A hierarchy of Dispersive Layerwise models for free surface flow

Derivation and Simulation

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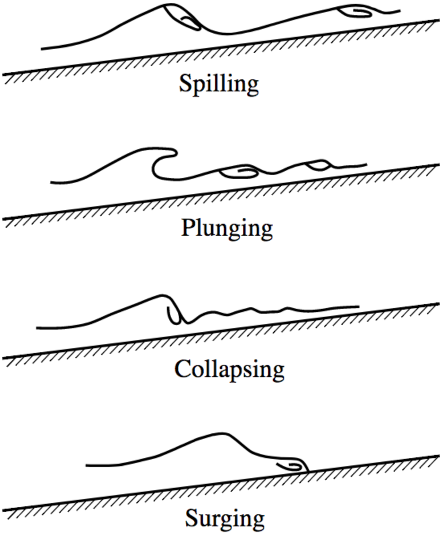
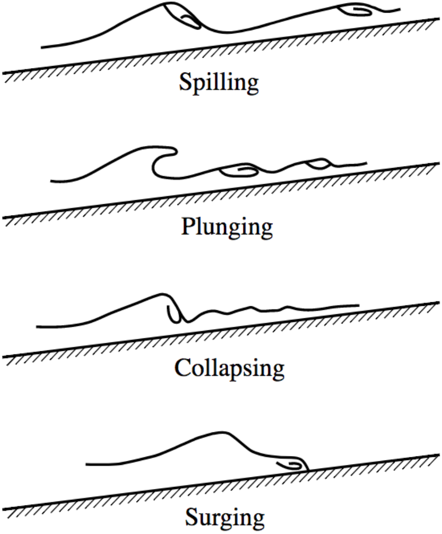
Yohan Penel (Ange)
and Jacques Saint-Marie (Ange)

NumWave - 11 december 2017



Vertical profile in breaking wave (left) and hydraulic jump (right) [Chow'59]



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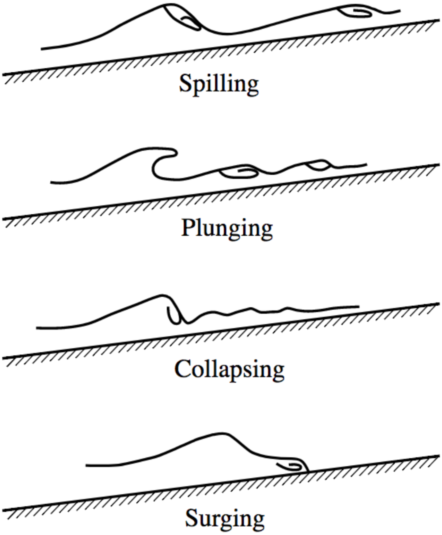

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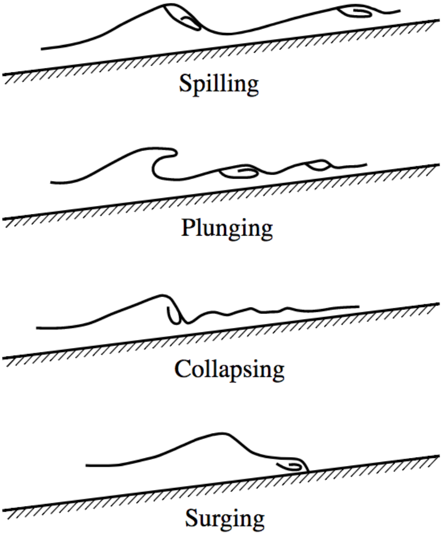

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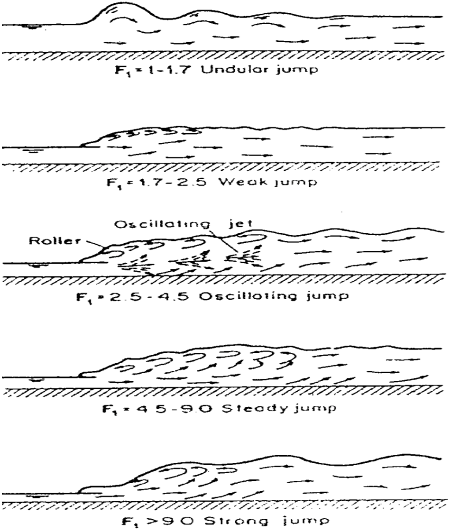

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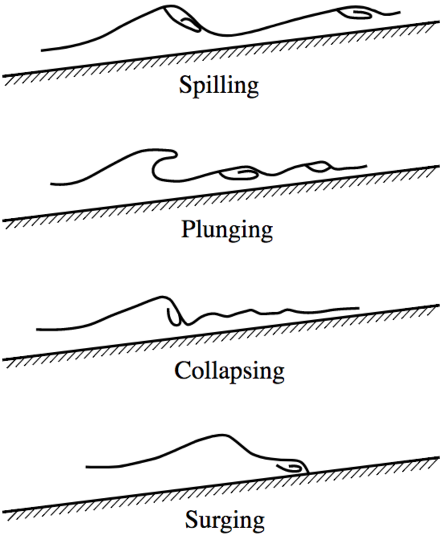
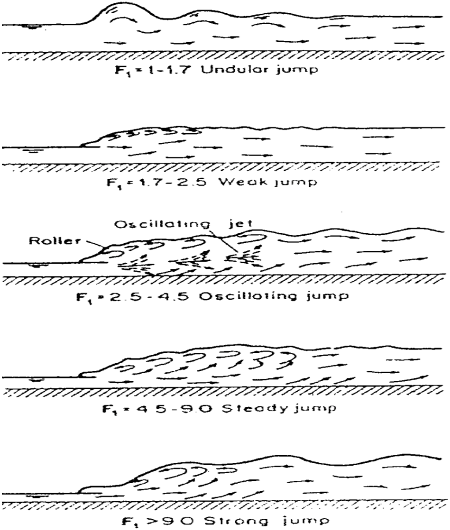
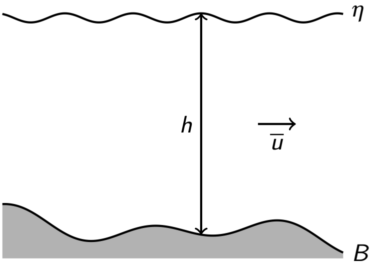
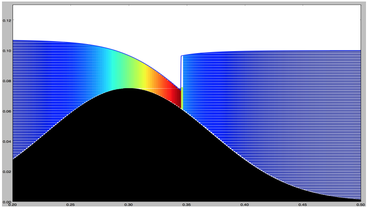




Vertical profile in breaking wave (left) and hydraulic jump (right) [Chow'59]

$(E) \xrightarrow{H^u_{yp})\text{ and }H^p_{yp})} (SW)$





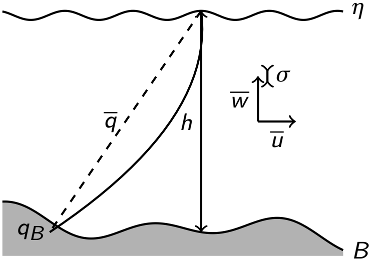
MODELS:

