf{w}}{\partial n_e^x} = 0, \quad \text{where} \quad A = \frac{\partial(\mathbf{F}\cdot\mathbf{n}_e)}{\partial \mathbf{w}} = \begin{pmatrix} 0 & 0 & gn_e^x \\ 0 & 0 & gn_e^y \\ Hn_e^x & Hn_e^y & 0 \end{pmatrix},$$

is A is the Jacobian matrix. Classical computations lead to

$$\begin{aligned} F(\mathbf{w}^*_{\text{Roe}})\cdot\mathbf{n}_{ed} &= A\{\mathbf{w}\} - \frac{1}{2\sqrt{gH}}A^2[\mathbf{w}], \\ F(\mathbf{w}^*_{\text{Rus}})\cdot\mathbf{n}_{ed} &= A\{\mathbf{w}\} - \tfrac{1}{2}\sqrt{gH}\,I_{3,3}[\mathbf{w}], \end{aligned}$$

where $\{\mathbf{w}\}$ and $[\mathbf{w}]$ denote the mean value and the jump of $\mathbf{w}$ between the right and left fields, respectively. In a more general (PVM) approach the numerical flux is defined as

$$F(\mathbf{w}^*)\cdot\mathbf{n} = A\{\mathbf{w}\} - \frac{1}{2}\sum_{j=0}^{N}\widetilde{\alpha}_j A^j[\mathbf{w}].$$

The coefficients $\widetilde{\alpha}_j$ are such that the polynomial interpolates the function “absolute value” at some eigenvalues of the Jacobian matrix (Castro and Fernandez-Nieto, SIAM 2012).

These coefficients are determined in order to guarantee the stability of the scheme.

Since we have the property $A^{2j-1} = (gH)^{j-1} A, \quad A^{2j} = (gH)^{j-1} A^2, \quad$ for $j = 1,2,3,\cdots$, we obtain

$$F(\mathbf{w}^*)\cdot\mathbf{n} = A\{\mathbf{w}\} - \frac{1}{2}\underbrace{\left(\alpha_0 I_{3,3}[\mathbf{w}] + \alpha_1 A[\mathbf{w}] + \alpha_2 A^2[\mathbf{w}]\right)}_{\text{numerical viscosity matrix}}.$$

with $\alpha_0 = p\sqrt{gH}$, $\alpha_1 = 0$ and $\alpha_2 = q/\sqrt{gH}$, and this leads to

$$F(\mathbf{w}^*)\cdot\mathbf{n} = \begin{pmatrix} g\eta^* n_x \\ g\eta^* n_y \\ H\mathbf{u}^*\cdot\mathbf{n} \end{pmatrix} = \begin{pmatrix} gn_{ed}^x\{\eta\} - \frac{\sqrt{gH}}{2}\left(p[u] + qn_{ed}^x\left(n_{ed}^x[u] + n_{ed}^y[v]\right)\right) \\ gn_{ed}^y\{\eta\} - \frac{\sqrt{gH}}{2}\left(p[v] + qn_{ed}^y\left(n_{ed}^x[u] + n_{ed}^y[v]\right)\right) \\ H\left(n_{ed}^x\{u\} + n_{ed}^y\{v\}\right) - \frac{\sqrt{gH}}{2}(p+q)[\eta] \end{pmatrix}.$$

The choice of p and q leads to a few classical schemes:

	Centered	Rusanov	Roe	PVM − 2	PVM − 4
p	0	1	0	1/2	3/8
q	0	0	1	1/2	5/8

To obtain the 2-D dispersion relations for the $P_1^{DG} - P_1^{DG}$ and $P_1^{NC} - P_1^{NC}$ pairs:

- Write the 18 TYPICAL (resp. 9) discrete equations for the $P_1^{DG} - P_1^{DG}$ (resp. $P_1^{NC} - P_1^{NC}$) pair
- Perform the 2-D Fourier analysis
- Compute the 18×18 (resp. 9×9) determinant for the $P_1^{DG} - P_1^{DG}$ (resp. $P_1^{NC} - P_1^{NC}$) pair
- “Solve“ equations of degree 18 and 9 in $\omega(kh, lh)$ and obtain the asymptotics as $h \to 0$

Theorem

In the limit as mesh spacing $h \to 0$ we obtain the asymptotic results

- *The* ***inertia-gravity modes*** *(for the 4 schemes): No spurious pressure and f-modes*

$$\begin{aligned} \omega^{DG} &= \omega^{AN} + i\,\mathscr{F}_1(k,l)\,h^3 \pm \mathscr{F}_2(k,l)\,h^4 + O(h^5) \\ \omega^{NC} &= \omega^{AN} \pm \mathscr{G}_1(k,l)\,h^4 + i\,\mathscr{G}_2(k,l)\,h^5 + O(h^6) \end{aligned}$$

