A modified phase field approximation for mean curvature flow with conservation of the volume

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Abstract

This paper is concerned with the motion of a time dependent hypersurface $\partial\Omega(t)$ in $\mathbf{R}^d$ that evolves with a normal velocity

$$V_n = \kappa - \fint_{\partial\Omega(t)} \kappa \, d\sigma,$$

where κ is the mean curvature of $\partial\Omega(t)$, and $\fint_I$ stands for $\frac{1}{|I|}\int_I$. Phase field approximation of this motion leads to the nonlocal Allen–Cahn equation

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2} W'(u) + \frac{1}{\epsilon^2} \fint_Q W'(u) \, dx,$$

where Q is an open box of $\mathbf{R}^d$ containing $\partial\Omega(t)$ for all t. We propose a modified version of this equation:

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2} W'(u) + \frac{1}{\epsilon^2} \sqrt{2W(u)} \left(\int_Q \sqrt{2W(u)} \, dx \right)^{-1} \int_Q W'(u) \, dx,$$

and we show that it has better volume preserving properties than the classical one, even in the presence of an additional forcing term g.

1 Introduction and motivation

In the last decades, a lot of work has been devoted to motions of interfaces, and particularly to motion by mean curvature. Applications concern image processing (denoising, segmentation), material sciences (motion of grain boundaries in alloys, crystal growth), biology (modelling of vesicles and blood cells).

In this paper, we are interested in phase field equations as an approximation to motion by mean curvature with a forcing term and a volume constraint.

For t in $[0,T]$, let $\Omega(t)$ denote the evolution by mean curvature with a forcing term of a smooth bounded domain Ω_0 in $\mathbf{R}^d$. More precisely, the normal velocity V_n, with normal n pointing towards the exterior of $\Omega(t)$, is given at a point x of $\partial\Omega(t)$ by

$$V_n = \kappa + g, \tag{1}$$

where κ denotes the mean curvature at x, with the convention that κ is negative if the set is convex, and where $g = g(x,t)$ is a given smooth forcing term. In this work, we only consider smooth motions, which are well-defined if T is sufficiently small [2]. Singularities may develop in finite time, however, and one may need to consider evolutions in the sense of viscosity solutions [3, 8].

The evolution of $\Omega(t)$ is closely related to the minimization of the following energy:

$$J(\Omega) = \int_{\partial\Omega} 1\, d\sigma - \int_{\Omega} g\, dx.$$

Indeed, one can view (1) as a first order optimality condition for this energy. The functional J can be approximated by a Ginzburg–Landau energy [10, 9]:

$$J_\epsilon(u) = \int_{\mathbf{R}^d} \left(\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u) \right) dx - c_W \int_{\mathbf{R}^d} gu\, dx,$$

where ϵ is a small parameter, W a double well potential with wells at 0 and 1, for example $W(s) = \frac{1}{2}s^2(1-s)^2$, and where

$$c_W = \int_0^1 \sqrt{2W(s)}\, ds.$$

Modica and Mortola [10, 9] have shown the Γ-convergence of J_ϵ to $c_W J$ in $L^1(\mathbf{R}^d)$ in the absence of forcing terms (see also [4]). The extension of these results to motions with bounded forcing terms is straightforward. The corresponding Allen–Cahn equation [1], obtained as the gradient flow of J_ϵ, reads

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2}W'(u) + \frac{1}{\epsilon}c_W g. \tag{2}$$

This equation is usually solved in a fixed box Q of $\mathbf{R}^d$, which contains the motion $\Omega(t)$ for all t in $[0,T]$. Existence, uniqueness and a comparison principle have been established for this equation (see for example chapters 14 and 15 in [2]). To this equation, one usually associates the profile

$$q = \arg\min \left\{ \int_{\mathbf{R}} \left(\frac{1}{2}\gamma'^2 + W(\gamma) \right) ds \ ; \ \gamma \in V \right\},$$

where V is the space of functions in $H^1_{loc}(\mathbf{R})$ that satisfies $\gamma(-\infty) = 1$, $\gamma(+\infty) = 0$, $\gamma(0) = \frac{1}{2}$. For t in $[0, T]$, the motion $\Omega(t)$ can be approximated by that of

$$\Omega_\epsilon(t) = \left\{ x \in \mathbf{R}^d \ ; \ u_\epsilon(x,t) \geq \frac{1}{2} \right\},$$

where u_ϵ solves (2) with the initial condition

$$u_\epsilon(x, 0) = q\left(\frac{d(x, \Omega_0)}{\epsilon} \right).$$

Here $d(x, \Omega)$ denotes the signed distance of a point x to the set Ω. The convergence of $\partial\Omega_\epsilon(t)$ to $\partial\Omega(t)$ has been proved for smooth motions [7, 5] and in the general case without fattening [3, 8]. The rate of convergence has been proven to be $O(\epsilon^2 |\log \epsilon|^2)$. Actually, a formal asymptotic expansion shows that u_ϵ behaves like

$$u_\epsilon(x,t) = q\left(\frac{d\big(x, \Omega_\epsilon(t)\big)}{\epsilon} \right) + \epsilon g(x,t) \eta\left(\frac{d\big(x, \Omega_\epsilon(t)\big)}{\epsilon} \right) + O(\epsilon^2), \tag{3}$$

where η is defined as the solution in $H^2_{loc}(\mathbf{R})$, with polynomial growth, of

$$\begin{cases} \eta'' - W''(q)\eta = -c_W + q', \\ \eta(0) = 0. \end{cases} \tag{4}$$

When $g = 1$, the modified profile $s \mapsto q_\epsilon(s) = q(s) + \epsilon\eta(s)$ (see figure 1) can be evaluated at $s = \pm\infty$, where it takes the respective values

$$\epsilon \frac{c_W}{W''(1)} \quad \text{and} \quad 1 + \epsilon \frac{c_W}{W''(0)}.$$

These values correspond to the positions of the wells of a modified double well potential $W_{\epsilon,g}$, defined by $W'_{\epsilon,g} = W' - \epsilon c_W g$ and $W_{\epsilon,g}(0) = 0$.

Our main interest is the numerical simulation of interfaces $\partial\Omega(t)$ evolving from $\partial\Omega_0$ with normal velocity given by

$$V_n = \kappa + g - \fint_{\partial\Omega(t)} (\kappa + g)\, d\sigma. \tag{5}$$

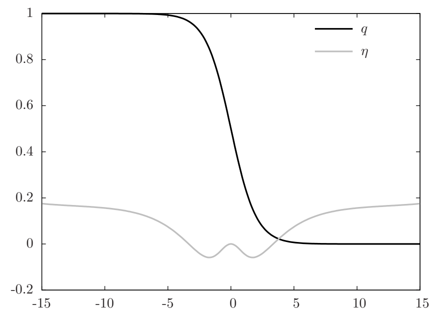

Figure 1: Profile of the function $s \mapsto q(s) + \epsilon\eta(s)$ when $W(s) = \frac{1}{2}s^2(1-s)^2$.

In this case, it is easy to see that the volume of $\Omega(t)$,

$$|\Omega(t)| = \int_{\Omega(t)} 1\,dx,$$

remains constant in time. For instance, using the results in [12], one may check that the shape derivative of the volume is zero. The usual strategy to approximate (5) is based on the remark that the mass

$$\int_{\mathbf{R}^d} u_\epsilon\,dx$$

is a good approximation of the volume $|\Omega(t)|$. One can then add to the Allen–Cahn equation an extra forcing term $\lambda(t)$, independent of x, in order to impose the conservation of mass. This leads to the following equation:

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2}\Big(W'(u) - \epsilon c_W g\Big) + \frac{1}{\epsilon} c_W \lambda.$$

The forcing term λ can be viewed as a Lagrange multiplier associated to the volume constraint. It can be determined by integrating the equation over Q, which gives

$$\lambda = \frac{\epsilon}{c_W} \fint_Q \frac{1}{\epsilon^2}\Big(W'(u) - \epsilon c_W g\Big)\,dx.$$

In the case where $g = 0$, the previous equation reduces to

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2}W'(u) + \fint_Q \frac{1}{\epsilon^2}W'(u)\,dx, \tag{6}$$

which is the classical Allen–Cahn conserved equation (see [11] and [6]). Formally, one can think of this equation as an approximation to motion by mean curvature with a modified forcing term $g_\epsilon(t)$, independent of x and given by

$$g_\epsilon = \frac{\epsilon}{c_W} \fint_Q \frac{1}{\epsilon^2} W'(u)\, dx.$$

In view of expansion (3), one expects solutions of (6) to behave like

$$u_\epsilon(x,t) = q\Big(\frac{d\big(x, \Omega_\epsilon(t)\big)}{\epsilon}\Big) + \epsilon g_\epsilon(t) \eta\Big(\frac{d\big(x, \Omega_\epsilon(t)\big)}{\epsilon}\Big) + O(\epsilon^2).$$

By integration over Q, one sees that (see proposition 1 further)

$$\int_Q u_\epsilon\, dx = |\Omega_\epsilon(t)| + \epsilon g_\epsilon \int_Q \eta\Big(\frac{d\big(x, \Omega_\epsilon(t)\big)}{\epsilon}\Big)\, dx + O(\epsilon^2).$$

As the mass of u_ϵ is conserved, as $g_\epsilon = O(1)$, and as

$$\int_Q \eta\Big(\frac{d\big(x, \Omega_\epsilon(t)\big)}{\epsilon}\Big)\, dx = O(1),$$

given the values of η at $\pm\infty$, one expects that

$$|\Omega_\epsilon(t)| = |\Omega_0| + O(\epsilon)$$

only. This is not satisfactory for many applications, where loss of volume during numerical computations strongly affects the dynamics.

The aim of this work is to propose another phase field model that has better volume conservation properties than the conserved Allen–Cahn equation. The paper is organised as follow:

In section 2, we introduce the following phase field approximation for mean curvature flow with a forcing term:

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2}\Big(W'(u) - \epsilon\sqrt{2W(u)}g\Big). \tag{7}$$

It can be seen as the gradient flow of

$$\tilde{J}_\epsilon(u) = \int_{\mathbf{R}^d} \Big(\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u)\Big)\, dx - \int_{\mathbf{R}^d} G(u) g\, dx,$$

with

$$G(s) = \int_0^s \sqrt{2W(t)}\, dt.$$

We first explain via a formal asymptotic analysis why solutions of (7) are expected to take the form

$$u_\epsilon(x,t) = q\Big(\frac{d\big(x, \Omega_\epsilon(t)\big)}{\epsilon}\Big) + O(\epsilon^2). \tag{8}$$

Then, following an argument due to [5], we rigorously prove the convergence of this phase field equation to the motion (1).

In section 3, we consider the evolution $\Omega(t)$ of a smooth bounded domain Ω_0 according to

$$V_n = \kappa + g - \fint_{\partial\Omega} (\kappa + g)\, d\sigma.$$

Let u_ϵ be a solution of

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2}\Big(W'(u) - \epsilon\sqrt{2W(u)}g\Big) + \frac{1}{\epsilon^2}\frac{\sqrt{2W(u)}}{\int_{\mathbf{R}^d}\sqrt{2W(u)}\,dx}\int_{\mathbf{R}^d}\Big(W'(u) - \epsilon\sqrt{2W(u)}g\Big)\,dx. \quad (9)$$

We show that if u_ϵ behaves like in expansion (8), then for all t in $[0,T]$,

$$\begin{aligned} |\Omega_0| &= \int_{\mathbf{R}^d} u_\epsilon(x,0)\,dx + O(\epsilon^2) \\ &= \int_{\mathbf{R}^d} u_\epsilon(x,t)\,dx + O(\epsilon^2) \\ &= |\Omega_\epsilon(t)| + O(\epsilon^2), \end{aligned}$$

while the solutions of (2) may only conserve volume up to order ϵ.