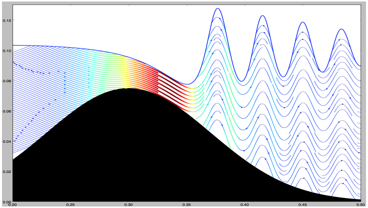
- (E) Euler eq. with **monovalued** free surface
- (SW) Shallow water model [Saint-Venant'1871]

ASSUMPTIONS:

- $H^p_{yp})$ **Hydrostatic** pressure: $p(t,x,z) = P(t,x) + g(\eta(t,x) - z)$
- $H^u_{yp})$ **Homogeneity** of the **horizontal** velocity in the column: $u(t,x,z) = \overline{u}(t,x)$

$$\begin{tikzcd}[column sep=large, row sep=large]
(E) \arrow[d, "H^u_{yp})"'] \arrow[rr, "H^u_{yp}) \text{ and } H^p_{yp})"] & & (SW) \\
(GN) \arrow[r, "H^w_{yp})"'] & (NH) \arrow[ur, "H^p_{yp})"'] &
\end{tikzcd}$$





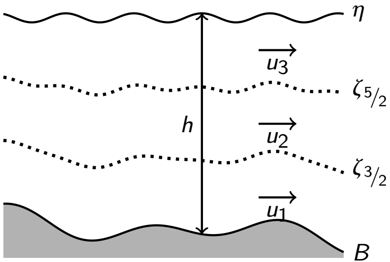
MODELS:

- (E) Euler eq. with **monovalued** free surface
- (SW) Shallow water model [Saint-Venant'1871]
- (NH) Non-hydrostatic model [Sainte-Marie'11]
- (GN) Green-Naghdi model [Serre'53]

ASSUMPTIONS:

- $H^p_{yp})$ **Hydrostatic** pressure: $p(t,x,z) = P(t,x) + g(\eta(t,x) - z)$
- $H^u_{yp})$ **Homogeneity** of the **horizontal** velocity in the column: $u(t,x,z) = \overline{u}(t,x)$
- $H^w_{yp})$ **Homogeneity** of the **vertical** velocity in the column: $w(t,x,z) = \overline{w}(t,x)$

$$
\begin{tikzcd}
(E) \arrow[rrr, "H^p_{yp})"] \arrow[d, "H^u_{yp})"] \arrow[drrr, "H^u_{yp}) \text{ and } H^p_{yp})"] & & & (SW_L) \arrow[d, "H^u_{yp})"] \\
(GN) \arrow[rr, "H^w_{yp})"'] & & (NH) \arrow[r, "H^p_{yp})"'] & (SW)
\end{tikzcd}
$$



MODELS:

- (E) Euler eq. with **monovalued** free surface
- (SW) Shallow water model
- (SW_L) Layerwise shallow water model [ABPSM'11]
- (NH) Non-hydrostatic model
- (GN) Green-Naghdi model

ASSUMPTIONS:

- $H^p_{yp})$ **Hydrostatic** pressure: $p(t,x,z) = P(t,x) + g(\eta(t,x) - z)$
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\begin{tikzcd}
 & (GN_L) \arrow[r, "H^w_{yp})"] & (NH_L) \arrow[dr, "H^p_{yp})"] & \\
(E) \arrow[ur, "\text{regul.}"] \arrow[rrr, "H^p_{yp})"] \arrow[d, "H^u_{yp})"] \arrow[drrr, "H^u_{yp}) \text{ and } H^p_{yp})"] & & & (SW_L) \arrow[d, "H^u_{yp})"] \\
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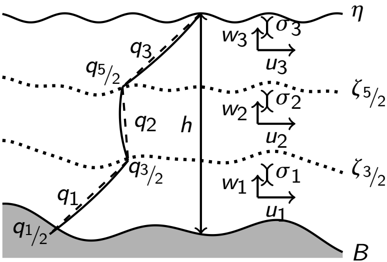

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(E)	Euler eq. with **monovalued** free surface		
(SW)	Shallow water model	(SW_L)	Layerwise shallow water model [ABPSM'11]
(NH)	Non-hydrostatic model	(NH_L)	Layerwise non-hydrostatic model [FNPPSM]
(GN)	Green-Naghdi model	(GN_L)	Layerwise Green-Naghdi model [FNPPSM]

ASSUMPTIONS:

$H^p_{yp})$ **Hydrostatic** pressure: $p(t,x,z) = P(t,x) + g(\eta(t,x) - z)$

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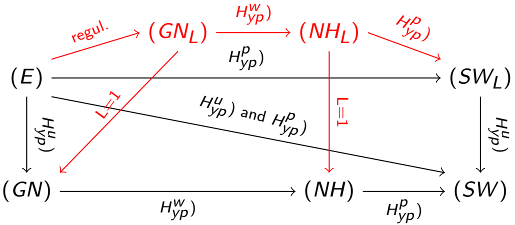

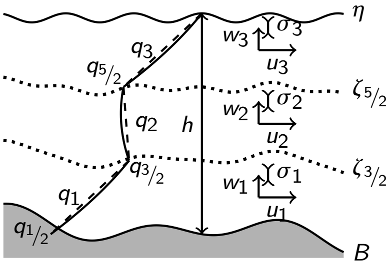

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ASSUMPTIONS:

$H^p_{yp})$ **Hydrostatic** pressure: $p(t,x,z) = P(t,x) + g(\eta(t,x) - z)$

$H^u_{yp})$ **Homogeneity** of the **horizontal** velocity in the column: $u(t,x,z) = \overline{u}(t,x)$

$H^w_{yp})$ **Homogeneity** of the **vertical** velocity in the column: $w(t,x,z) = \overline{w}(t,x)$

Euler equations (E):

Between a **given bottom** $B(t,x)$ and a **monovalued** free surface $\eta(t,x)$:

(1) $\nabla\cdot u + \partial_z w = 0$

(2) $\partial_t u + u\cdot\nabla u + w\partial_z u = -\nabla p$

(3) $\partial_t w + u\cdot\nabla w + w\partial_z w = -\partial_z p - g$

with **no-penetration** at bottom is assumed: (4) $\partial_t B + u_{|z=B}\cdot\nabla B - w_{|z=B} = 0$

at free surface, the **pressure is fixed**: (5) $p_{|z=\eta} = P(t,x)$

and the **kinematic condition** is considered: (6) $\partial_t \eta + u_{|z=\eta}\cdot\nabla\eta - w_{|z=\eta} = 0$

and compatible initial condition η^0, u^0 and w^0. $\nabla\cdot u^0 + \partial_z w^0 = 0$

$\partial_t B_{|t=0} + u^0_{|z=B}\cdot\nabla B_{|t=0} - w^0_{|z=B} = 0$

Energy conservation of (E)

For s.e.s., we have: $\partial_t\left(\mathscr{E}^{\eta-B} + \mathscr{K}^u + \mathscr{K}^w\right) + \nabla\cdot\mathscr{G} = (\eta - B)\partial_t P + p_{|z=B}\partial_t B$

with $\mathscr{E}^h = h\left(gB + P + g\frac{h}{2}\right)$ and $\mathscr{K}^\psi = \frac{1}{2}\int_B^\eta |\psi|^2\,\mathrm{d}z$.

