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Spatially Extended Differential Equations in a Potential: On the Borders of the Basins of Attraction Related to Local Minima

Emmanuel Risler

Institut Non Linéaire de Nice, UMR CNRS-UNSA 6618, 1361 route des Lucioles, F-06560 Valbonne, France

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We consider a class of non-linear parabolic or hyperbolic partial differential equations on unbounded domains. These equations can be viewed as spatial extensions of dissipative oscillators in a potential. We prove that the basins of attraction of the homogeneous stationary solutions corresponding to local minima of the potential are open, and we describe the asymptotic behavior of a class of solutions which belong to the borders of these basins. We also study the basin of attraction of a homogeneous stationary solution corresponding to a global minimum of the potential. © 2000 Academic Press

1. INTRODUCTION

1. This paper is concerned with the asymptotic behavior as $t \to +\infty$ of solutions $u(x, t)$ of the non-linear parabolic equation

$$u_t + V'(u) = u_{xx}, \qquad x \in \,]-\infty; +\infty[, \tag{1}$$

or of the non-linear hyperbolic equation

$$u_{tt} + v(u)\, u_t + V'(u) = u_{xx}, \qquad x \in \,]-\infty; +\infty[, \tag{2}$$

where the potential V and the damping v are smooth (C^∞) functions $\mathbf{R} \to \mathbf{R}$, and v takes only strictly positive values (Eq. (1) is in some sense a limit, when $v \to +\infty$, of (2); it is therefore natural to consider these two equations simultaneously).

These partial differential equations can be viewed as "spatial extensions" of the differential equations

$$u_t + V'(u) = 0 \tag{3}$$

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and

$$u_{tt} + v(u)\, u_t + V'(u) = 0. \tag{4}$$

All these equations are dissipative; namely, for Eqs. (1) and (2) one can write, at least formally, the following decreases of energies: in the parabolic case,

$$\partial_t \int \left(\frac{u_x^2}{2} + V(u) \right) dx = -\int u_t^2 \, dx, \tag{5}$$

and in the hyperbolic case,

$$\partial_t \int \left(\frac{u_x^2}{2} + \frac{u_t^2}{2} + V(u) \right) dx = -\int v(u)\, u_t^2 \, dx. \tag{6}$$

The consequence of this dissipativeness is that the dynamics of these partial differential equations is not truly infinite-dimensional (high-frequency Fourier modes are damped) and therefore there is some hope to understand it completely.

A natural approach is to try to establish "correspondence rules" between the dynamics of the differential equations (for which we have a complete understanding) and that of the partial differential equations. The present work is a attempt to provide some results in this spirit: we will show that to local minima of the potential V correspond attractive fixed points for Eqs. (1) and (2) (as is the case for the differential Eqs. (3) and (4)) and we will try to describe the asymptotic dynamics of (part of) the points which belong to the borders of the basins of attraction of these attractive fixed points.

The results presented here were at the origin motivated by the work [1] of Argentina *et al.* There these authors studied the transition between annihilation and reflection at the collision of two bistable fronts, for the damped sine-Gordon equation with a torque,

$$u_{tt} + v u_t + \sin u - \Omega = u_{xx} \tag{7}$$

(the damping $v > 0$ and the torque $0 < \Omega < 1$ being constants). They showed the existence of a critical damping $v_c > 0$ such that, for $v > v_c$, the fronts annihilate each other, for $v < v_c$, they reflect, and for $v = v_c$, the solution displays an intermediate behavior, and converges towards a solution which they called "nucleation solution" (meaning that it is the solution through which one can "nucleate" the next local minimum of the potential). The present paper originated from efforts towards a rigorous justification (still in progress) of this behavior. Roughly speaking, we will show here that this "nucleation solution" is in some sense the natural attractor of the border of the basin of attraction corresponding to a local minimum of the potential.

Of course the methods will rely strongly on the expressions (5) and (6) which enable to consider Eqs. (1) and (2) at least formally as gradient-like systems. Nevertheless, as we want to be able to consider front-like solutions, we will have to work with infinite extensions (the space-variable x will belong to $\mathbf{R}$) and uniformly local functional spaces (containing all uniformly bounded and uniformly sufficiently smooth functions); although natural, this framework yields certain difficulties due to the fact that the energy is in general not finite.

2. Let $\rho\colon \mathbf{R} \to \mathbf{R}$ be a weight function (ρ is of class C^∞, has compact support, takes only non-negative values, and is not identically zero); for x, y in $\mathbf{R}$, write $T_x\rho(y) = \rho(y-x)$, and for $k \in \mathbf{N}$, let $H^k_{\mathrm{ul}}(\mathbf{R})$ be the set of functions u in $H^k_{\mathrm{loc}}(\mathbf{R})$ such that

$$\|u\|_{H^k_{\mathrm{ul}}(\mathbf{R})} = \sup_{x \in \mathbf{R}} \|T_x \rho u\|_{H^k(\mathbf{R})} < \infty$$

(different choices of ρ give rise to equivalent norms on the same space $H^k_{\mathrm{ul}}(\mathbf{R})$). Let

$$X = H^1_{\mathrm{ul}}(\mathbf{R}); \qquad Y = H^1_{\mathrm{ul}}(\mathbf{R}) \times L^2_{\mathrm{ul}}(\mathbf{R}),$$

and write $\|\cdots\|_X = \|\cdots\|_{H^1_{\mathrm{ul}}(\mathbf{R})}$ and $\|\cdots\|_Y = \|\cdots\|_{H^1_{\mathrm{ul}}(\mathbf{R}) \times L^2_{\mathrm{ul}}(\mathbf{R})}$.

The following results about existence, uniqueness, and properties of solutions are standard (see for instance [15, 3]).

Parabolic case. For any u_0 in X, Eq. (1) has a unique solution $u \in C^0([0; T_{\max}[, X)$ satisfying $u(0) = u_0$, defined on a maximal time interval $[0; T_{\max}[$, $0 < T_{\max} \leqslant +\infty$; if $T_{\max} < +\infty$, then $\|u(t)\|_X \to +\infty$ when $t \to T_{\max}$; for any $k \in \mathbf{N}$, $u_{|]0;\, T_{\max}[}$ actually belongs to $C^\infty(]0; T_{\max}[, H^k_{\mathrm{ul}}(\mathbf{R}))$; finally, times of existence are locally bounded from below, and solutions depend locally Lipschitz-continuously on initial conditions.

Let $\mathscr{P}_t(u_0) = u(t)$, $0 \leqslant t < T_{\max}$ denote the solution of (1) with initial condition u_0.

Hyperbolic case. For any $U_0 = (u_0, v_0) \in Y$, Eq. (2) has a unique (mild) solution $U \in C^0(]-T'_{\max}, T_{\max}[, Y)$ satisfying $U(0) = U_0$, defined on a maximal time interval $]-T'_{\max}, T_{\max}[$, $0 < T_{\max}, T'_{\max} \leqslant +\infty$; if $T_{\max}$ (resp. $T'_{\max}$) $< +\infty$, then $\|U(t)\|_Y \to +\infty$ when $t \to T_{\max}$ (resp. when $t \to -T'_{\max}$); writing $U(t) = (u(t), v(t))$, we have $u \in C^0(]-T'_{\max}, T_{\max}[, H^1_{\mathrm{ul}}(\mathbf{R})) \cap C^1(]-T'_{\max}, T_{\max}[, L^2_{\mathrm{ul}}(\mathbf{R}))$ and $\frac{du}{dt} = v$; for any $-T'_{\max} < t < T_{\max}$ and for any interval $[a; b]$ of $\mathbf{R}$, $u(t)_{|[a;\, b]}$ and $v(t)_{|[a;\, b]}$ actually only depend on the restriction of the initial condition on the interval

$[a-|t|; b+|t|]$ (initial data propagate at a speed less or equal to one); finally, times of existence are locally bounded from below, and solutions depend locally Lipschitz-continuously on initial conditions.

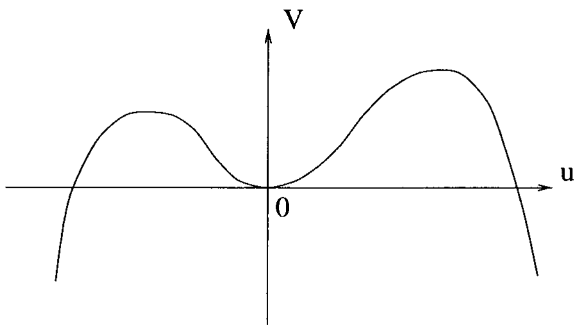
Let $\mathscr{H}_t(U_0) = U(t)$, $-T'_{\max} < t < T_{\max}$ denote the solution of (2) with initial condition U_0.

Global existence results require certain hypotheses on the behavior of V and v at infinity (for instance $V(u) > -C(1+|u|^2)$ and $v(u) > C^{-1}$, for a constant $C > 0$) which we don't want to make here. Also, local (similarly global) existence results can be obtained in both cases for more general (less regular) initial conditions, but then the use of the energy functionals we shall use later becomes certainly more delicate.

3. We now make, for the rest of the paper, the following hypothesis:

$$0 \text{ is a strict local minimum of } V \tag{8}$$

(in other words there exists $\varepsilon > 0$ such that, for $u \in]-\varepsilon; 0[\cup]0; \varepsilon[$, $V(u) > V(0)$); we have $V'(0) = 0$ and we will suppose (without loss of generality) that $V(0) = 0$. The point $u(x) \equiv 0$ of X (resp. $(u(x), v(x)) \equiv (0, 0)$ of Y) defines a (homogeneous) stationary solution of the parabolic equation (1) (resp. the hyperbolic equation (2)).



Notation. Write

$$\mathscr{B}_{0,\,\mathrm{par}} = \{u \in X \mid \|\mathscr{P}_t(u)\|_X \to 0 \text{ when } t \to +\infty\},$$
$$\mathscr{B}_{0,\,\mathrm{hyp}} = \{U \in Y \mid \|\mathscr{H}_t(U)\|_Y \to (0, 0) \text{ when } t \to +\infty\},$$

and denote by $\partial\mathscr{B}_{0,\,\mathrm{par}}$ (resp. $\partial\mathscr{B}_{0,\,\mathrm{hyp}}$) the border of $\mathscr{B}_{0,\,\mathrm{par}}$ (resp. of $\mathscr{B}_{0,\,\mathrm{hyp}}$) for the topology induced by $\|\cdots\|_X$ (resp. by $\|\cdots\|_Y$).

PROPOSITION 1. *Suppose that there exists* $\varepsilon > 0$ *such that, for* $u \in]-\varepsilon; \varepsilon[\setminus\{0\}$, $V'(u)\,u > 0$; *then,* $\mathscr{B}_{0,\,\mathrm{par}}$ *is open in* X *for* $\|\cdots\|_X$.

Suppose moreover that one of the two following hypotheses is satisfied:

- $V''(0) > 0$;
- *the damping* ν *is constant* (*it does not depend on* u);

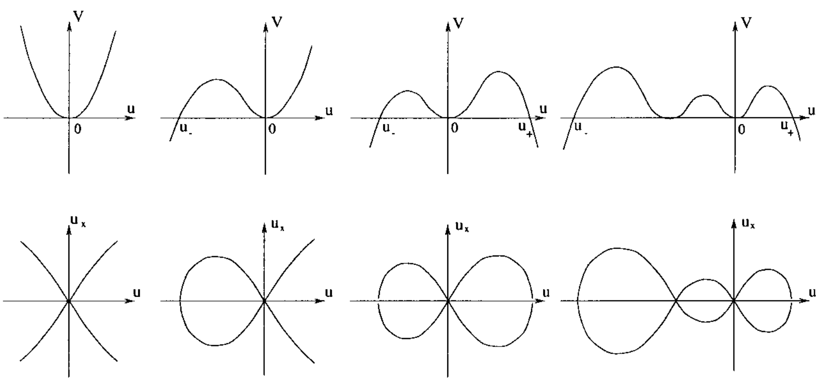
then $\mathscr{B}_{0,\,\text{hyp}}$ *is open in* Y *for* $\| \cdots \|_Y$.

Of course, one would like to prove that $\mathscr{B}_{0,\,\text{hyp}}$ is open even when $V''(0) = 0$ and the damping ν is non-constant, but for technical reasons, we have not been able to obtain this result.

4. The main goal of this paper is to try to describe the asymptotic dynamics of certain solutions which belong to $\partial\mathscr{B}_{0,\,\text{par}}$ or $\partial\mathscr{B}_{0,\,\text{hyp}}$. We will use the decreases of energy functionals in order to prove that these solutions converge locally to stationary solutions; in both cases (Eq. (1) or (2)) stationary solutions $u(x)$ obey the equation

$$u'' = V'(u), \tag{9}$$

which represents a conservative oscillator in the potential $-V$; of particular interest for us will be the energy-0 hypersurface in the phase space $\mathbf{R}^2 = \{(u, u')\}$ of this equation. Here are examples of the possible shapes of this energy hypersurface for various potentials V.



In order to state precise results, we will have to introduce some notations related to this hypersurface; let

- $N = \{u \in \mathbf{R} \mid V(u) < 0\}$;
- $u_+ = \inf N \cap \mathbf{R}_+$ if this set is non empty, $+\infty$ otherwise;
- $u_- = \sup N \cap \mathbf{R}_-$ if this set is non empty, $-\infty$ otherwise;
- $Z = \{u \in \,]u_-; u_+[\,$ s.t. $V(u) = 0\}$;

the set Z contains 0 and, for any $u \in Z$, we have $V'(u) = 0$ and $V''(u) \geqslant 0$. We will make the following hypotheses :

$$\text{the set } Z \text{ is a discrete subset of } \mathbf{R}, \tag{10}$$

$$\text{if } u_+ (\text{resp. } u_-) \text{ is finite, then } V'(u_+) < 0 \ (\text{resp. } V'(u_-) > 0) \tag{11}$$

(these hypotheses are not very restrictive: (10) is always satisfied if the potential is real-analytic, and (11) is generic; however, they could probably be released with a bit of additional work).

Denote by S the hypersurface of energy 0 in the phase space $\mathbf{R}^2$ for Eq. (9); the set S is the union of trajectories of solutions of this equation; let S_{b} be the subset of S which is the union of those of these trajectories which are bounded in $\mathbf{R}^2$, and let $S_{\text{b},0}$ be the connected component containing $(0, 0)$ of S_{b}. The set $S_{\text{b},0}$ is the union of:

- $\{(u, 0) \mid u \in Z\}$;
- if Z is not reduced to $\{0\}$, for each pair of consecutive points of Z, the trajectories of two solutions, heteroclinic to these two points (these two trajectories are symmetric with respect to the u-axis);
- if $u_+ < +\infty$, the trajectory of a solution homoclinic to $\max(Z)$;
- if $u_- > -\infty$, the trajectory of a solution homoclinic to $\min(Z)$

($S_{\text{b},0}$ is reduced to $\{(0, 0)\}$ if and only if 0 is a strict global minimum of V).

