Some Anisotropic Viscoelastic Green Functions

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Abstract. In this paper, we compute the closed form expressions of elastodynamic Green functions for three different viscoelastic media with simple type of anisotropy. We follow Burridge *et al.* [Proc. Royal Soc. of London. 440(1910): (1993)] to express unknown Green function in terms of three scalar functions ϕ_i, by using the spectral decomposition of the Christoffel tensor associated with the medium. The problem of computing Green function is, thus reduced to the resolution of three scalar wave equations satisfied by ϕ_i, and subsequent equations with ϕ_i as source terms. To describe viscosity effects, we choose an empirical power law model which becomes well known Voigt model for quadratic frequency losses.

1. Introduction

Numerous applications in biomedical imaging [**6, 14**], seismology [**2, 22**], exploration geophysics [**29, 30**], material sciences [**4, 15**] and engineering sciences [**1, 17, 32**] have fueled research and development in theory of elasticity. Elastic properties and attributes have gained interest in the recent decades as a diagnostic tool for non-invasive imaging [**28, 37**]. Their high correlation with the pathology and the underlying structure of soft tissues has inspired many investigations in biomedical imaging and led to many interesting mathematical problems [**7, 10, 9, 11, 8, 16, 38, 39**].

Biological materials are often assumed to be isotropic and inviscid with respect to elastic deformation. However, several recent studies indicate that many soft tissues exhibit anisotropic and viscoelastic behavior [**27, 35, 38, 39, 33, 47**]. Sinkus *et al.* have inferred in [**38**] that breast tumor tends to be anisotropic, while Weaver *et al.* [**46**] have provided an evidence that even non cancerous breast tissue is anisotropic. White matter in brain [**33**] and cortical bones [**47**] also exhibit similar behavior. Moreover, it has been observed that the shear velocities parallel and orthogonal to the fiber direction in forearm [**35**] and biceps [**27**] are different. This indicates that the skeletal muscles with directional structure are actually anisotropic.

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Thus, an assumption of isotropy can lead to erroneous forward-modelled wave synthetics, while an estimation of viscosity effects can be very useful in characterization and identification of anomaly [**16**].

A possible approach to handle viscosity effects on image reconstruction has been proposed in [**18**] using stationary phase theorem. It is shown that the ideal Green function (in an inviscid regime) can be approximated from the viscous one by solving an ordinary differential equation. Once the ideal Green function is known one can identify a possible anomaly using imaging algorithms such as time reversal, back-propagation, Kirchhoff migration or MUSIC [**7, 12, 14, 6**]. One can also find the elastic moduli of the anomaly using the asymptotic formalism and reconstructing a certain polarization tensor in the far field [**10, 12, 15, 13**].

The importance of Green function stems from its role as a tool for the numerical and asymptotic techniques in biomedical imaging. Many inverse problems involving the estimation and acquisition of elastic parameters become tractable once the associated Green function is computed [**5, 7, 12, 18**]. Several attempts have been made to compute Green functions in purely elastic and/or isotropic regime. (See *e.g.* [**18, 16, 19, 22, 36, 43, 44, 45**] and references therein). However, it is not possible to give a closed form expression for general anisotropic Green functions without certain restrictions on the media. In this work, we provide anisotropic viscoelastic Green function in closed form for three particular anisotropic media.

The elastodynamic Green function in isotropic media is calculated by separating wave modes using Helmholtz decomposition of the elastic wavefield [**2, 18, 16**]. Unfortunately, this simple approach does not work in anisotropic media, where three different waves propagate with different phase velocities and polarization directions [**22, 17, 23**]. A polarization direction of quasi-longitudinal wave that differs from that of wave vector, impedes Helmholtz decomposition to completely separate wave modes [**26**].

The phase velocities and polarization vectors are the eigenvalues and eigenvectors of the Christoffel tensor $\underline{\mathbf{\Gamma}}$ associated with the medium. So, the wavefield can always be decomposed using the spectral basis of $\underline{\mathbf{\Gamma}}$. Based on this observation, Burridge *et al.* [**19**] proposed a new approach to calculate elastodynamic Green functions. Their approach consists of finding the eigenvalues and eigenvectors of Christoffel tensor $\underline{\mathbf{\Gamma}}(\nabla_x)$ using the duality between algebraic and differential objects. Therefore it is possible to express the Green function $\underline{\mathbf{G}}$ in terms of three scalar functions ϕ_i satisfying partial differential equations with constant coefficients. Thus the problem of computing $\underline{\mathbf{G}}$ reduces to the resolution of three differential equations for ϕ_i and of three subsequent equations (which may or may not be differential equations) with ϕ_i as source terms. See [**19**] for more details.

Finding the closed form expressions of the eigenvalues of Christoffel tensor $\underline{\mathbf{\Gamma}}$ is usually not so trivial because its characteristic equation is a polynomial of degree six in the components of its argument vector. However, with some restrictions on the material, roots of the characteristic equation can be given [**36**]. In this article, we consider three different media for which not only the explicite expressions of the eigenvalues of $\underline{\mathbf{\Gamma}}$ are known [**19, 44**], but they are also quadratic homogeneous forms, in the components of the argument vector. As a consequence, equations satisfied by ϕ_i become scalar wave equations. Following Burridge *et al.* [**19**], we find the viscoelastic Green functions for each medium. It is important to note that the elastodynamic Green function in a purely elastic regime, for the media under consideration, are well known [**44, 19**]. Also, the expression of the Green function for viscoelastic isotropic medium, which is computed as a special case, matches the one provided in [**18**].

It has been shown in [**20**] that Voigt model is well adopted to describe the viscosity response of many soft tissues to low frequency excitations. In this work, we consider a more general model proposed by Szabo and Wu in [**40**], which describes an empirical power law behavior of many viscoelastic materials including human myocardium. This model is based on a time-domain statement of causality [**41, 42**] and reduces to Voigt model for the specific case of quadratic frequency losses.

We provide some mathematical notions, theme and the outlines of the article in the next section.

2. Mathematical Context and Paper Outlines

2.1. Viscoelastic Wave Equation. Consider an open subset Ω of $\mathbb{R}^3$, filled with a homogeneous anisotropic viscoelastic material. Let

$$\mathbf{u}(\mathbf{x},t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$$

be the displacement field at time t of the material particle at position $\mathbf{x} \in \Omega$ and $\nabla_x \mathbf{u}(\mathbf{x},t)$ be its gradiant.

Under the assumptions of linearity and small perturbations, we define the order two strain tensor by

$$\underline{\varepsilon} : (\mathbf{x},t) \in \Omega \times \mathbb{R}^+ \longmapsto \frac{1}{2}\left(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T\right)(\mathbf{x},t), \tag{2.1}$$

where the superscript T indicates a transpose operation.

Let $\underline{\underline{\mathbf{C}}} \in \mathcal{L}_s^2(\mathbb{R}^3)$ and $\underline{\underline{\mathbf{V}}} \in \mathcal{L}_s^2(\mathbb{R}^3)$ be the stiffness and viscosity tensors of the material respectively. Here $\mathcal{L}_s^2(\mathbb{R}^3)$ is the space of symmetric tensors of order four. These tensors are assumed to be positive definite, i.e. there exists a constant $\delta > 0$ such that

$$(\underline{\underline{\mathbf{C}}} : \underline{\xi}) : \underline{\xi} \geq \delta |\underline{\xi}|^2 \quad \text{and} \quad (\underline{\underline{\mathbf{V}}} : \underline{\xi}) : \underline{\xi} \geq \delta |\underline{\xi}|^2, \quad \forall \underline{\xi} \in \mathcal{L}_s(\mathbb{R}^d),$$

where $\mathcal{L}_s(\mathbb{R}^3)$ denotes the space of symmetric tensors of order two.

The generalized Hooke's Law [**40**] for power law media states that the stress distribution

$$\underline{\sigma} : \Omega \times \mathbb{R}^+ \to \mathcal{L}_s(\mathbb{R}^3)$$

produced by deformation $\underline{\varepsilon}$, satisfies:

$$\underline{\sigma} = \underline{\underline{\mathbf{C}}} : \underline{\varepsilon} + \underline{\underline{\mathbf{V}}} : \mathcal{A}[\underline{\varepsilon}] \tag{2.2}$$

where $\mathcal{A}$ is a causal operator defined as

$$\mathcal{A}[\varphi] = \left| \begin{array}{ll} -(-1)^{\gamma/2} \dfrac{\partial^{\gamma-1}\varphi}{\partial t^{\gamma-1}} & \gamma \text{ is an even integer,} \\[2ex] \dfrac{2}{\pi}(\gamma-1)!(-1)^{(\gamma+1)/2} \dfrac{H(t)}{t^\gamma} *_t \varphi & \gamma \text{ is an odd integer,} \\[2ex] -\dfrac{2}{\pi}\Gamma(\gamma)\sin(\gamma\pi/2) \dfrac{H(t)}{|t|^\gamma} *_t \varphi & \gamma \text{ is a non integer.} \end{array} \right. \tag{2.3}$$

Note that by convention,

$$\mathcal{A}[\mathbf{u}]_i = \mathcal{A}[u_i] \quad \text{and} \quad \mathcal{A}[\underline{\varepsilon}]_{ij} = \mathcal{A}[\varepsilon_{ij}] \quad 1 \leq i,j \leq 3.$$

Here $H(t)$ is the Heaviside funtion, Γ is the gamma function and $*_t$ represents convolution with respect to variable t. See [**3, 21, 40, 41, 42**] for comprehensive details and discussion on fractional attenuation models, causality and the loss operator $\mathcal{A}$.

