ELSEVIER

Physica D 146 (2000) 121–136

PHYSICA D

www.elsevier.com/locate/physd

Criteria for the stability of spatial extensions of fixed points and periodic orbits of differential equations in dimension 2

Emmanuel Risler*

Institut Non Linéaire de Nice, UMR CNRS-UNSA 6618, 1361 Route des Lucioles, F-06560 Valbonne, France

Received 1 November 1999; accepted 26 May 2000
Communicated by A.C. Newell

Abstract

We consider partial differential equations that can be viewed as spatial extensions of two-dimensional differential equations with respect to a coupling matrix. We provide analytical and geometrical criteria governing the stability of spatially homogeneous stationary solutions and the phase stability of spatially homogeneous time periodic solutions. We distinguish cases where the differential equation has a conservative or a dissipative behavior. The geometrical criteria roughly speaking relate to the "sense of rotation" of, on one hand, the coupling matrix, and, on the other, the flow of the differential equation around the spatially homogeneous solution. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Partial differential equations; Spatial extensions; Time periodic solutions; Fixed points; Periodic orbits

1. Introduction

Many partial differential equations arising in mechanics, physics, chemistry, biology, etc. can be viewed as *spatially extended differential equations*, i.e. can be written in the form

$$\partial_t u = f(u) + C\,\Delta u, \tag{1}$$

where $f(\cdot)$ is a smooth vector field on an open set of $\mathbf{R}^d$, $d \geq 1$, C is a *coupling map*, i.e. a linear map: $\mathbf{R}^d \to \mathbf{R}^d$ having no eigenvalue with strictly negative real part, and $\Delta u = (\Delta u_1, \dots, \Delta u_d)$ means, on each coordinate, the Laplace operator with respect to a space coordinate x varying in an open set Ω of $\mathbf{R}^n$, $n \geq 1$.

This is the case, for instance, for (nonlinear) heat equations ($C = \mathrm{Id}$), reaction–diffusion equations (C=diagonal), wave equations

$$\left(C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right),$$

* Tel.: +33-492967300; fax: +33-493652517
E-mail address: risler@inln.cnrs.fr (E. Risler).

0167-2789/00/$ – see front matter
PII: S0167-2789(00)00123-8

Schrödinger equations

$$\left(C = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right),$$

real and complex Ginzburg–Landau equations, etc. (see [5] for more details).

The reason for the terminology "spatially extended differential equations" is that these equations can be viewed as spatial extensions of the ordinary differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u). \tag{2}$$

We would like to understand the relations between the dynamics of the differential equation (2), and that of its spatial extension (1). We suppose that the conditions at the boundary of Ω are of Neumann type or periodic, so that, to any solution $t \mapsto u_0(t)$ of (2) canonically corresponds the spatially homogeneous solution $t \mapsto \bar{u}_0(t)$, $\bar{u}_0(t)(x) \equiv u_0(t)$, $x \in \Omega$, for (1); this solution $\bar{u}_0(\cdot)$ will be called the *spatial extension* of the solution $u_0(\cdot)$. A natural question concerns the relations between the stability properties of $u_0(\cdot)$ and $\bar{u}_0(\cdot)$.

Suppose that $u_0(\cdot)$ is a stable or neutral fixed (resp. periodic) solution of (2); then $\bar{u}_0(\cdot)$ is also fixed (resp. periodic), but might be unstable. This instability is at the origin of many phenomena displaying "patterns" or "spatio-temporal chaos" in nonlinear physics (see [2]). A first step towards a better understanding of such phenomena consists in analyzing the occurrence of this instability. Linearizing (1) around $\bar{u}_0(\cdot)$ formally gives

$$\partial_t u = (\mathrm{D}f(u_0(t)) + C\Delta)u, \tag{3}$$

which reduces in Fourier coordinates to

$$\partial_t \hat{u}(\vec{k}) = (\mathrm{D}f(u_0(t)) - |\vec{k}|^2 C)\hat{u}(\vec{k}), \tag{4}$$

which is a linear differential equation in dimension d with constant (resp. periodic) coefficients. Write $M(t) = Df(u_0(t))$, and consider the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (M(t) + \lambda C)u, \tag{5}$$

depending on the parameter λ (with corresponds to $-|\vec{k}|^2$). The linear stability of $\bar{u}_0$ for (3) reduces, at least formally, to the linear stability of 0 for the differential equation (5), for any λ in the spectrum of the Laplace operator on Ω. Remark that the same reduction could have been done for any equation involving instead of the Laplace operator a more general operator (any adjoint-operator acting on a Hilbert space H of functions over a metric space, vanishing on constant functions, and whose spectrum is bounded from above, see [5]; for instance, a differential operator like $-\sum_i \partial_{x_i}^4$, a discrete Laplace operator on a lattice, etc.).

In the following, we will forget about the precise nature of this operator, and just consider the family of differential equations (5), depending on the parameter λ.

We will say that the *spatial extension of the solution* $u_0(\cdot)$ *with respect to the coupling map C is*

- *linearly stable* (resp. *unstable*) *with respect to* λ_0 (for some $\lambda_0 \in]-\infty; 0]$) if 0 is linearly stable (resp. unstable) for the differential equation (5) with $\lambda = \lambda_0$;
- *linearly unstable* if it is linearly unstable (in the sense above) with respect to some $\lambda_0 \in]-\infty; 0]$.

If $u_0(\cdot)$ is a stable or neutral fixed point, instability of a spatial extension of $u_0(\cdot)$ is known as *Turing instability* [7]; if $u_0(\cdot)$ is linearly stable, this instability cannot occur with respect to values of λ close to 0. If on the other hand $u_0(\cdot)$ is a stable or neutral periodic orbit, then it always has a Floquet multiplier which is equal to 1 (in the direction of the flow). Thus, even small perturbations of the Floquet map can render this multiplier larger than 1, and instability of a spatial extension of $u_0(\cdot)$ might occur with respect to any value of λ negative and close enough to 0; if this occurs, one speaks of *phase instability* [1,4,8].