Let H be the set of functions $u(x)$, $x \in \mathbf{R}$ which are (homoclinic or heteroclinic) non-constant solutions of the stationary Eq. (9), whose trajectories in the phase space belong to $S_{\text{b},0}$, and which are normalized with respect to translation invariance (for instance the following way : if $h \in H$ is a heteroclinic solution between u_i and u_j, one requires that $h(0) = (u_i + u_j)/2$, and if it is a homoclinic solution, one requires that h be even). Remark that, if 0 is not a global minimum of V, H is generically reduced to one or two homoclinic solutions. For instance, in the case of the dampted sine-Gordon equation with torque (7), the set H corresponding to a local minimum of the potential is reduced to one homoclinic solution; it is this solution which was called in [1] "nucleation solution".

We are now in position to define the possible asymptotic behaviors for solutions in the border of $\mathscr{B}_{0,\text{par}}$ or $\mathscr{B}_{0,\text{hyp}}$. Let $\theta: \mathbf{R} \to \mathbf{R}$ be a C^∞ cut-off function satisfying $\theta(x) = 0$ for $x \leqslant -1$ and $\theta(x) = 1$ for $x \geqslant 1$.

Definition. Let $\mathscr{A}$ be the set of functions $u \in C^0([0; +\infty[, H^1_{\text{ul}}(\mathbf{R}))$ such that there exists $n \in \mathbf{N}$, $n \geqslant 1$, $h_1, ..., h_n$ in H, and functions $x_1, ..., x_n$ in $C^1([0; +\infty[, \mathbf{R})$, satisfying $x_j'(t) \to 0$ when $t \to +\infty$ for $j = 1, ..., n$, with the following properties:

- $u(t)(x)=h_1(x-x_1)$ for $x\in\,]-\infty;x_1]$.
- $u(t)(x)=h_n(x-x_n)$ for $x\in[x_n;+\infty[$.
- if $n\geqslant 2$, for $j=1, ..., n-1$,

— $x_{j+1}(t)-x_j(t)$ is not smaller than 2 and converges to $+\infty$ when $t\to+\infty$.

— $\lim_{x\to+\infty} h_j(x)=\lim_{x\to-\infty} h_{j+1}(x)$.

— $u(t)(x)=(1-\theta(x-(x_j+x_{j+1})/2))\, h_j(x-x_j)+\theta(x-(x_j+x_{j+1})/2)\, h_{j+1}(x-x_{j+1})$ for $x\in[x_j;x_{j+1}]$.

Thus $\mathscr{A}$ is the set of functions which asymptotically look like the pasting of a finite number of homoclinic or heteroclinic solutions of the 0-energy hypersurface which move slowly away the ones from the others.

5. We can now state our main results.

THEOREM 1. *Suppose that V satisfies* (8), (10), (11), *and that $V''(0)>0$; then, for any $u\in X$, if*:

- $u\in\partial\mathscr{B}_{0,\,\mathrm{par}}$,
- *writing* $l_+=\limsup_{|x|\to+\infty} u(x)$ *and* $l_-=\liminf_{|x|\to+\infty} u(x)$, *we have: if $l_+>0$, then $V'(v)>0$ for any $v\in\,]0;l_+]$ and if $l_-<0$, then $V'(v)<0$ for any $v\in[l_-;0[$,*
- *there exists $\varepsilon>0$ such that, for any $v\in X$ satisfying $\|v-u\|_X<\varepsilon$, $\|\mathscr{P}_t v\|_{L^\infty(\mathbf{R})}$ is bounded independently of v and of $t\geqslant 0$,*

then there exists $\tilde{u}\in\mathscr{A}$ such that $\|\mathscr{P}_t(u)-\tilde{u}(t)\|_X\to 0$ when $t\to+\infty$.

THEOREM 2. *Suppose that V satisfies* (8), (10), (11), *and that $V''(0)>0$; then, for any $U=(u,v)\in Y$, if*:

- $U\in\partial\mathscr{B}_{0,\,\mathrm{hyp}}$,
- $\limsup_{|x_0|\to+\infty}\int_{\mathbf{R}} T_{x_0}\rho(u^2+u_x^2+v^2)\,dx<\varepsilon$ ($T_x\rho$ *was defined earlier in number* 2), *where $\varepsilon>0$ is a small constant* (*depending on V, v, and ρ*),
- *writing $\mathscr{H}_t(U)=(u(t),v(t))$, $\|u(t)\|_{L^\infty(\mathbf{R})}$ is bounded independent of $t\geqslant 0$,*

then there exists $\tilde{u}\in\mathscr{A}$ such that $\|\mathscr{H}_t(U)-(\tilde{u}(t),0)\|_Y\to 0$ when $t\to+\infty$.

In the two following theorems, we get rid of the hypothesis $V''(0)>0$, but, on the other hand, we only deal with solutions which are of finite energy and which admit small energy perturbations in $\mathscr{B}_{0,\,\mathrm{par}}$ or $\mathscr{B}_{0,\,\mathrm{hyp}}$.

THEOREM 3. *Suppose that V satisfies* (8), (10), (11); *then, for any $u \in X$, if*:

- $u \in \partial\mathscr{B}_{0,\,\mathrm{par}}$ *and for any* $\alpha > 0$, *there exists* $v \in \mathscr{B}_{0,\,\mathrm{par}}$ *such that* $\|v-u\|_{H^1(\mathbf{R})} < \alpha$,
- $|u(x)| \to 0$ *when* $|x| \to +\infty$ *and* $\int_{\mathbf{R}}((u_x^2/2)+V(u))\,dx < +\infty$,
- $\|\mathscr{P}_t u\|_{L^\infty(\mathbf{R})}$ *is bounded independently of* $t \geq 0$,

then there exists $\tilde{u} \in \mathscr{A}$ *such that* $\|\mathscr{P}_t(u) - \tilde{u}(t)\|_X \to 0$ *when* $t \to +\infty$.

THEOREM 4. *Suppose that V satisfies* (8), (10), (11); *then, for any* $U=(u, v) \in Y$, *if*:

- $U \in \partial\mathscr{B}_{0,\,\mathrm{hyp}}$ *and for any* $\alpha > 0$, *there exists* $W \in \mathscr{B}_{0,\,\mathrm{hyp}}$ *such that* $\|W-U\|_{H^1(\mathbf{R})\times L^2(\mathbf{R})} < \alpha$,
- $|u(x)| \to 0$ *when* $|x| \to +\infty$ *and* $\int_{\mathbf{R}}((u_t^2/2)+(u_x^2/)2)+V(u))\,dx < +\infty$,
- *writing* $\mathscr{H}_t(U) = (u(t), v(t))$, $\|u(t)\|_{L^\infty(\mathbf{R})}$ *is bounded independently of* $t \geq 0$,

then there exists $\tilde{u} \in \mathscr{A}$ *such that* $\|\mathscr{H}_t(U) - (\tilde{u}(t), 0)\|_Y \to 0$ *when* $t \to +\infty$.

The method to prove these results is roughly the following. The fact that the solution we consider is accumulated by solutions which converge to 0 implies that its energy always remains non-negative. On the other hand, by (5), (6), energy decreases at a rate proportional to $\|u_t\|_{L^2(\mathbf{R})}$; this enables to prove that, roughly speaking, u_t converges to 0 (indeed, when the energy is infinite, we will use finiteness of a certain localized energy), and in view of Eqs. (1) and (2), this implies local convergence to stationary solutions. The more precise asymptotic behavior will be obtained by continuity (homotopy) arguments in the phase space of Eq. (9) (this part will be common to the parabolic and the hyperbolic cases).

A result very close to Theorem 1 was obtained by Fife in [11] (see also [10]); there he obtained even more precise results on the global dynamics of the parabolic Eq. (1), but for a less general (bistable) potential; some of our arguments are directly inspired by those of [11, Sect. 3.1.2]; however, the arguments used in [11] to obtain the precise asymptotic behavior corresponding to our results rely strongly on the maximum principle for parabolic equations, and thus do not extend to the hyperbolic case (contrarily to our arguments, see Sect. 4.1).

Other results in the same spirit, both for the parabolic and the hyperbolic equation, were obtained by Feireisl (see [7, 8]), Feireisl and Petzeltovà [9], including the situation where the dimension d in the space variable x is more than one (in this case, the object corresponding to a solution homoclinic to 0 for Eq. (9) is called a “ground state” for the elliptic stationary solution); these authors used finiteness of energy and a concentration-compactness principle to prove local convergence to ground state solutions for subsequences of times converging to infinity; in the case $d=1$ however, our results are more general and more precise.

It is likely that Theorems 1 and 2 remain true without the assumption $V''(0)>0$, but we were not able to prove this. Also, we were not able to get rid of the assumption about the bound in $L^\infty(\mathbf{R})$ on the solution. On the other hand, there are several situations where such a bound can be proved, either by using maximum principles (see Section 2) or by energy estimates; for instance we will prove the following result.

PROPOSITION 2. (1) *Suppose there exists a constant $C>0$ such that, for $u>C$, $V'(u)>0$, and for $u<-C$, $V'(u)<0$; then, the semi-flow $(\mathscr{P}_t)_{t\geqslant 0}$ associated to Eq.* (1) *on X is globally defined (one has a global existence result), and admits an attractive ball in $X=H^1_{\mathrm{ul}}(\mathbf{R})$.*

(2) *Suppose there exists a constant $C>0$ such that $C^{-1}<\nu(\cdot)<C$, and, for $|u|>C$, $C^{-1}<V''(u)<C$; then, the flow $(\mathscr{H}_t)_{t\in\mathbf{R}}$ associated to Eq.* (2) *on Y is globally defined, and admits an attractive ball in $Y=H^1_{\mathrm{ul}}(\mathbf{R})\times L^2_{\mathrm{ul}}(\mathbf{R})$.*

The conclusions of this proposition imply uniform asymptotic bounds in $L^\infty(\mathbf{R})$ for the solutions.

Finally, we consider the case of a global minimum.

THEOREM 5. *Suppose that 0 is a strict global minimum (i.e. $V(v)>0$ for $v\neq 0$) and that* $\liminf_{|v|\to+\infty} V(v)>0$; *then,*

(1) *for any $u\in X$, write $l_+=\limsup_{|x|\to+\infty} u(x)$ and $l_-=\liminf_{|x|\to+\infty} u(x)$; suppose that the following holds:*

- *if $l_+>0$, then for any $v\in\,]0;l_+]$, $V'(v)>0$*
- *if $l_-<0$, then for any $v\in[l_-;0[$, $V'(v)<0$*

then $\|\mathscr{P}_t(u)\|_X\to 0$ when $t\to+\infty$.

(2) *for any $U=(u,v)\in Y$, if one of the two following conditions holds:*

(a) $\int_{\mathbf{R}}((v^2/2)+(u_x^2/2)+V(u))\,dx<+\infty$,

(b) $V''(0)>0$ *and* $\limsup_{|x_0|\to+\infty}\int_{\mathbf{R}} T_{x_0}\rho(u^2+u_x^2+v^2)\,dx<\varepsilon$, *where $\varepsilon>0$ is a small constant (depending on V, ν, and ρ),*

then $\|\mathscr{H}_t(U)\|_Y\to(0,0)$ when $t\to+\infty$.

It is likely that this theorem remains true without assuming that the global minimum 0 is strict (i.e., just assuming that $V(u) \geqslant 0$ for any $u \in \mathbf{R}$), without the assumptions on $\liminf_{v \to \pm\infty} V(v)$, and finally, for the second part (case (b)), without the assumption $V''(0) > 0$.

6. These results (Theorems 1 to 4) raise questions about the dynamics of the "metastable patterns" in the set $\mathscr{A}$ describing the possible asymptotic shapes of the solutions considered. The dynamics of metastable patterns ("multi-kinks," which we called: pasting of "heteroclinic solutions between local minima at the same depth") has been precisely investigated by Carr and Pego (on a finite interval, see [2], and see [6] for the extension of their method to the real line); these authors characterized accurately the dynamics by a simple potential model. It is natural to wonder whether their method applies to the patterns which appear in the set $\mathscr{A}$. The main difference (regardless the fact that they considered only the parabolic case) is that the patterns in $\mathscr{A}$ can be (locally), codimension-one unstable (when solutions homoclinic to a local minimum occur).

Results in this direction could enable to eliminate a priori certain patterns in the definition of the set $\mathscr{A}$ of "possible asymptotic patterns", precisely all those whose prescribed dynamics would be in contradiction with the fact that the distance between two stationary solutions appearing consecutively in a pattern must tend to $+\infty$ (see the definition of $\mathscr{A}$). Actually, even for the particular problem considered in [1] (transition between annihilation and reflection at the collision of two fronts for Eq. (7)) we were not able to prove that the solution converges towards one nucleation solution (but only towards a "pasting" of a finite number of nucleation solutions)).

7. To finish this introduction, let us rapidly review some points where, from our point of view, certain statements should be improved, or certain hypotheses should be weakened.

In Proposition 1, in the hyperbolic case, one should get rid of the hypothesis $V''(0) > 0$ or $v(\cdot)$ constant. This is related to the maximum principle for hyperbolic equations (see Sect. 2.2), which we were not able to use when $v(\cdot)$ is non-constant. In Theorems 1 and 2, one should get rid of the hypothesis $V''(0) > 0$, and, in Theorems 1, 2, 3, 4, one should get rid of hypothesis (11) on $V(\cdot)$. This requires a better understanding of the propagation of energy in space. Recent ideas of Slijepcevic (see [14]) could be helpful. One should be more precise, as explained in the previous paragraph, on the possible asymptotic behaviors of the considered solutions (i.e. one should provide a more restrictive definition of the set $\mathscr{A}$). Finally, in the hyperbolic case, Proposition 2 and Theorem 5 should be improved (in Theorem 5, b), one should get rid of the hypothesis $V''(0) > 0$).

8. The paper is organized as follows. In Section 2, we use maximum principles and energy functionals to prove Propositions 1 and 2. In Section 3, we prove that the solutions considered converge locally towards stationary solutions (Sect. 3.1 for the parabolic equation and Sect. 3.2 for the hyperbolic equation). We complete the proof of the main results (precise asymptotic behavior) in Section 4.1, and we prove Theorem 5 in Section 4.2.

2. ENERGY ESTIMATES AND MAXIMUM PRINCIPLES

Notation. Throughout the rest of the paper, integrals $\int \cdots$ without more specifications will always mean $\int_{\mathbf{R}} \cdots dx$. Solutions of Eqs. (1) and (2) will be in general introduced directly as functions $u(x, t)$ or $u(\cdot, t)$ of the two variables x and t, rather than as functions of t with values in the space X or Y.

2.1. *The Parabolic Equation*

Functions from the space $X = H^1_{\mathrm{ul}}(\mathbf{R})$ in general do not have a finite energy; we will thus use localized energy functionals (this method was extensively used by Collet in [4]).

