On the Green Function in Visco-Elastic Media Obeying a Frequency Power-Law

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Abstract

In this work, we present an explicite expression for the Green function in a viscoelastic medium. We choose Szabo and Wu's frequency power law model to describe the viscoelastic properties and derive a generalized viscoelastic wave equation. We express the ideal Green function (without any viscus effect) in terms of the viscus Green function using an attenuation operator. By means of an approximation of the ideal Green function, we address the problem of reconstructing a small anomaly in a viscoelastic medium from wavefield measurements.

1 Introduction

The elastic properties of human soft tissues have been exploited in a number of imaging modalities in recent past, because the elasticity varies significantly in order of magnitude with different tissue types and is closely linked with the pathology of the tissue.

Most of the time, medium is considered to be ideal (without any viscus effect), neglecting the fact that a wave losses some of its energy to the medium and its amplitude decreases with time due to viscosity. While, an estimation of the viscosity effects can some times be very useful in the characterization and identification of the anomaly [9].

To address the problem of reconstructing a small anomaly in viscoelastic media from wavefield measurements, it is important to first model the mechanical response of such media to excitations .

The Voigt model is a common model to describe the viscoelastic properties of tissues. Catheline *et al.* [10] have shown that this model is well adapted to describe the viscoelastic response of tissues to low-frequency excitations. However, we choose a more general model derived by Szabo and Wu in [16] that describes observed power-law behavior of many viscoelastic materials including human myocardium. This model is based on a time-domain statement of causality [15] and reduces to the Voigt model for the specific case of quadratic frequency losses.

Expressing the ideal elastic field without any viscous effect in terms of the measured field in a viscous medium, one can generalize the methods described in [2, 3, 4, 5, 8], namely the time reversal, back-propagation and Krichhoff Imaging, to recover the viscoelastic and geometric properties of an anomaly from wavefield measurements. To achieve this goal, we focus on the Green function in this article. We present a relationship between the ideal Green function and the viscoelastic

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Green function in the limiting case when the shear modulus $\lambda \to \infty$, in a quasi-incompressible medium. We also provide an approximation of this relationship using the stationary phase theorem.

The article is organized as follows. In section 2, we introduce a general visco-elastic wave equation based on Szabo and Wu's power law model. Section 3 is devoted to the derivation of the Green function in the viscoelastic medium. In section 4, we present an approximation of the ideal green function in the case of quadratic losses and provide a procedure of image reconstruction in viscoelastic media. We support our work with numerical illustrations, which are presented in section 5.

2 General Visco-Elastic Wave Equation

When a wave travels through a biological medium, its amplitude decreases with time due to attenuation. The attenuation coefficient for biological tissue may be approximated by a power-law over a wide range of frequencies. Measured attenuation coefficients of soft tissues typically have linear or greater than linear dependence on frequency. [11, 15, 16]

In an ideal elastic medium; without attenuation, Hooke's law gives the following relationship between stress and strain tensors:

$$\mathcal{T} = \mathcal{C} : \mathcal{S} \tag{1}$$

where $\mathcal{T}$, $\mathcal{C}$ and $\mathcal{S}$ are respectively stress, stiffness and strain tensors of orders 2, 4 and 2 and $:$ represents tensorial product.

Consider a viscoelastic medium. Suppose that the medium is homogeneous and isotropic. We write

$$\mathcal{C} = [\mathcal{C}_{ijkl}] = [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})],$$
$$\eta = [\eta_{ijkl}] = [\eta_s\delta_{ij}\delta_{kl} + \eta_p(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})],$$

where δ_{ab} is the Kronecker delta function, μ, λ are the Lamé parameters, and η_s, η_p are the shear and bulk viscosities, respectively. Here we have adopted the generalized summation convention over the repeated index.

Throughout this work we suppose that

$$\eta_p, \eta_s << 1. \tag{2}$$

For a medium obeying a power-law attenuation model and under the smallness condition (2), a generalized Hooke's law reads [16]

$$\mathcal{T}(x,t) = \mathcal{C} : \mathcal{S}(x,t) + \eta : \mathcal{M}(\mathcal{S})(x,t) \tag{3}$$

where $\mathcal{M}$ is the convolution operator given by

$$\mathcal{M}(\mathcal{S}) = \begin{cases} -(-1)^{\gamma/2}\frac{\partial^{\gamma-1}\mathcal{S}}{\partial t^{\gamma-1}} & \gamma \text{ is an even integer,} \\ \frac{2}{\pi}(\gamma-1)!(-1)^{(\gamma+1)/2}\frac{H(t)}{t^\gamma} * \mathcal{S} & \gamma \text{ is an odd integer,} \\ -\frac{2}{\pi}\Gamma(\gamma)\sin(\gamma\pi/2)\frac{H(t)}{|t|^\gamma} * \mathcal{S} & \gamma \text{ is a non integer.} \end{cases} \tag{4}$$

Here $H(t)$ is the Heaviside function and Γ denotes the gamma function.

Note that for the common case when, $\gamma = 2$, the generalized Hooke's law (3) reduces to the Voigt model,

$$\mathcal{T} = \mathcal{C} : \mathcal{S} + \eta : \frac{\partial \mathcal{S}}{\partial t}. \tag{5}$$

Taking the divergence of (3) we get

$$\nabla \cdot \mathcal{T} = \left(\bar{\lambda} + \bar{\mu}\right) \nabla(\nabla \cdot \mathbf{u}) + \bar{\mu}\Delta \mathbf{u}, \tag{6}$$

where

$$\bar{\lambda} = \lambda + \eta_p \mathcal{M}(\cdot) \quad \text{and} \quad \bar{\mu} = \mu + \eta_s \mathcal{M}(\cdot).$$

Consider the equation of motion for the system,

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{F} = \nabla \cdot \mathcal{T}, \tag{7}$$

with ρ being the constant density and $\mathbf{F}$ the applied force. Using the expression (6) for $\nabla \cdot \mathcal{T}$ in (7), we obtain the generalized viscoelastic wave equation

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{F} = \left(\bar{\lambda} + \bar{\mu}\right) \nabla(\nabla \cdot \mathbf{u}) + \bar{\mu}\Delta \mathbf{u}. \tag{8}$$

3 Green Function

In this section we find the Green function of the viscoelastic wave equation (8). We first provide the following Helmholtz decomposition:

3.1 Helmholtz Decomposition

The following lemma holds.