The viscoelastic wave equation satisfied by the displacement field $\mathbf{u}(\mathbf{x}, t)$ reads now

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{F} \quad = \quad \nabla_x \cdot \underline{\sigma} = \nabla_x \cdot \left(\underline{\underline{\mathbf{C}}} : \underline{\varepsilon} + \underline{\underline{\mathbf{V}}} : \mathcal{A}[\underline{\varepsilon}] \right),$$

where $\mathbf{F}(\mathbf{x}, t)$ is the applied force and ρ is the density (supposed to be constant) of the material.

REMARK 2.1. For quadratic frequency losses, i.e, when $\gamma = 2$, operator $\mathcal{A}$ reduces to a first order time derivative. Therefore, power-law attenuation model turns out to be the Voigt model in this case.

2.2. Spectral decomposition by Christoffel tensors. We introduce now the Christoffel tensors $\underline{\mathbf{\Gamma}}^c, \underline{\mathbf{\Gamma}}^v : \mathbb{R}^3 \to \mathcal{L}_s(\mathbb{R}^3)$ associated respectively with $\underline{\underline{\mathbf{C}}}$ and $\underline{\underline{\mathbf{V}}}$ defined by:

$$\Gamma^c_{ij}(\mathbf{n}) = \sum_{k,l=1}^{3} C_{kilj} n_k n_j, \quad \Gamma^v_{ij}(\mathbf{n}) = \sum_{k,l=1}^{3} V_{kilj} n_k n_j, \qquad \forall \mathbf{n} \in \mathbb{R}^3, \quad 1 \le i, j \le 3.$$

Remark that the viscoelastic wave equation can be rewritten in terms of Christoffel tensors as :

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{F} = \underline{\mathbf{\Gamma}}^c[\nabla_x]\mathbf{u} + \underline{\mathbf{\Gamma}}^v[\nabla_x]\mathcal{A}[\mathbf{u}]. \tag{2.4}$$

Note that $\underline{\mathbf{\Gamma}}^c$ and $\underline{\mathbf{\Gamma}}^v$ are symmetric and positive definite as $\underline{\underline{\mathbf{C}}}$ and $\underline{\underline{\mathbf{V}}}$ are already symmetric positive definite.

Let L^c_i be the eigenvalues and $\mathbf{D}^c_i$ be the associated eigenvectors of $\underline{\mathbf{\Gamma}}^c$ for $i = 1, 2, 3$. We define the quantities M^c_i and $\underline{\mathbf{E}}^c_i$ by

$$M^c_i = \mathbf{D}^c_i \cdot \mathbf{D}^c_i, \quad \text{and} \quad \underline{\mathbf{E}}^c_i = (M^c_i)^{-1} \mathbf{D}^c_i \otimes \mathbf{D}^c_i. \tag{2.5}$$

As $\underline{\mathbf{\Gamma}}^c$ is symmetric, the eigenvectors $\mathbf{D}^c_i$ are orthogonal and the spectral decomposition of the Christoffel tensor $\underline{\mathbf{\Gamma}}^c$ can be given as:

$$\underline{\mathbf{\Gamma}}^c = \sum_{i=1}^{3} L^c_i \underline{\mathbf{E}}^c_i \quad \text{with} \quad \underline{\mathbf{I}} = \sum_{i=1}^{3} \underline{\mathbf{E}}^c_i \tag{2.6}$$

where $\underline{\mathbf{I}} \in \mathcal{L}_s(\mathbb{R}^3)$ is the identity tensor.

Similarly, consider $\underline{\mathbf{\Gamma}}^v$ the Christoffel tensor associated with $\underline{\underline{\mathbf{V}}}$ and define the quantities L^v_i, $\mathbf{D}^v_i$, M^v_i and $\underline{\mathbf{E}}^v_i$ such as

$$\underline{\mathbf{\Gamma}}^v = \sum_{i=1}^{3} L^v_i \underline{\mathbf{E}}^v_i \quad \text{with} \quad \underline{\mathbf{I}} = \sum_{i=1}^{3} \underline{\mathbf{E}}^v_i. \tag{2.7}$$

We assume that the tensors $\underline{\mathbf{\Gamma}}^c$ and $\underline{\mathbf{\Gamma}}^v$ have the same structure in the sense that the eigenvectors $\mathbf{D}^c_i$ and $\mathbf{D}^v_i$ are equal. In the sequel, by abuse of notation, we use $\mathbf{D}$ instead of $\mathbf{D}^c$ or $\mathbf{D}^v$ and same for $\underline{\mathbf{E}}$ and M.

2.3. Paper Outline. The aim of this work is to compute the elastodynamic Green function $\underline{\mathbf{G}}$ associated to viscoelastic wave equation (2.4). More presisely, $\underline{\mathbf{G}}$ is the solution of the equation

$$(\underline{\mathbf{\Gamma}}^c[\nabla_x]\underline{\mathbf{G}}(\mathbf{x},t) + \underline{\mathbf{\Gamma}}^v[\nabla_x]\mathcal{A}[\underline{\mathbf{G}}](\mathbf{x},t)) - \rho\frac{\partial^2\underline{\mathbf{G}}(\mathbf{x},t)}{\partial t^2} = \delta(t)\delta(\mathbf{x})\underline{\mathbf{I}}, \tag{2.8}$$

The idea is to use the spectral decomposition of $\underline{\mathbf{G}}$ of the form

$$\underline{\mathbf{G}} = \sum_{i=1}^{3}\underline{\mathbf{E}}_i(\nabla_x)\phi_i = \sum_{i=1}^{3}(\mathbf{D}_i\otimes\mathbf{D}_i)M_i^{-1}\phi_i, \tag{2.9}$$

where ϕ_i are three scalar functions satifying

$$(L_i^c(\nabla_x)\phi_i + L_i^v(\nabla_x)\mathcal{A}[\phi_i]) - \rho\frac{\partial^2\phi_i}{\partial t^2} = \delta(t)\delta(\mathbf{x}) \tag{2.10}$$

(See apendix A for more details about this decomposition.)

Therefore, to obtain an expression of $\underline{\mathbf{G}}$, we need to:

1- solve three partial differential equations (2.10) in ϕ_i

2- subsequent equations

$$\psi_i = M_i^{-1}\phi_i \tag{2.11}$$

3- and calculate second order derivatives of ψ_i to compute

$$(\mathbf{D}_i\otimes\mathbf{D}_i)\psi_i$$

In the following section, we give simple examples of anisotropic media which satisfy some restrictive properties and assumptions (see subsection 3.4) defining the limits of our approach. In section 4, we derive the solutions ϕ_i of equations (2.10). In section 5, we give an explicite resolution of $\psi_i = M_i^{-1}\phi_i$ and $(\mathbf{D}_i\otimes\mathbf{D}_i)\psi_i$. Finally, in the last section, we compute the Green function for three simple anisotropic media.

3. Some Simple Anisotropic Viscoelastic Media

In this section, we present three viscoelastic media with simple type of anisotropy. We also describe some important properties of the media and our basic assumptions in this article.

Definition 3.1. We will call a tensor $\underline{\mathbf{c}} = (c_{mn}) \in \mathcal{L}_s(\mathbb{R}^6)$ the Voigt representation of an order four tensor $\underline{\underline{\mathbf{C}}} \in \mathcal{L}_s^2(\mathbb{R}^3)$ if

$$c_{mn} = c_{p(i,j)p(k,l)} = C_{ijkl} \quad 1 \le i,j,k,l \le 3$$

where

$$p(i,i) = i, \quad p(i,j) = p(j,i), \quad p(2,3) = 4, \quad p(1,3) = 5, \quad p(1,2) = 6.$$

We will use $\underline{\mathbf{c}}$ and $\underline{\mathbf{v}}$ for the Voigt representations of stiffness tensor $\underline{\underline{\mathbf{C}}}$ and viscosity tensor $\underline{\underline{\mathbf{V}}}$ respectively.

We will let tensors $\underline{\mathbf{c}}$ and $\underline{\mathbf{v}}$ to have a same structure. For each media, the expressions for $\underline{\mathbf{\Gamma}}^c$, $L_i^c(\nabla_x)$, $\mathbf{D}_i^c(\nabla_x)$ and $M_i^c(\nabla_x)$ are provided [**19, 44**]. Throughout this section, μ_{pq} will assume the value c_{pq} for $\underline{\mathbf{c}}$ and v_{pq} for $\underline{\mathbf{v}}$ where the subscripts $p,q \in \{1,2,\cdots,6\}$. Moreover, we assume that the axes of material are identical with the Cartesian coordinate axes $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ and $\partial_i = \dfrac{\partial}{\partial x_i}$.