Consider a solution $u(\cdot, t)$ of Eq. (1) on a maximal interval $[0; T_{\max}[$. For any smooth function φ such that φ and φ' belong to $L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, we have

$$\partial_t \int \varphi \left(\frac{u_x^2}{2} + V(u) \right) = -\int \varphi u_t^2 - \int \varphi' u_t u_x,$$

$$\partial_t \int \varphi \frac{u^2}{2} = -\int \varphi (u_x^2 + V'(u)\, u) - \int \varphi' u u_x$$

(these equalities are obtained by multiplying Eq. (1) respectively by φu_t and φu and integrating on $\mathbf{R}$).

Let ϕ be any smooth function in $L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ satisfying $|\phi'(x)| \leqslant \phi(x)$, $x \in \mathbf{R}$ (for instance we could take $\phi(x) = \frac{1}{1+x^2}$). Take $\varepsilon > 0$ to be chosen later and $x_0 \in \mathbf{R}$; write $\phi_\varepsilon(x) = \phi(\varepsilon x)$, $T_{x_0} \phi_\varepsilon(x) = \phi_\varepsilon(x - x_0)$, and let φ be the function $T_{x_0} \phi_\varepsilon$; remark that $|\varphi'(x)| \leqslant \varepsilon \varphi(x)$.

Define the function $\tilde{E}_{\mathrm{par}}(x, t)$ by $\tilde{E}_{\mathrm{par}} = u^2/2 + u_x^2/2 + V(u)$. The two equalities above yield

$$\partial_t \int \varphi \tilde{E}_{\mathrm{par}} \leqslant -\int \varphi \left(\left(1 - \frac{\varepsilon}{2}\right) u_t^2 + (1 - \varepsilon)\, u_x^2 + \left(V'(u)\, u - \frac{\varepsilon}{2} u^2 \right) \right). \tag{12}$$

An important tool for the study of Eq. (1) is the well-known maximum principle (see [16]); the following statement is taken from [5].

PROPOSITION 3. *Suppose that u_1 and u_2 are bounded continuous functions in $(x_1; x_2)\times[0; t_0]$ for some $t_0>0$, the quantities x_1 and x_2 being finite or infinite. Suppose that*

$$\partial_t u_1 + V'(u_1) - \partial_x^2 u_1 \leqslant \partial_t u_2 + V'(u_2) - \partial_x^2 u_2$$

and

$$u_1(x,0) \leqslant u_2(x,0), \qquad x\in\mathbf{R};$$

if x_i, $i=1, 2$ is finite, assume furthermore

$$u_1(x_i,t) \leqslant u_2(x_i,t), \qquad t\in[0; t_0];$$

then, for any $t\in[0; t_0]$,

$$u_1(x,t) \leqslant u_2(x,t), \qquad x\in\mathbf{R}.$$

In the rest of this paper, we will call *super-solutions* (resp. *sub-solutions*) for Eq. (1) functions $u(x,t)$ satisfying $\partial_t u + V'(u) - \partial_x^2 u \geqslant 0$ (resp. $\leqslant 0$).

Proof of Proposition 2 for the parabolic equation. Let $u(\cdot, t)$ be any solution of (1) on a maximal interval $[0; T_{\max}[$. By hypothesis, there exists a constant $C>0$ such that $V'(v)<0$ for $v\leqslant -C$ and $V'(v)>0$ for $v\geqslant C$. Thus, by the maximum principle stated above, we have $T_{\max}=+\infty$, and, for any $t\geqslant 0$, $\|u(\cdot,t)\|_{L^\infty(\mathbf{R})} \leqslant \max(C, \|u(\cdot,0)\|_{L^\infty(\mathbf{R})})$; moreover, there exists $\tau>0$ (depending on $u(\cdot,\cdot)$) such that, for $t>\tau$, $\|u(\cdot,t)\|_{L^\infty(\mathbf{R})} \leqslant C$.

For $\varepsilon=1/2$, inequality (12) yields

$$\partial_t \int \varphi \tilde{E}_{\text{par}} \leqslant -\int \varphi \left(\frac{u_x^2}{2} + V'(u)\,u - \frac{u^2}{4}\right).$$

There exists a constant $C_1>0$ such that, for $u\in[-C; C]$, $-(V'(u)\,u - \frac{u^2}{4}) \leqslant C_1 - (V(u) + \frac{u^2}{2})$; thus, for $t>\tau$,

$$\partial_t \int \varphi \tilde{E}_{\text{par}} \leqslant -\int \varphi \tilde{E}_{\text{par}} + C_1 \int \varphi.$$

Recall that $\varphi = T_{x_0}\phi_\varepsilon$. This inequality shows that $\sup_{x_0\in\mathbf{R}} \int T_{x_0}\phi_\varepsilon \tilde{E}_{\text{par}}$ eventually becomes smaller than $1 + C_1\int\varphi$. Thus, as $\|u(\cdot,t)\|_{L^\infty(\mathbf{R})}$ is bounded, an analogous bound holds for $\sup_{x_0\in\mathbf{R}} \int T_{x_0}\phi_\varepsilon u_x^2$, and Proposition 2 follows. ∎

Proof of Proposition 1 *for the parabolic equation.* Let $u(\cdot, t)$ be a solution of Eq. (1) on a maximal interval $[0; T_{\max}[$. By the maximum principle stated above, and by the fact that $V'(v)\, v > 0$ for $v \neq 0$ small, if $\|u(\cdot, 0)\|_X$ is sufficiently small, then $\|u(\cdot, t)\|_{L^\infty(\mathbf{R})} \to 0$ when $t \to +\infty$.

If $V''(0) > 0$, (12) forces exponential convergence to 0 for $\|u(\cdot, t)\|_X$, and the result follows.

If $V''(0) = 0$, (12) yields

$$\partial_t \int \varphi \tilde{E}_{\text{par}} \leqslant -(1-\varepsilon) \int \varphi u_x^2 + \frac{\varepsilon}{2} \int \varphi u^2,$$

and thus (for say $\varepsilon = 1/2$),

$$\partial_t \int \varphi \tilde{E}_{\text{par}} \leqslant -\int \varphi \tilde{E}_{\text{par}} + \int \varphi(u^2 + V(u)).$$

Denote by ψ the function $t \mapsto \int \varphi \tilde{E}_{\text{par}}$ and by θ the function $t \mapsto \int \varphi(u^2 + V(u))$. By the variation of the constant formula, we have, for any $t > 0$ and $s > 0$,

$$\psi(t+s) \leqslant e^{-s} \theta(t) + \int_0^s e^{-(s-r)} \theta(t+r)\, dr.$$

As $\theta(t) \to 0$ when $t \to +\infty$, we have $\psi(t) \to 0$ when $t \to +\infty$, and the result follows. ∎

2.2. *The Hyperbolic Equation*

Write $\mathscr{V}(u) = \int_0^u v(v)\, dv$, $u \in \mathbf{R}$.

We proceed like in the parabolic case. Consider a solution $u(\cdot, t)$ of Eq. (2) on a maximal time interval $]-T'_{\max}; T_{\max}[$. For any smooth function φ such that φ and φ' belong to $L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, the functions

$$t \mapsto \int \varphi \left(\frac{u_t^2}{2} + \frac{u_x^2}{2} + V(u) \right) \qquad \text{and} \qquad t \mapsto \int \varphi(u u_t + \mathscr{V}(u)\, u)$$

are of class C^1, and we have

$$\partial_t \int \varphi \left(\frac{u_t^2}{2} + \frac{u_x^2}{2} + V(u) \right) = -\int \varphi v(u)\, u_t^2 - \int \varphi' u_t u_x$$

and

$$\partial_t \int \varphi(u u_t + \mathscr{V}(u)\, u) = \int \varphi(-u_x^2 + u_t^2 + \mathscr{V}(u)\, u_t - V'(u)\, u) - \int \varphi' u u_x$$

(these equalities are obtained by multiplying Eq. (2) respectively by φu_t and φu and integrating on $\mathbf{R}$).

As in Section 2.1, we suppose that $\varphi = T_{x_0}\phi_\varepsilon$, $\varepsilon > 0$, $x_0 \in \mathbf{R}$. Take any $\gamma > 0$, and write

$$\tilde{E}_{\text{hyp}} = \frac{u_t^2}{2} + \frac{u_x^2}{2} + V(u) + \gamma(uu_t + \mathscr{V}(u)\, u).$$

The preceding equalities yield

$$\partial_t \int \varphi \tilde{E}_{\text{hyp}} \leqslant -\int \varphi \left(\left(\nu(u) - \gamma - \frac{\varepsilon}{2} \right) u_t^2 + \left(\gamma \left(1 - \frac{\varepsilon}{2} \right) - \frac{\varepsilon}{2} \right) u_x^2 \right.$$
$$\left. - \gamma \mathscr{V}(u)\, u_t + \gamma \left(V'(u)\, u - \frac{\varepsilon}{2} u^2 \right) \right). \tag{13}$$

Proof of Proposition 2 *for the hyperbolic equation.* Let $u(\cdot, t)$ be any solution of Eq. (2) on a maximal time interval $]-T'_{\max}; T_{\max}[$. In the following estimates, $C_1, C_2, \ldots$ denote (large) positive constants which are independent of $u(\ ,\ )$.

According to the hypotheses, we have $C_1^{-1} \leqslant \nu(\cdot) \leqslant C_1$ and, for any $v \in \mathbf{R}$, $V'(v)\, v \geqslant C_1^{-1} v^2 - C_1$. Thus we have

$$\gamma \mathscr{V}(u)\, u_t \leqslant C_1^{-1} \frac{u_t^2}{2} + C_1 \frac{\gamma^2 \mathscr{V}(u)^2}{2} \leqslant C_1^{-1} \frac{u_t^2}{2} + C_1^3 \frac{\gamma^2 u^2}{2}.$$

According to (13), we thus have, for γ and ε sufficiently small (γ depending on C_1, and ε depending on C_1 and γ),

$$\partial_t \int \varphi \tilde{E}_{\text{hyp}} \leqslant -\frac{1}{C_2} \int \varphi (u_t^2 + u_x^2 + u^2) + \gamma C_1 \int \varphi, \tag{14}$$

where C_2 depends on the choice of γ. On the other hand, the hypotheses show that

$$\tilde{E}_{\text{hyp}} \leqslant C_3(1 + u_t^2 + u_x^2 + u^2).$$

Thus $-(u_t^2 + u_x^2 + u^2) \leqslant 1 - C_3^{-1} \tilde{E}_{\text{hyp}}$ and (14) yields

$$\partial_t \int \varphi \tilde{E}_{\text{hyp}} \leqslant -\frac{1}{C_4} \int \varphi \tilde{E}_{\text{hyp}} + C_4, \tag{15}$$

which shows that $\sup_{x_0 \in \mathbf{R}} \int T_{x_0} \phi_\varepsilon \tilde{E}_{\text{hyp}}$ eventually becomes smaller than $1 + C_4^2$. Finally, according to the hypotheses, we have (for γ sufficiently small, depending on C_1)

$$\tilde{E}_{\text{hyp}} \geqslant \frac{1}{C_5} (u_t^2 + u_x^2 + u^2) - C_5$$

and the result follows. ∎

Proof of Proposition 1 *for the hyperbolic equation when* $V''(0) > 0$. Consider a solution $u(\cdot, t)$ of Eq. (2) on a maximal time interval $]-T'_{\max}; T_{\max}[$. At a time t_0 where $\|u(\cdot, t_0)\|_{L^\infty(\mathbf{R})}$ is sufficiently small, we have, for γ sufficiently small,

$$C^{-1}(u_t^2 + u_x^2 + u^2) \leqslant \tilde{E}_{\text{hyp}} \leqslant C(u_t^2 + u_x^2 + u^2), \tag{16}$$

and, according to (13) (for γ and ε sufficiently small, ε depending on γ),

$$\partial_t \int \varphi \tilde{E}_{\text{hyp}} \leqslant -\frac{1}{C} \int \varphi (u_t^2 + u_x^2 + u^2) \leqslant -\frac{1}{C^2} \int \varphi \tilde{E}_{\text{hyp}}, \tag{17}$$

where C is a positive constant.

Now, suppose that $\|(u(\cdot, 0), u_t(\cdot, 0))\|_Y$ is small; then, at $t = 0$, $\sup_{x_0 \in \mathbf{R}} \int T_{x_0} \phi_\varepsilon \tilde{E}_{\text{hyp}}$ is small, and $\|u(\cdot, 0)\|_{L^\infty(\mathbf{R})}$ is small. As long as $\|u(\cdot, t)\|_{L^\infty(\mathbf{R})}$ remains small, inequality (17) holds and shows that $\sup_{x_0 \in \mathbf{R}} \int T_{x_0} \phi_\varepsilon \tilde{E}_{\text{hyp}}$ decreases (in particular it remains small); but, in turn, the smallness of this supremum implies, by (16), that $\|u(\cdot, t)\|_{L^\infty(\mathbf{R})}$ is small. This shows that $\|u(\cdot, t)\|_{L^\infty(\mathbf{R})}$ remains small for all $t > 0$, and inequality (17) shows that $\sup_{x_0 \in \mathbf{R}} \int T_{x_0} \phi_\varepsilon \tilde{E}_{\text{hyp}}$ converges exponentially to 0, uniformly with respect to x_0. According to (16), the same is true for $\|(u(\cdot, t), u_t(\cdot, t))\|_Y$. ∎

Proof of Proposition 1 *for the hyperbolic equation when* $V''(0) = 0$ *and the damping* v *is constant.* This proof will require the use of a maximum principle for hyperbolic equations. First, we state a general result.

PROPOSITION 4. *Consider Eq.* (2) *with a constant and strictly positive damping* v:

$$u_{tt} + v u_t + V'(u) = u_{xx}; \tag{18}$$

suppose that V'' *is bounded from above and that*

$$\frac{v^2}{4} \geqslant \sup V''; \tag{19}$$

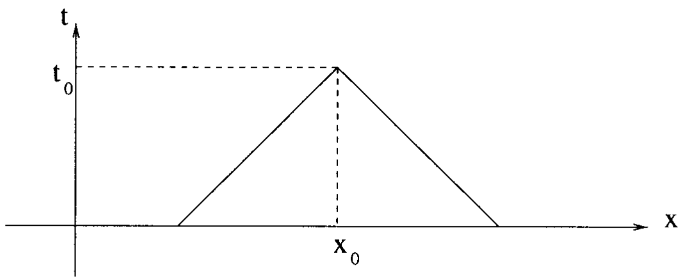
then, for any pair $u_1(\cdot, t)$ *and* $u_2(\cdot, t)$ *of smooth (classical) solutions of* (18) *on the triangle* $T = T(x_0, t_0) = \{(x, t) \in \mathbf{R}^2 \mid |x - x_0| \leqslant t_0 - t\}$, $x_0 \in \mathbf{R}$, $t_0 > 0$, *if*

- $u_1(x, 0) \leqslant u_2(x, 0)$, $x \in [x_0 - t_0; x_0 + t_0]$
- $u_{1,t}(x, 0) \leqslant u_{2,t}(x, 0) + \frac{\nu}{2}(u_2(x, 0) - u_1(x, 0))$, $x \in [x_0 - t_0; x_0 + t_0]$

then, for any $(x, t) \in T$, *we have*

$$u_1(x, t) \leqslant u_2(x, t).$$

Similar results for more general (mild) solutions can be deduced via density arguments.