In section 4, we present numerical evidence for the above claims, which show that the modified phase field model (9) has indeed better volume preservation properties.

2 A modified reaction–diffusion equation for mean curvature flow with a forcing term

Let $\partial\Omega(t)$ denote an evolving hypersurface of codimension 1 in $\mathbf{R}^d$, with velocity law $V_n = \kappa + g$. This motion can be interpreted as the energy gradient of

$$J(\Omega) = \int_{\partial\Omega} 1\, ds - \int_\Omega g\, dx.$$

Let W be a bounded double well potential. In this whole section, we will for convenience use a potential with wells at -1 and 1, for example $W(s) = \min\{\frac{1}{2}(1-s^2)^2, M\}$, where M is a given positive constant. Our strategy is to introduce a modified Ginzburg–Landau energy $\tilde{J}_\epsilon$ defined on $L^1(\mathbf{R}^d)$ by

$$\tilde{J}_\epsilon(u) = \begin{cases} \displaystyle\int_{\mathbf{R}^d}\Big(\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u)\Big)\,dx - \int_{\mathbf{R}^d} G(u) g\, dx & \text{if } u \in H^1(\mathbf{R}^d), \\ +\infty & \text{otherwise,} \end{cases}$$

with

$$G(s) = \int_0^s \sqrt{2W(t)}\,dt.$$

The function G is Lipschizt continuous. For g in $L^\infty(\mathbf{R}^d)$, the term $u \mapsto \int_{\mathbf{R}^d} G(u)g\,dx$ acts as a continuous perturbation in the $L^1(\mathbf{R}^d)$ topology of the classical Modica–Mortola energy. The stability of Γ-convergence with respect to continuous perturbations allows us to extend the Modica–Mortola result to the case at hand, and show that $\tilde{J}_\epsilon$ Γ-converges to $c_W J$. The gradient flow of $\tilde{J}_\epsilon$ should then provide a mean to approximate the motion of $\partial\Omega(t)$ via the resolution of the reaction–diffusion equation (7).

Remark 1. *In the simplest case where $g = 1$, equations* (2) *and* (7) *can be expressed as Allen–Cahn equations with particular double well potentials respectively equal to $W_{1,\epsilon}(s) = W(s) + \epsilon c_W s$, and $W_{2,\epsilon}(s) = W(s) + \epsilon G(s)$. These two potentials are related through the position and height of their wells, which are asymptotically equal as $\epsilon \to 0$. This explains why we expect that* (2) *and* (7) *converge to the same motion.*

2.1 Formal asymptotics for the modified Allen–Cahn equation

We denote by u_ϵ the solution of equation (7):

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2} W'(u) + \frac{1}{\epsilon}\sqrt{2W(u)}g,$$

with initial condition

$$u(x,0) = q\left(\frac{d(x,\Omega_0)}{\epsilon}\right).$$

Our aim is to propose an asymptotic analysis of u_ϵ in the simplest two-dimensional radial case. Using polar coordinates (r,θ), we consider a forcing term g which does not depend on θ: $g = g(r,t)$. The initial set Ω_0 is taken as a disk of radius 1:

$$\Omega_0 = \{(r,\theta) \in [0,+\infty) \times [0,2\pi)\ ;\ r \le 1\}.$$

Let $\Omega(t)$ be the mean curvature flow evolving from Ω_0 according to the law $V_n = \kappa + g$. It is well known that in this case, $\Omega(t)$ remains a circle for all t (recall that the forcing term g is supposed to be radial). We will denote by $R(t)$ the radius of $\Omega(t)$, solution of the following ODE:

$$R' + \frac{1}{R} = g(R,t),$$

with initial condition $R(0) = 1$. In this simple case, the solution u_ϵ is also radial and depends only on r. It satisfies

$$\partial_t u_\epsilon - \frac{1}{r}\partial_r(r\partial_r u_\epsilon) + \frac{1}{\epsilon^2}W'(u_\epsilon) - \frac{1}{\epsilon}\sqrt{2W(u_\epsilon)}g = 0.$$

As u_ϵ is radial, every of its level sets is circular, and we denote by $R_\epsilon(t)$ the radius of $\{u_\epsilon(r,t)=\frac{1}{2}\}$. We thus have $R_\epsilon(0)=1$ and $u_\epsilon(R_\epsilon,t)=\frac{1}{2}$. We introduce the classical stretched variable $y=\frac{r-R_\epsilon}{\epsilon}$ (see [5]), and we define U_ϵ by

$$U_\epsilon(y,t)=u_\epsilon(R_\epsilon+\epsilon y,t).$$

This new function U_ϵ satisfies

$$\partial_t U_\epsilon-\frac{1}{\epsilon}R'_\epsilon\partial_y U_\epsilon-\frac{1}{\epsilon r}\partial_y U_\epsilon-\frac{1}{\epsilon^2}\partial_{yy}U_\epsilon+\frac{1}{\epsilon^2}W'(U_\epsilon)-\frac{1}{\epsilon}\sqrt{2W(U_\epsilon)}g=0. \tag{10}$$

We now consider asymptotic developments of U_ϵ and R_ϵ as follow:

$$U_\epsilon(y,t)=\sum_{i=0}^{+\infty}\epsilon^i U_i(y,t),\quad R_\epsilon(t)=\sum_{i=0}^{+\infty}\epsilon^i R_i(t),$$

with $U_0(0,t)=\frac{1}{2}$, $R_0(0)=R(0)$, and $U_i(0,t)=0$, $R_i(0)=0$ for all $i\geq 1$. We have

$$\begin{aligned}
\frac{1}{r}&=\left(\epsilon y+\sum_{i=0}^{+\infty}\epsilon^i R_i\right)^{-1}=\frac{1}{R_0}-\epsilon\frac{y+R_1}{{R_0}^2}+O(\epsilon^2),\\
W'(U_\epsilon)&=W'(U_0)+\epsilon W''(U_0)U_1+\epsilon^2\Big(W'''(U_0)U_1+W''(U_0)U_2\Big)+O(\epsilon^3),\\
\sqrt{2W(U_\epsilon)}&=\sqrt{2W(U_0)}+\epsilon\frac{W'(U_0)}{\sqrt{2W(U_0)}}U_1+O(\epsilon^2),\\
g(r,t)&=g\left(\epsilon y+\sum_{i=0}^{+\infty}\epsilon^i R_i,t\right)=g(R_0,t)+\epsilon\partial_r g(R_0,t)(y+R_1)+O(\epsilon^2).
\end{aligned}$$

Using these equalities, (10) rewrites

$$\begin{aligned}
0=&\frac{1}{\epsilon^2}\Big(\partial_{yy}U_0-W'(U_0)\Big)\\
&+\frac{1}{\epsilon}\left(\partial_{yy}U_1-W''(U_0)U_1+\partial_y U_0\left(R'_0+\frac{1}{R_0}\right)+\sqrt{2W(U_0)}g(R_0,t)\right)\\
&-\partial_t U_0+R'_1\partial_y U_0+R'_0\partial_y U_1+\partial_{yy}U_2+\frac{1}{R_0}\partial_y U_1-\frac{y+R_1}{{R_0}^2}\partial_y U_0\\
&+g(R_0,t)\frac{W'(U_0)}{\sqrt{2W(U_0)}}U_1+\partial_r g(R_0,t)(y+R_1)\sqrt{2W(U_0)}\\
&-W'''(U_0)U_1-W''(U_0)U_2\\
&+O(\epsilon).
\end{aligned} \tag{11}$$

Following powers of ϵ, we will now identify each term to zero.

Terms in ϵ^{-2}. The first term U_0 satisfies $\partial_{yy} U_0 = W'(U_0)$ with initial condition $U_0(0,t) = \frac{1}{2}$ for all t in $[0,T]$. It can thus be identified to the profile q:

$$\forall y \in \mathbf{R}, \quad \forall t \in [0,T], \quad U_0(y,t) = q(y).$$

Terms in ϵ^{-1}. Knowing by definition of the profile that $q' = -\sqrt{2W(q)}$, it follows from $U_0(.,t) = q(.)$ that $\partial_y U_0 = -\sqrt{2W(U_0)}$. Equation (11) then gives

$$\partial_{yy} U_1 - W''(U_0) U_1 = -\partial_y U_0 \Big(R_0' + \frac{1}{R_0} - g(R_0,t) \Big).$$

Multiplying this equality by $\partial_y U_0$ and integrating over $\mathbf{R}$, we get

$$\begin{aligned}\Big(R_0' + \frac{1}{R_0} - g(R_0,t) \Big) \int_{\mathbf{R}} (\partial_y U_0)^2 \, dy &= -\int_{\mathbf{R}} \Big(\partial_{yy} U_1 - W''(U_0) U_1 \Big) \partial_y U_0 \, dy \\ &= -\int_{\mathbf{R}} \partial_y \Big(\partial_{yy} U_0 - W'(U_0) \Big) U_1 \, dy \\ &= 0.\end{aligned}$$

As $\int_{\mathbf{R}} (\partial_y U_0)^2 \, dy$ is strictly positive, we get the following equation on R_0:

$$R_0' + \frac{1}{R_0} = g(R_0,t),$$

with initial condition $R_0(0) = R(0)$. Hence R_0 can be identified to R since they both satisfy the same ODE with the same initial datum. It follows from (11) that U_1 is solution of

$$\partial_{yy} U_1 - W''(U_0) U_1 = 0.$$

We then know (see section 3 in [5]) that there exists $\alpha(t) \in \mathbf{R}$ such that $U_1(.,t) = \alpha(t) q'(.)$. Indeed, the kernel of the operator $A : H^1(\mathbf{R}) \to H^{-1}(\mathbf{R})$ defined by $A\zeta = \zeta'' - W''(q)\zeta$ can be identified to $\mathrm{span}(q')$. Using $U_1(0,t) = 0$, we conclude that $\alpha(t) = 0$ for all t, so that

$$\forall y \in \mathbf{R}, \quad \forall t \in [0,T], \quad U_1(y,t) = 0.$$

Terms in ϵ^0. Using $U_1 = 0$ and $\partial_t U_0 = 0$, we get from (11) that

$$\partial_{yy} U_2 - W''(U_0) U_2 = \partial_y U_0 \Big(\frac{y + R_1}{{R_0}^2} - R_1' + \partial_r g(R_0,t)(y + R_1) \Big).$$

Multiplying by $\partial_y U_0$ and integrating over $\mathbf{R}$, we have

$$\begin{aligned}&\Big(R_1' - \frac{R_1}{{R_0}^2} - \partial_r g(R_0,t) R_1 \Big) \int_{\mathbf{R}} (\partial_y U_0)^2 \, dy \\ &= -\int_{\mathbf{R}} \Big(\partial_{yy} U_2 - W''(U_0) U_2 \Big) \partial_y U_0 \, dy + \Big(\frac{1}{{R_0}^2} + \partial_r g(R_0,t) \Big) \int_{\mathbf{R}} y (\partial_y U_0)^2 \, dy.\end{aligned}$$

The first term in the right member vanishes as previously for U_1. The second term also vanishes since $\partial_y U_0(.,t) = q'(.)$ is even with our choice of W. We deduce that R_1 is solution of

$$R_1' = \left(\frac{1}{{R_0}^2} + \partial_r g(R_0,t)\right)R_1,$$

with initial condition $R_1(0) = 0$. Hence $R_1(t) = 0$ for all t in $[0,T]$. Finally, U_2 is obtained as the solution of

$$\partial_{yy}U_2 - W''(U_0)U_2 = y\partial_y U_0\left(\frac{1}{{R_0}^2} + \partial_r g(R_0,t)\right).$$

Introducing the solution ξ in $H^1(\mathbf{R})$ of

$$\xi''(y) - W''\big(q(y)\big)\xi(y) = yq'(y)$$

with $\xi(0) = 0$, we can express U_2:

$$\forall y \in \mathbf{R}, \quad \forall t \in [0,T], \quad U_2(y,t) = \xi(y)\left(\frac{1}{{R_0}^2} + \partial_r g(R_0,t)\right).$$

We finally conclude from this formal asymptotic analysis that u_ϵ, solution of (7), is expected of the form

$$u_\epsilon(x,t) = q\left(\frac{d\big(x,\Omega_\epsilon(t)\big)}{\epsilon}\right) + \epsilon^2\left(\frac{1}{{R_0}^2} + \partial_r g(R_0,t)\right)\xi\left(\frac{d\big(x,\Omega_\epsilon(t)\big)}{\epsilon}\right) + O(\epsilon^3), \quad (12)$$

where $\Omega_\epsilon(t)$ converge to $\Omega(t)$ in $O(\epsilon^2)$.