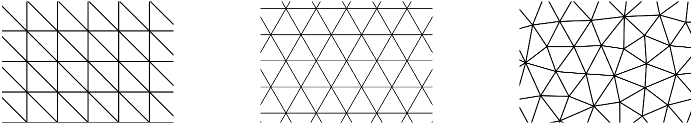
- *The* ***geostrophic mode****: No spurious geostrophic modes*

Rusanov, PVM−2, PVM−4	Roe
$\omega^{DG} = i\,\mathscr{F}_3(k,l)\,h^3 + O(h^5)$	$\omega^{DG} = 0$ *and* $i\,\mathscr{F}_3(k,l)\,h + O(h^2)$
$\omega^{NC} = i\mathscr{G}_3(k,l)\,h^5 + O(h^7)$	$\omega^{NC} = i\mathscr{G}_3(k,l)\,h + O(h^2)$

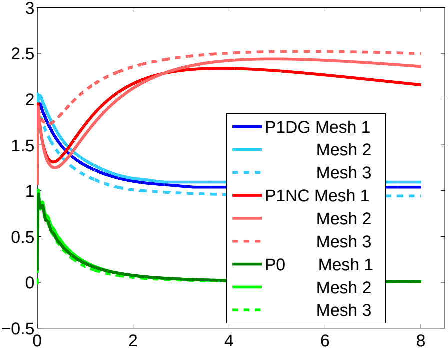
- *Existence of* ***high frequency modes****: $\omega_j = \frac{\alpha_j + i\,\beta_j}{h} + O(h)$, with $\beta_j > 0$.*

DG simulations (I): Balanced flow, Convergence rate for $\|\mathbf{u}\|$ after 8 weeks.

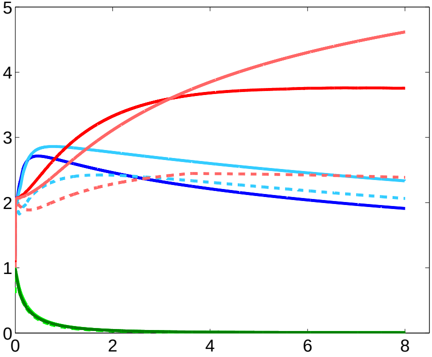
Mesh 1 Mesh 2 Mesh 3



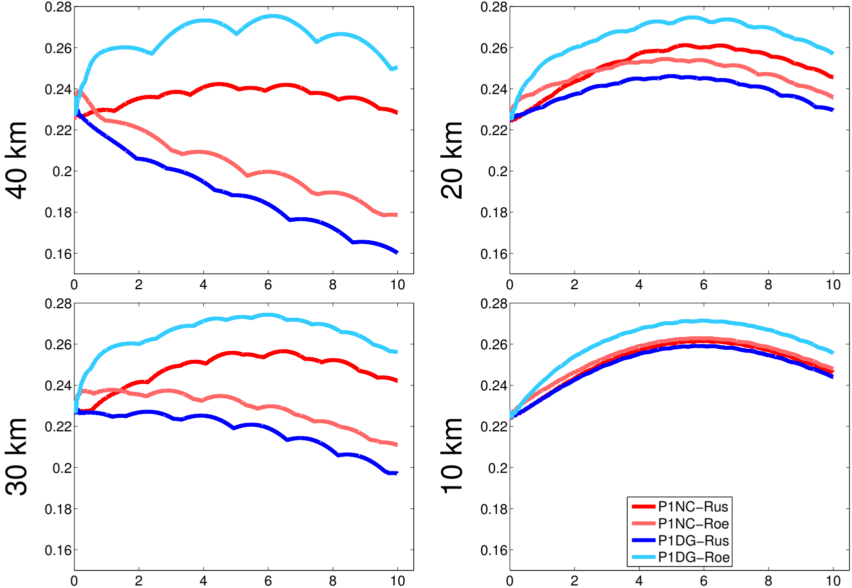
Roe



Rus



DG simulations (II): Propagating eddy: $\|\mathbf{u}\|_{L^\infty}$ after 10 weeks



Conclusions

- The discretization of the shallow-water equations usually leads to computational modes.
- We have proposed to study these problems by using Fourier (dispersion) analyses and the study of the kernels of the discrete operators.
- The cause of the computational solutions is mainly due to:
  - Wrong choice of discrete spaces for the variables u, v and η (spurion η modes).
  - An imbalance between the d.o.f. of u, v and η nodal values (inertial modes).
  - The use of normal velocities (f–modes).
- The Fourier analyses show that stabilized DG methods are free of spurious solutions:
  - The P_1^{NC} approximation with the Rusanov scheme is highly accurate for all modes.
  - Further, the P_1^{NC} scheme ”naturally“ discretizes the laplacian term without recouring to the LDG method for viscous flows.
  - Finally, we have obtained numerically a CFL limit of:
    - ★ 0.18 for the P_1^{DG} scheme
    - ★ 0.30 for the P_1^{NC} scheme.
- Fourier analysis should be performed for 3D models.