Euler equations (E):

Between a **given bottom** $B(t,x)$ and a **monovalued** free surface $\eta(t,x)$:

(1) $\nabla\cdot u+\partial_z w=0$

(2) $\partial_t u+u\cdot\nabla u+w\partial_z u=-\nabla(P+g\eta+q)$

(3) $\partial_t w+u\cdot\nabla w+w\partial_z w=-\partial_z q$

with **no-penetration** at bottom is assumed: (4) $\partial_t B+u_{|_{z=B}}\cdot\nabla B-w_{|_{z=B}}=0$

at free surface, the **pressure is fixed**: (5) $p_{|_{z=\eta}}=P(t,x)$

and the **kinematic condition** is considered: (6) $\partial_t\eta+u_{|_{z=\eta}}\cdot\nabla\eta-w_{|_{z=\eta}}=0$

and compatible initial condition η^0, u^0 and w^0. $\nabla\cdot u^0+\partial_z w^0=0$

$$\partial_t B_{|_{t=0}}+u^0_{|_{z=B}}\cdot\nabla B_{|_{t=0}}-w^0_{|_{z=B}}=0$$

- We use the **hydrodynamic** pressure: $q(t,x,z)=p(t,x,z)-(P(t,x)+g(\eta(t,x)-z))$

- For L fixed parameter $(\ell_i)_{1\le i\le L}$ such that $\sum_i\ell_i=1$, we **discretize** the column by layers of thickness $\ell_i h$ with $h=\eta-B$ and $\mathbb{L}_i=[\zeta_{i-½},\zeta_{i+½}]$ with $\zeta_{i+½}=\zeta_{i-½}+\ell_i h$ and $\zeta_{½}=B$.

- Assuming u is close enough to its average in $\mathbb{L}_i$: $u(t,x,z\in\mathbb{L}_i)=u_i(t,x)+O(\varepsilon)$

Euler equations (E):

Between a **given bottom** $B(t,x)$ and a **monovalued** free surface $\eta(t,x)$:

(1) $\nabla \cdot u + \partial_z w = 0$

(2) $\partial_t u + u \cdot \nabla u + w \partial_z u = -\nabla(P + g\eta + q)$

(3) $\partial_t w + u \cdot \nabla w + w \partial_z w = -\partial_z q$

with **no-penetration** at bottom is assumed: (4) $\partial_t B + u_{|_{z=B}} \cdot \nabla B - w_{|_{z=B}} = 0$

at free surface, the **pressure is fixed**: (5) $p_{|_{z=\eta}} = P(t,x)$

and the **kinematic condition** is considered: (6) $\partial_t \eta + u_{|_{z=\eta}} \cdot \nabla \eta - w_{|_{z=\eta}} = 0$

and compatible initial condition η^0, u^0 and w^0. $\nabla \cdot u^0 + \partial_z w^0 = 0$

$\partial_t B_{|_{t=0}} + u^0_{|_{z=B}} \cdot \nabla B_{|_{t=0}} - w^0_{|_{z=B}} = 0$

- Integrating (1) on $\mathbb{L}_i$: $\partial_t h + \nabla \cdot (h u_i) = \frac{1}{\ell_i} [G]_{i-½}^{i+½}$

with G is the mass exchanged between the layers $G_{i+½} = \partial_t \zeta_{i+½} + u_{|_{z=\zeta_{i+½}}} \cdot \nabla \zeta_{i+½} - w_{|_{z=\zeta_{i+½}}}$

Euler equations (E):

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and compatible initial condition η^0, u^0 and w^0. $\nabla \cdot u^0 + \partial_z w^0 = 0$

$$\partial_t B_{|_{t=0}} + u^0_{|_{z=B}} \cdot \nabla B_{|_{t=0}} - w^0_{|_{z=B}} = 0$$

- Integrating (1) on $\mathbb{L}_i$: $$\partial_t h + \nabla \cdot (h u_i) = \frac{1}{\ell_i} [G]_{i-1/2}^{i+1/2}$$
- Integrating (2) on $\mathbb{L}_i$: $$\partial_t (h u_i) + \nabla \cdot (h u_i \otimes u_i) = -h \nabla (P + g\eta) + \frac{1}{\ell_i} [uG]_{i-1/2}^{i+1/2} - \frac{1}{\ell_i} \int_{\mathbb{L}_i} \nabla q \, \mathrm{d}z$$

Euler equations (E):

Between a **given bottom** $B(t,x)$ and a **monovalued** free surface $\eta(t,x)$:

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with **no-penetration** at bottom is assumed: (4) $\partial_t B+u_{|_{z=B}}\cdot\nabla B-w_{|_{z=B}}=0$

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and the **kinematic condition** is considered: (6) $\partial_t\eta+u_{|_{z=\eta}}\cdot\nabla\eta-w_{|_{z=\eta}}=0$

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$$\partial_t B_{|_{t=0}}+u^0_{|_{z=B}}\cdot\nabla B_{|_{t=0}}-w^0_{|_{z=B}}=0$$

- Integrating (1) on $\mathbb{L}_i$: $$\partial_t h+\nabla\cdot(hu_i)=\frac{1}{\ell_i}\left[G\right]_{i-1/2}^{i+1/2}$$
- Integrating (2) on $\mathbb{L}_i$: $$\partial_t(hu_i)+\nabla\cdot(hu_i\otimes u_i)=-h\nabla(P+g\eta)+\frac{1}{\ell_i}\left[uG\right]_{i-1/2}^{i+1/2}\cancel{-\frac{1}{\ell_i}\int_{\mathbb{L}_i}\nabla q\,\mathrm{d}z}$$
  - Assuming H^p_{yp}), we have $q=0$, and it leads to (SW_L).
  - To close the system, we set $$u_{i+1/2}G_{i+1/2}=\frac{u_i+u_{i+1}}{2}G_{i+1/2}+\gamma\frac{u_{i+1}-u_i}{2}\left|G_{i+1/2}\right|.$$

Energy conservation of (SW_L)

Assuming $\gamma\geq 0$. For s.e.s., we have: $$\partial_t\left(\mathcal{E}^h+\sum_i\mathcal{K}_i^u\right)+\nabla\cdot\mathcal{G}\leq h\partial_t P+gh\partial_t B$$

with $\mathcal{E}^h=h\left(gB+P+g\frac{h}{2}\right)$ and $\mathcal{K}_i^\psi=\frac{\ell_i}{2}h|\psi_i|^2$.a̧

Euler equations (E):

Between a **given bottom** $B(t,x)$ and a **monovalued** free surface $\eta(t,x)$:	(1) $\nabla\cdot u + \partial_z w = 0$ (2) $\partial_t u + u\cdot\nabla u + w\partial_z u = -\nabla(P + g\eta + q)$ (3) $\partial_t w + u\cdot\nabla w + w\partial_z w = -\partial_z q$
with **no-penetration** at bottom is assumed:	(4) $\partial_t B + u_{\vert_{z=B}}\cdot\nabla B - w_{\vert_{z=B}} = 0$
at free surface, the **pressure is fixed**:	(5) $p_{\vert_{z=\eta}} = P(t,x)$
and the **kinematic condition** is considered:	(6) $\partial_t\eta + u_{\vert_{z=\eta}}\cdot\nabla\eta - w_{\vert_{z=\eta}} = 0$
and compatible initial condition η^0, u^0 and w^0.	$\nabla\cdot u^0 + \partial_z w^0 = 0$ $\partial_t B_{\vert_{t=0}} + u^0_{\vert_{z=B}}\cdot\nabla B_{\vert_{t=0}} - w^0_{\vert_{z=B}} = 0$