We will thus say that *the spatial extension of a periodic orbit* $u_0(\cdot)$ *with respect to the coupling map C is*

- *phase stable* (resp. *unstable*) if it is stable (resp. unstable) with respect to any $\lambda \in]-\varepsilon; 0[$ for some $\varepsilon > 0$.

This paper is concerned with the stability of spatial extensions of fixed points, and the phase stability of spatial extensions of periodic orbits, for fixed points and periodic orbit of differential equations in dimension $d = 2$. This is the first dimension where these stability questions arise, and many examples of partial differential equations correspond to this case (see the beginning of this section). But, above all, the dimension 2 will enable us to provide geometrical answers to these stability questions.

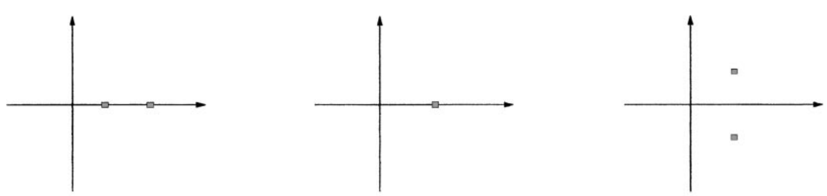
Let us describe rapidly the contents of the paper. Section 2 is devoted to some preliminaries. The case of fixed points is treated in Section 3; this case is very elementary, since it reduces to making the sum of two matrices. We provide geometrical criteria of stability or instability (Corollaries 1–3), and give some examples.

The main part is Section 4, which is devoted to periodic orbits. As $d = 2$, these periodic orbits have two Floquet multipliers, one of them (in the direction of the flow) being equal to 1. We distinguish two cases: the conservative case, when the second multiplier is also equal to 1, and the dissipative case, when it is strictly smaller than 1. An elementary perturbation argument enables to provide, in both cases, a general criterion governing the phase stability of spatial extensions of a periodic orbit (Corollaries 4 and 5). Next, we derive geometrical formulations of these criteria. This works excellent in the conservative case (Proposition 3), and becomes slightly more painful in the dissipative case (Proposition 4 and Corollary 6). The geometrical criteria roughly speaking relate to the “sense of rotation” of, on one hand, the coupling map, and, on the other, the flow of the differential equation around the periodic orbit. Several examples (nonlinear Schrödinger equation, sine-Gordon equation, complex Ginzburg–Landau equation) are presented.

2. Preliminaries

Definitions. We will say that the coupling map (matrix) C is

- *of type I*, or equivalently *purely diffusive*, if C is proportional to the identity;
- *of type II*, or equivalently *diffusive*, if the two eigenvalues of C are distinct and real;
- *of type III*, or equivalently *propagative*, if the two eigenvalues of C are equal and C is not diagonalizable;
- *of type IV*, or equivalently *dispersive*, if the two eigenvalues of C are nonreal.



II III IV

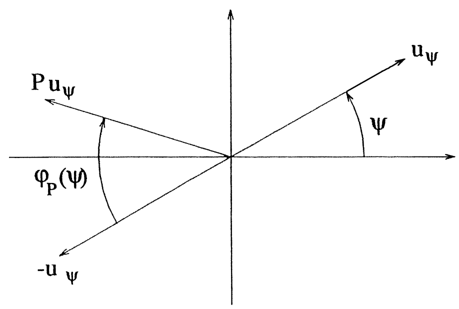
Denote by $\mathrm{Rot}_{\pi/2}$ the rotation of center 0 and of angle $\pi/2$ in $\mathbf{R}^2$. We will say that a linear map $L : \mathbf{R}^2 \to \mathbf{R}^2$ is

- *forward monotonic* (resp. *backward monotonic*) if, for any $u \in \mathbf{R}^2$, $(Lu, \mathrm{Rot}_{\pi/2}\, u) \geq 0$ (resp. ≤ 0) (where $(\cdot,\cdot)$ denotes the usual scalar product of $\mathbf{R}^2$);
- *strictly forward monotonic* (resp. *strictly backward monotonic*) if it is forward monotonic (resp. backward monotonic) and if moreover there are vectors u for which $(Lu, \mathrm{Rot}_{\pi/2}\, u) > 0$ (resp. < 0);
- (*strictly*) *monotonic* if it is (strictly) forward monotonic or (strictly) backward monotonic.

Remarks.
1. *The map C (and equivalently $-C$, which will be considered later) is monotonic (resp. strictly monotonic) if and only if C is of types I, III, or IV (resp. of types III or IV).*
2. *If C is of type I, then it commutes with any matrix, and we can see in Eq. (5) that the coupling has only a stabilizing influence. Thus, in this case, the spatial extension of u_0 is always linearly stable with respect to any $\lambda < 0$. Therefore, in the following, we will sometimes exclude this particular case.*

Notation. Let P be any 2×2 real matrix.

For $\psi \in \mathbf{R}/2\pi\mathbf{Z}$, let u_ψ denote the vector $(\cos\psi,\ \sin\psi)$; if $u_\psi \notin \ker P$, we will write $\varphi_P(\psi)$ for the number in $]-\pi;\pi]$ corresponding to the angle $(-u_\psi, Pu_\psi)$.

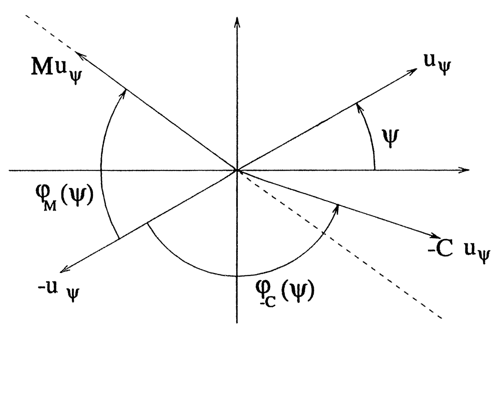


3. Fixed points

Here we suppose that u_0 is a fixed point for $f(\cdot)$; then the matrix $M(t)$ defined above is independent of t, we denote it by M. We suppose that the fixed point u_0 is not linearly unstable (i.e. M has no eigenvalue of strictly positive real part). Then, for any $\lambda < 0$, the trace of $M + \lambda C$ is negative, and thus the spatial extension of u_0 with respect to C is linearly unstable with respect to λ if and only if $M + \lambda C$ has a real and strictly positive eigenvalue. The following lemma provides a geometric criterion for this to occur.