Lemma 3.1 *If the displacement field* $\mathbf{u}(x,t)$ *satisfy (8),* $\frac{\partial \mathbf{u}(x,0)}{\partial t} = \nabla A + \nabla \times B$ *and* $\mathbf{u}(x,0) = \nabla C + \nabla \times D$ *and if the body force* $\mathbf{F} = \nabla \varphi_f + \nabla \times \psi_f$ *then there exist potentials* φ_u *and* ψ_u *such that*

- $\mathbf{u} = \nabla \varphi_u + \nabla \times \psi_u; \qquad \nabla \cdot \psi_u = 0;$
- $\frac{\partial^2 \varphi_u}{\partial t^2} = \frac{\varphi_f}{\rho} + c_p^2 \Delta \varphi_u + \nu_p \mathcal{M}(\Delta \varphi_u) \approx \frac{\varphi_f}{\rho} - \frac{\nu_p \mathcal{M}(\varphi_f)}{\rho c_p^2} + c_p^2 \Delta \varphi_u + \frac{\nu_p}{c_p^2} \mathcal{M}(\partial_t^2 \varphi_u);$
- $\frac{\partial^2 \psi_u}{\partial t^2} = \frac{\psi_f}{\rho} + c_s^2 \Delta \psi_u + \nu_s \mathcal{M}(\Delta \psi_u) \approx \frac{\psi_f}{\rho} - \frac{\nu_s \mathcal{M}(\psi_f)}{\rho c_s^2} + c_s^2 \Delta \psi_u + \frac{\nu_s}{c_s^2} \mathcal{M}(\partial_t^2 \psi_u),$

with

$$c_p^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_s^2 = \frac{\mu}{\rho}, \quad \nu_p = \frac{\eta_p + 2\eta_s}{\rho} \quad \text{and} \quad \nu_s = \frac{\eta_s}{\rho}.$$

Proof. For φ_u and ψ_u defined as

$$\varphi_u(x,t) = \int_0^t \int_0^\tau \left[\frac{\varphi_f}{\rho} + (c_p^2 + \nu_p \mathcal{M})(\nabla \cdot u) \right] ds d\tau + tA + C \tag{9}$$

$$\psi_u(x,t) = \int_0^t \int_0^\tau \left[\frac{\vec{\psi}_f}{\rho} - (c_s^2 + \nu_s \mathcal{M})(\nabla \times u) \right] ds d\tau + t\vec{B} + \vec{D} \tag{10}$$

we have the required expression for $\mathbf{u}$. Moreover, it is evident from (10) that $\nabla \cdot \psi_u = 0$

Now, on differentiating φ_u and ψ_u twice with respect to time, we get

$$\frac{\partial^2 \varphi_u}{\partial t^2} = \frac{\varphi_f}{\rho} + c_p^2 \Delta \varphi_u + \nu_p \mathcal{M}(\Delta \varphi_u)$$

$$\frac{\partial^2\psi_u}{\partial t^2} = \frac{\psi_f}{\rho} + c_s^2\Delta\psi_u + \nu_s\mathcal{M}(\Delta\psi_u)$$

Finally, applying $\mathcal{M}$ on last two equations, neglecting the higher order terms in ν_s and ν_p and injecting back the expressions for $\mathcal{M}(\Delta\varphi_u)$ and $\mathcal{M}(\Delta\psi_u)$, we get the required differential equations for φ_u and ψ_u. □

Let

$$K_m(\omega) = \omega\sqrt{\left(1 - \frac{\nu_m}{c_m^2}\hat{\mathcal{M}}(\omega)\right)}, \quad m = s, p, \tag{11}$$

where the multiplication operator $\hat{\mathcal{M}}(\omega)$ is the Fourier transform of the convolution operator $\mathcal{M}$.

If φ_u and ψ_u are causal then it implies the causality of the inverse Fourier transform of $K_m(\omega), m = s, p$. Applying the Kramers-Krönig relations[1], it follows that

$$-\Im m K_m(\omega) = \mathcal{H}\Big[\Re e K_m(\omega)\Big] \quad \text{and} \quad \Re e K_m(\omega) = \mathcal{H}\Big[\Im m K_m(\omega)\Big], \quad m = p, s, \tag{12}$$

where $\mathcal{H}$ is the Hilbert transform. Note that $\mathcal{H}^2 = -I$. The convolution operator $\mathcal{M}$ given by (4) is based on the constraint that causality imposes on (3). Under the smallness assumption (2), the expressions in (4) can be found from the Kramers-Krönig relations (12). One drawback of (12) is that the attenuation, $\Im m K_m(\omega)$, must be known at all frequencies to determine the dispersion, $\Re e K_m(\omega)$. However, bounds on the dispersion can be obtained from measurements of the attenuation over a finite frequency range [13].

3.2 Solution of (8) with a Concentrated Force.

Let u_{ij} denote the i-th component of the solution $\mathbf{u}_j$ of the elastic wave equation related to a force $\mathbf{F}$ concentrated in the x_j-direction. Let $j = 1$ for simplicity and suppose that

$$\mathbf{F} = -T(t)\delta(x-\xi)\mathbf{e}_1 = -T(t)\delta(x-\xi)(1,0,0), \tag{13}$$

where ξ is the source point and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis of $\mathbb{R}^3$.

Let $\mathbf{Z}$ be the solution of the poisson equation

$$\nabla^2\mathbf{Z} = \mathbf{F}$$

Then

$$\mathbf{Z}(x,t;\xi) = \frac{T(t)}{4\pi}\frac{1}{r}\mathbf{e}_1.$$

As $\nabla^2\mathbf{Z} = \nabla(\nabla\cdot\mathbf{Z}) - \nabla\times(\nabla\times\mathbf{Z})$, the Helmholtz decomposition of the force $\mathbf{F}$ can be written [14] as

$$\begin{cases} \mathbf{F} = \nabla\varphi_f + \nabla\times\psi_f, \\ \varphi_f = \nabla\cdot\mathbf{Z} = \frac{T(t)}{4\pi}\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right), \\ \psi_f = -\nabla\times\mathbf{Z} = -\frac{T(t)}{4\pi}\left(0, \frac{\partial}{\partial x_3}\left(\frac{1}{r}\right), -\frac{\partial}{\partial x_2}\left(\frac{1}{r}\right)\right), \end{cases} \tag{14}$$

where $r = |x-\xi|$.