2.2 Proof of convergence for the modified phase field model

In this section, we closely follow the work of [5] to prove the following theorem:

Theorem 1. *Let $\Omega(t)$ be a regular mean curvature flow with a forcing term g that satisfies*

$$g(.,t) \in W^{3,\infty}(\mathbf{R}^d), \quad \partial_t g \in W^{1,\infty}\big(\mathbf{R}^d \times (0,T)\big). \quad (13)$$

Given $\epsilon > 0$, let u_ϵ be solution of (7):

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2}W'(u) + \frac{1}{\epsilon}\sqrt{2W(u)}g,$$

and let $\partial\Omega_\epsilon(t) = \left\{x \in \mathbf{R}^d \ ;\ u_\epsilon(x,t) = \frac{1}{2}\right\}$. Assume that the potential W is given by $W(s) = \frac{1}{2}(1-s^2)^2$. Then there exist $\epsilon_0 > 0$ and a constant C depending only on T such that for all ϵ in $(0,\epsilon_0]$, the following estimate holds:

$$\forall t \in [0,T], \quad \partial\Omega_\epsilon(t) \subseteq \left\{x \in \mathbf{R}^d \ ;\ \operatorname{dist}\big(x,\partial\Omega(t)\big) \le C\epsilon^2|\log\epsilon|^2\right\}. \quad (14)$$

Notations and assumptions. Let $T > 0$. For all t in $[0, T]$, let $\Omega(t)$ be a mean curvature flow with a forcing term g that satisfies (13). In the sequel, we will for convenience identify the signed distance to $\Omega(t)$ to a function $d\colon \mathbf{R}^d \times [0, T] \to \mathbf{R}$ defined by

$$d(x,t) = d\big(x, \Omega(t)\big) = \begin{cases} \operatorname{dist}\big(x, \Omega(t)\big) & \text{if } x \in \mathbf{R}^d \setminus \Omega(t), \\ 0 & \text{if } x \in \partial\Omega(t), \\ -\operatorname{dist}\big(x, \Omega(t)\big) & \text{if } x \in \Omega(t). \end{cases}$$

We assume that $\partial\Omega(t)$ is smooth enough so that d satisfies

$$d,\ \partial_t d,\ \partial_t \partial_{xx} d \in C^0(\bar{\Lambda}), \tag{15}$$

where $\bar{\Lambda}$ is a tubular neighborhood of $\partial\Omega(t)$. We assume that $\partial\Omega(t)$ is oriented by the outward normal vector n defined at a point x of $\partial\Omega(t)$ by $n(x,t) = \nabla d(x,t)$. We denote by $\kappa_1, \ldots, \kappa_{d-1}$ the principal curvatures of $\partial\Omega(t)$, and we set

$$\kappa(x,t) = \sum_{i=1}^{d-1} \kappa_i(x,t), \quad h(x,t) = \sum_{i=1}^{d-1} \kappa_i^2(x,t).$$

We choose κ to be negative for convex balls. The evolution of $\partial\Omega(t)$ is defined by $V_n(x,t) = \kappa(x,t) + g(x,t)$ for all (x,t) in $\partial\Omega(t) \times [0,T]$, where V_n denote the normal velocity.

Given $D > 0$, we define a tubular neighborhood $\Lambda(t)$ of $\partial\Omega(t)$ by

$$\Lambda(t) = \Big\{ x \in \mathbf{R}^d \ ;\ |d(x,t)| \le D \Big\}, \tag{16}$$

and we set

$$\Lambda = \bigcup_{t \in [0,T]} \Lambda(t) \times \{t\}.$$

If D is sufficiently small, one can associate to any point (x,t) of Λ a unique projection $s(x,t)$ on $\partial\Omega(t)$ such that

$$\operatorname{dist}\big(s(x,t), x\big) = |d(x,t)|.$$

For any scalar or vector function f defined on $\partial\Omega(t)$, we denote by $\bar{f}$ its extension on Λ, defined by $\bar{f}(x,t) = f\big(s(x,t), t\big)$. If f is real-valued, then we clearly have $\nabla d \cdot \nabla \bar{f} = 0$ on Λ. It follows from (15) that

$$\|\bar{h}\|_{L^\infty(\Lambda)},\ \|\partial_t \bar{h}\|_{L^\infty(\Lambda)},\ \|\nabla \bar{h}\|_{L^\infty(\Lambda)},\ \|\Delta \bar{h}\|_{L^\infty(\Lambda)} < +\infty. \tag{17}$$

Moreover, geometric properties of the distance function d imply

$$\Delta d(x,t) = \sum_{i=1}^{d-1} \frac{-\bar\kappa_i(x,t)}{1-d(x,t)\bar\kappa_i(x,t)} = -\bar\kappa(x,t) - d(x,t)\bar h(x,t) + O\big(d(x,t)^2\big),$$
$$\partial_t d(x,t) = -\bar V_n(x,t) = -\bar\kappa(x,t) - \bar g(x,t).$$

These estimates show that the motion of $\partial\Omega(t)$ can be described by an equation on d inside the whole Λ (see [2]):

$$\forall (x,t)\in\Lambda, \quad \partial_t d(x,t) - \Delta d(x,t) = -\bar g(x,t) + d(x,t)\bar h(x,t) + O\big(d(x,t)^2\big). \quad (18)$$

We denote by q the profile function associated with the double well potential W:

$$q = \arg\min_{\zeta} \left\{ \int_{\mathbf{R}} \left(\frac{1}{2}\zeta'^2 + W(\zeta)\right) ds \ ; \ \zeta\in H^1_{loc}(\mathbf{R}), \ \lim_{x\to\pm\infty} \zeta = \mp 1, \ \zeta(0)=0 \right\}.$$

The Euler equation for this problem writes $q'' = W'(q)$. More precisely, as W is smooth, q is strictly decreasing and we have $q' = -\sqrt{2W(q)}$. If W is defined by $W(s) = \frac{1}{2}(1-s^2)^2$, q is given by $q(s) = -\tanh(s)$. In this case, there exists a positive constant C such that $|q-1| \le -Cq'$.

Let ξ in $H^2(\mathbf{R})$ be solution of equation

$$\xi''(s) - W''\big(q(s)\big)\xi(s) = sq'(s) \quad (19)$$

with initial condition $\xi(0)=0$. Existence and uniqueness results for this equation may be found in section 3 of [5], along with the following estimate:

$$|\xi(s)|, \ |\xi'(s)| \le -C(1+s^2)q'(s). \quad (20)$$

Comparison lemma. Our proof of convergence relies on the following lemma:

Lemma 1. *Let $\epsilon>0$, and let u and v in $L^2\big(0,T;H^2(\mathbf{R}^d)\big)\cap H^1\big(0,T;L^2(\mathbf{R}^d)\big)$ be such that*

$$\partial_t u - \Delta u + \frac{1}{\epsilon^2}W'(u) - \frac{1}{\epsilon}\sqrt{2W(u)}g \ge \partial_t v - \Delta v + \frac{1}{\epsilon^2}W'(v) - \frac{1}{\epsilon}\sqrt{2W(v)}g \quad (21)$$

in $\mathbf{R}^d\times(0,T)$, and $u(x,0)\ge v(x,0)$ for x in $\mathbf{R}^d$. Then $u\ge v$ in $\mathbf{R}^d\times(0,T)$.

Proof. Let $e=\max(v-u,0)$. Multiplying (21) by e and integrating over $\mathbf{R}^d$, we get

$$\begin{aligned}\frac{d}{dt}\|e(.,t)\|^2_{L^2(\mathbf{R}^d)} &\le \frac{2}{\epsilon^2}\left\langle W'(u)-W'(v)\,,\,e\right\rangle_{L^2(\mathbf{R}^d)} \\ &\qquad - \frac{2}{\epsilon}\left\langle \Big(\sqrt{2W(u)}-\sqrt{2W(v)}\Big)g\,,\,e\right\rangle_{L^2(\mathbf{R}^d)} \\ &\le \frac{2}{\epsilon^2}\left\langle W'_{g,\epsilon}(u)-W'_{g,\epsilon}(v)\,,\,e\right\rangle_{L^2(\mathbf{R}^d)},\end{aligned}$$

where $W_{g,\epsilon}$ is defined by

$$W'_{g,\epsilon}(s) = W'(s) - \epsilon\sqrt{2W(s)}g, \quad W(0) = 0.$$

The idea is then to decompose $W'_{g,\epsilon}$ under the form

$$W'_{g,\epsilon} = W'_{g,\epsilon,L} + W'_{g,\epsilon,I},$$

where $W'_{g,\epsilon,L}$ is Lipschitz continuous on $\mathbf{R}$, and $W'_{g,\epsilon,I}$ is nondecreasing. More precisely, when $W(s) = \frac{1}{2}(1-s^2)^2$, we can use

$$W'_{g,\epsilon,L}(s) = W'_{g,\epsilon}(s)\chi_{[-1,1]}(s) \quad \text{and} \quad W'_{g,\epsilon,I}(s) = W'_{g,\epsilon}(s)\big(1-\chi_{[-1,1]}(s)\big),$$

which satisfy the previous assumption if $\|g\|_{L^\infty} \leq \frac{2}{\epsilon}$. Then, noticing that $e(x,0) = 0$ by assumption, we obtain

$$\begin{aligned}\|e(.,t)\|^2_{L^2(\mathbf{R}^d)} &\leq \frac{2}{\epsilon^2}\int_0^t \left|\big\langle W'_{g,\epsilon}(u) - W'_{g,\epsilon}(v)\,,\, e(.,\tau)\big\rangle_{L^2(\mathbf{R}^d)}\right| d\tau \\ &\leq \frac{2}{\epsilon^2}\sup_{x\in\mathbf{R}^d}\big\{\mathrm{lip}(W'_{g,\epsilon,L})\big\}\int_0^t \|e(.,\tau)\|^2_{L^2(\mathbf{R}^d)}\,d\tau.\end{aligned}$$

Note that the sup is bounded just as $\|g\|_{L^\infty}$. Gronwall's lemma implies that for almost every t in $(0,T)$, $\|e(.,t)\|_{L^2(\mathbf{R}^d)} = 0$, and $e = 0$ almost everywhere in $\mathbf{R}^d \times (0,T)$. □