Lemma 1. *The two following assertions are equivalent:*
1. *There exists $\lambda < 0$ such that $M + \lambda C$ has a real and strictly positive eigenvalue.*
2. *There exists $\psi \in \mathbf{R}/2\pi\mathbf{Z}$ such that $u_\psi \notin \ker M \cup \ker C$ and such that $|\varphi_M(\psi) - \varphi_{-C}(\psi)| > \pi$.*

The proof is elementary:



Definition. We will say that the fixed point u_0 is (*strictly*) *forward monotonic* (resp. *backward monotonic*) if the matrix M is (strictly) forward monotonic (resp. backward monotonic).

Corollary 1. *If the fixed point u_0 and the matrix $-C$ are both forward or backward monotonic, then the spatial extension of u_0 with respect to C is not linearly unstable.*

The proof of this corollary is an immediate consequence of Lemma 1. In the conservative case, we derive a generic instability result.

Corollary 2. *If u_0 is strictly forward* (*resp. backward*) *monotonic and $-C$ is strictly backward* (*resp. forward*) *monotonic, and if the eigenvalues of M and $-C$ all have vanishing real parts, then the spatial extension of u_0 with respect to C is generically* (*if M and $-C$ are not proportional*) *linearly unstable.*

Proof. As C and M are nonvanishing, $\ker C$ and $\ker M$ are at most one-dimensional. For any $\psi \in \mathbf{R}/2\pi\mathbf{Z}$ such that $u_\psi \notin \ker C \cup \ker M$, $\varphi_M(\psi)$ and $\varphi_{-C}(\psi)$ have opposite signs. Thus, there exists a unique $\lambda_0(\psi) \in]-\infty; 0[$ such that $\varphi_{M+\lambda C}(\psi)$ changes sign when $\lambda = \lambda_0(\psi)$. Generically (more precisely, if M and $-C$ are not proportional), we have $\inf_\psi \lambda_0(\psi) < \sup_\psi \lambda_0(\psi)$; in this generic case, for any $\lambda \in]\inf_\psi \lambda_0(\psi); \sup_\psi \lambda_0(\psi)[$, the two eigenvalues of $M + \lambda C$ are real and distinct, and, as the trace of $M + \lambda C$ vanishes, the result follows. □

The following corollary shows that, as soon as C is not of type I, the Turing instability may occur.

Corollary 3. *If C is not proportional to the identity, then there exists a matrix M' whose two eigenvalues have strictly negative real parts, such that there exists a $\lambda < 0$ for which $M' + \lambda C$ has a real and strictly positive eigenvalue.*

Proof. As C is not proportional to the identity, there exists a $v \in \mathbf{R}^2$ such that Cv is not proportional to v. Up to a change of basis, we can suppose that $v = (1,0)$ and that $\varphi_{-C}(0) > 0$. Choose for M' the matrix

$$\begin{pmatrix} 1 & \varepsilon \\ -3/\varepsilon & -2 \end{pmatrix}, \quad \varepsilon > 0.$$

This matrix has two complex conjugated eigenvalues of strictly negative real part, and $\varphi_{M'}(0)$ converges towards $-\pi$ when $\varepsilon \to 0$. Thus, according to Lemma 1, if ε is sufficiently small, M' has the desired property. □

We illustrate the foregoing arguments using two examples.

Example 1 (Nonlinear wave equations). The nonlinear wave equation

$$u_{tt} = g(u, u_t) + \Delta u$$

can be viewed as a spatial extension of the two-dimensional differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ g(u, v) \end{pmatrix} \tag{6}$$

with respect to the coupling matrix

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

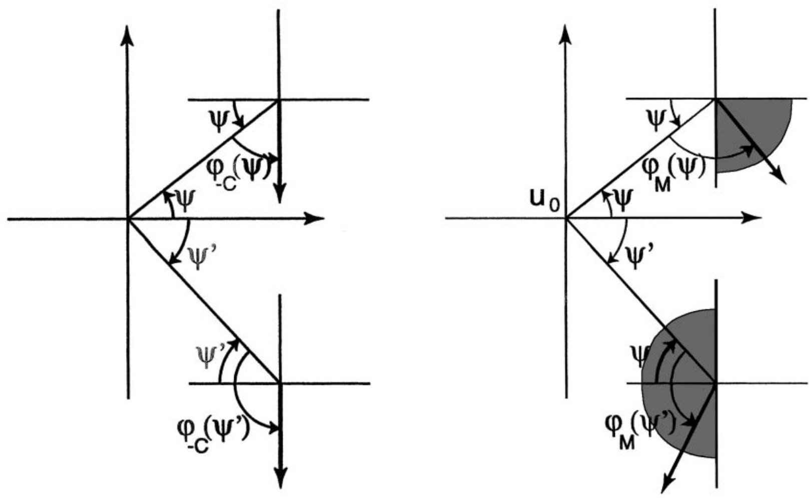
If u_0 is a fixed point for (6), and if u_0 is not linearly unstable, then the spatial extension of u_0 with respect to C is never linearly unstable. This is elementary to prove by computation, but it is also possible to give a geometrical proof in the spirit of the foregoing discussion, as follows.

We have

$$M = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$$

with $a \leq 0$ and $b \leq 0$ and $-C$ is backward monotonic. If moreover $b^2 + 4a \leq 0$, then u_0 is also backward monotonic (indeed $\varphi_M(\pi/2) > 0$) and the result follows by Corollary 1. Without this last hypothesis, let us remark (see illustration) that

- for $\psi \in [-\pi/2; 0]$, $\varphi_M(\psi) + \psi \in [-\pi/2; \pi/2]$ and $\varphi_{-C}(\psi) + \psi = \pi/2$;
- for $\psi \in [0; \pi/2]$, as $a \leq 0$ and $b \leq 0$, $\varphi_M(\psi) + \psi \in [\pi/2; \pi]$ and $\varphi_{-C}(\psi) + \psi = \pi/2$.