[1] see [15, 16, 17] for more details on causality and KKR

Consider the Helmholtz decomposition for $\mathbf{u}_1$ as

$$\mathbf{u}_1 = \nabla\varphi_1 + \nabla \times \vec{\psi}_1 \tag{15}$$

then, according to lemma 3.1, φ_1 and ψ_1 are the solutions of the equations

$$\Delta\varphi_1 - \frac{1}{c_p^2}\frac{\partial^2\varphi_1}{\partial t^2} + \frac{\nu_p}{c_p^4}\mathcal{M}(\partial_t^2\varphi_1) = \frac{\nu_p\mathcal{M}(\varphi_f)}{\rho c_p^4} - \frac{\varphi_f}{c_p^2\rho}, \tag{16}$$

$$\Delta\psi_1 - \frac{1}{c_s^2}\frac{\partial^2\psi_1}{\partial t^2} + \frac{\nu_s}{c_s^4}\mathcal{M}(\partial_t^2\psi_1) = \frac{\nu_s\mathcal{M}(\psi_f)}{\rho c_s^4} - \frac{\psi_f}{c_s^2\rho}. \tag{17}$$

Taking the Fourier transform of (15),(16) and (17) with respect to t we get

$$\hat{\mathbf{u}}_1 = \nabla\hat{\varphi}_1 + \nabla \times \hat{\psi}_1 \tag{18}$$

$$\Delta\hat{\varphi}_1 + \frac{K_p^2(\omega)}{c_p^2}\hat{\varphi}_1 = \frac{\nu_p\hat{\mathcal{M}}(\omega)\hat{\varphi}_f}{\rho c_p^4} - \frac{\hat{\varphi}_f}{\rho c_p^2}, \tag{19}$$

$$\Delta\hat{\psi}_1 + \frac{K_s^2(\omega)}{c_s^2}\hat{\psi}_1 = \frac{\nu_s\hat{\mathcal{M}}(\omega)\hat{\psi}_f}{\rho c_s^4} - \frac{\hat{\psi}_f}{\rho c_s^2}, \tag{20}$$

where $K_m(\omega), m = p, s$, are defined in (11).

It is well known that the Green functions of the Helmholtz equations (19) and (20) are

$$\hat{g}^m(x,\omega) = \frac{e^{\sqrt{-1}\frac{K_m(\omega)}{c_m}|x|}}{4\pi|x|}, \quad m = s, p.$$

We closely follow the argument in [14], and write $\hat{\varphi}_1$ as

$$\begin{aligned}\hat{\varphi}_1(x,\omega;\xi) &= \hat{g}^m(x,\omega) *_x \left(\frac{\nu_p\hat{\mathcal{M}}(\omega)\varphi_f}{\rho c_p^4} - \frac{\varphi_f}{c_p^2\rho}\right) \\ &= -\left(1 - \frac{\nu_p\hat{\mathcal{M}}(\omega)}{c_p^2}\right)\frac{\hat{T}(\omega)}{\rho(4\pi c_p)^2}\int_{\mathbb{R}^3}\hat{g}^p(x-z,\omega)\frac{\partial}{\partial z_1}\frac{1}{|z-\xi|}dz.\end{aligned}$$

Note that $z \to \hat{g}^p(x-z,\omega)$ is constant on each sphere $\partial B(x,h)$, centered on x with radius h. Use of spherical coordinates leads to

$$\hat{\varphi}_1(x,\omega;\xi) = -(1 - \frac{\nu_p\hat{\mathcal{M}}(\omega)}{c_p^2})\frac{\hat{T}(\omega)}{\rho(4\pi c_p)^2}\int_0^\infty \hat{g}^p(h,\omega)\int_{\partial B(x,h)}\frac{\partial}{\partial z_1}\left(\frac{1}{|z-\xi|}\right)d\sigma(z)dh.$$

From [1], it follows that

$$\int_{\partial B(x,h)}\frac{\partial}{\partial z_1}\left(\frac{1}{|z-\xi|}\right)d\sigma(z) = \begin{cases}0 & \text{if } h > r \\ 4\pi h^2\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right) & \text{if } h < r.\end{cases}$$

Therefore, we have following expression for $\hat{\varphi}_1$:

$$\begin{aligned}\hat{\varphi}_1(x,\omega;\xi) &= -\left(1 - \frac{\nu_p\hat{\mathcal{M}}(\omega)}{c_p^2}\right)\frac{\hat{T}(\omega)}{4\pi\rho c_p^2}\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)\int_0^r he^{\sqrt{-1}\frac{K_p(\omega)}{c_p}h}dh, \\ &= -\left(1 - \frac{\nu_p\hat{\mathcal{M}}(\omega)}{c_p^2}\right)\frac{\hat{T}(\omega)}{4\pi\rho}\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)\int_0^{r/c_p}\zeta e^{\sqrt{-1}K_p(\omega)\zeta}d\zeta. \end{aligned}\tag{21}$$

In the same way, the vector $\hat{\psi}_1$ is given by

$$\hat{\psi}_1(x,\omega;\xi) = (1-\frac{\nu_s\hat{\mathcal{M}}(\omega)}{c_s^2})\frac{\hat{T}(\omega)}{4\pi\rho}\left(0, \frac{\partial}{\partial x_3}\left(\frac{1}{r}\right), -\frac{\partial}{\partial x_2}\left(\frac{1}{r}\right)\right)\int_0^{r/c_s}\zeta e^{\sqrt{-1}K_s(\omega)\zeta}\,d\zeta. \tag{22}$$

We Introduce following notation for simplicity:

$$I_m(r,\omega) = A_m\int_0^{r/c_m}\zeta e^{\sqrt{-1}K_m(\omega)\zeta}\,d\zeta \tag{23}$$

$$E_m(r,\omega) = A_m e^{\sqrt{-1}K_m(\omega)\frac{r}{c_m}}, \tag{24}$$

$$A_m(\omega) = \left(1-\frac{\nu_m\hat{\mathcal{M}}(\omega)}{c_m^2}\right), \quad m = p, s. \tag{25}$$