Construction of a subsolution. Using our previous asymptotic expansion of u_ϵ, we now build a subsolution to problem (7). Let $\delta \geq 3$ be a fixed integer. For all $\epsilon > 0$, we set $s_\epsilon = \delta|\log\epsilon|$. Since $q(s) = -\tanh(s)$, we have

$$q(s_\epsilon) = -1 + \frac{2\epsilon^{2\delta}}{1+\epsilon^{2\delta}} = -1 + O(\epsilon^{2\delta}), \quad q'(s_\epsilon) = -\big(1 - q(s_\epsilon)^2\big) = O(\epsilon^{2\delta}),$$

and it follows from (20) that

$$|\xi(s_\epsilon)| = O(\epsilon^{2\delta}|\log\epsilon|^2), \quad |\xi'(s_\epsilon)| = O(\epsilon^{2\delta}|\log\epsilon|^2).$$

We define two auxiliary functions q_ϵ and ξ_ϵ by

$$q_\epsilon(s) = \begin{cases} q(s) & \text{if } 0 \leq s \leq s_\epsilon, \\ P_q(s) & \text{if } s_\epsilon \leq s \leq 2s_\epsilon, \\ -1 & \text{if } s > 2s_\epsilon, \\ -q_\epsilon(-s) & \text{if } s < 0, \end{cases}$$

and

$$\xi_\epsilon(s) = \begin{cases} \xi(s) & \text{if } 0 \le s \le s_\epsilon, \\ P_\xi(s) & \text{if } s_\epsilon \le s \le 2s_\epsilon, \\ 0 & \text{if } s > 2s_\epsilon, \\ -\xi_\epsilon(-s) & \text{if } s < 0, \end{cases}$$

where P_q and P_ξ are polynomials of degree 3 defined in such a way that q_ϵ and ξ_ϵ are in $C^1(\mathbf{R})$. It follows that

$$\|P_q + 1\|_{L^\infty(I_\epsilon)} + s_\epsilon \|P_q'\|_{L^\infty(I_\epsilon)} + s_\epsilon^2 \|P_q''\|_{L^\infty(I_\epsilon)} \le C\Big(|q(s_\epsilon) + 1| + s_\epsilon |q'(s_\epsilon)|\Big),$$
$$\|P_\xi\|_{L^\infty(I_\epsilon)} + s_\epsilon \|P_\xi'\|_{L^\infty(I_\epsilon)} + s_\epsilon^2 \|P_\xi''\|_{L^\infty(I_\epsilon)} \le C\Big(|q(s_\epsilon)| + s_\epsilon |q'(s_\epsilon)|\Big),$$

with $I_\epsilon = [s_\epsilon, 2s\epsilon]$. Then we easily check that

$$\|q_\epsilon - q\|_{L^\infty(\mathbf{R})} = o(\epsilon^{2\delta-1}), \quad \|\xi_\epsilon - \xi\|_{L^\infty(\mathbf{R})} = o(\epsilon^{2\delta-1}),$$

together with

$$q_\epsilon'' - W'(q_\epsilon) = o(\epsilon^{2\delta-1}), \quad q_\epsilon' + \sqrt{2W(q_\epsilon)} = o(\epsilon^{2\delta-1}), \tag{22}$$

and

$$\xi_\epsilon'' - W''(q_\epsilon)\xi_\epsilon - sq_\epsilon' = o(\epsilon^{2\delta-1}).$$

For $\epsilon > 0$, we introduce the modified distance function d_ϵ^- defined by:

$$\forall (x,t) \in \mathbf{R}^d \times [0,T], \quad d_\epsilon^-(x,t) = d(x,t) + c_1(t)\epsilon^2 |\log \epsilon|^2,$$

where c_1 is a positive continuous function, independent of ϵ, that will be determined later. For t in $[0,T]$, we introduce the sets

$$\Lambda_\epsilon^-(t) = \Big\{x \in \mathbf{R}^d \ ; \ |d_\epsilon^-(x,t)| < 2\delta\epsilon|\log\epsilon|\Big\}$$

and

$$\Lambda_\epsilon^- = \bigcup_{t\in[0,T]} \Lambda_\epsilon^-(t) \times \{t\}.$$

It is then possible to find $\epsilon_0 > 0$ depending only on δ, c_1, D, such that

$$\forall \epsilon \le \epsilon_0, \quad \forall t \in [0,T], \quad \Lambda_\epsilon^-(t) \subset \Lambda(t), \tag{23}$$

where $\Lambda(t)$ is the tubular neighborhood defined in (16). In particular, we see that

$$\forall (x,t) \in \Lambda_\epsilon^-, \quad d(x,t) = O(\epsilon|\log\epsilon|).$$

Noticing that $\nabla d_\epsilon^- = \nabla d$ and $\nabla d_\epsilon^- \cdot \nabla \bar{h} = 0$ in Λ_ϵ^-, it follows from (18) that

$$\begin{aligned}\partial_t d_\epsilon^- - \Delta d_\epsilon^- &= \partial_t d - \Delta d + c_1' \epsilon^2 |\log \epsilon|^2 \\ &= -\bar{g} + d_\epsilon^- \bar{h} + (c_1' - c_1 \bar{h})\epsilon^2 |\log \epsilon|^2 + O(\epsilon^2 |\log \epsilon|^2). \end{aligned} \tag{24}$$

Setting $y = \frac{d_\epsilon^-}{\epsilon}$, we define v_ϵ^- on $\mathbf{R}^d \times [0,T]$ by

$$v_\epsilon^- = \begin{cases} q_\epsilon(y) + \epsilon^2(\bar{h} + \nabla d \cdot \nabla g)\xi_\epsilon(y) - c_2 \epsilon^3 |\log \epsilon|^2 & \text{in } \Lambda_\epsilon^-, \\ -1 - c_2 \epsilon^3 |\log \epsilon|^2 & \text{in } \{d_\epsilon^- \geq 2\delta\epsilon |\log \epsilon|\}, \\ +1 - c_2 \epsilon^3 |\log \epsilon|^2 & \text{in } \{d_\epsilon^- \leq -2\delta\epsilon |\log \epsilon|\}, \end{cases}$$

where c_2 is a constant independent of ϵ that we will be determined later. In view of (15), we easily check that v_ϵ^- belongs to $L^2\big(0,T; H^1_{loc}(\mathbf{R}^d)\big) \cap H^1\big(0,T; L^2_{loc}(\mathbf{R}^d)\big)$. Our goal is to show that v_ϵ^- is a subsolution of (7).

Let u_ϵ be solution of (7). We will first prove that

$$\forall x \in \mathbf{R}^d, \quad v_\epsilon^-(x,0) \leq u_\epsilon(x,0). \tag{25}$$

To this end, we introduce w_ϵ defined by

$$w_\epsilon = q(y) + \epsilon^2(\bar{h} + \nabla d \cdot \nabla g)\xi(y) - \frac{c_2}{2}\epsilon^3 |\log \epsilon|^2,$$

and we note that when ϵ is sufficiently small,

$$\begin{aligned} v_\epsilon(x,0) &\leq w_\epsilon(x,0) - \frac{c_2}{2}\epsilon^3 |\log \epsilon|^2 + o(\epsilon^{2\delta - 1}) \\ &\leq w_\epsilon(x,0), \end{aligned}$$

so that (25) follows from showing that

$$\begin{aligned} w_\epsilon(x,0) - u_\epsilon(x,0) &= q\big(y(x,0)\big) - q\left(\frac{d(x,0)}{\epsilon}\right) \\ &\quad + \epsilon^2\Big(\bar{h}(x,0) + \nabla d(x,0) \cdot \nabla g(x,0)\Big)\xi\big(y(x,0)\big) - \frac{c_2}{2}\epsilon^3 |\log \epsilon|^2 \end{aligned}$$

is non-positive. We define for convenience

$$\begin{aligned} I_1 &= q\big(y(x,0)\big) - q\left(\frac{d(x,0)}{\epsilon}\right), \\ I_2 &= \epsilon^2\Big(\bar{h}(x,0) + \nabla d(x,0) \cdot \nabla g(x,0)\Big)\xi\big(y(x,0)\big). \end{aligned}$$

The following lemma is proved in section 6 of [5] (recall that q' is negative).

Lemma 2. *Let* $z = \frac{d(x,0)}{\epsilon}$, *and let* $y = \frac{d_\epsilon^-(x,0)}{\epsilon} = z + c_1(0)\epsilon|\log\epsilon|^2$. *Then for* ϵ *sufficiently small,*

$$2q'(y) \le q'(s) \le \frac{1}{2}q'(y)$$

for all s *in* $[z, y]$.

This lemma implies that

$$\begin{aligned} I_1 &= q\left(\frac{d_\epsilon^-(x,0)}{\epsilon}\right) - q\left(\frac{d_\epsilon^-(x,0) - c_1(0)\epsilon^2|\log\epsilon|^2}{\epsilon}\right) \\ &\le \frac{1}{2}q'\left(\frac{d_\epsilon^-(x,0)}{\epsilon}\right)c_1(0)\epsilon|\log\epsilon|^2, \end{aligned}$$

and using (19), it follows that

$$I_2 \le -Kc\epsilon^2\big(1 + y(x,0)^2\big)q'\big(y(x,0)\big),$$

where $K = \|h(.,0)\|_{L^\infty(\Lambda(0))} + \|\nabla g(.,0)\|_{L^\infty(\mathbf{R}^d)}$. We then distinguish two cases. If $|y| > |\log\epsilon|$, then $(1+y^2)|q'(y)| < O(\epsilon^2|\log\epsilon|^2)$ and I_2 is controlled by the negative term $-\frac{c_2}{2}\epsilon^3|\log\epsilon|^2$, so that

$$\begin{aligned} w_\epsilon(x,0) - u_\epsilon(x,0) &= I_1 + I_2 - \frac{c_2}{2}\epsilon^3|\log\epsilon|^2 \\ &\le I_2 - \frac{c_2}{2}\epsilon^3|\log\epsilon|^2 \\ &\le O(\epsilon^4|\log\epsilon|^2) - \frac{c_2}{2}\epsilon^3|\log\epsilon|^2. \end{aligned}$$

If $|y| < |\log\epsilon|$, then I_2 is controlled by I_1, and

$$\begin{aligned} w_\epsilon(x,0) - u_\epsilon(x,0) &= I_1 + I_2 - \frac{c_2}{2}\epsilon^3|\log\epsilon|^2 \\ &\le q'\left(\frac{d_\epsilon^-(x,0)}{\epsilon}\right)\left(\frac{1}{2}c_1(0)\epsilon|\log\epsilon|^2 - O(\epsilon^2|\log\epsilon|^2)\right). \end{aligned}$$

Thus, choosing $c_1(0)$ and c_2 sufficiently large, we get the desired estimate (25).

Let us now check that

$$\partial_t v_\epsilon^- - \Delta v_\epsilon^- + \frac{1}{\epsilon^2}W'(v_\epsilon^-) - \frac{1}{\epsilon}\sqrt{2W(v_\epsilon^-)}g \le 0 \tag{26}$$

in $\mathbf{R}^d \times (0,T)$.