Thus, we always have $|\varphi_M(\psi) - \varphi_{-C}(\psi)| \leq \pi$, and, according to Lemma 1, the result follows.

Example 2 (A nonlinear Schrödinger equation). Consider the celebrated cubic nonlinear Schrödinger equation

$$\mathrm{i}u_t + u_{xx} + \sigma(1 - |u|^2)u = 0, \quad u \in \mathbf{C} \tag{7}$$

in the focusing ($\sigma = -1$) or defocusing ($\sigma = +1$) case. This equation can be viewed as a spatial extension of the differential equation

$$u_t = \mathrm{i}\sigma(1 - |u|^2)u, \quad u \in \mathbf{C} \simeq \mathbf{R}^2 \tag{8}$$

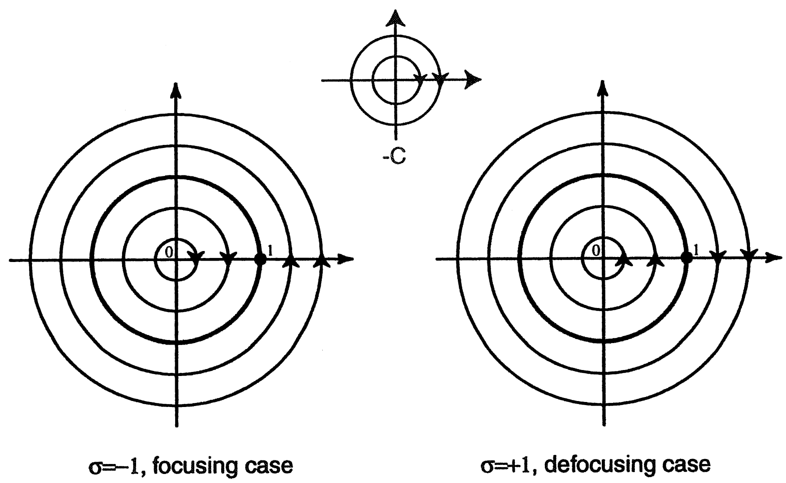
with respect to the coupling matrix

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The points on the circle of center 0 and of radius 1 are fixed for this differential equation and, if $\sigma = -1$ (resp. $\sigma = +1$), they are all strictly forward (resp. backward) monotonic; actually, the differential of the vector field at any of these points is conjugated to the matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \left(\text{resp.} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right).$$

On the other hand, the matrix $-C$ is strictly backward monotonic. Thus, according to Corollary 2, the spatial extension of any of these fixed points is linearly unstable in the focusing case, and is not linearly unstable in the defocusing case.



4. Periodic orbits

4.1. Preliminaries

Here we suppose that $u_0(\cdot)$ is a periodic orbit for $f(\cdot)$; denote by T its (smallest) period. For $t \in \mathbf{R}$, let $e_1(t) = f(u_0(t))$, and let $e_2(t)$ be any vector of $\mathbf{R}^2$ linearly independent of $e_1(t)$ and depending smoothly and T-periodically on t (the simplest choice being $e_2(t) = \mathrm{Rot}_{\pi/2}\, e_1(t)$, where $\mathrm{Rot}_{\pi/2}$ denotes the rotation of angle $\pi/2$).

For $t \in \mathbf{R}$, let $P(t)$ be the matrix whose columns are the respective coordinates of the vectors $e_1(t)$ and $e_2(t)$ in the canonical basis of $\mathbf{R}^2$, let $\hat{M}(t) = -P(t)^{-1}(\mathrm{d}P/\mathrm{d}t)(t) + P(t)^{-1}M(t)P(t)$, and let $\hat{C}(t) = P(t)^{-1}CP(t)$. Eq. (5) reads, in the moving frame $(e_1(\cdot), e_2(\cdot))$,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (\hat{M}(t) + \lambda\hat{C}(t))u. \tag{9}$$

Let us write

$$\hat{M}(t) = \begin{pmatrix} 0 & a(t) \\ 0 & b(t) \end{pmatrix}, \qquad \hat{C}(t) = \begin{pmatrix} c_1(t) & c_2(t) \\ c_3(t) & c_4(t) \end{pmatrix}, \qquad B_s^t = \exp\int_s^t b(\tau)\,\mathrm{d}\tau.$$

For $\lambda, t \in \mathbf{R}$, let $\phi_\lambda(t)$ denote the (linear) flow of the differential equation (9) between the times 0 and t. The two eigenvalues of $\phi_0(T)$ are 1 (the neutral Floquet multiplier in the direction of the flow) and B_0^T (the transverse Floquet multiplier).

We will suppose that $B_0^T \leq 1$, i.e. that the periodic orbit is not linearly unstable, and we will study the linear stability (i.e. the position of the eigenvalues) of $\phi_\lambda(T)$, for small negative λ, considering Eq. (9) as a perturbation of the case $\lambda = 0$. We will distinguish two cases: the dissipative case (i.e. $B_0^T < 1$), and the conservative case (i.e. $B_0^T = 1$).

4.2. Analytical criteria

Writing

$$u(t)=\begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

the differential equation (9) reads

$$\frac{\mathrm{d}x}{\mathrm{d}t}=ay+\lambda(c_1x+c_2y), \tag{10}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=by+\lambda(c_3x+c_4y). \tag{11}$$

4.2.1. Dissipative case

Suppose $B_0^T<1$. Then 1 is an isolated eigenvalue for the linear map $\phi_0(T)$. Thus, for any λ sufficiently close to 0, $\phi_\lambda(T)$ admits a unique eigenvalue close to 1 (denote it by $\mu(\lambda)$), the other eigenvalue being close to B_0^T. The phase stability is therefore governed by the position of $\mu(\lambda)$ with respect to 1.