3.1. Medium I. The first medium for which we present a closed form elastodynamic Green function is an orthorhombic medium with the tensors $\underline{\mathbf{c}}$ and $\underline{\mathbf{v}}$ of the form:

$$\begin{pmatrix} \mu_{11} & -\mu_{66} & -\mu_{55} & 0 & 0 & 0 \\ -\mu_{66} & \mu_{22} & -\mu_{44} & 0 & 0 & 0 \\ -\mu_{55} & -\mu_{44} & \mu_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_{66} \end{pmatrix}$$

The Christoffel tensor is given by

$$\underline{\mathbf{\Gamma}}^c = \begin{pmatrix} c_{11}\partial_1^2 + c_{66}\partial_2^2 + c_{55}\partial_3^2 & 0 & 0 \\ 0 & c_{66}\partial_1^2 + c_{22}\partial_2^2 + c_{44}\partial_3^2 & 0 \\ 0 & 0 & c_{55}\partial_1^2 + c_{44}\partial_2^2 + c_{33}\partial_3^2 \end{pmatrix}$$

Its eigenvalues $L_i^c(\nabla_x)$ and the associated eigenvectors $\mathbf{D}_i^c(\nabla_x)$ are:

$$\begin{aligned} L_1^c(\nabla_x) &= c_{11}\partial_1^2 + c_{66}\partial_2^2 + c_{55}\partial_3^2 \\ L_2^c(\nabla_x) &= c_{66}\partial_1^2 + c_{22}\partial_2^2 + c_{44}\partial_3^2 \\ L_3^c(\nabla_x) &= c_{55}\partial_1^2 + c_{44}\partial_2^2 + c_{33}\partial_3^2 \\ \mathbf{D}_i^c = \mathbf{e}_i \quad \text{with} \quad M_i^c &= 1 \quad \forall i = 1,2,3 \end{aligned}$$

3.2. Medium II. The second medium which we consider is a transversely isotropic medium having symmetry axis along $\mathbf{e}_3$ and defined by the stiffness and the viscosity tensors $\underline{\mathbf{c}}$ and $\underline{\mathbf{v}}$ of the form:

$$\begin{pmatrix} \mu_{11} & \mu_{12} & -\mu_{44} & 0 & 0 & 0 \\ \mu_{12} & \mu_{11} & -\mu_{44} & 0 & 0 & 0 \\ -\mu_{44} & -\mu_{44} & \mu_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_{66} \end{pmatrix}$$

with $\mu_{66} = (\mu_{11} - \mu_{12})/2$. Here

$$\underline{\mathbf{\Gamma}}^c = \begin{pmatrix} c_{11}\partial_1^2 + c_{66}\partial_2^2 + c_{44}\partial_3^2 & (c_{11} - c_{66})\,\partial_1\partial_2 & 0 \\ (c_{11} - c_{66})\,\partial_1\partial_2 & c_{66}\partial_1^2 + c_{11}\partial_2^2 + c_{44}\partial_3^2 & 0 \\ 0 & 0 & c_{44}\partial_1^2 + c_{44}\partial_2^2 + c_{33}\partial_3^2 \end{pmatrix}$$

The eigenvalues $L_i^c(\nabla_x)$ of $\underline{\mathbf{\Gamma}}^c(\nabla_x)$ in this case are

$$\begin{aligned} L_1^c(\nabla_x) &= c_{44}\partial_1^2 + c_{44}\partial_2^2 + c_{33}\partial_3^2 \\ L_2^c(\nabla_x) &= c_{11}\partial_1^2 + c_{11}\partial_2^2 + c_{44}\partial_3^2 \\ L_3^c(\nabla_x) &= c_{66}\partial_1^2 + c_{66}\partial_2^2 + c_{44}\partial_3^2 \end{aligned}$$

and the associated eigenvectors $\mathbf{D}_i^c(\nabla_x)$ are:

$$\mathbf{D}_1^c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{D}_2^c = \begin{pmatrix} \partial_1 \\ \partial_2 \\ 0 \end{pmatrix}, \quad \mathbf{D}_3^c = \begin{pmatrix} \partial_2 \\ -\partial_1 \\ 0 \end{pmatrix}.$$

Thus $M_1^c = 1, \quad \text{and} \quad M_2^c = M_3^c = \partial_1^2 + \partial_2^2$

3.3. Medium III. Finally, we will present the elastodynamic Green function for another transversely isotropic media with the axis of symmetry along $\mathbf{e}_3$ and having $\underline{\mathbf{c}}$ and $\underline{\mathbf{v}}$ of the form:

$$\begin{pmatrix} \mu_{11} & \mu_{11}-2\mu_{66} & \mu_{11}-2\mu_{44} & 0 & 0 & 0 \\ \mu_{11}-2\mu_{66} & \mu_{11} & \mu_{11}-2\mu_{44} & 0 & 0 & 0 \\ \mu_{11}-2\mu_{44} & \mu_{11}-2\mu_{44} & \mu_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_{66} \end{pmatrix}$$

The Christoffel tensor in this case is

$$\underline{\mathbf{\Gamma}}^c = \begin{pmatrix} c_{11}\partial_1^2 + c_{66}\partial_2^2 + c_{44}\partial_3^2 & (c_{11}-c_{66})\,\partial_1\partial_2 & (c_{11}-c_{44})\,\partial_1\partial_3 \\ (c_{11}-c_{66})\,\partial_1\partial_2 & c_{66}\partial_1^2 + c_{11}\partial_2^2 + c_{44}\partial_3^2 & (c_{11}-c_{44})\,\partial_2\partial_3 \\ (c_{11}-c_{44})\,\partial_1\partial_3 & (c_{11}-c_{44})\,\partial_2\partial_3 & c_{44}\partial_1^2 + c_{44}\partial_2^2 + c_{11}\partial_3^2 \end{pmatrix}$$

Its eigenvalues $L_i^c(\nabla_x)$ are:

$$\begin{aligned} L_1^c(\nabla_x) &= c_{11}\partial_1^2 + c_{11}\partial_2^2 + c_{11}\partial_3^2 = c_{11}\Delta_x \\ L_2^c(\nabla_x) &= c_{66}\partial_1^2 + c_{66}\partial_2^2 + c_{44}\partial_3^2 \\ L_3^c(\nabla_x) &= c_{44}\partial_1^2 + c_{44}\partial_2^2 + c_{44}\partial_3^2 = c_{44}\Delta_x \end{aligned}$$

and the eigenvectors $\mathbf{D}_i^c(\nabla_x)$ are:

$$\mathbf{D}_1^c = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}, \quad \mathbf{D}_2^c = \begin{pmatrix} \partial_2 \\ -\partial_1 \\ 0 \end{pmatrix}, \quad \mathbf{D}_3^c = \begin{pmatrix} -\partial_1\partial_3 \\ -\partial_2\partial_3 \\ \partial_1^2+\partial_2^2 \end{pmatrix} \tag{3.1}$$

In this case, $M_1^c = \Delta_x \quad M_2^c = \partial_1^2 + \partial_2^2 \quad$ and $\quad M_3^c = (\partial_1^2 + \partial_2^2)\Delta_x$

3.4. Properties of the Media and Main Assumptions. In all anisotropic media discussed above, it holds that

- The Christoffel tensors $\underline{\mathbf{\Gamma}}^c$ and $\underline{\mathbf{\Gamma}}^v$ have the same structure in the sense that
$$\mathbf{D}_i^c = \mathbf{D}_i^v, \quad \forall i = 1,2,3.$$
- The eigenvalues $L_i^c(\nabla_x)$ are homogeneous quadratic forms in the components of the argument vector ∇_x *i.e.*
$$L_i^c[\nabla_x] = \sum_j^3 a_{ij}^2 \frac{\partial^2}{\partial x_j^2},$$
and therefore equations (2.10) are actually scalar wave equations.
- In all the concerning cases, the operator $M_i^c(\nabla_x)$ is either constant or has a homogeneous quadratic form
$$M_i^c = \sum_j^3 m_{ij}^2 \frac{\partial^2}{\partial x_j^2}.$$

In addition, we assume that

- the eigenvalues of $\underline{\mathbf{\Gamma}}^c$ and $\underline{\mathbf{\Gamma}}^v$ satisfy
$$L_i^v(\nabla_x) = \beta_i L_i^c(\nabla_x).$$

- and the loss per wave length is small, i.e.

$$\beta_i << 1.$$

REMARK 3.2. The expression $M_3^c = (\partial_1^2 + \partial_2^2)\Delta_x$ will be avoided in the construction of the Green function by using the expression

$$\underline{\mathbf{G}} = \phi_3 \underline{\mathbf{I}} + \underline{\mathbf{E}}_1(\nabla_x)(\phi_1 - \phi_3) + \underline{\mathbf{E}}_2(\nabla_x)(\phi_2 - \phi_3)$$

for the elastodynamic Green function.

4. Solution of the Model Wave Problem

Let us now study the scalar wave problems (2.10). We consider a model problem and drop the subscript for brevity in this section as well as in the next section. Consider

$$(L^c[\nabla_x]\phi + L^v[\nabla_x]\mathcal{A}[\phi]) - \rho \frac{\partial^2 \phi}{\partial t^2} = \delta(t)\delta(\mathbf{x}). \tag{4.1}$$

Our assumptions on the media imply that L^c and L^v have the following form;

$$L^c[\nabla_x] = \sum_{j=1}^{3} a_j^2 \frac{\partial^2}{\partial x_j^2} \quad \text{and} \quad L^v[\nabla_x] = \beta L^c[\nabla_x] = \sum_{j=1}^{3} \beta a_j^2 \frac{\partial^2}{\partial x_j^2}$$

Therefore, the model equation (4.1) can be rewritten as:

$$\sum_{j=1}^{3}\left(a_j^2 \frac{\partial^2 \phi}{\partial x_j^2} + \beta a_j^2 \mathcal{A}\left[\frac{\partial^2 \phi}{\partial x_j^2}\right]\right) - \rho \frac{\partial^2 \phi}{\partial t^2} = \delta(t)\delta(\mathbf{x}),$$

By a change of variables $x_j = \dfrac{a_j}{\sqrt{\rho}}\xi_j$, we obtain in function $\widetilde{\phi}(\xi) = \phi(\mathbf{x})$ the following transformed equation :

$$\Delta_\xi \widetilde{\phi} + \beta \mathcal{A}\left[\Delta_\xi \widetilde{\phi}\right] - \frac{\partial^2 \widetilde{\phi}}{\partial t^2} = \frac{\sqrt{\rho}}{a}\delta(t)\delta(\xi). \tag{4.2}$$

where the constant $a = a_1 a_2 a_3$.