- Integrating (1) on $\mathbb{L}_i$: $\partial_t h + \nabla\cdot(hu_i) = \frac{1}{\ell_i}[G]_{i-1/2}^{i+1/2}$
- Integrating (2) on $\mathbb{L}_i$: $\partial_t(hu_i) + \nabla\cdot(hu_i\otimes u_i) = -h\nabla(P+g\eta) + \frac{1}{\ell_i}[uG]_{i-1/2}^{i+1/2} - \nabla(hq_i) + \frac{1}{\ell_i}[q\nabla\zeta]_{i-1/2}^{i+1/2}$

Euler equations (E):

Between a **given bottom** $B(t,x)$ and a **monovalued** free surface $\eta(t,x)$:

$$(1)\ \nabla\cdot u+\partial_z w=0$$
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and compatible initial condition η^0, u^0 and w^0. $\nabla\cdot u^0+\partial_z w^0=0$

$$\partial_t B_{|_{t=0}}+u^0_{|_{z=B}}\cdot\nabla B_{|_{t=0}}-w^0_{|_{z=B}}=0$$

- Integrating (1) on $\mathbb{L}_i$: $\partial_t h+\nabla\cdot\left(hu_i\right)=\frac{1}{\ell_i}\left[G\right]_{i-½}^{i+½}$
- Integrating (2) on $\mathbb{L}_i$: $\partial_t\left(hu_i\right)+\nabla\cdot\left(hu_i\otimes u_i\right)=-h\nabla\left(P+g\eta\right)+\frac{1}{\ell_i}\left[uG\right]_{i-½}^{i+½}-\nabla\left(hq_i\right)+\frac{1}{\ell_i}\left[q\nabla\zeta\right]_{i-½}^{i+½}$
- Integrating (3) on $\mathbb{L}_i$: $\partial_t\left(hw_i\right)+\nabla\cdot\left(hw_i\ u_i\right)=\frac{1}{\ell_i}\left[wG\right]_{i-½}^{i+½}-\frac{1}{\ell_i}\left[q\right]_{i-½}^{i+½}$

Euler equations (E):

Between a **given bottom** $B(t,x)$ and a **monovalued** free surface $\eta(t,x)$:

(1) $\nabla\cdot u + \partial_z w = 0$

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$\partial_t B_{|_{t=0}} + u^0_{|_{z=B}}\cdot\nabla B_{|_{t=0}} - w^0_{|_{z=B}} = 0$

- Integrating (1) on $\mathbb{L}_i$: $\partial_t h + \nabla\cdot(hu_i) = \frac{1}{\ell_i}[G]_{i-1/2}^{i+1/2}$
- Integrating (2) on $\mathbb{L}_i$: $\partial_t(hu_i) + \nabla\cdot(hu_i\otimes u_i) = -h\nabla(P+g\eta) + \frac{1}{\ell_i}[uG]_{i-1/2}^{i+1/2} - \nabla(hq_i) + \frac{1}{\ell_i}[q\nabla\zeta]_{i-1/2}^{i+1/2}$
- Integrating (3) on $\mathbb{L}_i$: $\partial_t(hw_i) + \nabla\cdot(hw_i\ u_i) = \frac{1}{\ell_i}[wG]_{i-1/2}^{i+1/2} - \frac{1}{\ell_i}[q]_{i-1/2}^{i+1/2}$
- Integrating **twice** (1) on $\mathbb{L}_i$: $w_i = \partial_t\zeta_i + u_i\cdot\nabla\zeta_i - G_i$

with $\zeta_i = \frac{\zeta_{i-1/2}+\zeta_{i+1/2}}{2}$ and $G_i = \frac{G_{i-1/2}+G_{i+1/2}}{2}$

Euler equations (E):

Between a **given bottom** $B(t,x)$ and a **monovalued** free surface $\eta(t,x)$:

with **no-penetration** at bottom is assumed:
at free surface, the **pressure is fixed**:
and the **kinematic condition** is considered:
and compatible initial condition η^0, u^0 and w^0.

$$
\begin{aligned}
&(1)\ \nabla\cdot u + \partial_z w = 0\\
&(2)\ \partial_t u + u\cdot\nabla u + w\partial_z u = -\nabla(P + g\eta + q)\\
&(3)\ \partial_t w + u\cdot\nabla w + w\partial_z w = -\partial_z q\\
&(4)\ \partial_t B + u_{|_{z=B}}\cdot\nabla B - w_{|_{z=B}} = 0\\
&(5)\ p_{|z=\eta} = P(t,x)\\
&(6)\ \partial_t\eta + u_{|_{z=\eta}}\cdot\nabla\eta - w_{|_{z=\eta}} = 0\\
&\nabla\cdot u^0 + \partial_z w^0 = 0\\
&\partial_t B_{|_{t=0}} + u^0_{|_{z=B}}\cdot\nabla B_{|_{t=0}} - w^0_{|_{z=B}} = 0
\end{aligned}
$$

- Integrating (1) on $\mathbb{L}_i$: $\partial_t h + \nabla\cdot(hu_i) = \frac{1}{\ell_i}[G]_{i-½}^{i+½}$
- Integrating (2) on $\mathbb{L}_i$: $\partial_t(hu_i) + \nabla\cdot(hu_i\otimes u_i) = -h\nabla(P+g\eta) + \frac{1}{\ell_i}[uG]_{i-½}^{i+½} - \nabla(hq_i) + \frac{1}{\ell_i}[q\nabla\zeta]_{i-½}^{i+½}$
- Integrating (3) on $\mathbb{L}_i$: $\partial_t(hw_i) + \nabla\cdot(hw_i\ u_i) = \frac{1}{\ell_i}[wG]_{i-½}^{i+½} - \frac{1}{\ell_i}[q]_{i-½}^{i+½}$
- Integrating **twice** (1) on $\mathbb{L}_i$: $w_i = \partial_t\zeta_i + u_i\cdot\nabla\zeta_i - G_i$
  - Assuming H^w_{yp}), we have $q_i = \frac{q_{i-½}+q_{i+½}}{2}$, and it leads to (NH_L).

Energy conservation of (NH_L)

Assuming $\gamma \geq 0$. For s.e.s., we have: $\partial_t\left(\mathscr{E}^h + \sum_i\left(\mathscr{K}_i^u + \mathscr{K}_i^w\right)\right) + \nabla\cdot\mathscr{G} \leq h\partial_t P + (gh + q_{½})\partial_t B$
with $\mathscr{E}^h = h\left(gB + P + g\frac{h}{2}\right)$ and $\mathscr{K}_i^\psi = \frac{\ell_i}{2}h|\psi_i|^2$.