Now, we calculate $\hat{u}_{i1} = \nabla\varphi_1 + \nabla\times\vec{\psi}_1$. For all $i = 1:3$

$$\begin{aligned}
(\nabla\hat{\varphi}_1)_i &= -\frac{\partial}{\partial x_i}\left[\left(1-\frac{\nu_p\hat{\mathcal{M}}(\omega)}{c_p^2}\right)\frac{\hat{T}(\omega)}{4\pi\rho}\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)\int_0^{r/c_p}\zeta e^{\sqrt{-1}K_p(\omega)\zeta}\,d\zeta\right], \\
&= -\left(1-\frac{\nu_p\hat{\mathcal{M}}(\omega)}{c_p^2}\right)\frac{\hat{T}(\omega)}{4\pi\rho}\frac{\partial^2}{\partial x_1 x_i}\left(\frac{1}{r}\right)\int_0^{r/c_p}\zeta e^{\sqrt{-1}K_p(\omega)\zeta}\,d\zeta \\
&\quad -\left(1-\frac{\nu_p\hat{\mathcal{M}}(\omega)}{c_p^2}\right)\frac{\hat{T}(\omega)}{4\pi\rho}\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x_i}\left(\frac{r}{c_p^2}e^{\sqrt{-1}K_p(\omega)\frac{r}{c_p}}\right), \\
&= -\frac{\hat{T}(\omega)}{4\pi\rho}\frac{\partial^2}{\partial x_i\partial x_1}\left(\frac{1}{r}\right)I_p(r,\omega) + \frac{\hat{T}(\omega)}{4\pi\rho}\frac{1}{rc_p^2}\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_i}E_p(r,\omega),
\end{aligned}$$

where we have used the equality $\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x_1}$. In the same way, the value $\left(\nabla\times\vec{\hat{\psi}}_1\right)_i$ is given by

$$\left(\nabla\times\vec{\hat{\psi}}_1\right)_i = \frac{\hat{T}(\omega)}{4\pi\rho}\frac{\partial^2}{\partial x_i\partial x_1}\left(\frac{1}{r}\right)I_s(r,\omega) + \frac{\hat{T}(\omega)}{4\pi\rho c_s^2 r}\left(\delta_{i1}-\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_1}\right)E_s(r,\omega).$$

Therefore

$$\begin{aligned}
\hat{u}_{i1} &= \frac{\hat{T}(\omega)}{4\pi\rho}\frac{\partial^2}{\partial x_i x_1}\left(\frac{1}{r}\right)[I_s(r,\omega)-I_p(r,\omega)] + \frac{\hat{T}(\omega)}{4\pi\rho c_p^2 r}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_1}E_p(r,\omega) \\
&\quad +\frac{\hat{T}(\omega)}{4\pi\rho c_s^2 r}\left(\delta_{i1}-\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_1}\right)E_s(r,\omega).
\end{aligned}$$

Hence, $\hat{u}_{ij}$, the i-th component of the solution $\hat{\mathbf{u}}_j$ for an arbitrary j, is

$$\begin{aligned}
\hat{u}_{ij} &= \frac{\hat{T}(\omega)}{4\pi\rho}\left(3\gamma_i\gamma_j-\delta_{ij}\right)\frac{1}{r^3}[I_s(r,\omega)-I_p(r,\omega)] + \frac{\hat{T}(\omega)}{4\pi\rho c_p^2}\gamma_i\gamma_j\frac{1}{r}E_p(r,\omega) \\
&\quad +\frac{\hat{T}(\omega)}{4\pi\rho c_s^2}\left(\delta_{ij}-\gamma_i\gamma_j\right)\frac{1}{r}E_s(r,\omega),
\end{aligned}$$

where $\gamma_i = \frac{\partial r}{\partial x_i} = (x_i-\xi_i)/r$ and I_m and E_m are given by equations (23) and (24).

3.3 Viscoelastic Green function

If we substitute $T(t) = \delta(t)$, where delta is the Dirac mass, then the function $u_{ij} = G_{ij}$ is the i-th component of the Green function related to the force concentrated in the x_j-direction. In this case, we have $\hat{T}(\omega) = 1$. Therefore, we have following expression for $\hat{G}_{ij}$:

$$\hat{G}_{ij}(x,\omega;\xi) = \tfrac{1}{4\pi\rho}\left(3\gamma_i\gamma_j - \delta_{ij}\right)\tfrac{1}{r^3}\left[I_s(r,\omega) - I_p(r,\omega)\right] + \tfrac{1}{4\pi\rho c_p^2}\gamma_i\gamma_j\tfrac{1}{r}E_p(r,\omega)$$
$$+\tfrac{1}{4\pi\rho c_s^2}\left(\delta_{ij} - \gamma_i\gamma_j\right)\tfrac{1}{r}E_s(r,\omega),$$

or equivalently,

$$\hat{G}_{ij}(x,\omega;\xi) = \hat{g}^p_{ij}(x,\omega;\xi) + \hat{g}^s_{ij}(x,\omega;\xi) + \hat{g}^{ps}_{ij}(x,\omega;\xi), \tag{26}$$

where

$$\hat{g}^{ps}_{ij}(x,\omega;\xi) = \frac{1}{4\pi\rho}\left(3\gamma_i\gamma_j - \delta_{ij}\right)\frac{1}{r^3}\left[I_s(r,\omega) - I_p(r,\omega)\right], \tag{27}$$

$$\hat{g}^p_{ij}(x,\omega;\xi) = \frac{A_p(\omega)}{\rho c_p^2}\gamma_i\gamma_j\hat{g}^p(r,\omega), \tag{28}$$

and

$$\hat{g}^s_{ij}(x,\omega;\xi) = \frac{A_s(\omega)}{\rho c_s^2}\left(\delta_{ij} - \gamma_i\gamma_j\right)\hat{g}^s(r,\omega). \tag{29}$$

Let $G(x,t;\xi) = (G_{ij}(x,t;\xi))$ denote the transient Green function of (8) associated with the source point ξ. Let $G^m(r,t)$ and $W_m(x,t)$ be the inverse Fourier transforms of $A_m(\omega)\hat{g}^m(r,\omega)$ and $I_m(r,\omega), m = p, s$, respectively. Then, from (26-29), we have

$$\begin{aligned} G_{ij}(x,t;\xi) &= \tfrac{1}{\rho c_p^2}\gamma_i\gamma_j G^p(r,t) + \tfrac{1}{\rho c_s^2}\left(\delta_{ij} - \gamma_i\gamma_j\right)G^s(r,t) \\ &\quad+\tfrac{1}{4\pi\rho}\left(3\gamma_i\gamma_j - \delta_{ij}\right)\tfrac{1}{r^3}\left[W_s(r,t) - W_p(r,t)\right]. \end{aligned} \tag{30}$$