Now, we apply $\mathcal{A}$ on both sides of the equation (4.2), and replace the resulting expression for $\mathcal{A}\left[\Delta_\xi \widetilde{\phi}\right]$ back into the equation (4.2). This yields:

$$\Delta_\xi \widetilde{\phi} + \beta \mathcal{A}\left[\frac{\partial^2 \widetilde{\phi}}{\partial t^2}\right] - \beta^2 \mathcal{A}^2\left[\Delta_\xi \widetilde{\phi}\right] - \frac{\partial^2 \widetilde{\phi}}{\partial t^2} = \frac{\sqrt{\rho}}{a}\delta(\xi)\left\{\delta(t) - \beta\mathcal{A}[\delta(t)]\right\}$$

Recall that $\beta << 1$ and the term in β^2 is negligible. Therefore, it holds

$$\Delta_\xi \widetilde{\phi} + \beta \mathcal{A}\left[\frac{\partial^2 \widetilde{\phi}}{\partial t^2}\right] - \frac{\partial^2 \widetilde{\phi}}{\partial t^2} \simeq \frac{\sqrt{\rho}}{a}\delta(\xi)\left\{\delta(t) - \beta\mathcal{A}[\delta(t)]\right\}. \tag{4.3}$$

Finally, taking temporal Fourier transform on both sides of (4.3), we obtain the corresponding Helmholtz equation:

$$\Delta_\xi \widetilde{\Phi} + \omega^2\left(1 - \beta\widehat{\mathcal{A}}(\omega)\right)\widetilde{\Phi} = \left(1 - \beta\widehat{\mathcal{A}}(\omega)\right)\frac{\sqrt{\rho}}{a}\delta(\xi) \tag{4.4}$$

where $\widetilde{\Phi}(\xi,\omega)$ and $\widehat{\mathcal{A}}(\omega)$ are the Fourier transforms of $\widetilde{\phi}(\xi,t)$ and the kernel of the convolution operator $\mathcal{A}$ respectively. Let

$$\kappa(\omega)=\sqrt{\omega^2\left(1-\beta\widehat{\mathcal{A}}(\omega)\right)}.$$

Then the solution of the Helmholtz equation (4.4) (see for instance [**25, 34**]) is expressed as

$$\Phi(\mathbf{x},\omega)=\sqrt{\rho}\left(1-\beta\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa(\omega)\tau(\mathbf{x})}}{4a\pi\tau(\mathbf{x})}.$$

where

$$\tau(\mathbf{x})=\sqrt{\rho}\sqrt{\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}}$$

Using density normalized constants $b_j=\dfrac{a_j}{\sqrt{\rho}}$, we have

$$\Phi(\mathbf{x},\omega)=\left(1-\beta\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa(\omega)\tau(\mathbf{x})}}{4b\rho\pi\tau(\mathbf{x})}. \tag{4.5}$$

where constant $b=b_1b_2b_3$ and

$$\tau(\mathbf{x})=\sqrt{\frac{x_1^2}{b_1^2}+\frac{x_2^2}{b_2^2}+\frac{x_3^2}{b_3^2}}$$

5. Solution of the Model Potential Problem

In this section, we find the solution of equation (2.11). We once again proceed with a model problem. Once the solution is obtained, we will aim to calculate, its second order derivatives for the evaluation of $\mathbf{D}\otimes\mathbf{D}\psi$.

5.1. Solution of the Potential Problem. Let $\psi(\mathbf{x},t)$, be the solution of equation (2.11) and $\Psi(\mathbf{x},\omega)$ be its Fourier transform with respect to variable t. Then $\Psi(\mathbf{x},\omega)$ satisfies,

$$M\Psi(\mathbf{x},\omega)=\Phi(\mathbf{x},\omega)=\left(1-\beta\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa(\omega)\tau(\mathbf{x})}}{4b\rho\pi\tau(\mathbf{x})}. \tag{5.1}$$

When M is constant, the solution of this equation is directly calculated. As $M=(\partial_1^2+\partial_2^2)\Delta_x$ will not be used in the construction of Green function, we are only interested in the case where M is a homogeneous quadratic form in the component of ∇_x *i.e.*

$$M=\sum_{j=1}^{3}m_j^2\frac{\partial^2}{\partial x_j^2}.$$

So, the model equation (5.1) can be rewritten as:

$$\sum_{j=1}^{3}m_j^2\frac{\partial^2\Psi}{\partial x_j^2}=\left(1-\beta\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa(\omega)\tau(\mathbf{x})}}{4b\rho\pi\tau(\mathbf{x})}\quad m_j\neq 0,\quad\forall j \tag{5.2}$$

By a change of variables $x_j=m_j\eta_j$, equation (5.2) becomes Poisson equation in $\overline{\Psi}(\eta,\omega)=\Psi(\mathbf{x},\omega)$ *i.e.*

$$\Delta_\eta\overline{\Psi}=\left(1-\beta\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa(\omega)\overline{\tau}(\eta)}}{4b\rho\pi\overline{\tau}(\eta)}=\overline{\Phi}(\eta,\omega) \tag{5.3}$$

where,

$$\overline{\tau}(\eta) = \sqrt{\frac{m_1^2\eta_1^2}{b_1^2} + \frac{m_2^2\eta_2^2}{b_2^2} + \frac{m_3^2\eta_3^2}{b_3^2}} = \tau(\mathbf{x}) \quad \text{and} \quad \overline{\Phi}(\eta,\omega) = \Phi(\mathbf{x},\omega)$$

Notice that the source $\overline{\Phi}(\eta,\omega)$ is symmetric with respect to ellipsoid $\overline{\tau}$, *i.e.*

$$\overline{\Phi}(\eta,\omega) = \overline{\Phi}(\overline{\tau},\omega).$$

Therefore, the solution $\overline{\Psi}$ of the Poisson equation (5.3) is the potential field of a uniformly charged ellipsoid due to a charge density $\overline{\Phi}(\overline{\tau},\omega)$. The potential field $\overline{\Psi}$ can be calculated with a classical approach using ellipsoidal coordinates. (See for example [**24, 31**] for the theory of potential problems in ellipsoidal coordinates.)

For the solution of the Poisson equation (5.3) we recall following result from [**31**, Ch. 7, Sec.6].

Proposition 5.1. Let

$$f(z) = \sum_{j=1}^{3} \frac{\zeta_j^2}{(\alpha_j h)^2 + z} - 1 \qquad \text{and} \quad g(z) = \Pi_{j=1}^3\left[(\alpha_j h)^2 + z\right]$$

and let $Z(h,\zeta)$ be the largest algebraic root of $f(z)g(z) = 0$. Then the solution of the Poisson equation

$$\Delta^2 Y(\zeta) = 4\pi\chi\left(\frac{\zeta_1^2}{\alpha_1^2} + \frac{\zeta_2^2}{\alpha_2^2} + \frac{\zeta_2^2}{\alpha_1^2}\right) \qquad \zeta \in \mathbb{R}^3$$

is given by

$$Y(\zeta) = 2\pi\alpha_1\alpha_2\alpha_3 \int_0^\infty \chi(h) I(h,\zeta) dh.$$

The integrand $I(h,\zeta)$ is defined as

$$I(h,\zeta) = \left| \begin{array}{ll} h^2 \displaystyle\int_{Z(h,\zeta)}^\infty \frac{1}{\sqrt{g(z)}} dz & Z > 0 \\ h^2 \displaystyle\int_0^\infty \frac{1}{\sqrt{g(z)}} dz & Z < 0 \end{array} \right.$$

Hence, the solution of (5.3) can be given as

$$\overline{\Psi}(\eta,\omega) = \frac{2\pi b}{m}\left(1 - \beta\widehat{\mathcal{A}}(\omega)\right)\frac{1}{4\pi}\int_0^\infty \frac{e^{\sqrt{-1}\kappa(\omega)h}}{4b\rho\pi h} I(h,\eta) dh$$

or equivalently,

$$\Psi(\mathbf{x},\omega) = \frac{1}{8\rho\pi m}\left(1 - \beta\widehat{\mathcal{A}}(\omega)\right)\int_0^\infty \frac{e^{\sqrt{-1}\kappa(\omega)h}}{h} I(h,\mathbf{x}) dh, \quad m = m_1 m_2 m_3 \tag{5.4}$$

By a change of variable $s = h^{-2}z$, we can write $I(h,\mathbf{x})$ as:

$$I(h,\mathbf{x}) = \left| \begin{array}{ll} mh \displaystyle\int_{S(h,\mathbf{x})}^\infty \frac{1}{\sqrt{G(s)}} ds & h < \tau \\ mh \displaystyle\int_0^\infty \frac{1}{\sqrt{G(s)}} ds & h > \tau \end{array} \right. \tag{5.5}$$

with $S(h,\mathbf{x}) = h^{-2}Z(h,\mathbf{x})$ being the largest algebraic root of the equation

$$F(s)G(s) = 0$$

where

$$(5.6)\quad \left| \begin{array}{l} F(s) = h^2 f(h^2 s) = \sum_{j=1}^{3} \{V_j(s)\}^{-1} x_j^2 - h^2 \\ \\ G(s) = \dfrac{m^2}{h^6} g(h^2 s) = \Pi_{j=1}^3 \{V_j(s)\} \\ \\ \text{with} \quad V_j(s) = b_j^2 + m_j^2 s \end{array} \right.$$

REMARK 5.2. Note that, $F(s) \equiv 0$ corresponds to a set of confocal ellipsoids

$$(5.7)\qquad s \longmapsto h^2(s) = \sum_{j=1}^{3} \{V_j(s)\}^{-1} x_j^2$$

such that $\tau(\mathbf{x}) = h(0)$ *i.e.* $S(\tau) = 0$. Moreover, $S > 0$ if the ellipsoid h lies inside τ and $S < 0$ if the ellipsoid h lies outside τ.