Euler equations (E):

Between a **given bottom** $B(t,x)$ and a **monovalued** free surface $\eta(t,x)$:

(1) $\nabla\cdot u + \partial_z w = 0$

(2) $\partial_t u + u\cdot\nabla u + w\partial_z u = -\nabla(P + g\eta + q)$

(3) $\partial_t w + u\cdot\nabla w + w\partial_z w = -\partial_z q$

with **no-penetration** at bottom is assumed: (4) $\partial_t B + u_{|z=B}\cdot\nabla B - w_{|z=B} = 0$

at free surface, the **pressure is fixed**: (5) $p_{|z=\eta} = P(t,x)$

and the **kinematic condition** is considered: (6) $\partial_t\eta + u_{|z=\eta}\cdot\nabla\eta - w_{|z=\eta} = 0$

and compatible initial condition η^0, u^0 and w^0. $\nabla\cdot u^0 + \partial_z w^0 = 0$

$\partial_t B_{|t=0} + u^0_{|z=B}\cdot\nabla B_{|t=0} - w^0_{|z=B} = 0$

- Integrating (1) on $\mathbb{L}_i$: $\partial_t h + \nabla\cdot(hu_i) = \frac{1}{\ell_i}[G]_{i-½}^{i+½}$
- Integrating (2) on $\mathbb{L}_i$: $\partial_t(hu_i) + \nabla\cdot(hu_i\otimes u_i) = -h\nabla(P+g\eta) + \frac{1}{\ell_i}[uG]_{i-½}^{i+½} - \nabla(hq_i) + \frac{1}{\ell_i}[q\nabla\zeta]_{i-½}^{i+½}$
- Integrating (3) on $\mathbb{L}_i$: $\partial_t(hw_i) + \nabla\cdot(hw_i\ u_i) = \frac{1}{\ell_i}[wG]_{i-½}^{i+½} - \frac{1}{\ell_i}[q]_{i-½}^{i+½}$
- Integrating **twice** (1) on $\mathbb{L}_i$: $w_i = \partial_t\zeta_i + u_i\cdot\nabla\zeta_i - G_i$
- Multiplying (3) by $(z-\zeta_i)$ and integrating (**weak derivation**) on $\mathbb{L}_i$:

$$\partial_t(h\sigma_i) + \nabla\cdot(h\sigma_i\ u_i) = -\frac{1}{\ell_i}\left(\left(\sigma_i - \sqrt{3}(w_{i+½} - w_i)\right)G_{i+½} - \left(\sigma_i + \sqrt{3}(w_{i-½} - w_i)\right)G_{i-½}\right) + \frac{2\sqrt{3}}{\ell_i}\left(q_i - \frac{q_{i-½}+q_{i+½}}{2}\right)$$

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Between a **given bottom** $B(t,x)$
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- Integrating (1) on $\mathbb{L}_i$: $\partial_t h+\nabla\cdot(hu_i)=\frac{1}{\ell_i}[G]_{i-½}^{i+½}$
- Integrating (2) on $\mathbb{L}_i$: $\partial_t(hu_i)+\nabla\cdot(hu_i\otimes u_i)=-h\nabla(P+g\eta)+\frac{1}{\ell_i}[uG]_{i-½}^{i+½}-\nabla(hq_i)+\frac{1}{\ell_i}[q\nabla\zeta]_{i-½}^{i+½}$
- Integrating (3) on $\mathbb{L}_i$: $\partial_t(hw_i)+\nabla\cdot(hw_i\ u_i)=\frac{1}{\ell_i}[wG]_{i-½}^{i+½}-\frac{1}{\ell_i}[q]_{i-½}^{i+½}$
- Integrating **twice** (1) on $\mathbb{L}_i$: $w_i=\partial_t\zeta_i+u_i\cdot\nabla\zeta_i-G_i$
- $\partial_t(h\sigma_i)+\nabla\cdot(h\sigma_i\ u_i)=-\frac{1}{\ell_i}\left(\left(\sigma_i-\sqrt{3}(w_{i+½}-w_i)\right)G_{i+½}-\left(\sigma_i+\sqrt{3}(w_{i-½}-w_i)\right)G_{i-½}\right)+\frac{2\sqrt{3}}{\ell_i}\left(q_i-\frac{q_{i-½}+q_{i+½}}{2}\right)$
- Looking at the **oriented standard variation** in (1) on $\mathbb{L}_i$: $\sigma_i=-\frac{\ell_i h}{2\sqrt{3}}\nabla\cdot u_i$

Energy conservation of (GN_L)

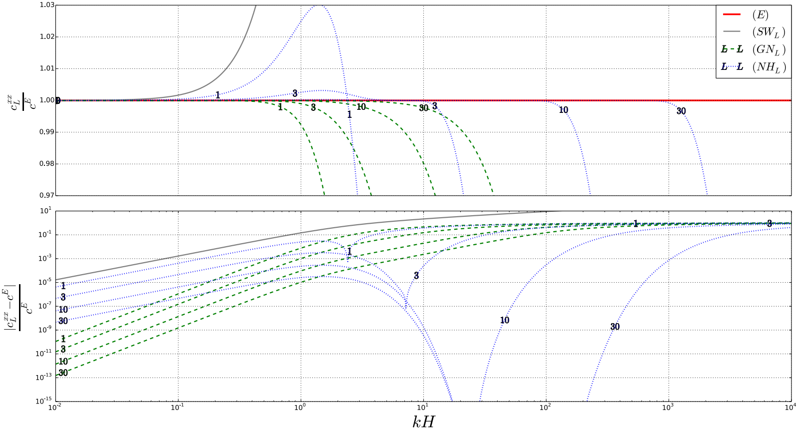
Assuming $\gamma\geq 0$. For s.e.s., we have: $\partial_t\left(\mathcal{E}^h+\sum_i\left(\mathcal{K}_i^u+\mathcal{K}_i^w+\mathcal{K}_i^\sigma\right)\right)+\nabla\cdot\mathcal{G}\leq h\partial_t P+(gh+q_{½})\partial_t B$
with $\mathcal{E}^h=h\left(gB+P+g\frac{h}{2}\right)$ and $\mathcal{K}_i^\psi=\frac{\ell_i}{2}h|\psi_i|^2$.

Dispersion relations of the layerwise models (NH_L) and (GN_L)

The dispersion relation around the state $(h,u_i)=(H,0)$ reads: $\left|c_L^{xx}\right| = \sqrt{\langle A_{kH}^{-1}e,\ell\rangle}\,c^{sh}$

with $c^{sh}=\sqrt{gH}$, $\ell=(\ell_i)_{1\le i\le L}$, $e=(1)_{1\le i\le L}$, $\lambda^{nh}=4$, $\lambda^{gn}=6$

$$\text{and } A_x = \mathrm{I}_d - x^2 B \text{ with } B_{i,j} = \frac{\ell_i^2}{\lambda^{xx}}\delta_{i,j} - \ell_j\left(\frac{\ell_{\max(i,j)}}{2} + \sum_{l=\max(i,j)+1}^{L}\ell_l\right)$$



Dispersion relations of the layerwise models (NH_L) and (GN_L)

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and $A_x = \mathrm{I}_d - x^2 B$ with $B_{i,j} = \dfrac{\ell_i^2}{\lambda^{xx}}\delta_{i,j} - \ell_j\left(\dfrac{\ell_{\max(i,j)}}{2} + \displaystyle\sum_{l=\max(i,j)+1}^{L}\ell_l\right)$

Theorem:

For any wave number $k>0$, the phase velocity of the dispersive models converges to the phase velocity of (E) when the number of layers goes to infinity.