Case 1: $(x,t) \in \Lambda_\epsilon^-$. In this case, (17) implies that

$$\begin{aligned}
\partial_t v_\epsilon^- &= \frac{1}{\epsilon} q_\epsilon'(y)\partial_t d_\epsilon^- + \epsilon^2\Big(\partial_t(\bar{h} + \nabla d\cdot\nabla g)\Big)\xi_\epsilon(y) + \epsilon(\bar{h} + \nabla d\cdot\nabla g)\xi_\epsilon'(y)\partial_t d_\epsilon^-,\\
\nabla v_\epsilon^- &= \frac{1}{\epsilon} q_\epsilon'(y)\nabla d_\epsilon^- + \epsilon^2\Big(\nabla(\bar{h} + \nabla d\cdot\nabla g)\Big)\xi_\epsilon(y) + \epsilon(\bar{h} + \nabla d\cdot\nabla g)\xi_\epsilon'(y)\nabla d_\epsilon^-,\\
\Delta v_\epsilon^- &= \frac{1}{\epsilon^2} q_\epsilon''(y) + \frac{1}{\epsilon} q_\epsilon'(y)\Delta d_\epsilon^- + (\bar{h} + \nabla d\cdot\nabla g)\xi_\epsilon''(y) + O(\epsilon).
\end{aligned}$$

Using these equalities with (24), we get

$$\begin{aligned}
\partial_t v_\epsilon^- - \Delta v_\epsilon^- &= -\frac{1}{\epsilon^2} q_\epsilon''(y) + \frac{1}{\epsilon} q_\epsilon'(y)(\partial_t d_\epsilon^- - \Delta d_\epsilon^-) - (\bar{h} + \nabla d\cdot\nabla g)\xi_\epsilon''(y) + O(\epsilon)\\
&= -\frac{1}{\epsilon^2} q_\epsilon''(y) + \frac{1}{\epsilon} q_\epsilon'(y)\Big(-\bar{g} + \epsilon y\bar{h} - \epsilon^2|\log\epsilon|^2(c_1\bar{h} - c_1')\Big)\\
&\qquad - (\bar{h} + \nabla d\cdot\nabla g)\xi_\epsilon''(y) + O(\epsilon|\log\epsilon|^2)\\
&= \frac{1}{\epsilon^2} q_\epsilon''(y) - \frac{1}{\epsilon} q_\epsilon'(y)\bar{g} + \Big(q_\epsilon'(y)y\bar{h} - (\bar{h} + \nabla d\cdot\nabla g)\xi_\epsilon''(y)\Big)\\
&\qquad - \epsilon|\log\epsilon|^2\Big(q_\epsilon'(y)(c_1\bar{h} - c_1')\Big) + O(\epsilon|\log\epsilon|^2).
\end{aligned}$$

Since (x,t) is in Λ_ϵ^-, we obtain the following estimates on the terms of order 0 in (26):

$$\begin{aligned}
g(x,t) &= g\big(s(x,t) + d\nabla d, t\big)\\
&= \bar{g} + d\nabla d\cdot\bar{\nabla} g + O(d^2)\\
&= \bar{g} + \epsilon y\nabla d_\epsilon^-\cdot\bar{\nabla} g - c_1\epsilon^2|\log\epsilon|^2\nabla d_\epsilon^-\cdot\bar{\nabla} g + O(\epsilon^2|\log\epsilon|^2),\\
\frac{1}{\epsilon^2}W'(v_\epsilon^-) &= \frac{1}{\epsilon^2}W'(q_\epsilon) + W''(q_\epsilon)(\bar{h} + \nabla d\cdot\nabla g)\xi_\epsilon - W''(q_\epsilon)c_2\epsilon|\log\epsilon|^2 + O(\epsilon),\\
\frac{1}{\epsilon}\sqrt{2W(v_\epsilon)}g &= \frac{1}{\epsilon}\sqrt{2W(q_\epsilon)}g + O(\epsilon)\\
&= \frac{1}{\epsilon}\sqrt{2W(q_\epsilon)}\Big(\bar{g} + \epsilon y\nabla d_\epsilon\cdot\bar{\nabla} g - c_1\epsilon^2|\log\epsilon|^2\nabla d_\epsilon^-\cdot\bar{\nabla} g\Big)\\
&\qquad + O(\epsilon|\log\epsilon|^2)\\
&= -\frac{1}{\epsilon}q_\epsilon'\Big(\bar{g} + \epsilon y\nabla d_\epsilon^-\cdot\bar{\nabla} g - c_1\epsilon^2|\log\epsilon|^2\nabla d_\epsilon^-\cdot\bar{\nabla} g\Big) + O(\epsilon|\log\epsilon|^2)\\
&= -\frac{1}{\epsilon}q_\epsilon'\Big(\bar{g} + \epsilon y\nabla d_\epsilon^-\cdot\nabla g - c_1\epsilon^2|\log\epsilon|^2\nabla d\cdot\bar{\nabla} g\Big) + O(\epsilon|\log\epsilon|^2).
\end{aligned}$$

Here, we used (22) and the fact that for (x,t) in Λ_ϵ^-,

$$y q_\epsilon'\nabla d\cdot(\nabla g - \bar{\nabla} g) = O(\epsilon|\log\epsilon|).$$

Summing the previous equalities, we obtain

$$\partial_t v_\epsilon - \Delta v_\epsilon + \frac{1}{\epsilon^2} W'(v_\epsilon) - \frac{1}{\epsilon}\sqrt{2W(v_\epsilon)}g = I_1 + I_2 + I_3 + I_4 + O(\epsilon|\log\epsilon|^2),$$

with

$$\begin{aligned}
I_1 &= -\frac{1}{\epsilon^2}\Big(q_\epsilon''(y) - W'\big(q_\epsilon(y)\big)\Big) = o(\epsilon^{2\delta-3}),\\
I_2 &= \frac{1}{\epsilon} q_\epsilon'(\bar{g} - \bar{g}) = 0,\\
I_3 &= -(\bar{h} + \nabla d\cdot\nabla g)\Big(\xi_\epsilon''(y) - W''(q_\epsilon)\xi_\epsilon - y q_\epsilon'\Big) = o(\epsilon^{2\delta-3}),\\
I_4 &= \epsilon|\log\epsilon|^2\Big(-q_\epsilon'(c_1\bar{h} - c_1' + c_1\nabla d\cdot\bar{\nabla} g) - c_2 W''(q_\epsilon)\Big).
\end{aligned}$$

We now determine the function c_1 and the constant c_2 so that I_4 is sufficiently negative to compensate the term of order $\epsilon|\log\epsilon|^2$. Letting $K = \|h\|_{L^\infty(\partial\Omega)} + \|\nabla g\|_{L^\infty(\mathbf{R}^d\times(0,T))}$, we set

$$c_1(t) = c\exp\big((1+K)t\big),$$

so that

$$-\epsilon|\log\epsilon|^2 q_\epsilon'\Big(c_1\bar{h} - c_1' + \nabla d\cdot\bar{\nabla} g c_1\Big) \le c_1\epsilon|\log\epsilon|^2 q_\epsilon'.$$

We thus have

$$I_4 \le -\epsilon|\log\epsilon|^2 c_2\bigg(-\frac{c_1}{c_2}q_\epsilon' + W''(q_\epsilon)\bigg).$$

Noticing that $-c_3 q_\epsilon' + W''(q_\epsilon)$ is uniformly positive for c_3 large enough, we can choose c and c_2 such that

$$\partial_t v_\epsilon^- - \Delta v_\epsilon^- + \frac{1}{\epsilon}\Big(W'(v_\epsilon^-) - g\sqrt{2W(v_\epsilon^-)}\Big) \le 0$$

in Λ_ϵ^-.

Case 2: $(x,t)\notin\Lambda_\epsilon^-$. Here, the function v_ϵ^- is given by $v_\epsilon = \pm 1 - c_2\epsilon^3|\log\epsilon|^2$, which implies $\partial_t v_\epsilon^- = 0$, $\Delta v_\epsilon^- = 0$,

$$\begin{aligned}
\frac{1}{\epsilon^2}W'(v_\epsilon^-) &= \frac{1}{\epsilon^2}\big(W'(\pm1) - W''(\pm1)c_2\epsilon^3|\log\epsilon|^2 + O(\epsilon^3)\big)\\
&= -W''(\pm1)c_2\epsilon|\log\epsilon|^2 + O(\epsilon),\\
\frac{1}{\epsilon}\sqrt{2W(v_\epsilon^-)}g &= \frac{1}{\epsilon}\big(\sqrt{2W(\pm1)}g - (\sqrt{2W})'(\pm1)gc_2\epsilon^3|\log\epsilon|^2 + O(\epsilon^3)\big)\\
&= O(\epsilon^2|\log\epsilon|^2).
\end{aligned}$$

Noticing that $W''(\pm 1) > 0$ and choosing c_2 large enough guarantees that in $\mathbf{R}^d \setminus \Lambda_\epsilon^-$,

$$\partial_t v_\epsilon^- - \Delta v_\epsilon^- + \frac{1}{\epsilon}\Big(W'(v_\epsilon^-) - g\sqrt{2W(v_\epsilon^-)}\Big) \le 0.$$

To conclude, we apply the comparison principle of lemma 1 and discover

$$\forall (x,t) \in \mathbf{R}^d \times [0,T], \quad v_\epsilon^-(x,t) \le u_\epsilon(x,t).$$

A similar argument can be applied to

$$v_\epsilon^+ = \begin{cases} q_\epsilon(y) + \epsilon^2(\bar{h} - \nabla d \cdot \nabla g)\xi_\epsilon(y) + c_2\epsilon^3|\log\epsilon|^2 & \text{in } \Lambda_\epsilon^+, \\ -1 + c_2\epsilon^3|\log\epsilon|^2 & \text{in } \{d_\epsilon^+ \ge 2\delta\epsilon|\log\epsilon|\}, \\ +1 + c_2\epsilon^3|\log\epsilon|^2 & \text{in } \{d_\epsilon^+ \le -2\delta\epsilon|\log\epsilon|\}, \end{cases}$$

with $y = \frac{d_\epsilon^+}{\epsilon}$ and

$$d_\epsilon^+(x,t) = d(x,t) - c_1(t)\epsilon^2|\log\epsilon|^2,$$

to show that v_ϵ^+ is a supersolution to (7):

$$\forall (x,t) \in \mathbf{R}^d \times [0,T], \quad v_\epsilon^+(x,t) \ge u_\epsilon(x,t).$$

Proof of the theorem.