This result has been pointed out to me by Th. Gallay (it is stated in a more general form in [13]); for completeness, we give a proof of it.

Proof. If $v(x, t)$ is a smooth function defined on the triangle $T = T(x, t) \subset T(x_0, t_0)$, we have the famous Ansatz (see [16])

$$v(x, t) = \tfrac{1}{2}(v(x - t, 0) + v(x + t, 0)) + \tfrac{1}{2}\int_I v_t\, dx + \tfrac{1}{2}\iint_T (v_{tt} - v_{xx})\, dx\, dt \tag{20}$$

where I denotes the segment between $(x - t, 0)$ and $(x + t, 0)$. On the other hand, if $u(x, t)$ is a smooth solution of (18), the change of variables $v(x, t) = e^{(\nu/2)t}u(x, t)$ yields

$$v_{tt} - v_{xx} = \frac{\nu^2}{4} v - e^{(\nu/2)t} V'(e^{-(\nu/2)t} v). \tag{21}$$

Let $u_1(x, t)$ and $u_2(x, t)$ be two smooth solutions of (18) on $T(x_0, t_0)$. Write $v_i(x, t) = e^{(\nu/2)t} u_i(x, t)$, $i = 1, 2$, and $\Delta v = v_2 - v_1$, $\Delta u = u_2 - u_1$. Equation (21) yields, if $\Delta v \geqslant 0$,

$$\Delta v_{tt} - \Delta v_{xx} \geqslant \left(\frac{\nu^2}{4} - \sup V''\right) \Delta v.$$

Thus, by (20), if $\Delta v \geqslant 0$ on T, we have

$$\Delta v(x, t) \geqslant \frac{1}{2}(\Delta u(x-t, 0) + \Delta u(x+t, 0)) + \frac{1}{2}\int_I \left(\Delta u_t + \frac{\nu}{2}\Delta u\right) dx$$
$$+\frac{1}{2}\left(\frac{\nu^2}{4} - \sup V''\right)\iint_T \Delta v\, dx\, dt, \tag{22}$$

which proves the proposition. ∎

We are now in position to prove Proposition 1 for the hyperbolic equation when $V''(0)=0$ and the damping ν is constant.

Let $u(\cdot, t)$ be a solution of (2) on a maximal interval $]-T'_{\max}; T_{\max}[$, and suppose that $\|(u(\cdot, 0)\, u_t(\cdot, 0))\|_Y \leqslant \alpha$, where $\alpha > 0$ has to be chosen later. The first step is to prove that, if α is sufficiently small, then $\|u(\cdot, t)\|_{L^\infty} \to 0$ when $t \to +\infty$. As the function $x \mapsto u_t(x, t)$ is a priori not regular (it belongs to $L^2_{\mathrm{ul}}(\mathbf{R})$), we will not be able to apply directly the proposition above, but we will use inequality (22), which remains true for mild solutions (in the space Y) of Eq. (2).

By hypothesis, there exists $w_0 > 0$ such that, for any $v \in [-w_0; 0[$, $V'(v) < 0$ and for any $v \in]0; w_0]$, $V'(v) > 0$; furthermore, as $V''(0) = 0$, we can choose w_0 small enough so that, for any $v \in [-w_0; w_0]$, $\frac{\nu^2}{4} \geqslant V''(v)$. Let $\bar{u}(t)$ (resp. $\underline{u}(t)$) be the solution of $u_{tt} + \nu u_t + V'(u) = 0$ with initial condition $\bar{u}(0) = w_0$ and $\bar{u}'(0) = 0$ (resp. $\underline{u}(0) = -w_0$ and $\underline{u}'(0) = 0$); then, $\bar{u}(t)$ (resp. $\underline{u}(t)$) converges to 0 when $t \to +\infty$.

Write $\Delta\bar{u}(x, t) = \bar{u}(t) - u(x, t)$, $\Delta\underline{u}(x, t) = u(x, t) - \underline{u}(x, t)$, $x \in \mathbf{R}$, $t \geqslant 0$, and write $\Delta\bar{v} = e^{(\nu/2)t}\Delta\bar{u}$, $\Delta\underline{v} = e^{(\nu/2)t}\Delta\underline{u}$. Fix $x \in \mathbf{R}$ and $t \geqslant 0$, and consider the triangle $T = T(x, t)$ defined above. Suppose that α is sufficiently small such that $\Delta\bar{u}(\cdot, 0) \geqslant w_0/2$ and $\Delta\underline{u}(\cdot, 0) \geqslant w_0/2$. Then, if $\Delta\bar{v} \geqslant 0$ on the triangle T, inequality (22) yields

$$\Delta\bar{v}(x, t) \geqslant \frac{1}{2}(\Delta\bar{u}(x-t, 0) + \Delta\bar{u}(x+t, 0))$$
$$+\frac{1}{2}\int_{x-t}^{x+t}\left(\Delta\bar{u}_t(y, 0) + \frac{\nu}{2}\Delta\bar{u}(y, 0)\right) dy,$$

and thus

$$\Delta\bar{v}(x, t) \geqslant \frac{w_0}{2} + \frac{\nu w_0}{4}t - \frac{1}{2}\int_{x-t}^{x+t} u_t(y, 0)\, dy$$
$$\geqslant \frac{w_0}{2} + \frac{\nu w_0}{4}t - \frac{1}{2}\sqrt{2t}\left(\int_{x-t}^{x+t} u_t^2(y, 0)\, dy\right)^{1/2}.$$

We see that, for α sufficiently small, the term $\frac{1}{2}\sqrt{2t}\,(\int_{x-t}^{x+t} u_t^2\,(y,0)\,dy)^{1/2}$ is dominated, for $t \leqslant 1$, by the term $\frac{w_0}{2}$, and, for $t > 1$, by the term $\frac{\nu w_0}{4}\,t$. This shows that $\Delta\bar{v}(\cdot,.)$ remains non-negative for all positive times. The same is true for $\Delta\underline{v}(\cdot,\cdot)$ by a similar argument. All this shows that $\|u(\cdot,t)\|_{L^\infty} \to 0$ when $t \to +\infty$.

Now, we see from the expression of $\tilde{E}_{\mathbf{hyp}}$ that, for γ sufficiently small and for $\|u(\cdot,t)\|_{L^\infty(\mathbf{R})}$ sufficiently small, we have

$$C^{-1}(u_t^2+u_x^2+u^2) \leqslant \tilde{E}_{\mathbf{hyp}} \leqslant (u_t^2+u_x^2+u^2),$$

and we deduce from (13) that, for γ and ε sufficiently small,

$$\partial_t \int \varphi \tilde{E}_{\mathbf{hyp}} \leqslant -C^{-1}\int \varphi(u_t^2+u_x^2)+\int \varphi u^2,$$

where C is a positive constant (depending on the choice of γ). As we know that $\|u(\cdot,t)\|_{L^\infty(\mathbf{R})} \to 0$ when $t \to +\infty$, the variation of the constant formula yields (as in Sect. 2.1) $\int \varphi \tilde{E}_{\mathbf{hyp}} \to 0$ when $t \to +\infty$, and the result follows. ∎

3. LOCAL CONVERGENCE TOWARDS STATIONARY SOLUTIONS

Denote by ρ any smooth function with compact support, non-negative values, satisfying $\rho(0)>0$. The aim of this section is to prove the following result.

PROPOSITION 5. *Consider a function $u(x,t)$, $x \in \mathbf{R}$, $t \geqslant 0$, with one of the two following hypotheses:*

1. $u(x,t) = \mathscr{P}_t(u_0)(x)$, $x \in \mathbf{R}$, $t \geqslant 0$, *where the potential V and the initial condition $u_0 \in X$ satisfy the assumptions of Theorems* 1 *or* 3.

2. $(u(x,t), u_t(x,t)) = \mathscr{H}_t(U_0)(x)$, $x \in \mathbf{R}$, $t \geqslant 0$, *where the potential V and the initial condition $U_0 \in Y$ satisfy the assumptions of Theorems* 2 *or* 4.

Write $E(x,t) = u_x^2/2 + V(u)$ in case 1 *above, and $E(x,t) = u_t^2/2 + u_x^2/2 + V(u)$ in case* 2. *Then, the following assertions hold.*

(1) *In the finite energy case* (*hypotheses of Theorem* 3 *or* 4), *for any $t \geqslant 0$, $\lim_{|x| \to +\infty} \int T_x\rho(u_x^2+u^2) = 0$, $\int E(x,t)\,dx$ converges to a non-negative limit when $t \to +\infty$, and the function $t \mapsto \int u_t^2$ is integrable on $\mathbf{R}_+$.*

In the infinite energy case (hypotheses of Theorem 1 *or* 2), *there exists a function* $L\colon \mathbf{R}_+ \to \mathbf{R}_+^*$, $L(t) \to +\infty$ *when* $t \to +\infty$, *such that* $\lim_{t\to+\infty} \sup_{|x|>L(t)} \int T_x\, \rho(u_x^2+u^2)=0$, $\int_{-L(t)}^{L(t)} E(x,t)\,dx$ *converges to a non-negative limit when* $t\to+\infty$, *and the function* $t \mapsto \int_{-L(t)}^{L(t)} u_t^2$ *is integrable on* $\mathbf{R}_+$.

(2) $\|u_t\|_{L^2_{ul}(\mathbf{R})}$ *converges to* 0 *when* $t\to+\infty$.

(3) *For any* $l>0$ *and* $\varepsilon>0$, *there exists* $T=T(\varepsilon,l)>0$ *such that, for any* $x_0\in\mathbf{R}$ *and* $t\geqslant T$, *there exists a stationary solution* $u_{\text{stat}}\colon \mathbf{R}\to\mathbf{R}$ *of Eq.* (9) *satisfying*

$$\|u(\cdot,t)-u_{\text{stat}}(\cdot)\|_{H^1([x_0-l;\,x_0+l])}\leqslant\varepsilon.$$

The proof of this proposition will be slightly different in the parabolic and in the hyperbolic case: we will take advantage, in the parabolic case, of the regularizing properties of the equation, and in the hyperbolic case, of the finite speed of propagation.

3.1. *Parabolic Equation*

The arguments in the two following sections are close to the ones of Fife in [11]. Estimates on u will be obtained using the maximum principle. For the estimates on derivatives (for instance Lemma 2), we will use localized energy estimates, but more general a priori estimates (for instance issued from [12]) could be used.

3.1.1. *Finite energy.* Suppose $u(\cdot,t)=\mathscr{P}_t(u_0)$, $t\geqslant 0$, where V and u_0 satisfy the hypotheses of Theorem 3.

We claim that, for any $t\geqslant 0$,

$$\lim_{|x|\to+\infty} |u(x,t)|=0 \qquad\text{and}\qquad \lim_{|x|\to+\infty} \int T_x\,\rho\, u_x^2=0. \tag{23}$$

Indeed, suppose that the converse is true and denote by t_0 the infimum of the set of positive times t for which (23) does not hold. By continuity of the trajectory $\mathscr{P}_t(u_0)$ in $H^1_{\text{ul}}(\mathbf{R})$, (23) holds for $t=t_0$. Take $t_1>t_0$ such that (23) does not hold at $t=t_1$. Again by continuity of the trajectory, the limit $\limsup_{|x|\to+\infty}|u(x,t_1)|$ is arbitrarily small if t_1 is chosen sufficiently close to t_0. If this limit is sufficiently small, then, in view of the shape of the potential V (hypothesis (8)), we get that the energy $\int E(x,t)\,dx$ is infinite at $t=t_1$, which is impossible (the energy is finite at $t=0$ and decreases afterwards by (5)).

The following Ansatz for Eq. (1),

$$v(\cdot,t)=e^{t\partial_{xx}}v_0+\int_0^t e^{(t-s)\,\partial_{xx}}(-V'(v(\cdot,s)))\,ds,$$

shows by Gronwall's lemma that the solutions depend locally Lipschitz-continuously, in the norm $H^1(\mathbf{R})$, on the initial conditions. As the initial condition u_0 is accumulated in $H^1(\mathbf{R})$ by solutions which converge to 0 as $t \to +\infty$, and thus whose energy remains finite and non-negative for all time, this implies that the energy of $\mathscr{P}_t(u_0)$ is actually non-negative for all time, and thus converges to a non-negative limit when $t \to +\infty$.

By (5), the function $t \mapsto \int u_t^2$ is integrable on $\mathbf{R}_+$, which completes the proof of assertion 1.

Thus, u_t roughly speaking converges to 0. We will establish this precisely.

Differentiating (1) yields (for $t > 0$)

$$u_{tt} = -V''(u)\, u_t + u_{xxt}, \tag{24}$$

and thus,

$$\partial_t \int \frac{u_t^2}{2} = -\int V''(u)\, u_t^2 - \int u_{xt}^2 \leqslant M \int u_t^2, \tag{25}$$

where $M = \max_{|u| \leqslant C} |V''(u)|$, C denoting the a priori bound on $\|u(\cdot, t)\|_{L^\infty(\mathbf{R})}$. This shows that $\int u_t^2$ converges to 0 when $t \to +\infty$, and proves assertion 2.

On the other hand, (25) also shows that $\int u_{xt}^2$ is integrable. Now, multiplying (24) by u_{tt} yields

$$\partial_t \int \frac{u_{xt}^2}{2} = -\int u_{tt}^2 - \int V''(u)\, u_t u_{tt} \leqslant -\int u_{tt}^2 + \int \left(\frac{V''(u)^2}{4} u_t^2 + u_{tt}^2 \right) \leqslant \frac{M^2}{4} \int u_t^2$$

which shows that $\int u_{xt}^2$ converges to 0, and thus that $\|u_t\|_{L^\infty(\mathbf{R})}$ converges to 0, when $t \to +\infty$.

Now, suppose that t is large; the preceding assertion shows that Eq. (1) can be seen as a perturbation of stationary Eq. (9) by a small forcing u_t. By continuity of the solutions of ordinary differential equations with respect to perturbation of the equation, this shows that $u(x, t)$ can be locally approximated for the C^1-norm (and thus for the H^1-norm) by solutions of the stationary equation.

On the other hand, we know, according to the remark before Section 3.1, that $\|u_x\|_{L^2_{\mathrm{ul}}(\mathbf{R})}$ is bounded independently of t. Besides, Eq. (1) itself shows that $\|u_{xx}(\cdot, t)\|_{L^\infty(\mathbf{R})}$ is bounded for t large, thus so is $\|u_x(\cdot, t)\|_{L^\infty(\mathbf{R})}$. Thus, the points $(u(x, t), u_x(x, t))$, which correspond to initial conditions for the differential equation argument stated above, belong to a bounded subset of $\mathbf{R}^2$. Thus, a compactness argument shows that the approximation by stationary solutions becomes uniformly precise on arbitrarily large intervals when $t \to +\infty$. This proves assertion 3 and completes the proof of Proposition 5. ∎

3.1.2. *Infinite energy.* Suppose $u(\cdot, t) = \mathscr{P}_t(u_0)$, $t \geqslant 0$, where V and u_0 satisfy the hypotheses of Theorem 1. Write $l_+ = \limsup_{|x| \to +\infty} u_0(x)$ and

$l_- = \liminf_{|x| \to +\infty} u_0(x)$. By hypothesis, if $l_+ > 0$ then $V'(w) > 0$ for $w \in]0; l_+]$ and if $l_- < 0$ then $V'(w) < 0$ for $w \in [l_-; 0[$.