5.2. Derivatives of the Potential field. Now we compute the derivatives of the potential Ψ. We note that $I(h,\mathbf{x})$ is constant with respect to $\mathbf{x}$ when $h > \tau$. So,

$$\frac{\partial I(h,\mathbf{x})}{\partial x_k} = \left| \begin{array}{ll} -mh \dfrac{\partial S(h,\mathbf{x})}{\partial x_k} \dfrac{1}{\sqrt{G(S(h,\mathbf{x}))}} & h < \tau \\ \\ 0 & h > \tau \end{array} \right.$$

for $k = 1,2,3$ and by consequence,

$$\frac{\partial \Psi}{\partial x_k} = -\frac{1}{8\rho\pi m}\left(1 - \beta\widehat{\mathcal{A}}(\omega)\right)\int_0^\infty \frac{e^{\sqrt{-1}\kappa(\omega)h}}{h}\frac{\partial I(h,\mathbf{x})}{\partial x_k} dh$$

or

$$(5.8)\quad \frac{\partial \Psi}{\partial x_k} = -\frac{1}{8\rho\pi}\left(1 - \beta\widehat{\mathcal{A}}(\omega)\right)\int_0^\tau \left[e^{\sqrt{-1}\kappa(\omega)h}\right]\frac{\partial S(h,\mathbf{x})}{\partial x_k}\frac{1}{\sqrt{G(S(h,\mathbf{x}))}} dh.$$

Now, we apply $\dfrac{\partial}{\partial x_l}$ for $l = 1,2,3$ on (5.8) to obtain the second order derivatives of Ψ:

$$\begin{aligned} -8\rho\pi \frac{\partial^2 \Psi}{\partial x_k x_l} &= \left(1 - \beta\widehat{\mathcal{A}}(\omega)\right)\frac{\partial}{\partial x_l}\left[\int_0^\tau \left[e^{\sqrt{-1}\kappa(\omega)h}\right]\frac{\partial S}{\partial x_k}\frac{1}{\sqrt{G(S)}} dh\right] \\ &= \left(1 - \beta\widehat{\mathcal{A}}(\omega)\right)\frac{\partial \tau}{\partial x_l}\left\{\left[e^{\sqrt{-1}\kappa(\omega)\tau}\right]\frac{\partial S(\tau)}{\partial x_k}\frac{1}{\sqrt{G(S(\tau))}}\right\} \\ &+ \left(1 - \beta\widehat{\mathcal{A}}(\omega)\right)\int_0^\tau \left[e^{\sqrt{-1}\kappa(\omega)h}\right]\frac{1}{\sqrt{G(S)}}\left\{\frac{\partial^2 S}{\partial x_k \partial x_l} - \frac{1}{2}\frac{\partial S}{\partial x_k}\frac{\partial S}{\partial x_l}\frac{G'(S)}{G(S)}\right\} dh \end{aligned}$$

As $F(S)G(S) = 0$ and $G(s)$ is normally non-zero on S, therefore by differentiating $F(S) = 0$, we obtain [**19**, eq. (5.21)-(5.23)]

$$(5.9)\qquad \frac{\partial S}{\partial x_k} = \frac{-2x_k}{V_k(S)F'(S)}$$

$$\frac{\partial^2 S}{\partial x_k x_l} = \frac{-4x_k x_l}{V_k(S)V_l(S)\left[F'(S)\right]^2}\left\{\frac{F''(S)}{F'(S)} + \frac{m_k^2}{V_k(S)} + \frac{m_l^2}{V_l(S)}\right\} - \frac{2\delta_{kl}}{V_k(S)F'(S)} \tag{5.10}$$

where,

$$F'(s) = \sum_{j=1}^{3} \frac{-m_j^2 x_j^2}{V_j^2(s)}, \quad F''(s) = \sum_{j=1}^{3} \frac{2m_j^4 x_j^2}{V_j^3(s)}, \quad G'(s) = G(s)\sum_{j=1}^{3} \frac{m_j^2}{V_j(s)} \tag{5.11}$$

and prime represents a derivative with respect to variable s.

Substituting the values from (5.9) and (5.10), the second order derivative of Ψ becomes

(5.12)

$$4\rho\pi \frac{\partial^2 \Psi}{\partial x_k x_l} = \left| \begin{array}{l} \frac{-x_k x_l \left(1 - \beta\widehat{\mathcal{A}}(\omega)\right)}{a a_k^2 a_l^2 F'(0)} \left\{ \frac{e^{\sqrt{-1}\kappa(\omega)\tau}}{\tau} \right\} \\ + \left(1 - \beta\widehat{\mathcal{A}}(\omega)\right) \int_0^\tau \left[e^{\sqrt{-1}\kappa(\omega)h}\right] \frac{1}{F'(S)\sqrt{G(S)}} \times \\ \left[\frac{2x_k x_l}{V_k(S)V_l(S)F'(S)} \left\{\frac{F''(S)}{F'(S)} + \frac{m_k^2}{V_k(S)} + \frac{m_l^2}{V_l(S)} + \frac{1}{2}\frac{G'(S)}{G(S)}\right\} + \frac{\delta_{kl}}{V_k(S)}\right] dh \end{array} \right.$$

REMARK 5.3. If for some $i \in \{1, 2, 3\}$, $m_i \to 0$ one semi axis of the ellipsoid τ tends to infinity but no singularity occurs. Therefore the results of this section are still valid in this case.

6. Elastodynamic Green Function

In this section we present the expressions for the elastodynamic Green functions for the media presented in section 3. Throughout this section $c_p = \sqrt{\frac{c_{pp}}{\rho}}$ with $p \in \{1, 2, \cdots, 6\}$. We recall that $\kappa_i(\omega) = \sqrt{\omega^2\left(1 - \beta_i\widehat{\mathcal{A}}(\omega)\right)}$.

Medium	b_1	b_2	b_3	m_1	m_2	m_3	M_i
I	c_1	c_6	c_5	1	0	0	M_1
	c_6	c_2	c_4	0	1	0	M_2
	c_5	c_4	c_3	0	0	1	M_3
II	c_4	c_4	c_3	0	0	1	M_1
	c_1	c_1	c_4	1	1	0	M_2
	c_6	c_6	c_4	*	*	*	M_3
III	c_1	c_1	c_1	1	1	1	M_1
	c_6	c_6	c_4	1	1	0	M_2
	c_4	c_4	c_4	*	*	*	M_3

TABLE 1. Values of b_i and m_i for different media. Here $*$ represents a value which is not used for reconstructing Green function.

6.1. Medium I. All the eigenvectors of $\underline{\mathbf{\Gamma}}$ are constants in this case *i.e.* $\mathbf{D}_i = \mathbf{e}_i$, therefore $M_i = 1$ and $\underline{\mathbf{E}}_i = \mathbf{e}_i \otimes \mathbf{e}_i$. If $\widehat{\underline{\mathbf{G}}}$ is the Fourier transform of the viscoelastic Green function $\underline{\mathbf{G}}$ with respect to variable t, then:
(6.1)

$$\widehat{\underline{\mathbf{G}}} = \sum_{i=1}^{3} \Phi_i(x,\omega)\mathbf{e}_i \otimes \mathbf{e}_i = \frac{1}{4\pi\rho}\sum_{i=1}^{3}\left[\frac{c_{i+3}\left(1-\beta_i\widehat{\mathcal{A}}(\omega)\right)}{c_i c_4 c_5 c_6 \tau_i}\exp(\sqrt{-1}\kappa_i(\omega)\tau_i)\right]\mathbf{e}_i \otimes \mathbf{e}_i$$

where

$$\tau_1 = \sqrt{\frac{x_1^2}{c_1^2}+\frac{x_2^2}{c_6^2}+\frac{x_3^2}{c_5^2}},\quad \tau_2 = \sqrt{\frac{x_1^2}{c_6^2}+\frac{x_2^2}{c_2^2}+\frac{x_3^2}{c_4^2}},\quad \tau_3 = \sqrt{\frac{x_1^2}{c_5^2}+\frac{x_2^2}{c_4^2}+\frac{x_3^2}{c_3^2}},$$