Proof of theorem 1. We choose ϵ_0 so that (23) holds. Let t in $[0,T]$ and x in $\partial\Omega_\epsilon(t)$ be given. We first show that x is in $\Lambda(t)$. Indeed, $u_\epsilon(x,t) = 0$ and

$$v_\epsilon^-(x,t) \le u_\epsilon(x,t) = 0 \le v_\epsilon^+(x,t). \tag{27}$$

Assume that $x \notin \Lambda(t)$. As $\Lambda_\epsilon^\pm(t) \subseteq \Lambda(t)$, we have $x \notin \Lambda_\epsilon^\pm(t)$, and thus, for ϵ sufficiently small, we deduce that $v_\epsilon^-(x,t)$ and $v_\epsilon^+(x,t)$ have the same sign. This contradicts (27), and we conclude that $x \in \Lambda(t)$. We then notice that

$$v_\epsilon^-(x,t) = q_\epsilon\left(\frac{d_\epsilon^-(x,t)}{\epsilon}\right) + O(\epsilon^2) \le 0$$

because

$$\epsilon^2(\bar{h} - \nabla d \cdot \nabla g)\xi_\epsilon(y) = O(\epsilon^2),$$

and hence

$$q_\epsilon\left(\frac{d_\epsilon^-(x,t)}{\epsilon}\right) \le O(\epsilon^2).$$

As $q'(0) = -1$, we get that

$$\frac{d_\epsilon^-(x,t)}{\epsilon} \ge O(\epsilon^2),$$

which shows that $d(x,t) \geq O(\epsilon^2 |\log \epsilon|^2)$. In a similar way, noticing that

$$v_\epsilon^+(x,t) = q_\epsilon\left(\frac{d_\epsilon^+(x,t)}{\epsilon}\right) + O(\epsilon^2) \geq 0,$$

we get that

$$\frac{d_\epsilon^+(x,t)}{\epsilon} \leq O(\epsilon^2),$$

and $d(x,t) \leq O(\epsilon^2 |\log \epsilon|^2)$. We conclude that

$$|d(x,t)| \leq O(\epsilon^2 |\log \epsilon|^2),$$

and (14) is proved. □

3 Application to mean curvature flow with conservation of the volume

In this section, we compare two phase field models for the approximation of motion by mean curvature with conservation of the volume:

$$V_n = \kappa - \fint_{\partial\Omega(t)} \kappa \, d\sigma. \tag{28}$$

As explained in the introduction, this motion is usually approximated by the following phase field equation (see [6]):

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2} W'(u) + \frac{1}{\epsilon^2} \fint_Q W'(u)\, dx. \tag{29}$$

The last term in this equation can be understood as a Lagrange multiplier for the mass constraint

$$\frac{d}{dt}\int_Q u \, dx = 0.$$

(Note that in this section, the potential W we consider has its wells at 0 and 1.) In the sequel, we compare this equation to

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2} W'(u) + \frac{1}{\epsilon^2} \frac{\sqrt{2W(u)}}{\int_{\mathbf{R}^d} \sqrt{2W(u)}\, dx} \int_{\mathbf{R}^d} W'(u)\, dx. \tag{30}$$

derived along the same lines as (7). The form of the last term is again related to conservation of mass, since the volume average of the right-hand side is easily seen to vanish.

There is no general proof of convergence of solutions of (29) and (30) to the motion (28). However, (29) is commonly used in computations. The numerical experiments presented further show that (30) conserves volume with a higher degree of accuracy than (29). Our aim in this section is to try so substantiate this claim, although our arguments are formal.

In both cases, the last term could be interpreted as a forcing term, by setting

$$g_\epsilon(t) = \frac{1}{\epsilon c_W} \fint_Q W'(u)\,dx$$

in the first model and

$$\tilde{g}_\epsilon(t) = \frac{1}{\epsilon} \frac{\int_{\mathbf{R}^d} W'(u)\,dx}{\int_{\mathbf{R}^d} \sqrt{2W(u)}\,dx}$$

in the second. Formally, one recovers the expressions of (2) and (7). However the forcing terms here depend on the solutions of (29) and (30). Assuming that one can generalize the results of section 2.1 (notwithstanding this dependence of g_ϵ and $\tilde{g}_\epsilon$), we expect solutions u_ϵ and $\tilde{u}_\epsilon$ of (29) and (30) to have the following asymptotic behavior:

$$u_\epsilon(x,t) = q\left(\frac{d\big(x, \Omega_\epsilon(t)\big)}{\epsilon}\right) + \epsilon g_\epsilon(t) \eta\left(\frac{d\big(x, \Omega_\epsilon(t)\big)}{\epsilon}\right) + O(\epsilon^2), \tag{31}$$

$$\tilde{u}_\epsilon(x,t) = q\left(\frac{d\big(x, \tilde{\Omega}_\epsilon(t)\big)}{\epsilon}\right) + \epsilon^2 h(x,t) \xi\left(\frac{d\big(x, \tilde{\Omega}_\epsilon(t)\big)}{\epsilon}\right) + O(\epsilon^3), \tag{32}$$

where $\Omega_\epsilon(t)$ (resp. $\tilde{\Omega}_\epsilon(t)$) denotes the set contained inside the level line $\{u_\epsilon(x,t) = \frac{1}{2}\}$ (resp. $\{\tilde{u}_\epsilon(x,t) = \frac{1}{2}\}$), and q, η, ξ are the profiles defined in (3), (4) and (19). We note that these profiles only depend on the choice of the potential W. Following (12), we see that as $\tilde{g}_\epsilon$ does not depend on x, only h appears in the term of order 2 of $\tilde{u}_\epsilon$.

We first establish the connection between the mass $\int_Q u_\epsilon\,dx$ (respectively $\int_Q \tilde{u}_\epsilon\,dx$) and the volume $|\Omega_\epsilon(t)|$ (respectively $|\tilde{\Omega}_\epsilon(t)|$).

Proposition 1. *Let E be a regular bounded domain of $\mathbf{R}^d$, and let*

$$v_\epsilon(x) = q\left(\frac{d(x,E)}{\epsilon}\right).$$

Assume that q is symmetric, i.e. $q(s) = 1 - q(-s)$, and that q decays exponentially to 0 as $s \to +\infty$. Then

$$|E| = \int_{\mathbf{R}^d} v_\epsilon\,dx + O(\epsilon^2).$$

Proof. Using the co-area formula,

$$\begin{aligned}\int_{\mathbf{R}^d} v_\epsilon\, dx &= \int_{\mathbf{R}^d} q\left(\frac{d(x,E)}{\epsilon}\right) dx \\ &= \int_{\mathbf{R}} h(s) q\left(\frac{s}{\epsilon}\right) ds \\ &= \int_{-\infty}^{0} h(s)\, ds + \int_{-\infty}^{0} h(s)\left(q\left(\frac{s}{\epsilon}\right) - 1\right) ds + \int_{0}^{+\infty} h(s) q\left(\frac{s}{\epsilon}\right) ds \\ &= |E| - \int_0^{+\infty} h(-s) q\left(\frac{s}{\epsilon}\right) ds + \int_0^{+\infty} h(s) q\left(\frac{s}{\epsilon}\right) ds \\ &= |E| + \int_0^{+\infty} \big(h(s) - h(-s)\big) q\left(\frac{s}{\epsilon}\right) ds \\ &= |E| + \epsilon \int_0^{+\infty} \big(h(s\epsilon) - h(-s\epsilon)\big) q(s)\, ds\end{aligned}$$

where $h(s) = |D\chi_{\{d(x,E)\leq s\}}|(\mathbf{R}^d)$ is the perimeter of the level line s of the signed distance function to E. Since E is smooth, one can estimate $h(s\epsilon) - h(-s\epsilon) = 2s\epsilon h'(0) + O(s^2\epsilon^2)$ for s in $(0, |\log\epsilon|)$. Furthermore, since q is exponentially decreasing to 0 as $s \to +\infty$, all the moments $\int_{s>0} s^n q(s)\, ds$ are finite. Thus, we can estimate

$$\left|\int_0^{|\log\epsilon|} \big(h(s\epsilon) - h(-s\epsilon)\big) q(s)\, ds\right| \leq \left|\int_0^{|\log\epsilon|} \big(2s\epsilon h'(0) + Cs^2\epsilon^2\big) q(s)\, ds\right| = O(\epsilon).$$

Moreover, since $h(s) \sim s^{d-1}$ as $s \to +\infty$, and since h is bounded on $(-\infty, 0)$, it is easy to check that

$$\int_{|\log\epsilon|}^{+\infty} h(s\epsilon) q(s)\, ds \leq C\epsilon^{d-1} \int_{|\log\epsilon|}^{+\infty} s^{d-1} q(s)\, ds = O(\epsilon^{d-1}),$$
$$\int_{|\log\epsilon|}^{+\infty} h(-s\epsilon) q(s)\, ds \leq C \int_{|\log\epsilon|}^{+\infty} q(s)\, ds = O(\epsilon).$$

It follows that

$$\int_{\mathbf{R}^d} v_\epsilon\, dx = |E| + O(\epsilon^2).$$

□

The result of proposition 1 still holds on a fixed bounded set Q that strictly contains E when ϵ is sufficiently small. This again is a consequence of the exponential decay of q. Recalling the asymptotic form of u_ϵ, it follows from the above proposition that, for the classical model (29),

$$\int_Q u_\epsilon\,dx = \int_Q q\left(\frac{d\big(x,\Omega_\epsilon(t)\big)}{\epsilon}\right)dx + \epsilon g_\epsilon \int_Q \eta\left(\frac{d\big(x,\Omega_\epsilon(t)\big)}{\epsilon}\right)dx + O(\epsilon^2).$$

In general, the term of order ϵ does not vanish, since

$$\lim_{s\to\pm\infty} \eta(s) = \frac{c_W}{W''(0)} \neq 0$$

when q is symmetric, and so

$$\int_Q u_\epsilon\,dx = |\Omega_\epsilon(t)| + O(\epsilon).$$

This explains why we cannot expect the model (29) to converge to the motion (28) with a better rate than $O(\epsilon)$. As for the model (30), we have

$$\int_Q \tilde{u}_\epsilon\,dx = \int_Q q\left(\frac{d\big(x,\tilde{\Omega}_\epsilon(t)\big)}{\epsilon}\right)dx + \epsilon^2 \int_Q h\xi\left(\frac{d\big(x,\tilde{\Omega}_\epsilon(t)\big)}{\epsilon}\right)dx + O(\epsilon^3),$$

that is

$$\int_Q \tilde{u}_\epsilon\,dx = |\tilde{\Omega}_\epsilon(t)| + O(\epsilon^2),$$

which presents a higher degree of accuracy on volume conservation.

We proved in the last section that solutions of (7) converge as $\epsilon \to 0$ to motion by mean curvature with a forcing term (1). Formally, the phase field equation (30) can be rewritten

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2}\left(W'(u) - \epsilon\sqrt{2W(u)}\tilde{g}_\epsilon\right).$$

The following property shows that, under the assumption (32), $\tilde{g}_\epsilon$ converges to $⨍_{\partial\Omega_\epsilon} \kappa\,d\sigma$, which is formally consistent to the limiting motion (1).

Proposition 2. *Let E be a regular bounded domain of $\mathbf{R}^d$, and let*

$$v_\epsilon(x) = q\left(\frac{d(x,E)}{\epsilon}\right).$$

Assume that q is symmetric, i.e. $q(s) = 1 - q(-s)$, and that q decays exponentially to 0 as $s \to +\infty$. Then

$$\frac{\int_{\mathbf{R}^d} g\sqrt{2W(v_\epsilon)}\,dx}{\int_{\mathbf{R}^d}\sqrt{2W(v_\epsilon)}\,dx} = ⨍_{\partial E} g\,d\sigma + O(\epsilon^2),$$

$$\frac{1}{\epsilon}\frac{\int_{\mathbf{R}^d} W'(v_\epsilon)\,dx}{\int_{\mathbf{R}^d}\sqrt{2W(v_\epsilon)}\,dx} = -⨍_{\partial E} \kappa\,d\sigma + O(\epsilon^2).$$

Proof. To prove the first equality, recall that q satisfies $\sqrt{2W(q)} = -q'$, and that q' is even. Let $h\colon \mathbf{R} \to \mathbf{R}$ be a continuous function, differentiable at $s = 0$, which grows polynomially in s. Since $\int_{\mathbf{R}} q'(s)\, ds = -1$, arguing as in the proof of proposition 1, it follows that

$$\begin{aligned}
\frac{1}{\epsilon}\int_{\mathbf{R}} \sqrt{2W\left(q\left(\frac{s}{\epsilon}\right)\right)} h(s)\, ds &= -\int_{\mathbf{R}} \frac{1}{\epsilon} q'\left(\frac{s}{\epsilon}\right) h(s)\, ds \\
&= -\int_{\mathbf{R}} q'(s) h(s\epsilon)\, ds \\
&= -\int_0^{+\infty} \big(h(s\epsilon) + h(-s\epsilon)\big) q'(s)\, ds \\
&= -\int_0^{3|\log\epsilon|} \big(h(s\epsilon) + h(-s\epsilon)\big) q'(s)\, ds + O(\epsilon^2) \\
&= -\int_0^{3|\log\epsilon|} \big(2h(0) + Cs^2\epsilon^2\big) q'(s)\, ds + O(\epsilon^2) \\
&= h(0) + O(\epsilon^2).
\end{aligned}$$