Let $Y(\cdot)$ be the unique T-periodic solution of the differential equation $\mathrm{d}Y/\mathrm{d}t=bY+c_3$, i.e.

$$Y(t)=B_0^tY_0+\int_0^t B_s^t c_3(s)\,\mathrm{d}s,\quad t\in\mathbf{R},\quad \text{where } Y_0=(1-B_0^T)^{-1}\int_0^T B_s^T c_3(s)\,\mathrm{d}s,$$

and let

$$\mathcal{I}=\int_0^T (a(s)Y(s)+c_1(s))\,\mathrm{d}s.$$

Proposition 1. *The derivative* $\mathrm{d}\mu/\mathrm{d}\lambda(0)$ *exists and is equal to* $\mathcal{I}$.

Corollary 4. *If* $\mathcal{I}>0$ (*resp.* $\mathcal{I}<0$), *then the spatial extension of* $u_0(\cdot)$ *with respect to C is phase stable* (*resp. phase unstable*).

Remarks.

1. *It is actually possible to choose the vectors* $e_2(\cdot)$ *involved in the moving frame in such a way that the matrix* $\hat{M}(t)$ *be constant with respect to t and equal to*

$$\begin{pmatrix} 0 & 0 \\ 0 & \log B_0^T \end{pmatrix}$$

(*see* [3]); *in this situation, the expression of* $\mathcal{I}$ *reduces to* $\int_0^T c_1(s)\,\mathrm{d}s$.
2. *Close to a supercritical Hopf bifurcation, this criterion reduces to the Benjamin–Feir criterion* (*see* [6]).

Using this criterion, one can provide an elementary rigorous justification of the Benjamin–Feir instability criterion for a spatially extended Hopf bifurcation (see [6]).

Proof of Proposition 1. For λ close to 0, the eigenspace of $\phi_\lambda(T)$ corresponding to the eigenvalue $\mu(\lambda)$ is close to the direction of the vector (1,0); thus, it contains a unique vector $\epsilon_\lambda(0)$ whose first coordinate is equal to 1.

Write

$$\epsilon_\lambda(0) = \begin{pmatrix} 1 \\ y_\lambda(0) \end{pmatrix}$$

for $t \in \mathbf{R}$, let $\epsilon_\lambda(t) = \phi_\lambda(t)(\epsilon_\lambda(0))$ and write

$$\epsilon_\lambda(t) = \begin{pmatrix} x_\lambda(t) \\ y_\lambda(t) \end{pmatrix}.$$

We have $\mu(\lambda) \to 1$ when $\lambda \to 0$, and, uniformly with respect to t, $x_\lambda(t) \to 1$ and $y_\lambda(t) \to 0$ when $\lambda \to 0$.

According to (11), we have

$$y_\lambda(T) = B_0^T y_\lambda(0) + \lambda \int_0^T B_s^T c_3(s)\,\mathrm{d}s + \mathrm{o}(\lambda),$$

and, as $y_\lambda(T) = \mu(\lambda) y_\lambda(0)$, this yields $y_\lambda(0) - \lambda Y_0 = \mathrm{o}(\lambda)$. Thus, $y_\lambda(t) - \lambda Y(t) = \mathrm{o}(\lambda)$, uniformly with respect to t.

Therefore, according to (10),

$$x_\lambda(T) - 1 - \lambda \mathcal{I} = \mathrm{o}(\lambda).$$

As $x_\lambda(T) = \mu(\lambda)$, this finishes the proof. □

Example. The complex Ginzburg–Landau equation

$$A_t = A - (1 + \mathrm{i}\alpha)|A|^2 A + (1 + \mathrm{i}\beta)\Delta A, \quad A \in \mathbf{C} \simeq \mathbf{R}^2 \tag{12}$$

can be viewed as a spatial extension of the differential equation $A_t = A - (1+\mathrm{i}\alpha)|A|^2 A$ with respect to the coupling matrix

$$C = \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix}.$$

This differential equation admits an attractive periodic solution $A_0(t) = \mathrm{e}^{-\mathrm{i}\alpha t + \varphi}$. With $e_2(\cdot) = R_{\pi/2}\, e_1(\cdot)$, the matrices $\hat{M}(\cdot)$ and $\hat{C}(\cdot)$ are constant and read, respectively,

$$\begin{pmatrix} 0 & 2\alpha \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix}.$$

Thus, $\mathcal{I} = T(1 + \alpha\beta)$ and the phase stability criterion $\mathcal{I} > 0$ stated above reduces to the celebrated Benjamin–Feir criterion

$$1 + \alpha\beta > 0$$

(nevertheless, in this case, this last criterion governs the linear stability of the spatial extension of the periodic orbit $A_0(\cdot)$ with respect to any (not only small) value of λ, as an immediate calculus shows).

4.2.2. Conservative case

Suppose $B_0^T = 1$. Then the two eigenvalues of $\phi_0(T)$ are equal to 1. For λ close to 0, the two eigenvalues of $\phi_\lambda(T)$ are thus close to 1, and can be real or complex conjugate. Denote them by $\mu_j(\lambda)$, $j = 1$ or 2 (with, for instance, the constraint that, if they are real, then $\mu_1(\lambda) \geq \mu_2(\lambda)$, and if they are complex conjugate, then $\mathrm{Im}\,\mu_1(\lambda) \geq 0$).

Write

$$\mathcal{J} = \int_0^T B_0^s a(s)\,\mathrm{d}s, \qquad \mathcal{K} = \int_0^T B_s^T c_3(s)\,\mathrm{d}s$$

$\left(\text{remark that } \phi_0(T) = \begin{pmatrix} 1 & \mathcal{J} \\ 0 & 1 \end{pmatrix}\right)$.