6.2. Medium II. According to section 4, the functions Φ_i have following expressions:

$$\Phi_1(\mathbf{x},\omega) = \left(1-\beta_1\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_1(\omega)\tau_1(\mathbf{x})}}{4c_4^2 c_3\rho\pi\tau_1(\mathbf{x})}$$
$$\Phi_2(\mathbf{x},\omega) = \left(1-\beta_2\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_2(\omega)\tau_2(\mathbf{x})}}{4c_1^2 c_4\rho\pi\tau_2(\mathbf{x})}$$
$$\Phi_3(\mathbf{x},\omega) = \left(1-\beta_3\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_3(\omega)\tau_3(\mathbf{x})}}{4c_6^2 c_4\rho\pi\tau_3(\mathbf{x})}$$

where

$$\tau_1(\mathbf{x}) = \sqrt{\frac{x_1^2}{c_4^2}+\frac{x_2^2}{c_4^2}+\frac{x_3^2}{c_3^2}},\quad \tau_2(\mathbf{x}) = \sqrt{\frac{x_1^2}{c_1^2}+\frac{x_2^2}{c_1^2}+\frac{x_3^2}{c_4^2}},\quad \tau_3(\mathbf{x}) = \sqrt{\frac{x_1^2}{c_6^2}+\frac{x_2^2}{c_6^2}+\frac{x_3^2}{c_4^2}}$$

To calculate Green function, we use the expression

$$\widehat{\underline{\mathbf{G}}} = \Phi_3\underline{\mathbf{I}} + \mathbf{D}_1\otimes\mathbf{D}_1 M_1^{-1}(\Phi_1-\Phi_3) + \mathbf{D}_2\otimes\mathbf{D}_2 M_2^{-1}(\Phi_2-\Phi_3).$$

$\mathbf{D}_1 = \mathbf{e}_3$ and $M_1 = 1$, yield

$$\mathbf{D}_1\otimes\mathbf{D}_1 M_1^{-1}(\Phi_1-\Phi_3) = (\Phi_1-\Phi_3)\,\mathbf{e}_3\otimes\mathbf{e}_3$$

To compute $\mathbf{D}_2\otimes\mathbf{D}_2 M_2^{-1}(\Phi_2-\Phi_3)$, suppose

$$\Psi_2 = M_2^{-1}\Phi_2 \quad\text{and}\quad \Psi_3 = M_2^{-1}\Phi_3$$

and notice that $m_1 = m_2 = 1$ and $m_3 = 0$. Moreover for Φ_2 and Φ_3, $b_1 = b_2$. (See Table 1). Thus, we have

$$\frac{4\rho\pi}{\left(1-\beta_2\widehat{\mathcal{A}}(\omega)\right)}\frac{\partial^2\Psi_2}{\partial x_k x_l} = \widehat{\mathbf{R}}_k\widehat{\mathbf{R}}_l\left\{\frac{e^{\sqrt{-1}\kappa_2(\omega)\tau_2}}{c_1^2 c_4\tau_2}\right\} - \frac{1}{c_4 R^2}(\delta_{kl}-2\widehat{\mathbf{R}}_k\widehat{\mathbf{R}}_l)\int_0^{\tau_2}\left[e^{\sqrt{-1}\kappa_2(\omega)h}\right]dh$$

$$\frac{4\rho\pi}{\left(1-\beta_3\widehat{\mathcal{A}}(\omega)\right)}\frac{\partial^2\Psi_3}{\partial x_k x_l} = \widehat{\mathbf{R}}_k\widehat{\mathbf{R}}_l\left\{\frac{e^{\sqrt{-1}\kappa_3(\omega)\tau_3}}{c_6^2 c_4\tau_3}\right\} - \frac{1}{c_4 R^2}(\delta_{kl}-2\widehat{\mathbf{R}}_k\widehat{\mathbf{R}}_l)\int_0^{\tau_3}\left[e^{\sqrt{-1}\kappa_3(\omega)h}\right]dh$$

where $\widehat{\mathbf{R}}_k = \dfrac{x_k}{R}$ for $k = 1,2$. See appendix C for the derivation of this result.

By using the second derivatives of Ψ_2 and Ψ_3 and the expression

$$\mathbf{D}_2\otimes\mathbf{D}_2 M_2^{-1}(\Phi_2-\Phi_3) = \sum_{k,l=1}^{2}\partial_k\partial_l(\Psi_2-\Psi_3)\,\mathbf{e}_k\otimes\mathbf{e}_l$$

we finally arrive at

$$\underline{\widehat{\mathbf{G}}} = \left(1-\beta_3\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_3(\omega)\tau_3(\mathbf{x})}}{4c_6^2c_4\rho\pi\tau_3(\mathbf{x})}\underline{\mathbf{J}} + \left(1-\beta_1\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_1(\omega)\tau_1(\mathbf{x})}}{4c_4^2c_3\rho\pi\tau_1(\mathbf{x})}\mathbf{e}_3\otimes\mathbf{e}_3$$

$$+\left[\left(1-\beta_2\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_2(\omega)\tau_2(\mathbf{x})}}{4c_1^2c_4\rho\pi\tau_2(\mathbf{x})} - \left(1-\beta_3\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_3(\omega)\tau_3(\mathbf{x})}}{4c_6^2c_4\rho\pi\tau_3(\mathbf{x})}\right]\widehat{\mathbf{R}}\otimes\widehat{\mathbf{R}}$$

$$-\frac{1}{4\rho\pi c_4R^2}(\underline{\mathbf{J}}-2\widehat{\mathbf{R}}\otimes\widehat{\mathbf{R}})\times$$

$$\left[\left(1-\beta_2\widehat{\mathcal{A}}(\omega)\right)\int_0^{\tau_2}\left[e^{\sqrt{-1}\kappa_2(\omega)h}\right]dh - \left(1-\beta_3\widehat{\mathcal{A}}(\omega)\right)\int_0^{\tau_3}\left[e^{\sqrt{-1}\kappa_3(\omega)h}\right]dh\right]$$

Or equivalently,

$$\underline{\widehat{\mathbf{G}}} = \Phi_1\mathbf{e}_3\otimes\mathbf{e}_3 + \Phi_2\widehat{\mathbf{R}}\otimes\widehat{\mathbf{R}} + \Phi_3(\underline{\mathbf{J}}-\widehat{\mathbf{R}}\otimes\widehat{\mathbf{R}})$$

$$-\frac{1}{R^2}\left[c_1^2\int_0^{\tau_2}h\Phi_2(h,\omega)dh - c_6^2\int_0^{\tau_3}h\Phi_3(h,\omega)dh\right](\underline{\mathbf{J}}-2\widehat{\mathbf{R}}\otimes\widehat{\mathbf{R}})$$

Here $\underline{\mathbf{J}} = \underline{\mathbf{I}} - \mathbf{e}_3\otimes\mathbf{e}_3$ and $\widehat{\mathbf{R}} = \widehat{\mathbf{R}}_1\mathbf{e}_1 + \widehat{\mathbf{R}}_2\mathbf{e}_2$

6.3. Medium III. The solutions of the wave equation Φ_i in this case are

$$\Phi_1(\mathbf{x},\omega) = \left(1-\beta_1\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_1(\omega)\tau_1(\mathbf{x})}}{4c_1^3\rho\pi\tau_1(\mathbf{x})}$$
$$\Phi_2(\mathbf{x},\omega) = \left(1-\beta_2\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_2(\omega)\tau_2(\mathbf{x})}}{4c_6^2c_4\rho\pi\tau_2(\mathbf{x})}$$
$$\Phi_3(\mathbf{x},\omega) = \left(1-\beta_3\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_3(\omega)\tau_3(\mathbf{x})}}{4c_4^3\rho\pi\tau_3(\mathbf{x})}$$

where

$$\tau_1(\mathbf{x}) = \frac{1}{c_1}\sqrt{x_1^2+x_2^2+x_3^2} = \frac{r}{c_1},\quad \tau_2(\mathbf{x}) = \sqrt{\frac{x_1^2}{c_6^2}+\frac{x_2^2}{c_6^2}+\frac{x_3^2}{c_4^2}},\quad \tau_3(\mathbf{x}) = \frac{r}{c_4}$$

To calculate Green function, we once again use the expression

$$\underline{\widehat{\mathbf{G}}} = \Phi_3\underline{\mathbf{I}} + \mathbf{D}_1\otimes\mathbf{D}_1M_1^{-1}(\Phi_1-\Phi_3) + \mathbf{D}_2\otimes\mathbf{D}_2M_2^{-1}(\Phi_2-\Phi_3).$$

Suppose $\Psi_1 = M_1^{-1}\Phi_1$ and $\Psi_3 = M_1^{-1}\Phi_3$. Notice that $m_1 = m_2 = m_3 = 1$ for M_1 and $b_1 = b_2 = b_3$ for Φ_1 as well as Φ_3 (see Table 1). Thus,

$$\frac{4\rho\pi}{\left(1-\beta_1\widehat{\mathcal{A}}(\omega)\right)}\frac{\partial^2\Psi_1}{\partial x_kx_l} = \widehat{\mathbf{r}}_k\widehat{\mathbf{r}}_l\left\{\frac{e^{\sqrt{-1}\kappa_1(\omega)\tau_1}}{c_1^3\tau_1}\right\} - \frac{1}{r^3}(\delta_{kl}-3\widehat{\mathbf{r}}_i\widehat{\mathbf{r}}_j)\int_0^{\tau_1}\left[he^{\sqrt{-1}\kappa_1(\omega)h}\right]dh$$

$$\frac{4\rho\pi}{\left(1-\beta_3\widehat{\mathcal{A}}(\omega)\right)}\frac{\partial^2\Psi_3}{\partial x_kx_l} = \widehat{\mathbf{r}}_k\widehat{\mathbf{r}}_l\left\{\frac{e^{\sqrt{-1}\kappa_3(\omega)\tau_1}}{c_1^3\tau_3}\right\} - \frac{1}{r^3}(\delta_{kl}-3\widehat{\mathbf{r}}_i\widehat{\mathbf{r}}_j)\int_0^{\tau_3}\left[he^{\sqrt{-1}\kappa_3(\omega)h}\right]dh$$