Next, the co-area formula yields

$$\frac{1}{\epsilon}\int_{\mathbf{R}^d} g\sqrt{2W(v_\epsilon)}\, dx = \frac{1}{\epsilon}\int_{\mathbf{R}} \left(\int_{d(x,E)=s} g\, d\sigma\right) \sqrt{2W\left(q\left(\frac{s}{\epsilon}\right)\right)}\, ds.$$

Since E is smooth, and since the forcing term g is bounded, the function

$$h\colon s \mapsto \int_{d(x,E)=s} g\, d\sigma$$

is continuous, differentiable at $s = 0$ and has polynomial growth at infinity: $h(s) \sim s^{d-1}$ when $s \to +\infty$. We can then apply the previous estimate to obtain

$$\frac{1}{\epsilon}\int_{\mathbf{R}^d} g\sqrt{2W(v_\epsilon)}\, dx = \int_{d(x,E)=0} g\, d\sigma + O(\epsilon^2) = \int_{\partial E} g\, d\sigma + O(\epsilon^2).$$

We notice that the same argument with $g = 1$ leads to

$$\frac{1}{\epsilon}\int_{\mathbf{R}^d} \sqrt{2W(v_\epsilon)}\, dx = \int_{d(x,E)=0} d\sigma + O(\epsilon^2) = |\partial E| + O(\epsilon^2),$$

so that combined with the previous equality, we obtain

$$\frac{\int_{\mathbf{R}^d} g\sqrt{2W(v_\epsilon)}\, dx}{\int_{\mathbf{R}^d} \sqrt{2W(v_\epsilon)}\, dx} = \fint_{\partial E} g\, d\sigma + O(\epsilon^2).$$

Let us now prove the second equality. Recall that

$$\begin{aligned}
v_\epsilon &= q\left(\frac{d(x,E)}{\epsilon}\right),\\
\nabla v_\epsilon &= \frac{1}{\epsilon} q'\left(\frac{d(x,E)}{\epsilon}\right)\nabla d(x,E),\\
\Delta v_\epsilon &= \frac{1}{\epsilon^2} q''\left(\frac{d(x,E)}{\epsilon}\right) + \frac{1}{\epsilon} q'\left(\frac{d(x,E)}{\epsilon}\right)\Delta d(x,E).
\end{aligned}$$

As $q'' = W'(q)$, it follows that

$$\begin{aligned}
\frac{1}{\epsilon^2}\int_{\mathbf{R}^d} W'(v_\epsilon)\,dx &= \int_{\mathbf{R}^d}\left(\frac{1}{\epsilon^2}W'(v_\epsilon) - \Delta v_\epsilon\right)dx\\
&= -\int_{\mathbf{R}^d}\frac{1}{\epsilon}q'\left(\frac{d(x,E)}{\epsilon}\right)\Delta d(x,E)\,dx\\
&= \int_{\mathbf{R}}\left(\int_{d(x,E)=s}\Delta d(x,E)\,d\sigma\right)\frac{1}{\epsilon}q'\left(\frac{s}{\epsilon}\right)ds.
\end{aligned}$$

The function

$$s \mapsto h(s) = \int_{d(x,E)=s}\Delta d(x,E)\,d\sigma$$

is not continuous on $\mathbf{R}$, but it is constant on a sufficiently small neighborhood of 0 (depending only on the topology of E) and grows polynomially like s^{d-1}. Arguing as in the first part of the proof, we obtain

$$\frac{1}{\epsilon^2}\int_{\mathbf{R}^d} W'(v_\epsilon)\,dx = h(0) + O(\epsilon^2) = -\int_{\partial E}\kappa\,d\sigma + O(\epsilon^2),$$

and

$$\frac{1}{\epsilon}\frac{\int_{\mathbf{R}^d} W'(v_\epsilon)\,dx}{\int_{\mathbf{R}^d}\sqrt{2W(v_\epsilon)}\,dx} = -⨍_{\partial E}\kappa\,d\sigma + O(\epsilon^2),$$

which completes the proof. ☐

Remark 2. *The first correcting term ξ vanishes at infinity in the expansion of $\tilde{u}_\epsilon$, and so would higher order terms. If we assume that* (32) *holds, a more careful analysis based on proposition 2 would show that*

$$\frac{1}{\epsilon}\frac{\int_{\mathbf{R}^d} W'(\tilde{u}_\epsilon)\,dx}{\int_{\mathbf{R}^d}\sqrt{2W(\tilde{u}_\epsilon)}\,dx} = -⨍_{\partial\tilde{\Omega}_\epsilon}\kappa\,d\sigma + O(\epsilon^2).$$

This heuristically justifies the use of (30) *as an approximation to the motion* (28).

Remark 3. *We can generalize our previous argument to the case of interfaces moving with normal velocity*

$$V_n = \kappa + g - \fint_{\partial\Omega} (\kappa + g)\, d\sigma. \tag{33}$$

The usual phase field approximation of such motions is based on the equation

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2}\Big(W'(u) - \epsilon c_W g\Big) + \frac{1}{\epsilon^2}\fint_Q \Big(W'(u) - \epsilon c_W g\Big)\, dx, \tag{34}$$

where the last term can be understood as a Lagrange multiplier. As explained above, one cannot expect that this model should converge to the motion (33) *with a better rate than* $O(\epsilon)$*. Generalizing our previous analysis, we may instead consider the following modified phase field model, which should improve the accuracy:*

$$\begin{aligned}\partial_t u = \Delta u - \frac{1}{\epsilon^2}\Big(W'(u) - \epsilon g\sqrt{2W(u)}\Big) \\ + \frac{1}{\epsilon^2}\frac{\sqrt{2W(u)}}{\int_{\mathbf{R}^d}\sqrt{2W(u)}\, dx}\int_{\mathbf{R}^d}\Big(W'(u) - \epsilon g\sqrt{2W(u)}\Big)\, dx.\end{aligned} \tag{35}$$

4 Numerical method and simulations

In this section, we describe the numerical method we use for solving

$$\begin{cases} \partial_t u = \Delta u - \frac{1}{\epsilon^2}F(u) & x \in Q \subset \mathbf{R}^d, \quad t \in [0, T], \\ u(x, 0) = u_0(x), & x \in Q, \end{cases} \tag{36}$$

where F takes one of the following forms:

$$\begin{aligned} W'_{\epsilon,g}(u) &= W'(u) - \epsilon c_W g, \\ \tilde{W}'_{\epsilon,g}(u) &= W'(u) - \epsilon g\sqrt{2W(u)}, \\ W'_{\epsilon,g,vol}(u) &= W'(u) - \epsilon c_W g - \fint_Q \Big(W'(u) - \epsilon c_W g\Big)\, dx, \\ \tilde{W}'_{\epsilon,g,vol}(u) &= W'(u) - \epsilon g\sqrt{2W(u)} \\ &\quad - \frac{\sqrt{2W(u)}}{\int_Q \sqrt{2W(u)}\, dx}\int_Q \Big(W'(u) - \epsilon g\sqrt{2W(u)}\Big)\, dx. \end{aligned}$$

The first form corresponds to the Allen–Cahn equation with a forcing term g. The second form corresponds to the modified approximation introduced in section 2.

Forms 3 and 4 are the respective forms when the volume is conserved (see (34) and (35)). We assume that

$$u_0 = q\left(\frac{d(x, \partial\Omega_0)}{\epsilon}\right),$$

where Ω_0 is a smooth bounded set of $\mathbf{R}^d$ strictly contained in the fixed box $Q = [-\frac{1}{2}, \frac{1}{2}]^d$, with $d = 2$ or 3. We assume also that during the evolution the sets $\Omega_\epsilon(t)$ remain within Q, so that we may impose periodic boundary conditions on ∂Q to the solutions of (36).

4.1 Numerical scheme

Equation (36) is numerically approximated via a splitting method between the diffusion and reaction terms. We take advantage of the periodicity to treat the diffusion part of the operator in the Fourier space. More precisely, the value $u_\epsilon(x, t_n)$ at time $t_n = t_0 + n\Delta t$ is approximated by

$$u_\epsilon^P(x, t_n) = \sum_{\max\limits_{1\le k\le d}|p_k|\le P} u_{\epsilon,p}(t_n)\exp(2i\pi p\cdot x).$$

In a first step, we set

$$u_\epsilon^P\left(x, t_n + \frac{1}{2}\right) = \sum_{\max\limits_{1\le k\le d}|p_k|\le P} u_{\epsilon,p}\left(t_n + \frac{1}{2}\right)\exp(2i\pi p\cdot x),$$

with

$$u_{\epsilon,p}\left(t_n + \frac{1}{2}\right) = u_{\epsilon,p}(t_n)\exp(-4\pi^2\Delta t|p|^2).$$

We then add the reaction term:

$$u_\epsilon^P(x, t_n + 1) = u_\epsilon^P\left(t_n + \frac{1}{2}\right) - \frac{\Delta t}{\epsilon^2}F\left(u_\epsilon^P\left(t_n + \frac{1}{2}\right)\right).$$

In practice, the first step is performed via a fast Fourier transform, with a computational cost of $O(P^d \log P)$. The corresponding numerical scheme turns out to be L^∞-stable for the standard Allen–Cahn equation with no forcing term, under the condition

$$\delta t \le M\epsilon^2,$$

where $M = \left(\sup_{t\in[0,1]} W''(t)\right)^{-1}$. It can be shown that this condition is also sufficient for the modified potential $\tilde{W}_{\epsilon,g}$. We impose this constraint in the following computations for all the choices of F. We use the double well potential $W(s) = \frac{1}{2}s^2(1-s)^2$, and P represents the number of Fourier modes in each dimension.

4.2 Numerical tests

Convergence test with no forcing term. This test illustrates the convergence of our numerical scheme when we consider the equation

$$\partial_t u = \Delta u - \frac{1}{\epsilon^2} W'(u),$$

with no forcing term, nor volume conservation. The initial set Ω_0 is taken as a circle of radius $R_0 = 0.25$. It should evolve as a circle, with radius $R(t) = \sqrt{{R_0}^2 - 2t}$, that decreases to a point at the extinction time $t_{ext} = \frac{1}{2} R_0^2$. Figure 2 represents $\Omega(t)$ at different times, for the choice of parameters $P = 2^8$, $\Delta t = 1/P^2$ and $\epsilon = 2/P$. Figure 3 shows the error between calculated and theoretical extinction times for different values of ϵ, in logarithmic scale. The error behaves like $O(\epsilon^2 |\log \epsilon|^2)$ as expected. This indicates that, with this choice of parameters, the error due to our numerical scheme is negligible compared to the 'modeling' error due to the approximation of the motion by the phase field equation.

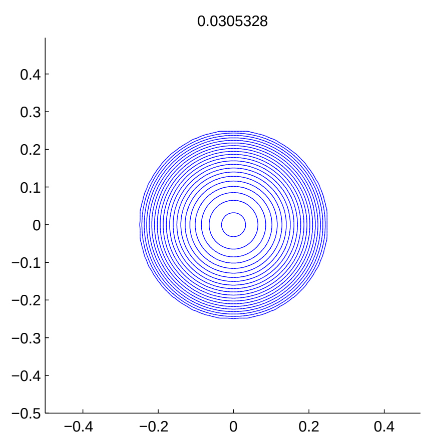

Figure 2: Mean curvature flow of a circle: level set $\{u_\epsilon(x,t) = \frac{1}{2}\}$ for different times.