$$c_L^{xx} \underset{L\to\infty}{\longrightarrow} c^E = \frac{tanh(kH)}{kH}$$

Layerwise Serre/Green-Naghdi model (GN_L):

$$\partial_t U_i + \nabla\cdot F(U_i) = S(U_i) + \frac{1}{\ell_i}\left[\Gamma\right]_{i-½\oplus}^{i+½\ominus} + D_i$$

$$\begin{aligned}
\partial_t h \ + \ \nabla\cdot(hu_i) \quad &= \quad \frac{1}{\ell_i}\left[G\right]_{i-½}^{i+½} \\
\partial_t (hu_i) \ + \ \nabla\cdot(hu_i\otimes u_i) + h\nabla(P+g\eta) &= \quad \frac{1}{\ell_i}\left[uG\right]_{i-½}^{i+½} - \nabla(hq_i) + \frac{1}{\ell_i}\left[q\nabla\zeta\right]_{i-½}^{i+½} \\
\partial_t (hw_i) \ + \ \nabla\cdot(hw_i\ u_i) \quad &= \quad \frac{1}{\ell_i}\left[wG\right]_{i-½}^{i+½} - \frac{1}{\ell_i}\left[q\right]_{i-½}^{i+½} \\
\partial_t (h\sigma_i) \ + \ \nabla\cdot(h\sigma_i\ u_i) \quad &= \quad \frac{1}{\ell_i}\left[\sigma G\right]_{i-½\oplus}^{i+½\ominus} + \frac{2\sqrt{3}}{\ell_i}\left(q_i - \frac{q_{i-½}+q_{i+½}}{2}\right) \\
w_i = \partial_t\zeta_i + u_i\cdot\nabla\zeta_i - G_i \qquad & \qquad \sigma_i = -\frac{\ell_i h}{2\sqrt{3}}\nabla\cdot u_i
\end{aligned}$$

Solved using a **splitting** approach:

Layerwise Serre/Green-Naghdi model (GN_L):

$$\partial_t U_i + \nabla \cdot F(U_i) = S(U_i) + \frac{1}{\ell_i}[\Gamma]_{i-½\oplus}^{i+½\ominus} + D_i$$

$$\begin{aligned}
\frac{h_i^{n\star} - h^n}{\delta_t^n} &+ \nabla \cdot \left(h^n u_i^n\right) &&= 0\\
\frac{h_i^{n\star} u_i^{n\star} - h^n u_i^n}{\delta_t^n} &+ \nabla \cdot \left(h^n u_i^n \otimes u_i^n\right) + h^n \nabla (P + g\eta^n) &&= 0\\
\frac{h_i^{n\star} w_i^{n\star} - h^n w_i^n}{\delta_t^n} &+ \nabla \cdot \left(h^n w_i^n \; u_i^n\right) &&= 0\\
\frac{h_i^{n\star} \sigma_i^{n\star} - h^n \sigma_i^n}{\delta_t^n} &+ \nabla \cdot \left(h^n \sigma_i^n \; u_i^n\right) &&= 0
\end{aligned}$$

Solved using a **splitting** approach:

1. **Advection and conservative forces**
   1.1 Horizontal advection
      using in each layer a **classical shallow water scheme** (Godunov, HLL, kinetic...)

Layerwise Serre/Green-Naghdi model (GN_L):

$$\partial_t U_i + \nabla \cdot F(U_i) = S(U_i) + \frac{1}{\ell_i}\left[\Gamma\right]_{i-½\oplus}^{i+½\ominus} + D_i$$

$$\begin{aligned}
\frac{h^{n+½} - h_i^{n\star}}{\delta_t^n} &= \frac{1}{\ell_i}\left[G^{n+½}\right]_{i-½}^{i+½}\\
\frac{h^{n+½}u_i^{n+½} - h_i^{n\star}u_i^{n\star}}{\delta_t^n} &= \frac{1}{\ell_i}\left[u^{n+½}G^{n+½}\right]_{i-½}^{i+½}\\
\frac{h_i^{n+½}w_i^{n+½} - h_i^{n\star}w_i^{n\star}}{\delta_t^n} &= \frac{1}{\ell_i}\left[w^{n+½}G^{n+½}\right]_{i-½}^{i+½}\\
\frac{h_i^{n+½}\sigma_i^{n+½} - h_i^{n\star}\sigma_i^{n\star}}{\delta_t^n} &= \frac{1}{\ell_i}\left[\sigma^{n+½}G^{n+½}\right]_{i-½\oplus}^{i+½\ominus}
\end{aligned}$$

Solved using a **splitting** approach:

1. **Advection and conservative forces**
   1.1 Horizontal advection
   1.2 Vertical advection
   implicit up-wind scheme but **solved explicitly**

Layerwise Serre/Green-Naghdi model (GN_L):

$$\partial_t U_i + \nabla\cdot F(U_i) = S(U_i) + \frac{1}{\ell_i}\left[\Gamma\right]_{i-½\oplus}^{i+½\ominus} + D_i$$

$$\begin{aligned}
\frac{h^{n+1} - h^{n+½}}{\delta_t^n} &= 0\\
\frac{h^{n+1} u_i^{n+1} - h^{n+½} u_i^{n+½}}{\delta_t^n} &= -\nabla\left(h^{n+1} q_i^{n+1}\right) + \frac{1}{\ell_i}\left[q^{n+1}\nabla\zeta^{n+1}\right]_{i-½}^{i+½}\\
\frac{h_i^{n+1} w_i^{n+1} - h_i^{n+½} w_i^{n+½}}{\delta_t^n} &= -\frac{1}{\ell_i}\left[q^{n+1}\right]_{i-½}^{i+½}\\
\frac{h_i^{n+1} \sigma_i^{n+1} - h_i^{n+½} \sigma_i^{n+½}}{\delta_t^n} &= \frac{2\sqrt{3}}{\ell_i}\left(q_i^{n+1} - \frac{q_{i-½}^{n+1} + q_{i+½}^{n+1}}{2}\right)
\end{aligned}$$

$$w_i^{n+1} = \partial_t^{n+1} B + u_i^{n+1}\cdot\nabla\zeta_{i-1/2}^{n+1} - \frac{\ell_i}{2} h^{n+1}\nabla_k\cdot u_i^{n+1} \quad ; \quad \sigma_i^{n+1} = -\frac{\ell_i h^{n+1}}{2\sqrt{3}}\nabla\cdot u_i^{n+1}$$

Solved using a **splitting** approach:

1. **Advection and conservative forces**
   1.1 Horizontal advection
   1.2 Vertical advection
2. **Dispersion and dissipative forces**
   Linear system but **very big** 1d: $(5LN)^2$-system with a 3-point stencil

Layerwise Serre/Green-Naghdi model (GN_L):

$$\partial_t U_i + \nabla \cdot F(U_i) = S(U_i) + \frac{1}{\ell_i}[\Gamma]_{i-½\oplus}^{i+½\ominus} + D_i$$

$$\sum_{j=1}^{L}\left(\alpha_{ij} u_j^{n+1} + \nabla\left(\mu_{ij}\cdot u_j^{n+1}\right) - \nu_{ij}\nabla\cdot u_j^{n+1} - \nabla\left(\kappa_{ij}\nabla\cdot u_j^{n+1}\right)\right) = \beta_i$$

$$w_i^{n+1} = \partial_t^{n+1} B + u_i^{n+1}\cdot\nabla\zeta_{i-1/2}^{n+1} - \frac{\ell_i}{2} h^{n+1}\nabla_k\cdot u_i^{n+1} \quad ; \quad \sigma_i^{n+1} = -\frac{\ell_i h^{n+1}}{2\sqrt{3}}\nabla\cdot u_i^{n+1}$$