See appendix B for the derivation of this result. It yields

$$\mathbf{D}_1 \otimes \mathbf{D}_1 M_1^{-1}\left(\Phi_1 - \Phi_3\right)$$

$$= \frac{1}{4\rho\pi}\left[\left(1-\beta_1\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_1(\omega)\tau_1(\mathbf{x})}}{c_1^3\tau_1(\mathbf{x})} + \left(1-\beta_3\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_3(\omega)\tau_3(\mathbf{x})}}{c_4^3\tau_3(\mathbf{x})}\right]\widehat{\mathbf{r}}\otimes\widehat{\mathbf{r}}$$

$$-\left[\left(1-\beta_1\widehat{\mathcal{A}}(\omega)\right)\int_0^{\tau_1}\left[he^{\sqrt{-1}\kappa_1(\omega)h}\right]dh - \left(1-\beta_3\widehat{\mathcal{A}}(\omega)\right)\int_0^{\tau_3}\left[he^{\sqrt{-1}\kappa_3(\omega)h}\right]dh\right]\times \quad .$$

$$\frac{1}{4\rho\pi r^3}\left(\underline{\mathbf{I}} - 3\widehat{\mathbf{r}}\otimes\widehat{\mathbf{r}}\right)$$

$$= \left[\Phi_1(x,\omega) - \Phi_3(x,\omega)\right]\widehat{\mathbf{r}}\otimes\widehat{\mathbf{r}} - \frac{1}{r^3}\left[\int_0^{\tau_1} h^2\Phi_1(h,\omega)dh - \int_0^{\tau_3} h^2\Phi_3(h,\omega)dh\right]\left(\underline{\mathbf{I}} - 3\widehat{\mathbf{r}}\otimes\widehat{\mathbf{r}}\right)$$

where $\widehat{\mathbf{r}} = \widehat{\mathbf{r}}_1\mathbf{e}_1 + \widehat{\mathbf{r}}_2\mathbf{e}_2 + \widehat{\mathbf{r}}_3\mathbf{e}_3$ with $\widehat{\mathbf{r}}_i = \dfrac{x_i}{r}$ for all $i = 1, 2, 3$.

To compute, $\mathbf{D}_2\otimes\mathbf{D}_2 M_2^{-1}\left(\Phi_2 - \Phi_3\right)$, suppose $\Psi_2 = M_2^{-1}\Phi_2$ and $\Psi_4 = M_2^{-1}\Phi_3$. By using formula (C.3) with $m_1 = m_2 = 1$ and $m_3 = 0$, we obtain:

$$\frac{4\rho\pi}{\left(1-\beta_2\widehat{\mathcal{A}}(\omega)\right)}\frac{\partial^2\Psi_2}{\partial x_k x_l} = \widehat{\mathbf{R}}_k\widehat{\mathbf{R}}_l\left\{\frac{e^{\sqrt{-1}\kappa_2(\omega)\tau_2}}{c_6^2 c_4\tau_2}\right\} - \frac{1}{c_4R^2}(\delta_{kl} - 2\widehat{\mathbf{R}}_k\widehat{\mathbf{R}}_l)\int_0^{\tau_2}\left[e^{\sqrt{-1}\kappa_2(\omega)h}\right]dh$$

$$\frac{4\rho\pi}{\left(1-\beta_3\widehat{\mathcal{A}}(\omega)\right)}\frac{\partial^2\Psi_4}{\partial x_k x_l} = \widehat{\mathbf{R}}_k\widehat{\mathbf{R}}_l\left\{\frac{e^{\sqrt{-1}\kappa_3(\omega)\tau_3}}{c_4^3\tau_3}\right\} - \frac{1}{c_4R^2}(\delta_{kl} - 2\widehat{\mathbf{R}}_k\widehat{\mathbf{R}}_l)\int_0^{\tau_3}\left[e^{\sqrt{-1}\kappa_3(\omega)h}\right]dh$$

with $\widehat{\mathbf{R}}_k = \dfrac{x_k}{R}$ and $k, l \in \{1, 2\}$. This allows us to write

$$\mathbf{D}_2 \otimes \mathbf{D}_2 M_2^{-1}\left(\Phi_2 - \Phi_3\right)$$

$$= \frac{1}{4\rho\pi}\left[\left(1-\beta_2\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_2(\omega)\tau_2(\mathbf{x})}}{c_1^3\tau_2(\mathbf{x})} + \left(1-\beta_3\widehat{\mathcal{A}}(\omega)\right)\frac{e^{\sqrt{-1}\kappa_3(\omega)\tau_3(\mathbf{x})}}{c_4^3\tau_3(\mathbf{x})}\right]\times$$

$$\left(\widehat{\mathbf{R}}_2^2\mathbf{e}_1\otimes\mathbf{e}_1 - \widehat{\mathbf{R}}_1\widehat{\mathbf{R}}_2[\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1] + \widehat{\mathbf{R}}_1^2\mathbf{e}_2\otimes\mathbf{e}_2\right)$$

$$-\frac{1}{4c_4\rho\pi R^2}\left[\left(1-\beta_2\widehat{\mathcal{A}}(\omega)\right)\int_0^{\tau_2}\left[e^{\sqrt{-1}\kappa_2(\omega)h}\right]dh - \left(1-\beta_3\widehat{\mathcal{A}}(\omega)\right)\int_0^{\tau_3}\left[e^{\sqrt{-1}\kappa_3(\omega)h}\right]dh\right]\times \quad \cdot$$

$$\left((1-2\widehat{\mathbf{R}}_2^2)\mathbf{e}_1\otimes\mathbf{e}_1 - 2\widehat{\mathbf{R}}_1\widehat{\mathbf{R}}_2[\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1] + (1-2\widehat{\mathbf{R}}_1^2)\mathbf{e}_2\otimes\mathbf{e}_2\right)$$

$$= \left[\Phi_2(x,\omega) - \Phi_3(x,\omega)\right]\widehat{\mathbf{R}}^\perp\otimes\widehat{\mathbf{R}}^\perp$$

$$-\frac{1}{R^2}\left[c_6^2\int_0^{\tau_2} h\Phi_2(h,\omega)dh - c_4^2\int_0^{\tau_3} h\Phi_3(h,\omega)dh\right]\left(\underline{\mathbf{J}} - 2\widehat{\mathbf{R}}^\perp\otimes\widehat{\mathbf{R}}^\perp\right)$$

where $\widehat{\mathbf{R}}^\perp = \widehat{\mathbf{R}}_2\mathbf{e}_1 - \widehat{\mathbf{R}}_1\mathbf{e}_2$ and $\underline{\mathbf{J}} = \underline{\mathbf{I}} - \mathbf{e}_3\otimes\mathbf{e}_3$.

Finally, we arrive at

$$\widehat{\underline{\mathbf{G}}} = \Phi_1 \widehat{\mathbf{r}} \otimes \widehat{\mathbf{r}} + \Phi_2 \widehat{\mathbf{R}}^{\perp} \otimes \widehat{\mathbf{R}}^{\perp} + \Phi_3 (\underline{\mathbf{I}} - \widehat{\mathbf{r}} \otimes \widehat{\mathbf{r}} - \widehat{\mathbf{R}}^{\perp} \otimes \widehat{\mathbf{R}}^{\perp})$$

$$-\frac{1}{r^3}\left[\int_0^{\tau_1} h^2 \Phi_1(h,\omega) dh - \int_0^{\tau_3} h^2 \Phi_3(h,\omega) dh\right](\underline{\mathbf{I}} - 3\widehat{\mathbf{r}} \otimes \widehat{\mathbf{r}})$$

$$-\frac{1}{R^2}\left[c_1^2 \int_0^{\tau_2} h \Phi_2(h,\omega) dh - c_6^2 \int_0^{\tau_3} h \Phi_3(h,\omega) dh\right](\underline{\mathbf{J}} - 2\widehat{\mathbf{R}}^{\perp} \otimes \widehat{\mathbf{R}}^{\perp})$$

6.4. Isotropic Medium. When $c_{66} = c_{44}$, medium III becomes isotropic. In this case

$$\Phi_2(\mathbf{x},\omega) = \Phi_3(\mathbf{x},\omega), \quad \beta_1 = \beta_2, \quad \tau_1(\mathbf{x}) = \frac{r}{c_1}, \quad \text{and} \quad \tau_2(\mathbf{x}) = \frac{r}{c_4} = \tau_3(\mathbf{x})$$

Thus, the Green function in an isotropic medium with independent elastic parameters c_{11} and c_{44} can be given in frequency domain as:

$$\widehat{\underline{\mathbf{G}}} = \Phi_2 \underline{\mathbf{I}} + \mathbf{D}_1 \otimes \mathbf{D}_1 M_1^{-1} (\Phi_1 - \Phi_2)$$

$$= \Phi_1 \widehat{\mathbf{r}} \otimes \widehat{\mathbf{r}} + \Phi_2 (\underline{\mathbf{I}} - \widehat{\mathbf{r}} \otimes \widehat{\mathbf{r}}) - \frac{1}{r^3}\left[\int_0^{\frac{r}{c_1}} h^2 \Phi_1(h,\omega) dh - \int_0^{\frac{r}{c_4}} h^2 \Phi_2(h,\omega) dh\right](\underline{\mathbf{I}} - 3\widehat{\mathbf{r}} \otimes \widehat{\mathbf{r}})$$

where Φ_1 and Φ_2 are the same as in medium III. This expression of the Green function has already been reported in a previous work [**18**].