Note that by a change of variables,

$$W_m(r,t) = \frac{4\pi}{c_m^2}\int_0^r \zeta^2 G^m(\zeta,t;\xi)d\zeta.$$

4 Approximate Green Function and the Imaging Problem

Consider the limiting case $\lambda \to +\infty$. The Green function for a quasi-incompressible visco-elastic medium is given by

$$G_{ij}(x,t;\xi) = \tfrac{1}{\rho c_s^2}\left(\delta_{ij} - \gamma_i\gamma_j\right)G^s(r,t) + \tfrac{1}{\rho c_s^2}\left(3\gamma_i\gamma_j - \delta_{ij}\right)\tfrac{1}{r^3}\int_0^r \zeta^2 G^s(\zeta,t)d\zeta.$$

To generalize the detection algorithms presented in [2, 3, 4, 5] to the visco-elastic case we shall express the ideal Green function without any viscous effect in terms of the Green function in a viscous medium. From

$$G^s(r,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\sqrt{-1}\omega t}A_s(\omega)g^s(r,\omega)\,d\omega,$$

it follows that

$$G^s(r,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} A_s(\omega)\frac{e^{\sqrt{-1}(-\omega t + \frac{K_s(\omega)}{c_s}r)}}{4\pi r}\,d\omega.$$

4.1 Approximation of the Ideal Green Function

Let us introduce the operator

$$L\phi(t) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{+\infty} A_s(\omega)\phi(\tau)e^{\sqrt{-1}K_s(\omega)\tau}e^{-\sqrt{-1}\omega t}\,d\tau\,d\omega,$$

for a causal function ϕ. We have

$$G^s(r,t;\xi) = L(\frac{\delta(\tau - r/c_s)}{4\pi r}),$$

and therefore,

$$L^*G^s(r,t) = L^*L(\frac{\delta(\tau - r/c_s)}{4\pi r}),$$

where L^* is the $L^2(0,+\infty)$-adjoint of L.

Consider for simplicity the Voigt model. Then, $\hat{\mathcal{M}}(\omega) = -\sqrt{-1}\omega$ and hence,

$$K_s(\omega) = \omega\sqrt{1+\frac{\sqrt{-1}\nu_s}{c_s^2}\omega} \approx \omega + \frac{\sqrt{-1}\nu_s}{2c_s^2}\omega^2,$$

under the smallness condition (2). The operator L can then be approximated by

$$\tilde{L}\phi(t) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{+\infty} A_s(\omega)\phi(\tau)e^{-\frac{\nu_s}{2c_s^2}\omega^2\tau}e^{\sqrt{-1}\omega(\tau-t)}\,d\tau\,d\omega.$$

Since

$$\int_{\mathbb{R}} e^{-\frac{\nu_s}{2c_s^2}\omega^2\tau}e^{\sqrt{-1}\omega(\tau-t)}\,d\omega = \frac{\sqrt{2\pi}c_s}{\sqrt{\nu_s\tau}}e^{-\frac{c_s^2(\tau-t)^2}{2\nu_s\tau}},$$

and

$$\sqrt{-1}\int_{\mathbb{R}} \omega e^{-\frac{\nu_s}{2c_s^2}\omega^2\tau}e^{\sqrt{-1}\omega(\tau-t)}\,d\omega = -\frac{\sqrt{2\pi}c_s}{\sqrt{\nu_s\tau}}\frac{\partial}{\partial t}e^{-\frac{c_s^2(\tau-t)^2}{2\nu_s\tau}},$$

it follows that

$$\tilde{L}\phi(t) = \int_0^{+\infty}\frac{t}{\tau}\phi(\tau)\frac{c_s}{\sqrt{2\pi\nu_s\tau}}e^{-\frac{c_s^2(\tau-t)^2}{2\nu_s\tau}}\,d\tau. \tag{31}$$

Analogously,

$$\tilde{L}^*\phi(t) = \int_0^{+\infty}\frac{\tau}{t}\phi(\tau)\frac{c_s}{\sqrt{2\pi\nu_s t}}e^{-\frac{c_s^2(\tau-t)^2}{2\nu_s t}}\,d\tau. \tag{32}$$

Since the phase in (32) is quadratic and ν_s is small then by consequence of the stationary phase theorem A.1, we have following result:

Theorem 4.1 (Approximation of operator L)

$$\tilde{L}^*\phi = \phi + \frac{\nu_s}{2c_s^2}\partial_{tt}(t\phi) + o\left(\frac{\nu_s}{c_s^2}\right), \quad \tilde{L}\phi = \phi + \frac{\nu_s}{2c_s^2}t\partial_{tt}\phi + o\left(\frac{\nu_s}{c_s^2}\right), \tag{33}$$

and therefore

$$\tilde{L}^*\tilde{L}\phi = \phi + \frac{\nu_s}{c_s^2}\partial_t(t\partial_t\phi) + o\left(\frac{\nu_s}{c_s^2}\right), \tag{34}$$

and,

$$(L^*\tilde{L})^{-1}\phi = \phi - \frac{\nu_s}{c_s^2}\partial_t(t\partial_t\phi) + o\left(\frac{\nu_s}{c_s^2}\right). \tag{35}$$

Proof.

1. *Proof of approximation (33)*:
Let us first consider the case of operator L^*. We have

$$\tilde{L}^*\phi(t) = \int_0^{+\infty} \frac{\tau}{t}\phi(\tau)\frac{c_s}{\sqrt{2\pi\nu_s t}}e^{-\frac{c_s^2(\tau-t)^2}{2\nu_s t}}d\tau = \frac{1}{t\sqrt{\epsilon}}\left(\int_0^{+\infty}\psi(\tau)e^{if(\tau)/\epsilon}\right),$$

with, $f(\tau) = i\pi(\tau-t)^2$, $\epsilon = \frac{2\pi\nu_s t}{c_s^2}$ and $\psi(\tau) = \tau\phi(\tau)$. Remark that the phase f satisfies at $\tau = t$, $f(t) = 0$, $f'(t) = 0$, $f''(t) = 2i\pi \neq 0$. Moreover, we have

$$\begin{cases} e^{if(t)/\epsilon}\left(\epsilon^{-1}f''(t)/2i\pi\right)^{-1/2} = \sqrt{\epsilon} \\ g_t(\tau) = f(\tau) - f(t) - \frac{1}{2}f''(t)(\tau-t)^2 = 0 \\ L_1\psi(t) = L_1^1\psi(t) = \frac{-1}{2i}f''(t)^{-1}\psi''(t) = \frac{1}{4\pi}(t\phi)''. \end{cases}$$

Thus, Theorem A.1 implies that

$$\left|\tilde{L}^*\phi(t) - \left(\phi(t) + \frac{\nu_s}{2c_s^2}(t\phi)''\right)\right| \leq \frac{C}{t}\epsilon^{3/2}\sum_{\alpha\leq 4}\sup|(t\phi)^{(\alpha)}|.$$