Proposition 2. *We have*

$$\frac{(\mu_j(\lambda)-1)^2}{\lambda} \to \mathcal{J}\mathcal{K} \quad \textit{when } \lambda \to 0,\ j = 1,2.$$

More precisely,

$$\mu_1(\lambda) = 1 + \sqrt{\sigma}\sqrt{|\lambda\mathcal{J}\mathcal{K}|} + \mathrm{o}(\sqrt{|\lambda|}), \qquad \mu_2(\lambda) = 1 - \sqrt{\sigma}\sqrt{|\lambda\mathcal{J}\mathcal{K}|} + \mathrm{o}(\sqrt{|\lambda|}),$$

where σ represents the sign of $\lambda\mathcal{J}\mathcal{K}$, i.e. $\sqrt{\sigma}$ is equal to 1 *if $\lambda\mathcal{J}\mathcal{K} \geq 0$, and to the complex number* i *if $\lambda\mathcal{J}\mathcal{K} < 0$.*

Corollary 5. *If $\mathcal{J}\mathcal{K} < 0$ (resp. if $\mathcal{J}\mathcal{K} > 0$) then the spatial extension of $u_0(\cdot)$ with respect to C is (resp. is not) phase unstable.*

Indeed, when $\lambda < 0$ and the eigenvalues of $\phi_\lambda(T)$ are complex conjugate, their modulus cannot be strictly greater than 1, because the trace of the coupling matrix C is nonnegative.

Remark. *It is actually possible to choose the vectors $e_2(\cdot)$ involved in the moving frame in such a way that the matrix $\hat{M}(t)$ be constant with respect to t and equal to*

$$\begin{pmatrix} 0 & T^{-1}\mathcal{J} \\ 0 & 0 \end{pmatrix}$$

(*see* [3]); *in this situation, the expression of $\mathcal{K}$ reduces to $\int_0^T c_3(s)\,\mathrm{d}s$.*

Proof of Proposition 2. For λ close to 0, the eigenspaces (in $\mathbf{C}^2$) of $\phi_\lambda(T)$ corresponding to the (possibly identical) eigenvalues $\mu_j(\lambda)$, $j = 1,2$, are close to the direction of the vector (1,0). Let $\epsilon_\lambda(0)$ be any eigenvector of $\phi_\lambda(T)$ having a first coordinate equal to 1, and denote by $\mu(\lambda)$ the corresponding eigenvalue of $\phi_\lambda(T)$.

Let us write, as in the dissipative case,

$$\epsilon_\lambda(0) = \begin{pmatrix} 1 \\ y_\lambda(0) \end{pmatrix},$$

and, for $t \in \mathbf{R}$, $\epsilon_\lambda(t) = \phi_\lambda(t)(\epsilon_\lambda(0))$ and

$$\epsilon_\lambda(t) = \begin{pmatrix} x_\lambda(t) \\ y_\lambda(t) \end{pmatrix}.$$

We have again $\mu(\lambda) \to 1$ when $\lambda \to 0$, and, uniformly with respect to t, $x_\lambda(t) \to 1$ and $y_\lambda(t) \to 0$ when $\lambda \to 0$.

According to (11), we have

$$y_\lambda(T) - y_\lambda(0) = \lambda \int_0^T B_s^T c_3(s)\,\mathrm{d}s + \mathrm{o}(\lambda) = \lambda\mathcal{K} + \mathrm{o}(\lambda).$$

Thus, $(\mu(\lambda)-1)y_\lambda(0) = \lambda\mathcal{K} + \mathrm{o}(\lambda)$. Besides, $\mu(\lambda)-1 = x_\lambda(T)-1$, and, according to (10),

$$x_\lambda(T) - 1 = \int_0^T a(s)y_\lambda(s)\,\mathrm{d}s + \mathcal{O}(\lambda) = y_\lambda(0)\int_0^T a(s)B_0^s\,\mathrm{d}s + \mathcal{O}(\lambda) = y_\lambda(0)\mathcal{J} + \mathcal{O}(\lambda).$$

Thus,

$$(\mu(\lambda)-1)^2 = (\mu(\lambda)-1)y_\lambda(0)\mathcal{J} + \mathrm{o}(\lambda) = \lambda\mathcal{J}\mathcal{K} + \mathrm{o}(\lambda),$$

which proves the first assertion of the proposition. The remaining assertions follow easily. □

4.3. Geometrical criteria

The geometrical criteria will involve the properties of forward or backward monotonicity for the periodic orbit $u_0(\cdot)$ (to be defined below). In order these formulations to make sense, we suppose from now on that the local frames $(e_1(t), e_2(t))$ have the usual orientation (the same as the canonical basis of $\mathbf{R}^2$). We begin with the conservative case, for which the geometrical criteria are the simplest and the most meaningful.

4.3.1. Conservative case

Definition. We will say that the periodic orbit $u_0(\cdot)$ is (*strictly*) *forward monotonic* (resp. *backward monotonic*) if the matrix of $\phi_0(T)$ is itself (strictly) forward monotonic (resp. backward monotonic), or equivalently if the number $\mathcal{J} = \int_0^T B_0^s a(s)\,\mathrm{d}s$ is (strictly) negative (resp. positive).

Remark. *The sign of $\mathcal{J}$, and consequently the monotonicity of $u_0(\cdot)$ are unchanged under conjugacy of the vector field $f(\cdot)$ by an orientation-preserving diffeomorphism of $\mathbf{R}^2$.*

Proposition 3. *If the periodic orbit $u_0(\cdot)$ and the matrix $-C$ are both strictly forward or backward monotonic, then the spatial extension of $u_0(\cdot)$ with respect to C is not phase unstable* (*and is even phase stable if* tr $C > 0$).

If on the other hand $u_0(\cdot)$ is strictly forward (*resp. backward*) *monotonic and $-C$ is strictly backward* (*resp. forward*) *monotonic, then the spatial extension of $u_0(\cdot)$ with respect to C is phase unstable*.

Proof. If $-C$ is strictly forward (resp. backward) monotonic, then $c_3(t) \le 0$ (resp. ≥ 0) for any $t \in \mathbf{R}$, and there are values of t for which the inequality is strict; thus, we have $\mathcal{K} < 0$ (resp. > 0), and the proposition follows from Corollary 5. □

Example.