In this paragraph, $C_0, C_1, \dots$ (resp. $\delta_1, \delta_2, \dots$) denote large (resp. small) positive constants which depend only on $u(\cdot, \cdot)$.

The first step is to control the behavior for x large of the solution u and of solutions v with initial conditions close to u_0. Take $l'_+ > \max(0, l_+)$ such that $V'(w) > 0$ for $w \in]0; l'_+]$ and $l'_- < \min(0, l_-)$ such that $V'(w) < 0$ for $w \in [l'_-; 0[$ (as $V''(0) > 0$, this is always possible).

Take any v_0 close to u_0 in X. Suppose that $\|v_0 - u_0\|_X$ is small enough so that, on one hand, $\limsup_{|x| \to +\infty} v_0(x) < l'_+$ and $\liminf_{|x| \to +\infty} v_0(x) > l'_-$, and, on the other hand (according to the hypotheses of Theorem 1), $\|\mathscr{P}_t(v_0)\|_{L^\infty(\mathbf{R})}$ is bounded by a constant C_0 (in particular, $\mathscr{P}_t(v_0)$ is defined for any $t \in \mathbf{R}_+$). Write $v(\cdot, t) = \mathscr{P}_t(v_0)$, $t \in \mathbf{R}_+$.

LEMMA 1. *There exist positive constants C_1, C_2, c, δ_1 such that*

$$|v(x, t)| \leqslant C_1 e^{-|x| + ct} + C_2 e^{-\delta_1 t}, \qquad x \in \mathbf{R}, \quad t \geqslant 0.$$

Proof. Let $\bar{v}(x, t) = C_1 e^{-x + ct} + l'_+ e^{-\delta_1 t}$. We will prove that one can choose the constants C_1, c, δ_1 such that $v(x, t) \leqslant \bar{v}(x, t)$, $x \in \mathbf{R}$, $t \geqslant 0$. Up to exchanging $v \to -v$ and $x \to -x$, this will prove the lemma.

Write $P(x, t) = \bar{v}_t + V'(\bar{v}) - \bar{v}_{xx}$. We want that $P(x, t) \geqslant 0$, i.e. that $\bar{v}$ be a super-solution. We have

$$P(x, t) = (c - 1)\, C_1 e^{-x + ct} + V'(\bar{v}) - \delta_1 l'_+ e^{-\delta_1 t}.$$

We suppose that $c > 1$. Let $l''_+ > l'_+$ such that $V'(w) > 0$ for $w \in]0; l''_+]$. As $V''(0) > 0$, there exists $\alpha > 0$ such that, for $w \in]0; l''_+]$, $V'(w) \geqslant \alpha w$.

Let us distinguish two cases. If $\bar{v} \leqslant l''_+$, then we have

$$P(x, t) \geqslant \alpha \bar{v} - \delta_1 l'_+ e^{-\delta_1 t} \geqslant (\alpha - \delta_1)\, l'_+ e^{-\delta_1 t}.$$

Let us choose $\delta_1 = \alpha$; then $P(x, t) \geqslant 0$ in this case.

If on the other hand $\bar{v} > l''_+$, then we have

$$P(x, t) \geqslant \bar{v} \left((c - 1) \frac{C_1 e^{-x + ct}}{\bar{v}} + \frac{V'(\bar{v})}{\bar{v}} - \delta_1 \right),$$

and $C_1 e^{-x + ct} / \bar{v} \geqslant 1 - l'_+ / l''_+ > 0$. Up to modifying $V(w)$ for large values of $|w|$ (larger than C_0, so that $v(\cdot, t)$ remains a solution of (1)), we can suppose that $w \mapsto \frac{V'(w)}{w}$ is bounded from below on $\mathbf{R}_+$; thus we can choose c sufficiently large so that the inequality above yields $P(x, t) \geqslant 0$.

Then $\bar{v}$ is a super-solution for (1). Finally, we can choose C_1 large enough so that $v_0(\cdot) \leqslant \bar{v}(\cdot, 0)$, and the conclusion follows by the maximum principle (Proposition 3). ∎

Fix any $c' > c$.

LEMMA 2. *There exist constants* $C_3 > 0$ *and* $\delta_2 > 0$ *such that*

$$\sup_{|x| > c't} \int T_x \rho(v_t^2 + v_x^2 + v^2) \leqslant C_3 e^{-\delta_2 t}, \qquad t \geqslant 0.$$

Proof. First, remark that, up to exchanging $x \to -x$, it is sufficient to prove the inequality for positive x. We will use exponential cutoff functions (the exponential decrease of the cutoff functions will yield the exponential decrease stated in the lemma). Let $\gamma: \mathbf{R} \to \mathbf{R}$ be a smooth function satisfying $\gamma(x) > 0$, $|\gamma'(x)| \leqslant \gamma(x)$ for any $x \in \mathbf{R}$, and $\gamma(x) = e^{-|x|}$ for $|x| \geqslant 1$.

Take any $\varepsilon > 0$ and $x_0 > 0$, and define the function $\varphi = \varphi_{\varepsilon, x_0, c'}$ by $\varphi(x, t) = \gamma(\varepsilon(x - x_0 - c't))$. The point is that all the following estimates will be uniform with respect to $x_0 > 0$. For ε sufficiently small (depending on c'), we have the following variant of (12),

$$\partial_t \int \varphi \left(\frac{v_x^2}{2} + \frac{v^2}{2} + V(v) \right) \leqslant -\int \varphi \left(v_t^2 \left(1 - \frac{\varepsilon}{2} \right) + v_x^2 \left(1 - \varepsilon - \frac{\varepsilon c'}{2} \right) + R(x, t) \right),$$

where $R(x, t) = V'(v)\, v - (\varepsilon + \varepsilon c') \frac{v^2}{2} - \varepsilon c' \, |V(v)|$.

Take and fix any c'' satisfying $c' > c'' > c$. We have the trivial decomposition

$$\int \varphi R = \int_{[-c''t;\, c''t]} \varphi R + \int_{\mathbf{R} \setminus [-c''t;\, c''t]} \varphi R.$$

According to the previous lemma, for $|x| \geqslant c''t$ and t sufficiently large (depending on $c'' - c$, C_1, C_2, δ_1), the quantity $|v(x, t)|$ is small, and thus, as $V''(0) > 0$, we have, for ε sufficiently small, $R(x, t) \geqslant C_4^{-1}(V(v) + \frac{v^2}{2})$.

On the other hand, for $|x| \leqslant c''t$ and t sufficiently large, we have $|\varphi(x, t)| = e^{-\varepsilon\, |x - x_0 - c't|} \leqslant e^{-\varepsilon(c' - c'')\, t}$. Thus, as $|v(x, t)|$ is bounded, the term $\int_{[-c''t;\, c''t]} \varphi R$ goes exponentially to 0. Finally we obtain, for ε sufficiently small,

$$\partial_t \int \varphi \left(\frac{v_x^2}{2} + \frac{v^2}{2} + V(v) \right) \leqslant C_5 e^{-\delta_3 t} - \int \varphi \frac{v_t^2}{2} - C_5^{-1} \int \varphi \left(\frac{v_x^2}{2} + \frac{v^2}{2} + V(v) \right). \tag{26}$$

On the other hand, we have

$$\partial_t \int \varphi v_t^2 = \int \varphi_t v_t^2 + 2 \int \varphi v_t(-V''(v)\, v_t + v_{xxt}),$$

and, arguing as in the previous paragraph, we get

$$\partial_t \int \varphi v_t^2 \leqslant C_6 \int \varphi v_t^2. \tag{27}$$

Combining (26) and (27), we obtain that $\int \varphi(v_x^2/2 + v^2/2 + V(v))$ and $\int \varphi v_t^2$ converge exponentially to 0 when $t \to +\infty$, the rate of convergence being uniform with respect to $x_0 > 0$. According to the bounds on $|v(\cdot, \cdot)|$, the result follows. ∎

We have

$$\partial_t \int_{-c't}^{c't} \left(\frac{v_x^2}{2} + V(v)\right) = -\int_{-c't}^{c't} v_t^2 + \left[\frac{v_x^2}{2} + V(v) + v_x v_t\right]_{-c't}^{c't}, \qquad t \geqslant 0.$$

Thus, according to the preceding lemma,

$$\partial_t \int_{-c't}^{c't} \left(\frac{v_x^2}{2} + V(v)\right) \leqslant -\int_{-c't}^{c't} v_t^2 + C_7 e^{-\delta_2 t}, \qquad t \geqslant 0. \tag{28}$$

We are now in position to argue as in the finite energy case. By hypothesis, u_0 is in the border of $\mathscr{B}_{0,\,\mathrm{par}}$, thus we can suppose that $v \in \mathscr{B}_{0,\,\mathrm{par}}$. In this case, $\|v(\cdot, t)\|_{H^1_{\mathrm{ul}}(\mathbf{R})}$ converges to 0 when $t \to +\infty$ and thus (as $V''(0) \geqslant 0$ around 0)

$$\liminf_{t \to +\infty} \int_{-c't}^{c't} \left(\frac{v_x^2}{2} + V(v)\right) \geqslant 0$$

(actually, there is convergence towards 0, but we will not need this). Inequality (28) then shows that

$$\int_{-c't}^{c't} \left(\frac{v_x^2}{2} + V(v)\right) \geqslant -\frac{C_7}{\delta_2}\, e^{-\delta_2 t}, \qquad t \geqslant 0.$$

By continuity when v_0 is chosen arbitrarily close to u_0, the same estimate holds for $u(\cdot, t)$, and we obtain

$$\liminf_{t \to +\infty} \int_{-c't}^{c't} \left(\frac{u_x^2}{2} + V(u)\right) \geqslant 0.$$

Now, inequality (28) applies to $u(\cdot, t)$ and shows that $\int_{-c't}^{c't} (u_x^2/2 + V(u))$ (being "almost decreasing") converges to a nonnegative limit when

$t \to +\infty$. This inequality also shows that the function $t \mapsto \int \eta u_t^2$ is integrable with respect to t on $\mathbf{R}_+$, and this completes the proof of assertion 1 of Proposition 5.

Let θ be a smooth function satisfying $0 \leqslant \theta \leqslant 1$, $\theta(x) = 1$ for $x \leqslant 0$, $\theta(x) = e^{-x}$ for $x \geqslant 1$, and $|\theta'(x)| \leqslant \theta(x)$, $x \in \mathbf{R}$. Define the function $\eta(x, t)$, $x \in \mathbf{R}$, $t \geqslant 0$ by $\eta(x, t) = \theta(x - c't)$ for $x \geqslant 0$, and $\eta(x, t) = \theta(-x - c't)$ for $x \leqslant 0$. The function $t \mapsto \int \eta u_t^2$ is integrable with respect to t on $\mathbb{R}_+$. Arguing as in Section 3.1.1, we get

$$\partial_t \int \eta u_t^2 \leqslant C_8 \int \eta u_t^2 - \int \eta u_{xt}^2 \tag{29}$$

and

$$\partial_t \int \eta u_{xt}^2 \leqslant C_9 \int \eta u_{xt}^2. \tag{30}$$

Inequality (29) shows that $\int \eta u_t^2$ converges to 0 when $t \to +\infty$; together with the preceding lemma, this proves assertion 2. This inequality also shows that the function $t \mapsto \int \eta u_{xt}^2$ is integrable on $\mathbf{R}_+$, and (30) shows that it converges to 0. We thus obtain $\limsup_{t \to +\infty} \sup_{|x| \leqslant c't} |u_t(x, t)| = 0$. Local convergence to stationary solutions (assertion 3) follows like in paragraph 3.1.1. ∎

3.2. *Hyperbolic Equation*

3.2.1. *Finite energy.* Suppose $(u(\cdot, t), u_t(\cdot, t)) = \mathscr{H}_t(U_0)$, $t \geqslant 0$, where V and U_0 satisfy the hypotheses of Theorem 4.

The fact that the energy $\int E(x, t)\, dx$ is bounded from above and the continuity of the trajectory $\mathscr{H}_t(U_0)$ in $H^1_{\mathrm{ul}}(\mathbf{R}) \times L^2_{\mathrm{ul}}(\mathbf{R})$ imply that, for any $t \geqslant 0$, we have $\lim_{|x| \to +\infty} |u(x, t)| = 0$ and $\lim_{|x| \to +\infty} \int T_x \rho\ u_x^2 = 0$ (the argument is the same as in Sect. 3.1.1).

Write

$$A = \begin{pmatrix} 0 & 0 \\ \partial_{xx} & 1 \end{pmatrix}.$$

The following Ansatz for Eq. (2):

$$(u(\cdot, t), u_t(\cdot, t)) = e^{tA} U_0 + \int_0^t e^{(t-s)A} (0, -v(u(\cdot, s))\, u_t(\cdot, s) - V'(u(\cdot, s)))\, ds, \tag{31}$$

shows that the solutions depend locally Lipschitz-continuously, in the norm $H^1(\mathbf{R}) \times L^2(\mathbf{R})$, on the initial conditions (apply Gronwall's lemma).

As U_0 is accumulated in $H^1(\mathbf{R}) \times L^2(\mathbf{R})$ by solutions which converge to $(0, 0)$, and thus whose energy remains finite and non-negative for all times, this implies that the energy of $\mathcal{H}_t(U_0)$ is actually strictly positive for all time, and thus converges to a non-negative limit when $t \to +\infty$. Then, by (6), the function $t \mapsto \int u_t^2$ is integrable on $\mathbf{R}_+$, which proves assertion 1 of Proposition 5.

For any $t_2 > t_1 > 0$, we have

$$\|u(\cdot, t_2) - u(\cdot, t_1)\|^2_{L^2(\mathbf{R})} \leqslant (t_2 - t_1) \int_{t_1}^{t_2} \left(\int u_t^2\,(x, s)\, dx \right) ds. \tag{32}$$

Thus the left-hand side of this inequality converges to 0 when $t_1 \to +\infty$ and $t_2 - t_1$ remains bounded. This will enable us, despite the absence of regularization, to prove local convergence towards stationary solutions. We will first prove this convergence for a norm weaker than $\| \cdots \|_{H^1_{\mathrm{ul}}(\mathbf{R})}$, and we will recover the convergence for $\| \cdots \|_{H^1_{\mathrm{ul}}(\mathbf{R})}$ in a second step.