The case of the operator $\tilde{L}$ is very similar. Note that

$$\tilde{L}\phi(t) = \int_0^{+\infty}\frac{t}{\tau}\phi(\tau)\frac{c_s}{\sqrt{2\pi\nu_s\tau}}e^{-\frac{c_s^2(\tau-t)^2}{2\nu_s\tau}}d\tau = \frac{t}{\sqrt{\epsilon}}\left(\int_0^{+\infty}\psi(\tau)e^{if(\tau)/\epsilon}\right),$$

with $f(\tau) = i\pi\frac{(\tau-t)^2}{\tau}$, $\epsilon = \frac{\nu_s}{2\pi c_s^2}$ and $\psi(\tau) = \phi(\tau)\tau^{-\frac{3}{2}}$. It follows that

$$f'(\tau) = i\pi\left(1 - \frac{t^2}{\tau^2}\right), \quad f''(\tau) = 2i\pi\frac{t^2}{\tau^3}, \quad f''(t) = 2i\pi\frac{1}{t},$$

and the function $g_t(\tau)$ is equal to

$$g_t(\tau) = i\pi\frac{(\tau-t)^2}{\tau} - i\pi\frac{(\tau-t)^2}{t} = i\pi\frac{(t-\tau)^3}{\tau t}.$$

We deduce that

$$\begin{cases} (g_t\psi)^{(4)}(t) &= \left(g_t^{(4)}(t)\psi(t) + 4g_t^{(3)}(t)\psi'(t)\right) = i\pi\left(\frac{24}{t^3}\psi(t) - \frac{24}{t^2}\psi'(t)\right) \\ (g_t^2\psi)^{(6)}(t) &= (g_t^2)^{(6)}(t)\psi(t) = -\pi^2\frac{6!}{t^4}\psi(t), \end{cases}$$

and then,

$$\begin{cases} L_1^1\psi = \frac{-1}{i}\left(\frac{1}{2}(f''(t))^{-1}\psi''(t)\right) = \frac{1}{4\pi}t\left(\frac{\tilde{\phi}}{\sqrt{t}}\right)'' = \frac{1}{4\pi}\left(\sqrt{t}\tilde{\phi}''(t) - \frac{\tilde{\phi}'(t)}{\sqrt{t}} + \frac{3}{4}\frac{\tilde{\phi}}{t^{3/2}}\right) \\ L_1^2\psi = \frac{1}{8i}f''(t)^{-2}\left(g_t^{(4)}(s)\psi(s) + 4g_t^{(3)}(t)\psi'(t)\right) = \frac{1}{4\pi}\left(3\left(\frac{\tilde{\phi}(t)}{\sqrt{t}}\right)' - 3\frac{\tilde{\phi}(t)}{t^{3/2}}\right) = \frac{1}{4\pi}\left(3\frac{\tilde{\phi}'(t)}{\sqrt{t}} - \frac{9}{2}\frac{\tilde{\phi}(t)}{t^{3/2}}\right) \\ L_1^3\psi = \frac{-1}{2^3 2!3!i}f''(t)^{-3}(g_t^2)^{(6)}(t)\psi(s) = \frac{1}{4\pi}\left(\frac{15}{4}\frac{\tilde{\phi}(t)}{t^{3/2}}\right), \end{cases}$$

where $\tilde{\phi}(\tau) = \phi(\tau)/\tau$. Therefore, we have

$$\begin{aligned} L^1\psi &= L_1^1\psi + L_1^2\psi + L_1^3\psi \\ &= \frac{1}{4\pi}\left(\sqrt{t}\tilde{\phi}''(t) + (3-1)\frac{\tilde{\phi}'(t)}{\sqrt{t}} + \left(\frac{3}{4} - \frac{9}{2} + \frac{15}{4}\right)\frac{\tilde{\phi}(t)}{t^{3/2}}\right) = \frac{1}{4\pi\sqrt{t}}\left(t\tilde{\phi}(t)\right)'' = \frac{1}{4\pi\sqrt{t}}\phi''(t), \end{aligned}$$

and again Theorem A.1 shows that

$$\left|\tilde{L}\phi(t) - \left(\phi(t) + \frac{\nu_s}{2c_s^2}t\phi''(t)\right)\right| \leq Ct\epsilon^{3/2}\sum_{\alpha\leq 4}\sup|\psi^{(\alpha)}(t)|.$$

2. *Proof of approximation* (34):
   Approximation (34) is evident and directly comes from (33).

3. *Proof of approximation* (35):
   Note that $\psi = (L^*\tilde{L})^{-1}\phi$ satisfies $(L^*\tilde{L})\psi = \phi$. As $\frac{\nu_s}{c_s^2} << 1$, we introduce the following asymptotic development of ψ,

$$\psi = \sum_{i=0}^{\infty} \left(\frac{\nu_s}{c_s^2}\right)^i \psi_i.$$

   From (34), it holds

$$\psi_0 + \left(\frac{\nu_s}{c_s^2}\right)((t\psi_0')' + \psi_1) + o\left(\frac{\nu_s}{c_s^2}\right) = \phi,$$

   and

$$\psi_0 = \phi, \quad \text{and} \quad \psi_1 = -\partial_t(t\partial_t\psi_0) = -\partial_t(t\partial_t\phi),$$

   and finally

$$(L^*\tilde{L})^{-1}\phi = \phi - \frac{\nu_s}{c_s^2}\partial_t(t\partial_t\phi) + o\left(\frac{\nu_s}{c_s^2}\right).$$

4.2 Imaging procedure

From the previous section, it follows that the ideal Green function, $\delta(\tau - r/c_s)/(4\pi r)$, can be approximately reconstructed from the viscous Green function, $G^s(r, t; \xi)$, by either solving the ODE

$$\phi + \frac{\nu_s}{c_s^2}\partial_t(t\partial_t\phi) = L^* G^s(r, t; \xi),$$

with $\phi = 0, t \ll 0$ or just making the approximation

$$\delta(\tau - r/c_s)/(4\pi r) \approx L^* G^s(r, t; \xi) - \frac{\nu_s}{c_s^2}\partial_t(t\partial_t L^* G^s(r, t; \xi)).$$

Once the ideal Green function $\delta(\tau - r/c_s)/(4\pi r)$ is reconstructed, one can find its source ξ using a time-reversal, a Kirchhoff or a backpropagation algorithm. See [2, 3, 4, 5].