1. Let us consider again the cubic nonlinear Schrödinger equation (7). The matrix $-C$ is strictly backward monotonic. The circles of center 0 and of radii in $]0;1[\cup]1;+\infty[$ are trajectories of periodic orbits for the corresponding differential equation (8). If $\sigma = -1$ (resp. $\sigma = +1$), all these periodic orbits are strictly forward (resp. backward) monotonic (see the figure in Section 3); their spatial extensions with respect to C are thus phase unstable in the focusing case, and not phase unstable in the defocusing case.
2. Consider the sine-Gordon equation

$$u_{tt} + \sin u = u_{xx}, \quad u \in R/2\pi\mathbf{Z}.$$

This equation can be viewed as a spatial extension of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\sin u \end{pmatrix}$$

with respect to the coupling matrix

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The matrix $-C$ is strictly backward monotonic, and this differential equation admits periodic orbits parameterized by their energy $E \in]-1;1[\cup]1;+\infty[$. The periodic orbits corresponding to $E \in]-1;1[$ are strictly forward monotonic, their spatial extensions with respect to C are thus phase unstable; the ones corresponding to $E \in]1;+\infty[$ are strictly backward monotonic, their spatial extensions with respect to C are thus not phase unstable.

Remark. *In the case of nonlinear wave equations viewed as spatial extensions of second-order systems, as for the sine-Gordon equation above, the periodic orbits, if they exist, always have the converse orientation. This enables to use a different terminology, presented below, to describe their monotonicity properties.*

Recall that an oscillator is called soft (*resp. hard*) *if the frequency of the oscillations decreases* (*resp. increases*) *with their amplitude. By analogy, we can say that a periodic orbit of a conservative second-order system is* (*strictly*) *soft* (*resp. hard*) *if it is* (*strictly*) *forward* (*resp. backward*) *monotonic in the sense defined above.*

With this definition, if this periodic orbit is strictly soft (*resp. hard*), *then its spatial extension is phase unstable* (*resp. is not phase unstable*). *For instance, in the example* (2) *above, the periodic orbits whose energy belongs to* $]-1;1[$ (*resp. to* $]1;+\infty[$) *are all strictly soft* (*resp. strictly hard*).

4.3.2. Dissipative case

In the dissipative case, the geometrical criteria are less natural and slightly more involved. They relate on quantitative estimates on the angles $\varphi_{-C}(\cdot)$, $\varphi_{-\hat{C}(t)}(\cdot)$, and $\varphi_{\hat{M}(t)}(\cdot)$. But these angles depend on the choice of the bases in which the matrices are computed, in particular on the local frames $(e_1(\cdot), e_2(\cdot))$. We thus fix this choice, imposing for the remaining that $e_2(t) = \mathrm{Rot}_{\pi/2}\, e_1(t)$, $t \in \mathbf{R}$.

Let

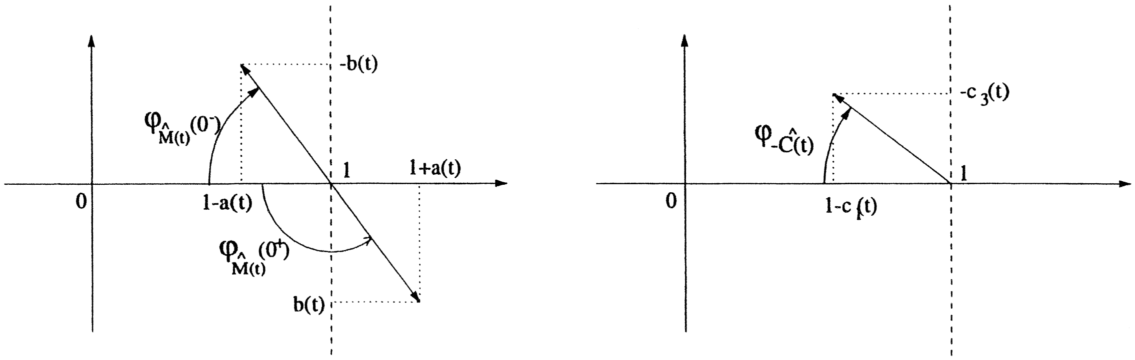
$$\Phi_{-C} = \bigcup_{\psi\in\mathbf{R}/2\pi\mathbf{Z},\, u_\psi\notin\ker C} \varphi_{-C}(\psi) = \bigcup_{t\in\mathbf{R},\, u_0\notin\ker\hat{C}(t)} \varphi_{-\hat{C}(t)}(0).$$

For $t \in \mathbf{R}$, if $\hat{M}(t) \neq 0$, then $\varphi_{\hat{M}(t)}(\psi)$ has a limit in $[-\pi;\pi]$ when $\psi \to 0^+$ (resp. when $\psi \to 0^-$); let us denote by $\varphi_{\hat{M}(t)}(0^+)$ (resp. $\varphi_{\hat{M}(t)}(0^-)$) this limit (remark that $|\varphi_{\hat{M}(t)}(0^+) - \varphi_{\hat{M}(t)}(0^-)| = \pi$), and let