Convergence test with a constant forcing term. Here we compare the two phase field models (2) and (7) as approximations to the motion (1). Theoretically, both give an approximation order of $O(\epsilon^2 |\log \epsilon|^2)$. We compare the numerical solutions in the simple case where the forcing term is a constant: $g = C_g$. The initial condition Ω_0 is a circle of radius R_0. During the evolution, $\Omega(t)$ also remains circular, and its radius R satisfies

$$\frac{dR}{dt} = -\frac{1}{R} + C_g.$$

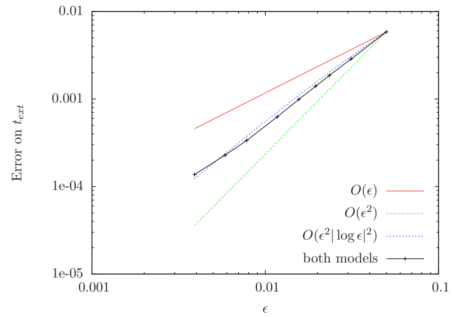

Figure 3: Mean curvature flow of a circle: error on the extinction time for different values of ϵ (logarithmic scale).

Assuming that $C_g < 1/R_0$, $\Omega(t)$ decreases to a point, with extinction at the time

$$t_{ext} = -\frac{1}{C_g}\left(\frac{1}{C_g}\ln(1 - C_g R_0) + R_0\right).$$

We represent on figure 4 the error on the extinction time for different values of ϵ, in logarithmic scale. We choose $C_g = 2$ and $C_g = -2$ respectively. Both models give comparable results, and as expected by the theory, we again observe a $O(\epsilon^2|\log\epsilon|^2)$ error.

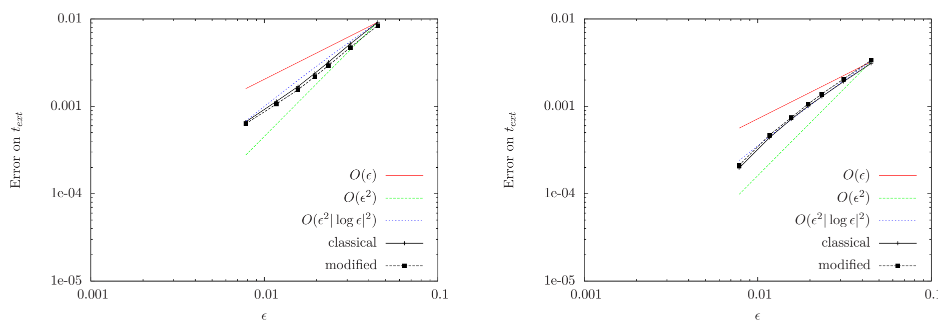

Figure 4: Mean curvature flow of a circle with a constant forcing term C_g: error on the extinction time for different values of ϵ (logarithmic scale). Left: $C_g = 2$. Right: $C_g = -2$.

Conservation of the volume with no forcing term. Here the initial configuration Ω_0 is the union of two disjoints circles of respective radii r_0 and R_0, with $r_0 < R_0$. As it evolves by conserved mean curvature flow (28), Ω remains the union of two circles, with radii r and R solutions of

$$\begin{cases} \dfrac{dr}{dt} = -\dfrac{1}{r} + \dfrac{2}{r+R}, \\ \dfrac{dR}{dt} = -\dfrac{1}{R} + \dfrac{2}{r+R}. \end{cases}$$

It is easy to check that the smallest circle decreases and disappears at extinction time

$$t_{ext} = -\frac{r_0 R_0}{2} + \frac{R_0^2 + r_0^2}{4} \ln\left(1 + \frac{2 r_0 R_0}{(R_0 - r_0)^2}\right).$$

Meanwhile, the radius of the initially larger circle grows to a maximal value

$$R_* = \sqrt{r_0^2 + R_0^2}$$

at extinction time. We presents results for $r_0 = 0.1$, $R_0 = 0.15$, $t_{ext} = 0.0133$, and for the choice of numerical parameters $P = 2^8$, $\Delta t = 2^{-16}$. The evolution of r and R is plotted on figure 5 for both models (29) and (30) and for different choices of ϵ:. Figure 6 depicts the error on extinction time in logarithmic scale. The graph clearly shows that the error on extinction time is of order ϵ for the classical model, while it scales like ϵ^2 for the modified model (30).

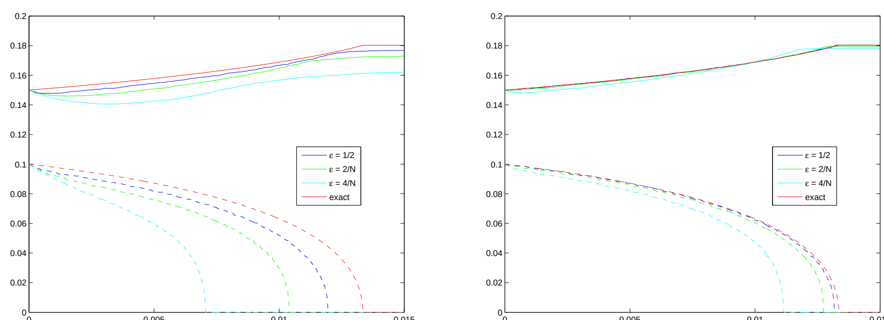

Figure 5: Conserved mean curvature flow of two disjoint circles: evolution of the radii against time, for different values of ϵ. Left: classical model (29). Right: modified model (30).

Conservation of the volume with a non-zero forcing term. Volume losses may become important when approximating forced mean curvature motion with the classical phase field model (29). The purpose of this test is to illustrate this point. We choose g to be an isotropic forcing term: $g(x) = c_g \cos(8\pi|x|)$. The initial configuration Ω_0 is the circle of radius $R_0 = 0.25$ centered at 0. It should

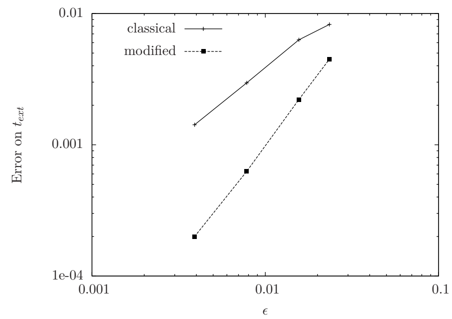

Figure 6: Conserved mean curvature flow of two disjoint circles: error on the extinction time against ϵ for the two models (29) and (30).

remain stationary (i.e. $\Omega(t) = \Omega_0$ for all t) whatever the value of the constant c_g. Figure 7 represents the computed evolutions using respectively (34) and (35). The numerical parameters are $P = 2^8$, $\epsilon = 2/P$ and $\Delta t = 1/P^2$. Clearly, the value of c_g has a significant impact on the results when using (34). Comparatively, the choice of c_g as a negligible impact on the evolutions computed with (35). This confirms the arguments developed in section 3.

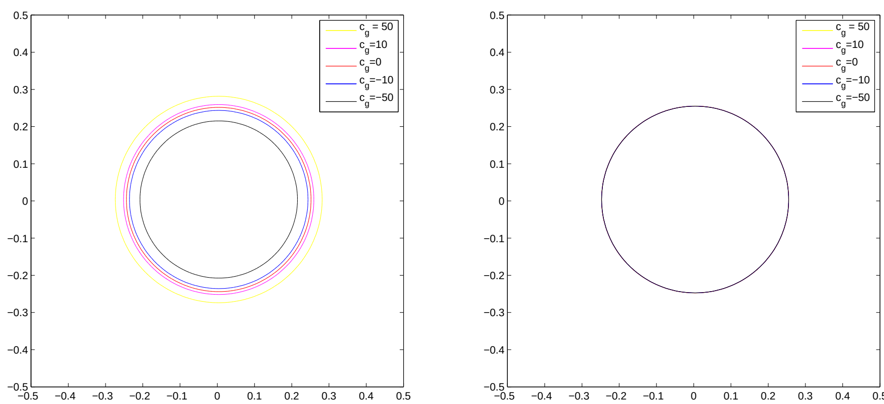

Figure 7: Conserved mean curvature flow of a circle with an additional isotropic forcing term $g(x) = c_g \cos(8\pi|x|)$: stationary shape for different choices of c_g. Left: classical model (34). Right: modified model (35).

An example in 3D. Here we illustrate the benefits of our approach on a classical three-dimensional example: the evolution of a torus with conservation of the volume and no additional forcing term. This example provides a good test case: because of the high values taken by the mean curvature, standard approaches may fail to reproduce the motion correctly. One also need to handle the topological change when we move from a toric shape to a spherical one.

We clearly observe on figure 8 that the classical model (34) leads to significant volume losses compared to our modified model (35). We plot on figure 9 the volume against time for both approaches. The volume error goes up to 30% for the classical model, whereas it is always strictly below 5% for ours. We notice that, in both cases, the error decreases in the second part of the evolution. Indeed, it is clear that the numerical error is maximal when the average mean curvature is maximal; when the topological change occurs, the average mean curvature instantly jumps to a smaller value, as the points where the mean curvature is the highest just disappear from the surface.

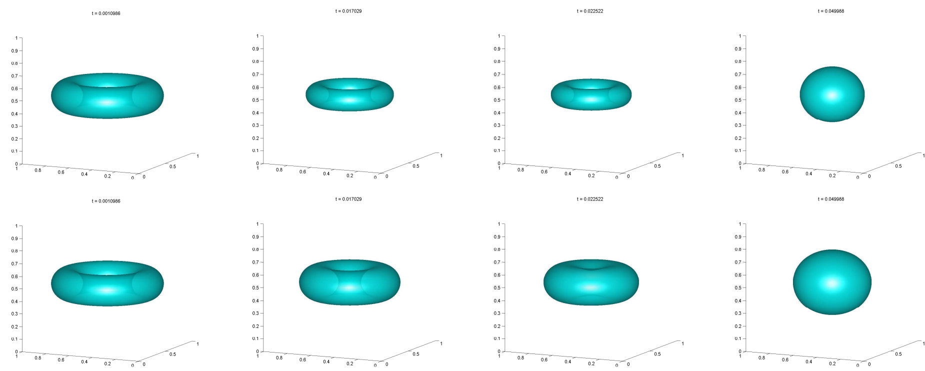

Figure 8: Evolution of a torus by mean curvature flow with conservation of the volume. First line: classical model (29). Second line: modified model (30).

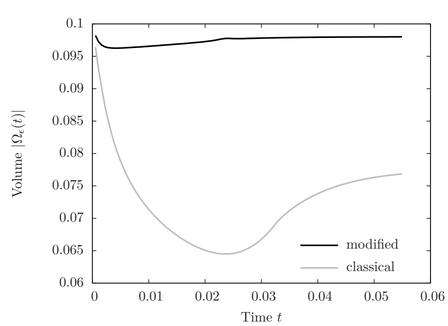

Figure 9: Torus example: volume against time for both models (29) and (30).

5 Conclusion

We introduced in this article a modified phase field model for the approximation of mean curvature flow with a forcing term. We rigorously proved its convergence with the same order as the classical Allen–Cahn equation: $O(\epsilon^2|\log\epsilon|^2)$.

We formally derived this model to the case of conserved mean curvature flow. We observed numerically an $O(\epsilon^2)$ error for the conservation of the volume, whereas the classical conserved Allen–Cahn equation just showed an $O(\epsilon)$ error in our simulations.

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