Using the asymptotic formalism developed in [5, 6, 7], one can also find the shear modulus of the anomaly using the ideal near-field measurements which can be reconstructed from the near-field measurements in the viscous medium. The asymptotic formalism reduces the anomaly imaging problem to the detection of the location and the reconstruction of a certain polarizability tensor in the far-field and separates the scales in the near-field.

5 Numerical Illustrations

5.1 Profile of the Green function

In this section, we illustrate the profile of the Green function for different values of the power law exponent γ. We choose parameters of simulation as in the work of Bercoff *et al.* [9]: we take $\rho = 1000$, $c_s = 1$, $c_p = 40$, $\eta_p = 0$.

In figure 2, we plot the first component G_{11} observed at the point $A = \frac{1}{\sqrt{2}}(r, r, 0)$ (see first image in figure 1) with $r = 0.015$ for three different pair of values for γ

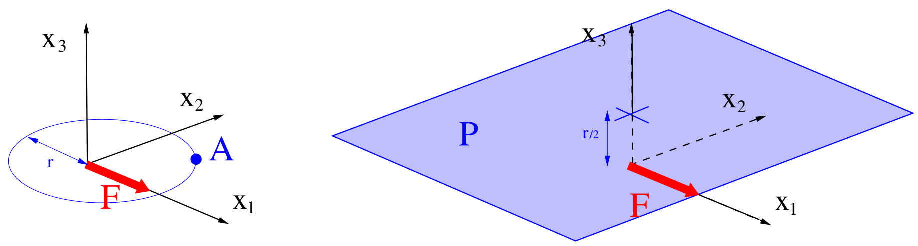

Figure 1:

and η_s. We can see that the attenuation behavior varies with respect to different choices of power law exponent γ. Moreover, we can clearly distinguish the three different terms of the Green function; *i.e.* G^s_{ij}, G^p_{ij} and G^{ps}_{ij}.

In figure 3, we plot the first component G_{11} of the green function, evaluated on the plane $P = \left\{x \in \mathbb{R}^3; x_3 = r/2\right\}$ (see second image in figure 1), and at time $t = r$. As expected, we get a diffusion of the wavefront with the increasing values of the power law exponent γ and depending on the choice of ν_s.

5.2 Approximation of attenuation operator L

Consider the limiting case when $\lambda \to +\infty$ with $\gamma = 2$. We take $\rho = 1000$, $c_s = 1$ and a concentrated force $\mathbf{F}$ of the form $\mathbf{F} = -T(t)\delta(x)\mathbf{e}_1$ where the time profile of the pulse, T, is a Gaussian with central frequency ω_0 and bandwidth ρ. Denote by $\vec{u}_{ideal}(x,t)$ the ideal response without attenuation and by $\vec{u}_{\nu_s}(x,t)$, the response associate to the attenuation coefficient ν_s. Following section 4.1, we have

$$\vec{u}_{\nu_s} \simeq L(\vec{u}_{ideal}).$$

In figure 4, we plot the first components of $t \to \vec{u}_{ideal}(A,t)$, $t \to \vec{u}_{\nu_s}(A,t)$ and $t \to L(\vec{u}_{ideal}(A,t))$ for different values of ω_0 whith $\eta_s = 0.02$. As expected, the function $t \to \vec{u}_{\nu_s}(A,t)$ and $t \to L(\vec{u}_{ideal}(A,t))$ are very similar which means that the operator L describes the effect of attenuation quite well.

Finally, in figure 5, we plot in logarithmic scale the error of approximation

$$\frac{\nu_s}{c_s^2} \to \|L\phi - \left(\phi + \frac{\nu_s}{2c_s^2} t\phi''\right)\|_\infty,$$

where $\phi(t)$ is the first component of $\vec{u}_{ideal}(x,t)$, computed at the point $x = A$ with $\omega_0 = \rho$. As expected, it clearly appears an approximation of order 2.

6 Conclusion

In this paper, we have computed the Green function in a visco-elastic medium obeying a frequency power-law. For the Voigt model, which corresponds to a quadratic frequency loss, we have used the stationary phase theorem A.1 to reconstruct the ideal Green function from the viscous one by solving an ODE. Once the ideal Green function is reconstructed, one can find its source ξ using the algorithms in [2, 3, 4, 5] such as time reversal, back-propagation, and Kirchhoff Imaging. For more general power-law media, one can recover the ideal Green function from the viscous one by inverting a fractional differential operator. This would be the subject of a forthcoming paper.

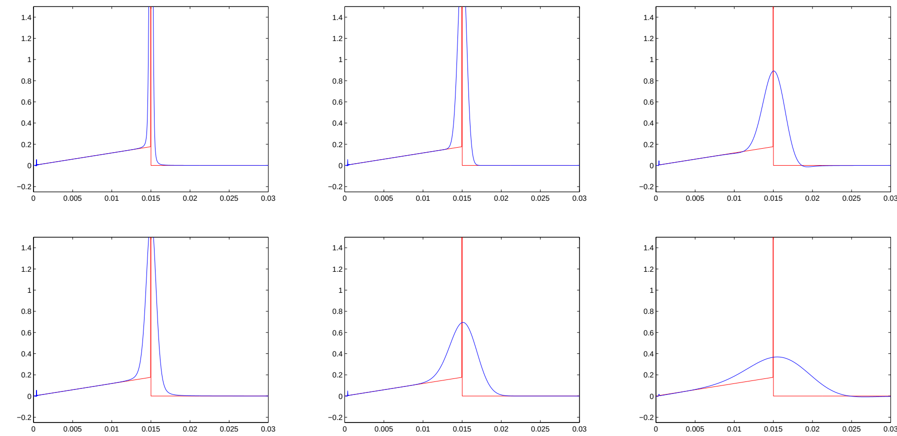

Figure 2: Temporal response $t \to G_{11}(A, t, 0)$ to a spatiotemporal delta function using a purely elastic Green's function (red line) and a viscous Green's function (blue line): First line : $\eta_s = 0.02$, Second line : $\eta_s = 0.2$; (left to right) $\gamma = 1.75$, $\gamma = 2$, $\gamma = 2.25$.