Solved using a **splitting** approach:

1. **Advection and conservative forces**
   - 1.1 Horizontal advection
   - 1.2 Vertical advection
2. **Dispersion and dissipative forces**
   - 2.1 Implicit elliptic scheme on u_i

     System smaller but **very large stencil** 1d: $(LN)^2$-system with a $5L$-point stencil
   - 2.1 Reconstruction of w_i and σ_i

     and eventually q_i and $q_{i-½}$

Layerwise Serre/Green-Naghdi model (GN_L):

$$\partial_t U_i + \nabla \cdot F(U_i) = S(U_i) + \frac{1}{\ell_i}[\Gamma]_{i-½\oplus}^{i+½\ominus} + D_i$$

$$\begin{aligned}
\partial_t h \;+\; \nabla\cdot(hu_i) &= \frac{1}{\ell_i}[G]_{i-½}^{i+½} \\
\partial_t(hu_i) \;+\; \nabla\cdot(hu_i \otimes u_i) + h\nabla(P+g\eta) &= \frac{1}{\ell_i}[uG]_{i-½}^{i+½} - \nabla(hq_i) + \frac{1}{\ell_i}[q\nabla\zeta]_{i-½}^{i+½} \\
\partial_t(hw_i) \;+\; \nabla\cdot(hw_i\ u_i) &= \frac{1}{\ell_i}[wG]_{i-½}^{i+½} - \frac{1}{\ell_i}[q]_{i-½}^{i+½} \\
\partial_t(h\sigma_i) \;+\; \nabla\cdot(h\sigma_i\ u_i) &= \frac{1}{\ell_i}[\sigma G]_{i-½\oplus}^{i+½\ominus} + \frac{2\sqrt{3}}{\ell_i}\left(q_i - \frac{q_{i-½}+q_{i+½}}{2}\right)
\end{aligned}$$

Theorem:

Using a numerical scheme of (SW) satisfying the **positivity** and the **entropy-dissipation** under a CFL condition

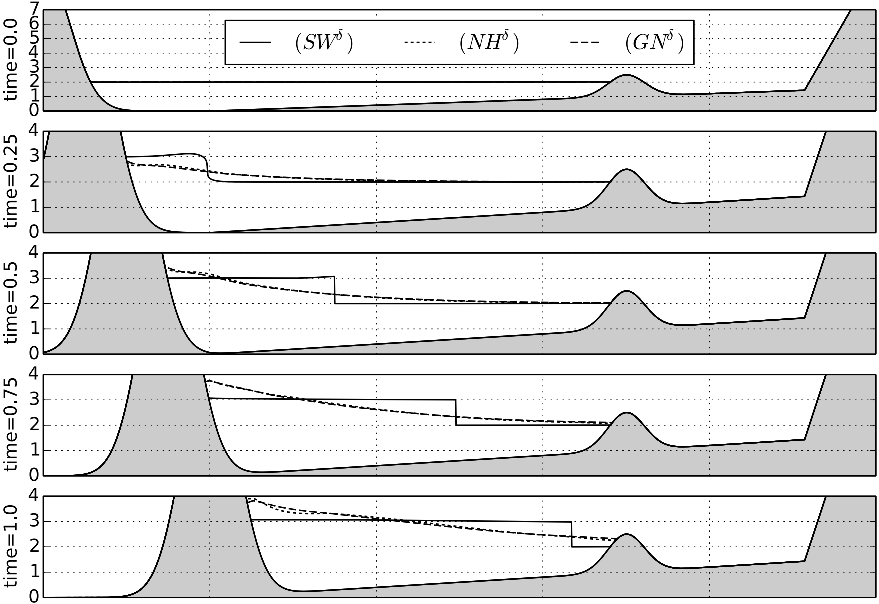
$$\lambda(h^n,\overline{u}^n)\delta_t^n \leq \delta_k$$

then under the CFL condition

$$\lambda\left(h^n, u_i^n\right)\delta_t^n \leq \delta_k$$

the proposed scheme satisfies the **positivity** and the **entropy-dissipation** even at **dry front**.

Seawall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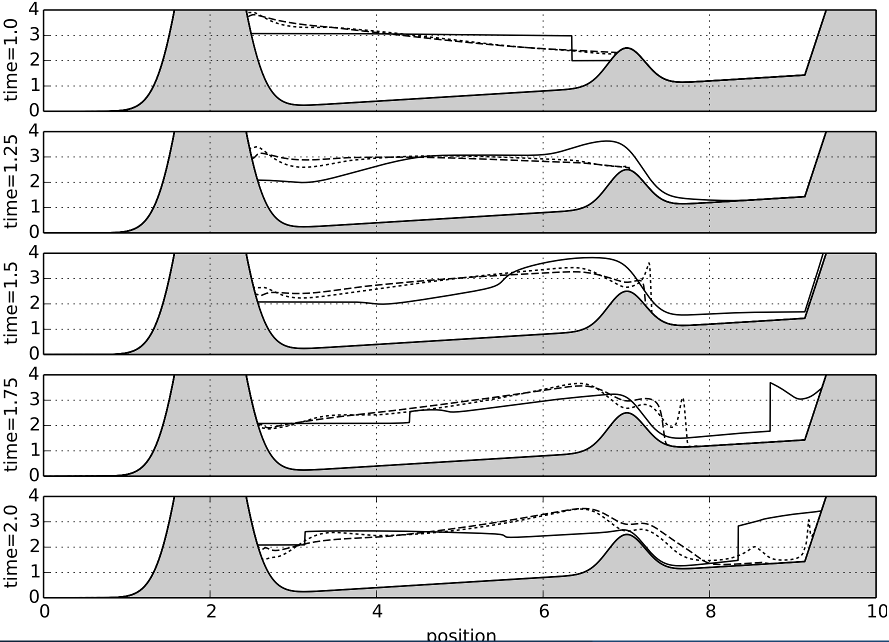
time=0.25
time=0.5
time=0.75
time=1.0
(SW^δ)
(NH^δ)
(GN^δ)
0
1
2
3
4
5
6
7