LEMMA 3. *Take any δ satisfying $\frac{1}{2} > \delta > 0$. For any $\varepsilon > 0$ and $l > 0$, there exists $T = T(\varepsilon, l)$ such that, for any $t \geqslant T$ and $x_0 \in \mathbf{R}$, $\|u_t(\cdot, t)\|_{H^{-\delta}([x_0 - l;\, x_0 + l])} < \varepsilon$ and there exists a stationary solution u_{stat} of Eq. (9) such that $\|u(\cdot, t) - u_{\mathrm{stat}}(\cdot)\|_{H^{1-\delta}([x_0 - l;\, x_0 + l])} < \varepsilon$.*

Proof. Suppose the converse, i.e. that there exists $\varepsilon > 0$ and $l > 0$ such that, for any $n \in \mathbf{N}$, there exists $t_n \geqslant n$ and $x_n \in \mathbf{R}$ such that, either $\|u_t(\cdot, t)\|_{H^{-\delta}([x_0 - l;\, x_0 + l])} \geqslant \varepsilon$, or, for any stationary solution u_{stat} of Eq. (9), $\|u_n(\cdot, t_n) - u_{\mathrm{stat}}(\cdot)\|_{H^{1-\delta}([x_n - l;\, x_n + l])} \geqslant \varepsilon$.

Up to extracting a subsequence, we can suppose that the functions $x \mapsto u_n(x_n + x, t_n)$ (resp. $x \mapsto \partial_t u_n(x_n + x, t_n)$) converge on $[-l; l]$ to a function $u_\infty \in H^1([-l; l])$ for the $H^{1-\delta}([-l; l])$-norm (resp. to a function $v_\infty \in L^2([-l; l])$ for the $H^{-\delta}([-l; l])$-norm).

Write $\tilde{u}_n(x, t) = u(x_n + x, t_n + t)$, $\partial_t \tilde{u}_n(x, t) = \partial_t u(x_n + x, t_n + t)$ and for any $x \in [-l; l]$ and $0 \leqslant t \leqslant \min(|x + l|, |l - x|)$, denote by $T(x, t)$ the triangle $\{(y, s) \mid |y - x| \leqslant t - s\}$. The following Ansatz (see [16] or Proposition 4) holds:

$$\tilde{u}_n(x, t) = \tfrac{1}{2}(\tilde{u}_n(x - t, 0) + \tilde{u}_n(x + t, 0)) + \tfrac{1}{2} \int_{x-t}^{x+t} \partial_t \tilde{u}_n(y, 0)\, dy$$

$$+ \tfrac{1}{2} \iint_{T(x,t)} l(-v(\tilde{u}_n(y, s))\, \partial_t \tilde{u}_n(y, s) - V'(\tilde{u}_n(y, s)))\, dy\, ds.$$

On the other hand, by local existence results, there exists $t' > 0$ and a function $w \in C^0(\{(x, t) \in T(0, l) \mid t \leqslant t'\})$, satisfying the Ansatz:

$$w(x,t)=\tfrac{1}{2}(u_\infty(x-t)+u_\infty(x+t))+\tfrac{1}{2}\int_{x-t}^{x+t} v_\infty(y)\,dy$$
$$+\tfrac{1}{2}\iint_{T(x,t)}(-V'(w(y,s)))\,dy\,ds. \tag{33}$$

We deduce from these expressions of $\tilde{u}_n$ and w that

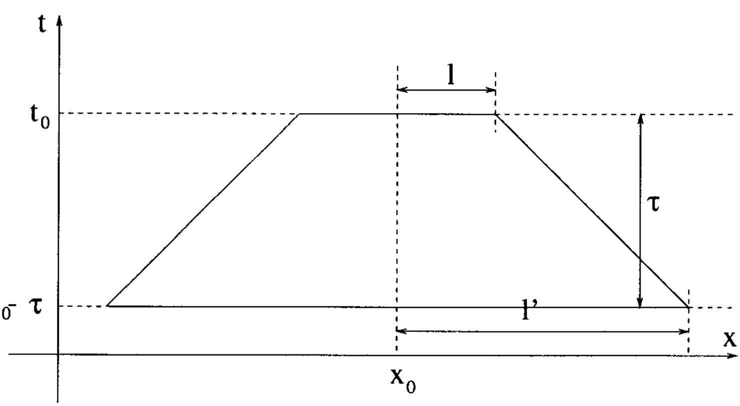
$$|w(x,t)-\tilde{u}_n(x,t)|\leqslant \alpha_n+\tfrac{1}{2}\iint_{T(x,t)}|V'(w(y,s))-V'(\tilde{u}_n(y,s))|\,dy\,ds,$$

where $\alpha_n\to 0$ when $n\to+\infty$ (indeed, as $\delta<\frac{1}{2}$, $\tilde{u}_n\to u_\infty$ uniformly on $[-l;l]$). By Gronwall's lemma, this implies that w is actually defined on $T(0,l)$ and that $\sup_{(x,t)\in T(0,l)}|w(x,t)-\tilde{u}_n(x,t)|$ converges to 0 when $n\to+\infty$.

Now, the fact that $\tilde{u}_n$ is "almost constant with respect to t" (inequality (32)) shows that, for any $0<t<l$, $w(\cdot,t)-w(\cdot,0)$ is arbitrarily small in $L^2([-(l-t);\ l-t])$, and thus actually vanishes identically. Then, (33) shows that $v_\infty(\cdot)=0$ in $L^2([-l;l])$ and that, for any $x\in]-l;l[$, $\varepsilon^{-2}(u_\infty(x-\varepsilon)+u_\infty(x+\varepsilon)-2u_\infty(x))$ converges to $-V'(u_\infty(x))$ when $\varepsilon\to 0$. Thus u_∞ is actually of class C^2 and is a solution of the stationary Eq. (9), which yields a contradiction. ∎

Lemma 4. *The preceding lemma holds with $\delta=0$.*

Proof. Fix $\frac{1}{2}>\delta>0$ and take any $\varepsilon>0$ and $l>0$ (we will prove the assertion of the lemma for ε and l). Take $\varepsilon'>0$ and $\tau>0$ to be chosen later, and write $l'=l+\tau$. By the preceding lemma, for any $t_0>T(\varepsilon',l')+\tau$ and $x_0\in\mathbf{R}$, there exists a stationary solution u_{stat} such that $\|u(\cdot,t_0-\tau)-u_{\text{stat}}(\cdot)\|_{H^{1-\delta}([x_0-l';x_0+l'])}<\varepsilon'$ and $\|u_t(\cdot,.t_0-\tau)\|_{H^{-\delta}([x_0-l';x_0+l'])}<\varepsilon'$. For $t\geqslant t_0-\tau$ and $x\in\mathbf{R}$, write $\Delta U(t)(x)=(u(x,t)-u_{\text{stat}}(x),u_t(x,t))$.



Fix any $v_0 > 0$ and write

$$\tilde{A} = \begin{pmatrix} 0 & 1 \\ -\frac{v_0^2}{4} + \partial_{xx} & -v_0 \end{pmatrix}.$$

Then we have

$$e^{t\tilde{A}} = e^{-(v_0/2)\,t} \begin{pmatrix} 1 & 0 \\ -\frac{v}{2} & 1 \end{pmatrix} e^{tA}, \qquad t \in \mathbf{R}.$$

As e^{tA} defines, for any $t \in \mathbf{R}$, an isometry of $H^1(\mathbf{R}) \times L^2(\mathbf{R})$, this shows that the norm of $e^{t\tilde{A}}$, viewed as a (bounded) operator on $H^1(\mathbf{R}) \times L^2(\mathbf{R})$, converges exponentially to 0 when $t \to +\infty$.

For $t > 0$, we have

$$\begin{aligned} \Delta U(t_0 - \tau + t) &= e^{t\tilde{A}} \Delta U(t_0 - \tau) \\ &\quad + \int_0^t e^{(t-s)\,\tilde{A}}(0, \; -(v(u(\cdot, t_0 - \tau + s)) - v_0)\, u_t(\cdot, t_0 - \tau + s) \\ &\quad + \frac{v_0^2}{4}(u(\cdot, t_0 - \tau + s) - u_{\text{stat}}(\cdot)) \\ &\quad - (V'(u(\cdot, t_0 - \tau + s)) - V'(u_{\text{stat}}(\cdot))))\, ds. \end{aligned}$$

By hypothesis, $\|u(t, \cdot)\|_{L^\infty(\mathbf{R})}$ is bounded independently of $t \geqslant 0$. Thus, Proposition 2 shows that $\|(u(t, \cdot), u_t(t, \cdot))\|_{H^1_{\text{ul}}(\mathbf{R}) \times L^2_{\text{ul}}(\mathbf{R})}$ is bounded independently of $t \geqslant 0$. Thus, according to the finite propagation property, we can choose τ sufficiently large (independently of $u(\cdot, \cdot)$) so that

$$\|e^{\tau\tilde{A}} \Delta U(t_0)\|_{H^1([x_0 - l;\, x_0 + l]) \times L^2([x_0 - l;\, x_0 + l])} < \frac{\varepsilon}{2}.$$

Then, the finite propagation property and the fact that $t \mapsto \int u_t^2$ is integrable on $\mathbf{R}_+$ show that, for t_0 sufficiently large and for ε' sufficiently small (depending on the previous choice of τ),

$$\|\Delta U(t_0)\|_{H^1([x_0 - l;\, x_0 + l]) \times L^2([x_0 - l;\, x_0 + l])} < \varepsilon,$$

which completes the proof. ∎

This proves assertions 2 and 3 of Proposition 5.

3.2.2. *Infinite energy.* Suppose $(u(\cdot, t), u_t(\cdot, t)) = \mathscr{H}_t(U_0)$, $t \geqslant 0$, where V and U_0 satisfy the hypotheses of Theorem 2.

As in the parabolic case, we will have to control the behavior of solutions with initial conditions close to U_0, for x large. By hypothesis, $\limsup_{|x| \to +\infty} \int T_x \rho(u(y,0)^2 + u_x(y,0)^2 + u_t(y,0)^2)\, dy < \varepsilon$, where $\varepsilon > 0$ has to be chosen.

Take any $V_0 \in Y$ with $\| V_0 - U_0 \|_Y$ small, and suppose that $\mathscr{H}_t(V_0)$ is defined for any $t \in \mathbf{R}_+$ (actually, we will only need to consider the cases where $V_0 \in \mathscr{B}_{0,\,\mathrm{hyp}}$ or $V_0 = U_0$). Write $\mathscr{H}_t(V_0) = (v(\cdot, t), v_t(\cdot, t))$, $t \geqslant 0$.

For ε and $\| V_0 - U_0 \|_Y$ sufficiently small, Proposition 1 and the finite propagation property show that there exists $x_0 > 0$ such that

$$\lim_{t \to +\infty} \sup_{|x_1| > x_0 + t} \int T_{x_1} \rho(v^2 + v_x^2 + v_t^2) = 0,$$

and this convergence is exponential (see the Proof of Proposition 1). Now, we proceed like in Section 3.1.2. Let θ be a smooth function satisfying $0 \leqslant \theta \leqslant 1$, $\theta(x) = 1$ for $x \leqslant 0$, $\theta(x) = 0$ for $x \geqslant 1$, and define the function $\eta(x, t)$, $x \in \mathbf{R}$, $t \geqslant 0$, by $\eta(x, t) = \theta(x - x_0 - t)$ for $x \geqslant 0$, and $\eta(x, t) = \theta(-x - x_0 - t)$ for $x \leqslant 0$. Then, for t large, we obtain

$$\partial_t \int \eta \left(\frac{v_t^2}{2} + \frac{v_x^2}{2} + V(v) \right) \leqslant e^{-\delta t} - \int \eta v(v)\, v_t^2,$$

where $\delta > 0$ is independent of V_0. Now, arguing like in the parabolic case (Section 3.1.2), we obtain that $\int_{-x_0 - t}^{x_0 + t} (u_t^2/2 + u_x^2/2 + V(u))$ converges to a non-negative limit and that $t \mapsto \int_{-x_0 - t}^{x_0 + t} u_t^2$ is integrable with respect to t on $\mathbf{R}_+$ (assertion 1). The proofs of assertion 2 and 3 (i.e. $\| u_t \|_{L^2_{\mathrm{ul}}(\mathbf{R})} \to 0$ when $t \to +\infty$ and $u(\cdot, \cdot)$ converges locally towards stationary solutions) are the same as in the finite-energy case of the preceding paragraph 3.2.1. ∎

4. GLOBAL SHAPE AT LARGE TIME

4.1. *End of the Proof of Theorems* 1–4

We consider a function $u(x, t)$, $x \in \mathbf{R}$, $t \geqslant 0$, and we suppose that the hypotheses of Proposition 5 are satisfied, i.e. u is a solution of Eq. (1) or (2), and the initial condition and the potential V satisfy the hypotheses of one of the Theorems 1, 2, 3, 4. To complete the proof of these theorems, we have to show that, for large t, the solution is actually close (in X or in Y) to a function in $\mathscr{A}$.

Write $W(\cdot) = -V(\cdot)$. Stationary solutions $v(x)$ of Eq. (1) or (2) are solutions of Eq. (9), which represents a conservative order 2 oscillator in the potential $-V = W$, and which we can rewrite

$$v'' = -W'(v). \tag{35}$$

Recall (see the introduction) that $S_{b,0}$ denotes the connected component containing $(0, 0)$ of the set S_b, which is the union of those trajectories $\bigcup_{x \in \mathbf{R}} \{(v(x), v'(x))\}$ of energy-0 solutions of Eq. (35) which are bounded in $\mathbf{R}^2$.

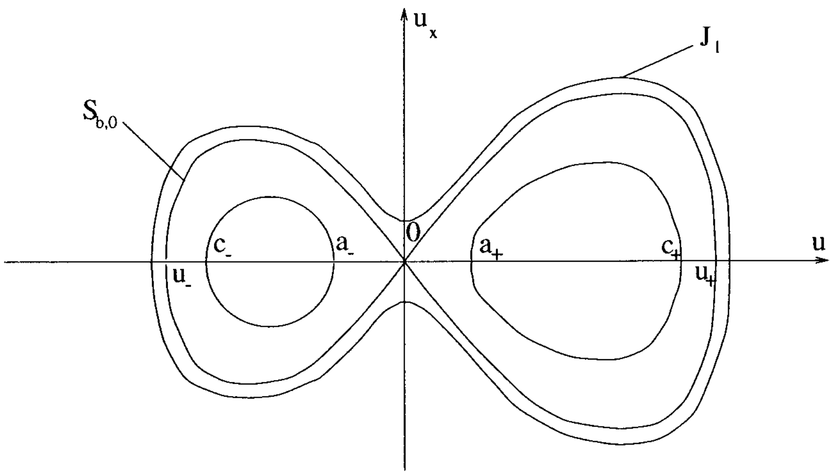
Let $\hat{S}_{b,0}$ denote the union of $S_{b,0}$ and of the bounded connected components of $\mathbf{R}^2 \backslash S_{b,0}$.