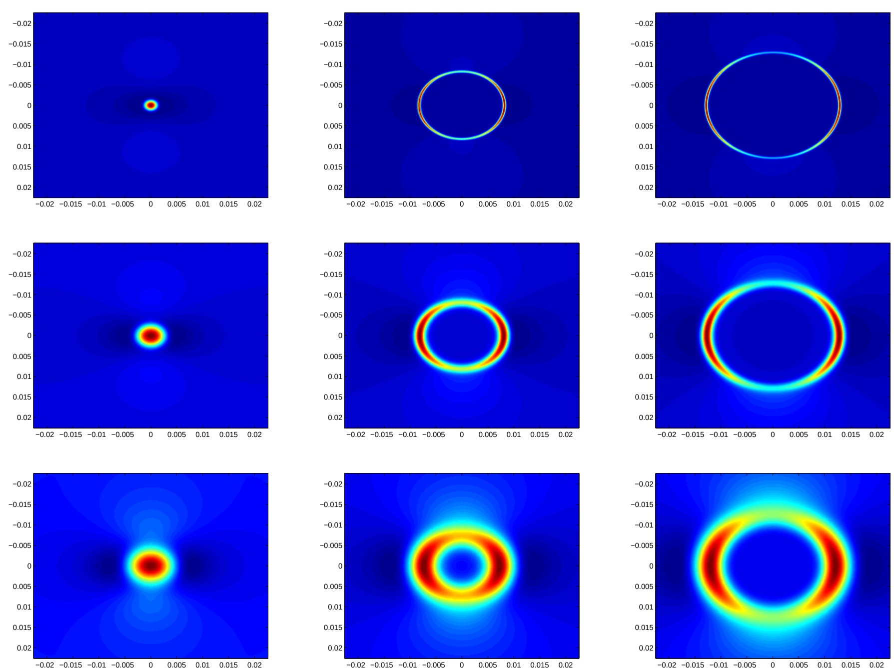

Figure 3: $2D$ spatial response $x \to G_{11}(x, t, 0)$ on the plan P to a spatiotemporal delta function with (up to down): a purely elastic Green's function, a viscous Green's function with ($\gamma = 1.75, \eta_s = 0.2$) and ($\gamma = 2, \eta_s = 0.2$). Left to right : $t = 0.0075$, $t = 0.0112$ and $t = 0.015$

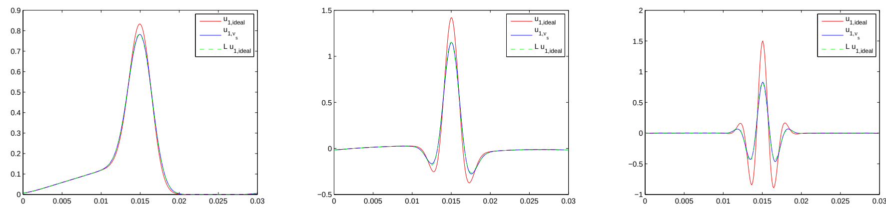

Figure 4: Comparison between $u_{1,\nu_s}(x,t)$ and $L(u_{1,ideal}(x,t))$ observed at $x = A$ with $\gamma = 2$ and $\eta_s = 0.02$; Left; $\omega_0 = 0$; Center, $\omega_0 = \rho$; Right, $\omega_0 = 2\rho$.

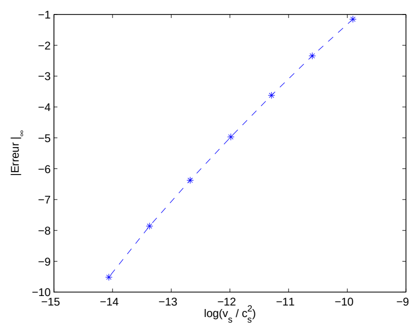

Figure 5: Approximation of operator L : Error $\frac{\nu_s}{c_s^2} \to \|L\phi - \left(\phi + \frac{\nu_s}{2c_s^2} t\phi''\right)\|_\infty$ in logarithmic scale in the case when $\phi(t) = u_{1,ideal}(A,t)$ with $\omega_0 = \rho$.

A Stationary Phase method

The proof of the following theorem is established in [12, Theorem 7.7.1].

Theorem A.1 (Stationary Phase) *Let $K \subset [0,\infty)$ be a compact set, X an open neighborhoud of K and k a positive integer. If $\psi \in C_0^{2k}(K)$, $f \in C^{3k+1}(X)$ and $Im(f) \geq 0$ in X, $Im(f(t_0)) = 0$, $f'(t_0) = 0$, $f''(t_0) \neq 0$, $f' \neq 0$ in $K \setminus \{t_0\}$ then for $\epsilon > 0$*

$$\left| \int_K \psi(t) e^{if(t)/\epsilon} dx - e^{if(t_0)/\epsilon} \left(\lambda f''(t_0)/2\pi i\right)^{-1/2} \sum_{j<k} \epsilon^j L_j \psi \right| \leq C\epsilon^k \sum_{\alpha \leq 2k} \sup |\psi^{(\alpha)}(x)|.$$

Here C is bounded when f stays in a bounded set in $C^{3k+1}(X)$ and $|t - t_0|/|f'(t)|$ has a uniform bound. With,

$$g_{t_0}(t) = f(t) - f(t_0) - \frac{1}{2} f''(t_0)(t - t_0)^2,$$

which vanishes up to third order at t_0, we have

$$L_j \psi = \sum_{\nu - \mu = j} \sum_{2\nu \geq 3\mu} i^{-j} \frac{2^{-\nu}}{\nu! \mu!} (-1)^\nu f''(t_0)^{-\nu} (g_{t_0}^\mu \psi)^{(2\nu)}(t_0).$$

□

Note that L_1 can be expressed as the sum $L_1\psi = L_1^1\psi + L_1^2\psi + L_1^3\psi$, where L_1^j is respectively associate to the pair $(\nu_j, \mu_j) = (1,0), (2,1), (3,2)$ and is identified to

$$\begin{cases} L_1^1\psi &= \frac{-1}{2i} f''(t_0)^{-1}\psi^{(2)}(t_0), \\ L_1^2\psi &= \frac{1}{2^2 2! i} f''(t_0)^{-2}(g_{t_0}u)^{(4)}(t_0) = \frac{1}{8i} f''(t_0)^{-2}\left(g_{t_0}^{(4)}(t_0)\psi(t_0) + 4g_{t_0}^{(3)}(t_0)\psi'(t_0)\right), \\ L_1^3\psi &= \frac{-1}{2^3 2! 3! i} f''(t_0)^{-3}(g_{t_0}^2\psi)^{(6)}(t_0) = \frac{-1}{2^3 2! 3! i} f''(t_0)^{-3}(g_{t_0}^2)^{(6)}(t_0)\psi(t_0). \end{cases}$$

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