Seawall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0
1
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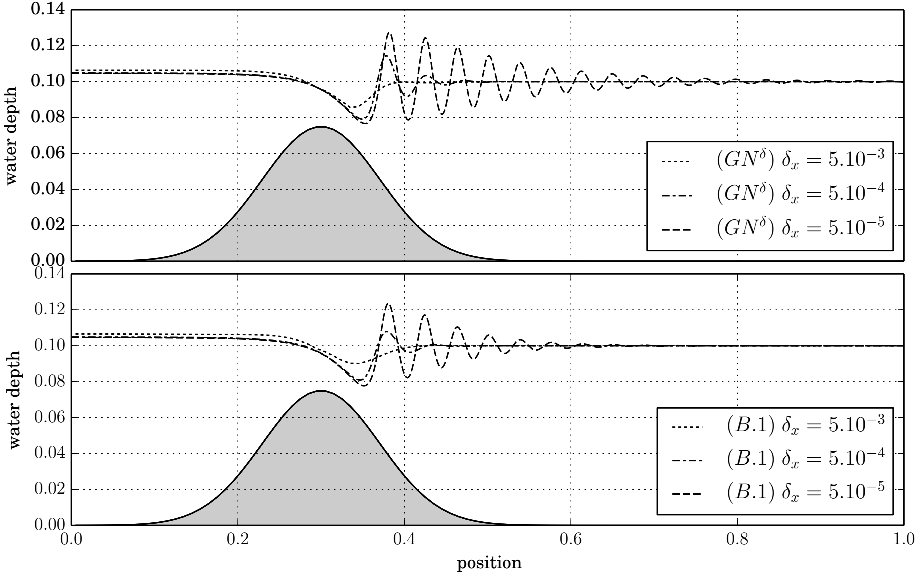
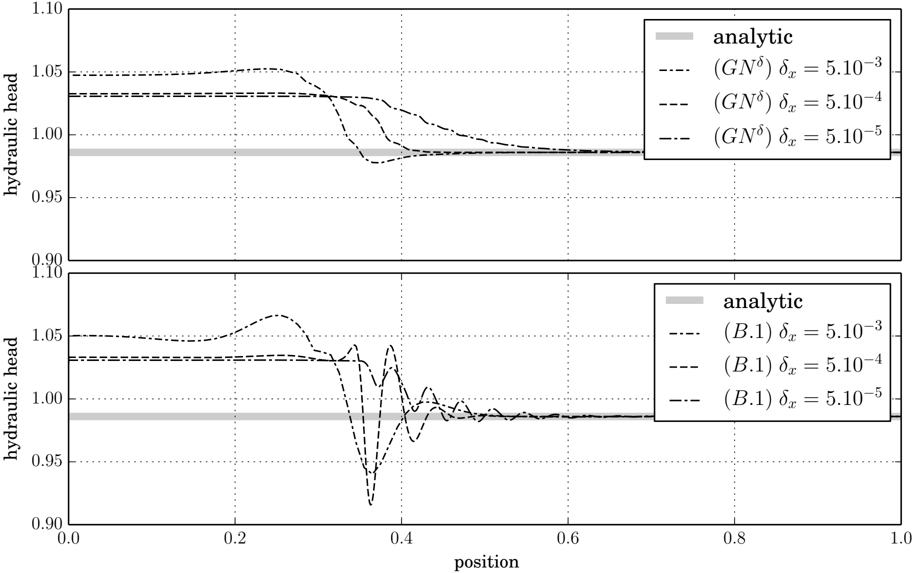
0
2
4
6
8
10
position

Dispersive jump

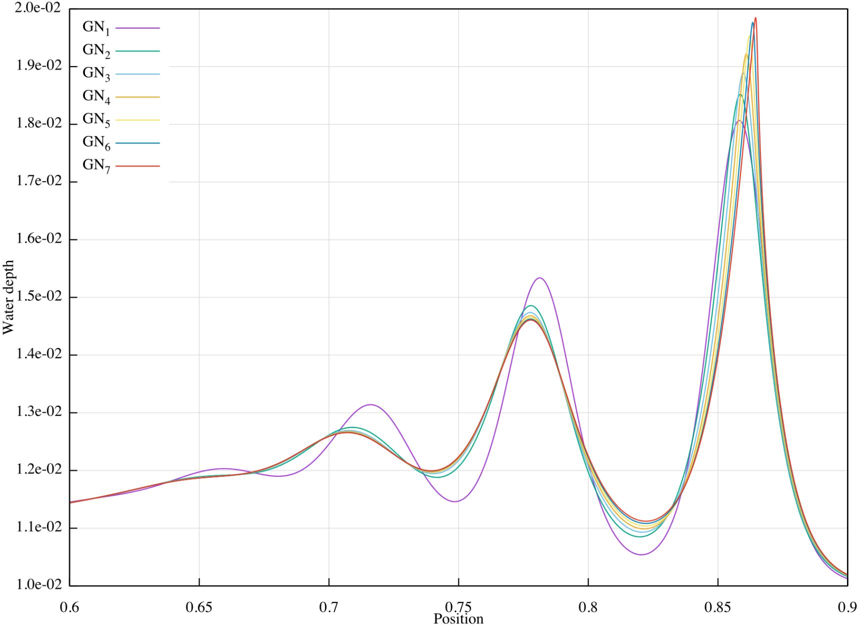


Dispersive jump

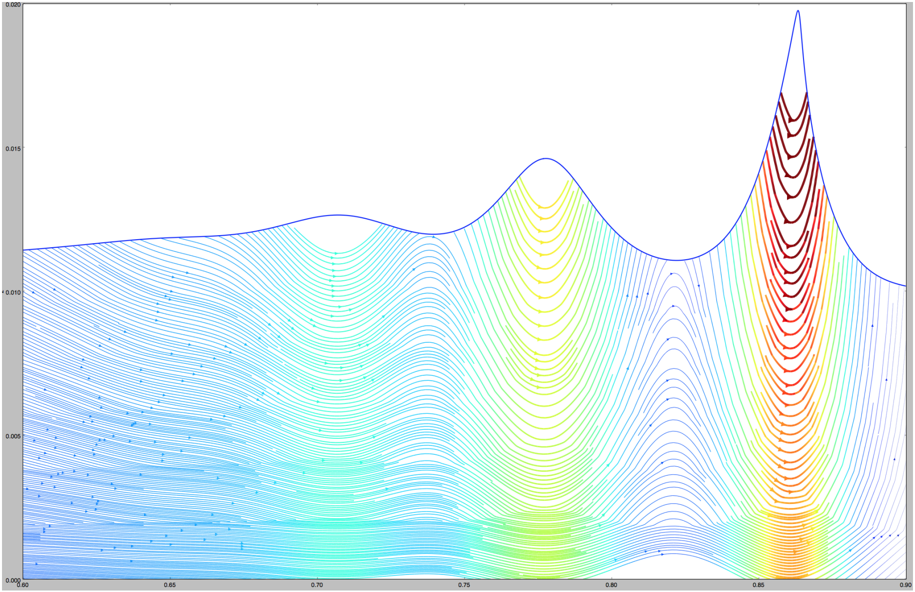


Water drop

▸ video



Water drop ▸ video



Realizations:

- A **dispersive layerwise** hierarchy of models was proposed

  [Fernandez-Nieto, Parisot, Penel and Sainte-Marie] hal-01324012
  Extendable to: viscosity, friction, Corriolis, capillary [Marche'07].

- An **entropy-satisfying** numerical scheme was designed

  [Parisot] hal-01242128.

Perspectives:

- Link with other models
  - the σ-**transform** [Castro]
  - (GN) with vorticity [Castro, Lannes'14]
- Boundary condition
  - **transparent boundary** for layerwise model [Kazakova, Noble]
  - imposed **hydraulic head** (for hydraulic jump analysis)
- Breaking waves
  - numerical investigation of the **energy dissipation**
  - **adaptative** layer discretization
- Considering other transformation (for plunging wave [Murashige])