LEMMA 5. *Assertion* 3 *of Proposition* 5 (*approximation by stationary solutions*) *holds with the supplementary assertion that the stationary solutions* u_{stat} *which approximate* u *have trajectories in* $\mathbf{R}^2$ *which belong to* $\hat{S}_{b,0}$.

Proof. Up to changing the potential $V(v)$ for large values of v (so that $t \mapsto u(\cdot, t)$ be still a solution of Eq. (1) or (2)), we can suppose that u_- and u_+ are finite (i.e. that $V(\cdot)$ takes strictly negative values somewhere at the left and at the right of 0).

By Sard's theorem, there exist a_-, b_-, a_+, b_+ in $\mathbf{R}$ satisfying (see the picture):

- $u_- < b_- < a_- < 0 < a_+ < b_+ < u_+$;
- $V(b_-) = V(a_-) > 0$, $V'(b_-) > 0$, $V'(a_-) < 0$, and for $v \in]b_-; a_-[$, $V(v) > V(a_-)$;
- $V(a_+) = V(b_+) > 0$, $V'(a_+) > 0$, $V'(b_+) < 0$, and for $v \in]a_+; b_+[$, $V(v) > V(a_+)$.



The solution of Eq. (35) passing through $(a_+, 0)$ and $(b_+, 0)$ (resp. through $(a_-, 0)$ and $(b_-, 0)$) is periodic; denote by τ_+ (resp. τ_-) its period. If v is a solution of Eq. (35) satisfying, say, $v(x_0) = a_+$, then:

• if $v'(x_0) \geqslant 0$, then there exists $x_1 > x_0$ such that, for any $x \in [x_0; x_1]$, $a_+ \leqslant v(x) \leqslant b_+$, and $v(x_1) = b_+$; furthermore, $x_1 - x_0 \leqslant \tau_+/2$;

• if $v'(x_0) \leqslant 0$, then there exists $x_1 < x_0$ such that, for any $x \in [x_1; x_0]$, $a_+ \leqslant v(x) \leqslant b_+$, and $v(x_1) = b_+$; furthermore, $x_0 - x_1 \leqslant \tau_+/2$.

Of course, a symmetric assertion holds if $v(x_0) = a_-$ instead of $v(x_0) = a_+$.

Let $m(t) = \sup\{m \in \mathbf{N} \mid \text{there exist } y_1 < \cdots < y_m \text{ such that, for each } i \in \{1, ..., m-1\}, \text{ either } u(y_i, t) = a_- \text{ and } u(y_{i+1}, t) = a_+, \text{ or } u(y_i, t) = a_+ \text{ and } u(y_{i+1}, t) = a_-\}$.

Remark that $m(t)$ is finite for all $t \geqslant 0$, since $|u(x, t)|$ is small when x is large and $\|u_x\|_{L^2_{\mathrm{ul}}(\mathbf{R})}$ is bounded.

Lemma 6. *For t sufficiently large, $m(t)$ is constant.*

Proof. We will show that, for t sufficiently large, the map $t \mapsto m(t)$ is continuous, and this will prove the lemma.

Upper-semi-continuity is a direct consequence of the definition of $m(t)$, of the continuity of $u(\cdot, t)$ in $H^1_{\mathrm{ul}}(\mathbf{R})$ with respect to t, and of the fact that $u(x, t)$ is small when x is large.

To prove the lower semi-continuity, the argument consists in showing that, for each y_i such that $u(y_i, t)$ takes the value a_+ (resp. a_-), it takes in a neighborhood values larger than a_+ (resp. smaller than a_-). The reason is that, according to the remark above, this is true for solutions of Eq. (35).

Let us show this precisely. Take a parameter $T > 0$ to be chosen later, take any time t larger than T, and take $y_1 < \cdots < y_{m(t)}$ satisfying the assertion in the definition of $m(t)$. In the following, we will denote by $\varepsilon_j(T)$, $j = 1, 2, ...$ functions depending only on T and on the solution $u(\cdot, \cdot)$ we are considering, and satisfying $\varepsilon_j(T) \to 0$ when $T \to +\infty$.

Take $i \in \{1, ..., m(t)\}$ and suppose for instance that $u(y_i, t) = a_+$. According to assertion 3 of Proposition 5, there exists a stationary solution u_{stat} such that $\|u(\cdot, t) - u_{\mathrm{stat}}\|_{H^1([y_i - \tau_+, y_i + \tau_+])} < \varepsilon_1(T)$; in particular, $|a_+ - u_{\mathrm{stat}}(y_i)| < \varepsilon_2(T)$. Let $\tilde{u}_{\mathrm{stat}}(\cdot)$ be the solution of Eq. (35) satisfying $\tilde{u}_{\mathrm{stat}}(y_i) = a_+$ and $\tilde{u}'_{\mathrm{stat}}(y_i) = u'_{\mathrm{stat}}(y_i)$; by continuous dependence of the solutions of (35) with respect to initial conditions, we have $\|u(\cdot, t) - \tilde{u}_{\mathrm{stat}}(\cdot)\|_{H^1([y_i - \tau_+; y_i + \tau_+])} < \varepsilon_3(T)$.

Besides, according to what we mentioned above after introducing a_-, b_-, a_+, b_+, we know that there exists y'_i satisfying $|y'_i - y_i| < \tau_+/2$ and such that $\tilde{u}_{\mathrm{stat}}(y'_i) = b_+$ and for any y in the interval between y_i and y'_i, $b_+ \geqslant \tilde{u}_{\mathrm{stat}}(y) \geqslant a_+$. This yields $u(y'_i, t) \geqslant b_+ - \varepsilon_4(T)$ and for any y in the interval between y_i and y'_i, $u(y, t) \geqslant a_+ - \varepsilon_4(T)$.

For indices k for which $u(y_k, t) = a_-$, we proceed the same way, and we obtain a point y'_k with $u(y'_k, t) \leqslant b_- + \varepsilon_5(t)$. We can see that, if T is sufficiently large, the points $y'_1, ..., y'_{m(t)}$ constructed this way satisfy $y'_1 < \cdots < y'_{m(t)}$; the lower semi-continuity of $t \mapsto m(t)$ follows. ∎

Continuation of proof of Lemma 5. Let $\alpha > 0$ be a small parameter to be chosen later. By definition of u_-, u_+, and Sard's theorem, there exists $v_- \in]u_- - \alpha; u_-[$ and $v_+ \in]u_+; u_+ + \alpha[$ such that

$$W(v_-) = W(v_+) > 0, \qquad W'(v_-) < 0, \quad W'(v_+) > 0,$$

and

$$W(v_\pm) > W(v) \qquad \text{for} \quad v \in]v_-; v_+[.$$

Denote by v_{stat} the periodic solution of Eq. (35) with initial conditions $v_{\text{stat}}(0) = v_-$, $v'_{\text{stat}}(0) = 0$; this solution is periodic, its trajectory in the phase space $\mathbf{R}^2$ is a Jordan curve J which contains $(v_-, 0)$ and $(v_+, 0)$; its energy is equal to $W(v_\pm)$, and thus $J \cap S_{\mathrm{b},0} = \varnothing$ (see previous figure). Denote by $\hat{J}$ the union of J and of the bounded connected component of $\mathbf{R}^2 \backslash J$.

Claim 1. $S_{\mathrm{b},0} \subset \hat{J} \backslash J$.

By connectedness and energy arguments, we have $S_{\mathrm{b},0} \subset [u_-; u_+] \times \mathbf{R}$ and $J \subset [v_-; v_+] \times \mathbf{R}$. Thus, we can change the potential W outside of $[v_-; v_+]$ without modifying $S_{\mathrm{b},0}$ and J, in order to have: for any $v > v_+$, $W'(v) > 0$ (and thus $W(v) > W(v_\pm)$) and for any $v < v_-$, $W'(v) < 0$ (and thus $W(v) > W(v_\pm)$); furthermore, we can suppose that $W(v) \to +\infty$ when $|v| \to +\infty$. Under these conditions, J is exactly the set of points $(x, y) \in \mathbf{R}^2$ for which the energy $\frac{y^2}{2} + W(x)$ is equal to $W(v_\pm) > 0$. Thus, any continuous path connecting $S_{\mathrm{b},0}$ to infinity in $\mathbf{R}^2$ must cross J, and claim 1 follows.

Claim 2. $\hat{S}_{\mathrm{b},0} \subset \hat{J} \backslash J$.

Indeed, the set $J \cup (\mathbf{R}^2 \backslash \hat{J})$ is connected and does not intersect $S_{b,0}$. Thus a bounded connected component B of $\mathbf{R}^2 \backslash S_{\mathrm{b},0}$ cannot intersect $J \cup (\mathbf{R}^2 \backslash \hat{J})$ (or thus it would be unbounded), and thus always belongs to $\hat{J} \backslash J$.

Claim 3. $J \subset \text{Neighb}(\hat{S}_{\mathrm{b},0}, \varepsilon(\alpha))$, where $\varepsilon(\alpha) \to 0$ when $\alpha \to 0$.

This claim follows from the fact that the value of the energy on J is close to 0 if α is close to 0.

Now, the idea of the following argument is that, if Lemma 5 was false, then there would exist large values of t for which the function $x \mapsto (u(x, t), u_t(x, t))$ remains close to the function $x \mapsto (v_{\text{stat}}(x), v'_{\text{stat}}(x))$, (i.e. to the trajectory J), on a large interval. This is incompatible with the fact that, for t large, $m(t)$ is constant, and in particular bounded.

Suppose Lemma 5 is false. Then, there exists $\varepsilon_0 > 0$ and $l_0 > 0$ such that, for any $T > 0$, there exists $\tau > T$ and $x_0 \in \mathbf{R}$ such that, for any solution u_{stat} of (35) whose trajectory in $\mathbf{R}^2$ belongs to $\hat{S}_{\text{b},0}$,

$$\|u(\cdot, \tau) - u_{\text{stat}}(\cdot)\|_{H^1([x_0 - l_0; x_0 + l_0])} > \varepsilon_0. \tag{36}$$

Fix such ε_0 and l_0. Let $T > 0$ to be chosen later, and let τ and x_0 (depending on T) be as in the preceding sentence.

Let $\beta > 0$ and $L > 1$ to be chosen later. According to assertion 3 of Proposition 5, we can suppose that T is sufficiently large (depending on β and L) so that, for any $z \in \mathbf{R}$, there exists a solution u_{stat} of (35) satisfying

$$\|u(\cdot, \tau) - u_{\text{stat}}(\cdot)\|_{H^1([z - L; z + L])} < \beta. \tag{37}$$

For $k \in \mathbf{N}$, denote by $u_{\text{stat}, k}$ the solution of (35) satisfying inequality (37) for $z = x_0 + k$. Let

$$\Gamma = \bigcup_{k \in \mathbf{N}} \bigcup_{x \in [x_0 + k; x_0 + k + 1]} \{(u_{\text{stat}, k}(x), u'_{\text{stat}, k}(x))\} \subset \mathbf{R}^2.$$

Because $L > 1$, this set is $\varepsilon_1(\beta)$-connected (this means that $\text{Neighb}(\Gamma, \varepsilon_1(\beta))$ is connected) with $\varepsilon_1(\beta) \to 0$ when $\beta \to 0$. Suppose $L \geqslant l_0$; then, if β is sufficiently small and if J is close enough to $\hat{S}_{\text{b},0}$ (i.e. α is small enough, depending on ε_0), inequalities (36) and (37) show that the trajectory of $u_{\text{stat},0}$ belongs to $\mathbf{R}^2 \backslash \hat{J}$. On the other hand, according to assertion 1 of Proposition 5, for k large, the trajectory of $u_{\text{stat}, k}$ is close to $(0, 0)$, in particular belongs to $\hat{J}$. This shows that the set Γ contains points $\varepsilon_1(\beta)$-close to J; in particular, there exists $k_0 \in \mathbf{N}$ such that

$$\|u(\cdot, \tau) - v_{\text{stat}}(\cdot)\|_{H^1([x_0 + k_0 - L; x_0 + k_0 + L])} < \varepsilon_2(\beta),$$

with $\varepsilon_2(\beta) \to 0$ when $\beta \to 0$ (recall that the trajectory in $\mathbf{R}^2$ of v_{stat} is J).

This last approximation shows that, for β sufficiently small and L sufficiently large, $m(\tau)$ is arbitrarily large. This contradicts Lemma 6, and the result follows. ∎

COROLLARY 1. *For t sufficiently large, we have, for all $x \in \mathbf{R}$, $\int_{x-1}^{x+1} V(u(x, t))\, dx \geqslant 0$.*

Proof. Suppose the converse, i.e. for any $p \in \mathbf{N}$, there exists $t_p > p$ and $x_p \in \mathbf{R}$ such that $\int_{x_p - 1}^{x_p + 1} V(u(x, t_p))\, dx < 0$. Then, for any $p \in \mathbf{N}$, there exists $y_p \in [x_p - 1; x_p + 1]$ such that $u(y_p, t_p) < u_-$ or $u(y_p, t_p) > u_+$ (indeed, on $[u_-; u_+]$ we have $V(\cdot) \geqslant 0$). Suppose for instance that $u(y_p, t_p) < u_-$. Then, the previous lemma shows that $u(y_p, t_p)$ is actually close to u_-; thus, if we denote by $u_{\text{stat}, -}$ the solution of (35) with initial condition

$(u_-, 0)$ at $t=0$, we can see that $\|u(\cdot, t_p) - u_{\text{stat},-}(\cdot - y_p)\|_{H^1([x_p-1; x_p+1])} < \varepsilon_1(p)$, with $\varepsilon_1(p) \to 0$ when $p \to +\infty$; this yields

$$\int_{x_p - y_p - 1}^{x_p - y_p + 1} V(u_{\text{stat},-})\, dx < \varepsilon_2(p)$$

(with $\varepsilon_2(p) \to 0$ when $p \to +\infty$), which is impossible, because this integral is bounded from below by a strictly positive constant which does not depend on p (remark that this last assertion would be false without the hypothesis (11)). ∎

Corollary 2. *Assertion 3 of Proposition 5 (approximation by stationary solutions) holds with the supplementary assertion that the stationary solutions u_{stat} which approximate u have trajectories in $\mathbf{R}^2$ which belong to $S_{b,0}$.*

Proof. Suppose the converse is true. Then there are arbitrarily large values of t for which the function $x \mapsto V(u(x,t))$ is larger than a fixed strictly positive constant α on arbitrarily large intervals. Thus, in view of Corollary 1, the integral $\int E(x,t)\, dx$ takes arbitrarily large values, which is impossible if we are in the finite energy case. Suppose we are in the infinite energy case. Then, as $\lim_{t\to+\infty} \sup_{|x|>L(t)} \int T_x\rho(u_x^2 + u^2) = 0$ (assertion 1 of Proposition 5), the intervals on which $V(u(x,t)) \geqslant \alpha$ must belong (for t large enough) almost entirely to $[-L(t); L(t)]$; this shows that $\int_{-L(t)}^{L(t)} E(x,t)\, dx$ takes arbitrarily large values, which is in contradiction with the fact that this last integral converges when $t \to +\infty$ (assertion 1 of Proposition 5). ∎

We can now easily complete the proofs of Theorems 1, 2, 3, 4. We know that, for large t, the function $u(\cdot, t)$ can be locally approximated by stationary solutions whose trajectories belong to $S_{b,0}$. More precisely, we deduce from assertion 3 of Proposition 5 that, for t sufficiently large, there exists $n(t) \in \mathbf{N}$ such that, if $n(t) = 0$, then $u(\cdot, t)$ is close to 0 in $H^1_{\text{ul}}(\mathbf{R})$; if $n(t) \geqslant 1$, then there exist non-constant functions $h_1^{(t)}, ..., h_{n(t)}^{(t)}$ in H (H was defined in introduction) and points $y_1(t), ..., y_{n(t)}(t)$ in $\mathbf{R}$ such that, if $n(t) = 1$, then $u(\cdot, t)$ is H^1_{ul}-close to $h_1^{(t)}(\cdot - y_1(t))$ on $\mathbf{R}$, and if $n(t) \geqslant 2$, then $u(\cdot, t)$ is H^1_{ul}-close to

- $h_1^{(t)}(\cdot - y_1(t))$ on $]-\infty; (y_1(t) + y_2(t))/2]$;
- $h_{n(t)}^{(t)}(\cdot - y_{n(t)}(t))$ on $[(y_{n(t)-1}(t) + y_{n(t)}(t))/2; +\infty[$;
- $h_j^{(t)}(\cdot - y_j(t))$ on $[(y_{j-1}(t) + y_j(t))/2; (y_j(t) + y_{j+1}(t))/2]$ for $2 \leqslant j \leqslant n(t) - 1$;

and, for each $j \in \{1, ..., n(t) - 1\}$, $y_{j+1}(t) - y_j(t)$ is large.