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Appendix A. Decomposition of the Green function

Consider the elastic equation satisfied by $\underline{\mathbf{G}}$:

$$(\underline{\mathbf{\Gamma}}^c(\nabla_x) \underline{\mathbf{G}}(\mathbf{x},t) + \underline{\mathbf{\Gamma}}^v(\nabla_x) \mathcal{A}[\underline{\mathbf{G}}](\mathbf{x},t)) - \rho \frac{\partial^2 \underline{\mathbf{G}}(\mathbf{x},t)}{\partial t^2} = \delta(t)\delta(\mathbf{x})\underline{\mathbf{I}}. \tag{A.1}$$

If $\underline{\mathbf{G}}$ is given in the form

$$\underline{\mathbf{G}} = \sum_{i=1}^{3} \underline{\mathbf{E}}_i(\nabla_x) \phi_i \tag{A.2}$$

Then substituting (A.2) in (A.1) yields:

$$\begin{aligned} \delta(t)\delta(\mathbf{x})\underline{\mathbf{I}} &= (\underline{\mathbf{\Gamma}}^c(\nabla_x) \underline{\mathbf{G}}(\mathbf{x},t) + \underline{\mathbf{\Gamma}}^v(\nabla_x) A[\underline{\mathbf{G}}](\mathbf{x},t)) - \rho \frac{\partial^2 \underline{\mathbf{G}}(\mathbf{x},t)}{\partial t^2} \\ &= \sum_{i,j=1}^{3} \left(L_j^c(\nabla_x)\phi_i + L_j^v(\nabla_x)\mathcal{A}[\phi_i]\right) \underline{\mathbf{E}}_j(\nabla_x) \underline{\mathbf{E}}_i(\nabla_x) - \rho \sum_{i=1}^{3} \underline{\mathbf{E}}_i(\nabla_x) \frac{\partial^2 \phi_i(\mathbf{x},t)}{\partial t^2} \end{aligned}$$

By definition $\underline{\mathbf{E}}_i(\nabla_x)$ is a projection operator which satisfies

$$\underline{\mathbf{E}}_i(\nabla_x) \underline{\mathbf{E}}_j(\nabla_x) = \delta_{ij} E_j(\nabla_x)$$

Consequently, we can have

$$\begin{aligned}\delta(t)\delta(\mathbf{x})\underline{\mathbf{I}} &= \sum_{i,j=1}^{3}\underline{\mathbf{E}}_j(\nabla_x)\delta_{ij}\rho^{-1}\left(L_j^c(\nabla_x)\phi_i + L_j^v(\nabla_x)\mathcal{A}[\phi_i]\right) - \rho\sum_{i=1}^{3}\underline{\mathbf{E}}_i(\nabla_x)\frac{\partial^2\phi_i(\mathbf{x},t)}{\partial t^2}\\ &= \sum_{i=1}^{3}\underline{\mathbf{E}}_i(\nabla_x)\left(\left(L_i^c(\nabla_x)\phi_i + L_i^v(\nabla_x)\mathcal{A}[\phi_i]\right) - \rho\frac{\partial^2\phi_i(\mathbf{x},t)}{\partial t^2}\right).\end{aligned}$$

Moreover $\underline{\mathbf{I}} = \sum_{i=1}^{3}\underline{\mathbf{E}}_i(\nabla_x)$, therefore

$$\sum_{i=1}^{3}\underline{\mathbf{E}}_i(\nabla_x)\left(\left(L_i^c(\nabla_x)\phi_i + L_i^v(\nabla_x)\mathcal{A}[\phi_i]\right) - \rho\frac{\partial^2\phi_i(\mathbf{x},t)}{\partial t^2} - \delta(t)\delta(\mathbf{x})\right) = 0$$

Finally, remark that $\underline{\mathbf{G}}$ we can express in the form (2.8) if the functions ϕ_i satisfy equation (2.10).

Appendix B. Derivative of Potential: Case I

If $b_1 = b_2 = b_3$ and $m_1 = m_2 = m_3$, we have

$$\text{(B.1)}\qquad \left|\begin{array}{l} V_1(s) = V_2(s) = V_3(s) = b_1^2 + m_1^2 s\\[2mm] F(s) = \displaystyle\sum_{j=1}^{3}\frac{x_j^2}{V_1(s)} - h^2 = \frac{r^2}{V_1(s)} - h^2\\[2mm] F'(s) = \displaystyle\sum_{j=1}^{3}\frac{-m_1^2x_j^2}{V_1^2(s)} = \frac{-m_1^2r^2}{V_1^2(s)} \quad\text{and}\quad F'(0) = \frac{-m_1^2r^2}{b_1^4}\\[2mm] F''(s) = \displaystyle\sum_{j=1}^{3}\frac{2m_1^4x_j^2}{V_1^3(s)} = \frac{2m_1^4r^2}{V_1^3(s)}\\[2mm] G(s) = (V_1(s))^3 \quad\text{and}\quad G'(s) = G(s)\dfrac{3m_1^2}{V_1(s)}\end{array}\right.$$

with $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. When $F(S) = 0$, we have

$$\text{(B.2)}\qquad \left|\begin{array}{l} V_1(S) = \dfrac{r^2}{h^2},\\[2mm] \left[\dfrac{1}{V_k(S)V_l(S)F'(S)}\right] = \dfrac{-1}{m_1^2r^2} \quad\text{and}\quad \dfrac{1}{F'(S)\sqrt{G(S)}} = \dfrac{-1}{m_1^2rh},\\[2mm] \left\{\dfrac{F''(S)}{F'(S)} + \dfrac{m_k^2}{V_k(S)} + \dfrac{m_l^2}{V_l(S)} + \dfrac{1}{2}\dfrac{G'(S)}{G(S)}\right\} = \dfrac{3}{2}\dfrac{m_1^2}{V_1(S)} = \dfrac{3}{2}\dfrac{m_1^2h^2}{r^2}\end{array}\right.$$

Substituting (B.1) and (B.2) in (5.12) we finally arrive at:
(B.3)

$$\frac{4\rho m_1^2\pi}{\left(1-\beta\widehat{\mathcal{A}}(\omega)\right)}\frac{\partial^2\Psi}{\partial x_k x_l} = \widehat{\mathbf{r}}_k\widehat{\mathbf{r}}_l\left\{\frac{e^{\sqrt{-1}\kappa(\omega)\tau}}{b\tau}\right\} - \frac{1}{r^3}(\delta_{kl} - 3\widehat{\mathbf{r}}_i\widehat{\mathbf{r}}_j)\int_0^\tau\left[he^{\sqrt{-1}\kappa(\omega)h}\right]dh$$

where $\widehat{\mathbf{r}}_j = \dfrac{x_j}{r}$ for all $j = 1, 2, 3$.

Appendix C. Derivative of Potential: Case II

If $b_1 = b_2$, $m_1 = m_2$ and $m_3 = 0$, we have

$$\text{(C.1)}\qquad \left|\begin{array}{l} V_1(s) = V_2(s) = b_1^2 + m_1^2 s \quad \text{and} V_3(s) = b_3^2 \\[2ex] F'(s) = \displaystyle\sum_{j=1}^{2}\frac{-m_1^2x_j^2}{V_1^2(s)} = \frac{-m_1^2R^2}{V_1^2(s)} \quad \text{and} \quad F'(0) = \frac{-m_1^2R^2}{b_1^4} \\[2ex] F''(s) = \displaystyle\sum_{j=1}^{2}\frac{2m_1^4x_j^2}{V_1^3(s)} = \frac{2m_1^4R^2}{V_1^3(s)} \\[2ex] G(s) = b_3^2\left(V_1(s)\right)^2 \quad \text{and} \quad G'(s) = G(s)\dfrac{2m_1^2}{V_1(s)} \end{array}\right.$$

with $R = \sqrt{x_1^2 + x_2^2}$. For all $l, k \in \{1, 2\}$, we have

$$\text{(C.2)}\qquad \left|\begin{array}{l} \left[\dfrac{1}{V_k(S)V_l(S)F'(S)}\right] = \dfrac{-1}{m_1^2R^2} \quad \text{and} \\[2ex] \dfrac{1}{F'(S)\sqrt{G(S)}} = \dfrac{-V(S)}{m_1^2b_3R^2}, \\[2ex] \left\{\dfrac{F''(S)}{F'(S)} + \dfrac{m_k^2}{V_k(S)} + \dfrac{m_l^2}{V_l(S)} + \dfrac{1}{2}\dfrac{G'(S)}{G(S)}\right\} = \dfrac{m_1^2}{V_1(S)} \end{array}\right.$$

Substituting (C.1) and (C.2) in (5.12) and simple calculations, we finally arrive at:
(C.3)

$$\frac{4\rho m_1^2\pi}{\left(1-\beta\widehat{\mathcal{A}}(\omega)\right)}\frac{\partial^2\Psi}{\partial x_k x_l} = \widehat{\mathbf{R}}_k\widehat{\mathbf{R}}_l\left\{\frac{e^{\sqrt{-1}\kappa(\omega)\tau}}{b\tau}\right\} - \frac{1}{b_3R^2}(\delta_{kl} - 2\widehat{\mathbf{R}}_k\widehat{\mathbf{R}}_l)\int_0^\tau\left[e^{\sqrt{-1}\kappa(\omega)h}\right]dh$$

where $\widehat{\mathbf{R}}_k = \dfrac{x_k}{R}$ for $k = 1, 2$.

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