The continuity of $u(\cdot, t)$ in $H^1_{\mathrm{ul}}(\mathbf{R})$ with respect to t shows that (for t sufficiently large) $n(t)$, and, if $n(t) \geqslant 1$, the family $h_1^{(t)}, ..., h_{n(t)}^{(t)}$ are actually independent of t (denote by n and by $h_1, ..., h_n$ these objects for t large). As by hypothesis $u(\cdot, t)$ does not converge to 0, $n(t)$ cannot be equal to 0.

The fact that the local approximation by stationary solutions becomes more and more accurate on arbitrarily large intervals when $t \to +\infty$ shows that, if $n \geqslant 2$, for $j = 1, ..., n-1$, $y_{j+1}(t) - y_j(t) \to +\infty$ when $t \to +\infty$.

For $j \in \{1, ..., n\}$, write

$$F_j(x, t) = \int_{x-1}^{x+1} (u(z, t) - h_j(z-x))\, h_j'(z-x)\, dz.$$

Remark that each function $F_j(\cdot, \cdot)$ is C^1, satisfies $F_j(y_j(t), t) \simeq 0$, and, for y not too far from $y_j(t)$,

$$\int_{y-1}^{y+1} h_j'^2(z-y)\, dz > 0.$$

Thus, the implicit function theorem shows that, for t sufficiently large and for any $j \in \{1, ..., n\}$, there exists a unique $x_j(t) \simeq y_j(t)$ such that $F_j(x_j(t), t) = 0$; moreover, the map $t \mapsto x_j(t)$ is C^1 and $x_j'(t) \to 0$ when $t \to +\infty$. Finally, the properties of approximation of $u(\cdot, t)$ by the functions h_j stated above remain true with the $y_j(\cdot)$ replaced by the $x_j(\cdot)$.

The proof of Theorems 1, 2, 3, 4 is complete.

4.2. *Proof of Theorem* 5

Roughly speaking, the same arguments as for the proofs of Theorems 1, 2, 3, 4 enable to prove Theorem 5 (the slight difference is that the a priori estimate on $\|u(\cdot, t)\|_{L^\infty(\mathbf{R})}$ is not in the hypotheses any more); nevertheless, in the parabolic case (assertion 1), we will provide a more direct proof based on the maximum principle.

4.2.1. *Parabolic equation.* We prove assertion 1 of Theorem 5.

Let $u(\cdot, t)$, $t \geqslant 0$ be a solution of Eq. (1) with initial condition $u(x, 0) = u_0(x)$. Write $l_+ = \limsup_{|x| \to +\infty} u_0(x)$, $l_- = \liminf_{|x| \to +\infty} u_0(x)$, and suppose that l_+, l_-, and V satisfy the hypotheses of assertion 1 of Theorem 5.

LEMMA 7. *We have*

$$\lim_{t \to +\infty} \limsup_{|x| \to +\infty} |u(x, t)| = 0.$$

Proof. The proof is somehow similar to that of Lemma 1. Take any $\varepsilon > 0$; we are going to prove that $\limsup_{t \to +\infty} \limsup_{x \to -\infty} |u(x, t)| \leqslant \varepsilon$. Up to exchanging $u \leftrightarrow -u$ and $x \leftrightarrow -x$, this will prove the lemma.

Let $c > 0$ to be chosen later, and let l'_+ be a real number satisfying $l'_+ > l_+$ and $V'(\cdot) > 0$ on $]0; l'_+]$. Let ϕ denote the solution of the differential equation

$$\phi'' = V'(\phi) + c\phi'$$

with initial condition $\phi(0) = l'_+$ and $\phi'(0) = 0$.

Claim. If c is sufficiently large, then $\phi'(\cdot) > 0$ on $[0; +\infty[$ and $\phi(x) \to +\infty$ when $x \to +\infty$.

The claim clearly holds if $V'(\cdot) > 0$ on $[0; +\infty[$. If $V'(\cdot)$ changes sign on $[0; +\infty[$, then let $v_+ = \inf\{v > 0 \mid V'(v) = 0\}$. Then $V'(\cdot) > 0$ on $[l'_+; v_+]$, thus there exists a smallest $x > 0$ such that $\phi(x) = v_+$, and $\phi'(x)$ is positive and arbitrarily large if c is sufficiently large. As $V(\cdot) \geqslant 0$, the claim follows by an energy argument.

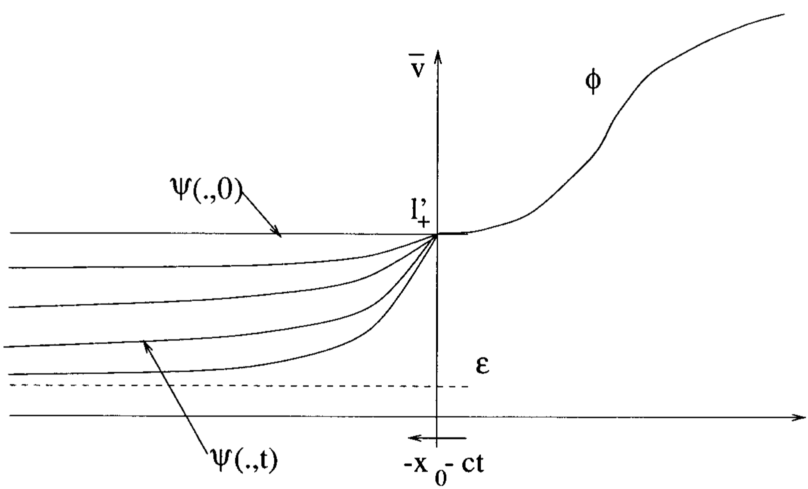
We suppose that c is sufficiently large so that the previous claim holds. Let $\alpha > 0$ and $x_0 > 0$ to be chosen later. Up to diminishing ε, we suppose that $\varepsilon < l'_+$. Write

$$\psi(x, t) = \varepsilon + (l'_+ - \varepsilon)(e^{-\alpha t} + (1 - e^{-\alpha t})\, e^x),$$

and define the function $\bar{v}(\cdot, \cdot)$ by

$$\bar{v}(x, t) = \phi(x + x_0 + ct) \qquad \text{if} \quad x \geqslant -x_0 - ct,$$

$$\bar{v}(x, t) = \psi(x + x_0 + ct, t) \qquad \text{if} \quad x \leqslant -x_0 - ct.$$



Let $P(x,t)=\bar{v}_t+V'(\bar{v})-\bar{v}_{x}x$. We want that $P(x,t)\geqslant 0$, i.e. that $\bar{v}$ be a supersolution. For $x>-x_0-ct$, according to the definition of ϕ, we have $P(x,t)=0$. For $x<-x_0-ct$, we have

$$P(x,t)=(l'_+-\varepsilon)(-\alpha e^{-\alpha t}+e^{x+x_0+ct}((c-1)(1-e^{-\alpha t})+\alpha e^{-\alpha t}))+V'(\bar{v}),$$

and, as $\varepsilon\leqslant\bar{v}\leqslant l'_+$, we have $V'(\bar{v})\geqslant\beta\bar{v}$, where β is a positive constant. We suppose that $c>1$ and we choose $\alpha=\beta/(l'_+-\varepsilon)$. Then $P(x,t)\geqslant 0$.

Finally, for $x=-x_0-ct$, $\bar{v}$ is not differentiable but $\bar{v}_{xx}$ is "infinitely negative"; thus $P(x,t)$ is "infinitely positive", and, finally, $\bar{v}$ is a supersolution.

Now, we can choose x_0 sufficiently large so that $\bar{v}(x,0)\geqslant u(x,0)$. Then, for any $t\geqslant 0$, we have $\bar{v}(x,t)\geqslant u(x,t)$, and the result follows. ∎

Again, take $\varepsilon>0$ arbitrarily small. We are going to prove that

$$\limsup_{t\to+\infty}\ \sup_{x\in\mathbf{R}} u(x,t)\leqslant\varepsilon.$$

Up to changing $u\leftrightarrow -u$, this will imply that $\sup_{x\in\mathbf{R}}|u(x,t)|\to 0$ when $t\to+\infty$, and, by the same arguments as for the proof of Proposition 1, that $\|u(\cdot,t)\|_X\to 0$ when $t\to+\infty$.

According to the hypotheses, if ε is sufficiently small, we have $V'(\varepsilon)>0$, $V(v)>V(\varepsilon)$ for $v>\varepsilon$, and $\liminf_{v\to+\infty}V(v)>V(\varepsilon)$. Let $w_0(\cdot)$ be the solution of $w''-V'(w)$ (i.e. of the stationary Eq. (9)) with initial condition $w_0(0)=\varepsilon$ and $w_0'(0)=0$. By an energy argument, we have $w_0'(x)>0$ for $x>0$ and $w_0(x)\to+\infty$ when $x\to+\infty$ (indeed, for $x\geqslant 1$, $w_0'(x)$ is bounded from below by a strictly positive constant).

Define the function $\phi_0(\cdot)$ by: $\phi_0(x)=\varepsilon$ for $x\leqslant 0$, and $\phi_0(x)=w_0(x)$ for $x\geqslant 0$. According to the preceding lemma, there exists $T>0$ such that $\limsup_{|x|\to+\infty}|u(x,t)|<\varepsilon/2$. Then, for $x_1>0$ sufficiently large, we have

$$u(x,T)<\phi_0(x+x_1)\qquad\text{and}\qquad u(x,T)<\phi_0(-x+x_1),\quad x\in\mathbf{R}.$$

As $(x,t)\mapsto\phi_0(x+x_1)$ and $(x,t)\mapsto\phi_0(-x+x_1)$ define supersolutions for Eq. (1), this shows that $\|u(\cdot,t)\|_{L^\infty(\mathbf{R})}$ is bounded independently of $t\geqslant 0$.

Now let $c_1>0$ to be chosen later, and let $w_1(\cdot)$ be the solution of the differential equation

$$w''+c_1w'-V'(w)=0$$

with initial condition $w_1(0)=\varepsilon$ and $w_1'(0)=0$. As $\|u(\cdot,t)\|_{L^\infty(\mathbf{R})}$ is bounded independently of $t\geqslant 0$, we can change the shape of the potential $V(\cdot)$ without changing the solution $u(\cdot,\cdot)$. In particular, we can make the hypothesis that $V'(v)>0$ for $v>0$ sufficiently large. With this hypothesis, we can choose c_1 sufficiently small so that $w_1'(\cdot)>0$ for $x>0$ and $w_1(x)\to+\infty$ when $x\to+\infty$.

Define the function $\phi_1(\cdot)$ by: $\phi_1(x)=\varepsilon$ for $x\leqslant 0$ and $\phi_1(x)=w_1(x)$ for $x\geqslant 0$. For $x_2>0$ sufficiently large, we have

$$u(x, T)<\phi_1(x+x_2) \qquad \text{and} \qquad u(x, T)<\phi_1(-x+x_2), \qquad x\in\mathbf{R}.$$

As $(x, t)\mapsto\phi_1(x+x_1-c_1(t-T))$ and $(x, t)\mapsto\phi_0(-x+x_2-c_1(t-T))$ define supersolutions for Eq. (1), this shows the desired result. ∎

4.2.2. *Hyperbolic equation.* We prove assertion 2 of Theorem 5. Consider a solution $u(\cdot, t)$, $t\geqslant 0$ of Eq. (2).

(a) Suppose first that $u(\cdot, 0)$ satisfies hypothesis (a) (see Theorem 5), i.e. the energy is finite at $t=0$. The energy decreases with time; as $V(\cdot)\geqslant 0$, it always remains non-negative, thus it converges to a non-negative limit, and the function $t\mapsto\int u_t^2$ is integrable on $\mathbf{R}_+$. On the other hand, finiteness of the energy and the fact that $\liminf_{|v|\to+\infty} V(v)>0$ imply that $\|u(\cdot, t)\|_{L^\infty(\mathbf{R})}$ is bounded independently of $t\geqslant 0$. Thus, by the same arguments as in Section 3.2.1, we obtain that, for t large, the functions $x\mapsto u(x, t)$ are locally uniformly approximated by stationary solutions of Eq. (2). But all stationary solutions, except the solution $u\equiv 0$, have an infinite energy. As the energy of our solution is finite, the result follows. ∎

(b) Now, suppose that hypothesis (b) is satisfied, i.e. $V''(0)>0$ and $\limsup_{|x|\to+\infty}\int T_x\rho(u^2+u_x^2+u_t^2)<\varepsilon$ at $t=0$, where ε is a constant to be chosen. By the same arguments as in Section 3.2.2, we obtain that, if ε is sufficiently small and for x_0 sufficiently large,

$$\lim_{t\to+\infty}\ \sup_{|x|>x_0+t}\int T_x\rho(u^2+u_x^2+u_t^2)=0,$$

and the convergence is exponential. Proceeding as in Section 3.2.2, we deduce from this and from the hypothesis $V(\cdot)\geqslant 0$ that $\int_{x_0-t}^{x_0+t}(u_t^2/2+u_x^2/2+V(u))$ converges to a non-negative limit when $t\to+\infty$. Again, this implies that $\|u(\cdot, t)\|_{L^\infty(\mathbf{R})}$ is bounded independently of $t\geqslant 0$. The remaining arguments are the same as in (a) above. ∎

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