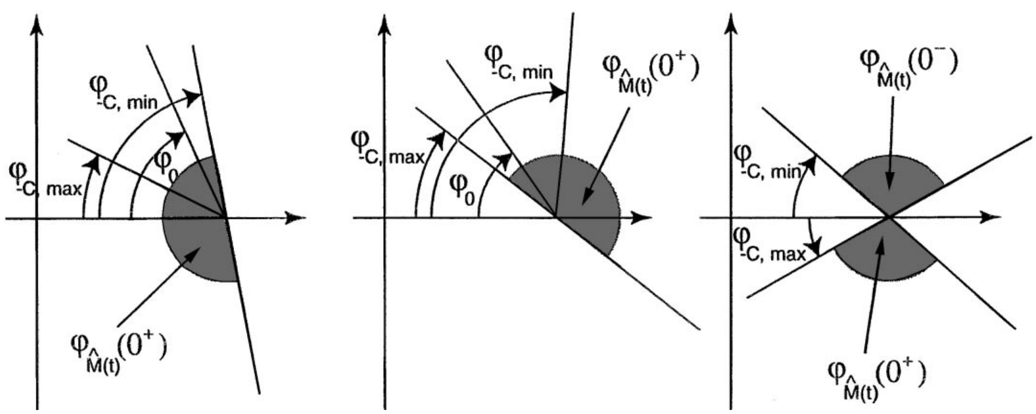
$$\Phi_{\hat{M}}(0^+) = \bigcup_{t\in\mathbf{R},\, \hat{M}(t)\neq 0} \varphi_{\hat{M}(t)}(0^+), \qquad \Phi_{\hat{M}}(0^-) = \bigcup_{t\in\mathbf{R},\, \hat{M}(t)\neq 0} \varphi_{\hat{M}(t)}(0^-).$$

In the proposition below, hypotheses (1a) and (1b) are symmetrical, as are hypotheses (2a) and (2b).

Proposition 4. *If one of the hypotheses* (1a), (1b), *and* (3) *below is satisfied, then the spatial extension of* $u_0(\cdot)$ *with respect to C is phase stable; if one of the hypotheses* (2a) *and* (2b) *below is satisfied, then it is phase unstable.*



1. (a) *The matrix $-C$ is forward monotonic, and there exists a $\varphi_0 \in]-\pi;0]$ such that $\Phi_{-C} \subset [\varphi_0;0]$ and $\Phi_{\hat{M}}(0^+) \subset]\varphi_0;\varphi_0+\pi[$.* (b) *The matrix $-C$ is backward monotonic, and there exists a $\varphi_0 \in [0;\pi[$ such that $\Phi_{-C} \subset [0;\varphi_0]$ and $\Phi_{\hat{M}}(0^-) \subset]\varphi_0-\pi;\varphi_0[$.*
2. (a) *The matrix $-C$ is forward monotonic, and there exists a $\varphi_0 \in]-\pi;0[$ such that $\Phi_{-C} \subset]-\pi;\varphi_0]$ and $\Phi_{\hat{M}}(0^+) \subset [-\pi;\varphi_0[\cup]\varphi_0+\pi;\pi]$.* (b) *The matrix $-C$ is backward monotonic, and there exists a $\varphi_0 \in]0;\pi[$ such that $\Phi_{-C} \subset [\varphi_0;\pi[$ and $\Phi_{\hat{M}}(0^-) \subset [-\pi;\varphi_0-\pi[\cup]\varphi_0;\pi]$.*
3. *The matrix $-C$ is neither forward nor backward monotonic, but there exists a $\varphi_0 \in]-\pi;0[$ and $\varphi_0' \in]0;\pi[$ such that $\Phi_{-C} \subset [\varphi_0;\varphi_0']$ and $\Phi_{\hat{M}}(0^+) \subset [\varphi_0';\varphi_0+\pi]$ (or equivalently $\Phi_{\hat{M}}(0^-) \subset [\varphi_0'-\pi;\varphi_0]$)).*



Example. Consider again the Ginzburg–Landau equation (12). For the periodic orbit $A_0(t) = \mathrm{e}^{-\mathrm{i}\alpha t+\varphi}$, the matrices $\hat{M}(t)$ and $\hat{C}(t)$ are constant with respect to t, and, respectively, equal to

$$\begin{pmatrix} 0 & 2\alpha \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix}.$$

We thus have

$$\Phi_{-C} = \{\arctan\beta\}, \qquad \Phi_{\hat{M}}(0^+) = \{\mathrm{arccot}(-\alpha)\}, \qquad \Phi_{\hat{M}}(0^-) = \{\mathrm{arccot}(-\alpha)-\pi\}.$$

We can thus translate the Benjamin–Feir criterion $1+\alpha\beta>0$ or <0 in geometrical terms, and make the correspondence with the various cases distinguished in Proposition 4. This gives

$$1+\alpha\beta>0 \Leftrightarrow \begin{cases} \text{if } \beta\le 0, & 0\le \Phi_{\hat M}(0^+)<\Phi_{-C}+\pi, \quad \text{case (1a)},\\ \text{if } \beta\ge 0, & \Phi_{-C}-\pi<\Phi_{\hat M}(0^-)\le 0, \quad \text{case (1b)},\end{cases}$$

$$1+\alpha\beta<0 \Leftrightarrow \begin{cases} \text{if } \beta< 0, & \Phi_{-C}+\pi<\Phi_{\hat M}(0^+)\le \pi, \quad \text{case (2a)},\\ \text{if } \beta> 0, & -\pi\le\Phi_{\hat M}(0^-)<\Phi_{-C}-\pi, \quad \text{case (2b)}.\end{cases}$$

Proof of Proposition 4. We exclude the case where C is proportional to the identity, where the spatial extension of $u_0(\cdot)$ with respect to C is always phase stable. According to Corollary 4, if $\mathcal{I}=\int_0^T(a(s)Y(s)+c_1(s))\,\mathrm{d}s$ is strictly positive (resp. negative), then we have phase stability (resp. instability). Remark that, if $c_3(t)\neq 0$, then $\cot\varphi_{-\hat C(t)}(0)=c_1(t)/c_3(t)$ and, if $b(t)\neq 0$, then $\cot\varphi_{\hat M(t)}(0^+)=\cot\varphi_{\hat M(t)}(0^-)=a(t)/b(t)$.

Suppose that (1a) holds. Then $c_3(\cdot)\le 0$; moreover, as C is not proportional to the identity, we have $\varphi_0<0$ and there are values of t for which $c_3(t)<0$. By definition of $Y(\cdot)$, this yields $Y(t)>0,\ t\in\mathbf{R}$. If $\varphi_{-\hat C(t)}(0)$ is defined, then it belongs to $[\varphi_0;0]$. Thus, if $c_3(t)<0$, then $\cot\varphi_{-\hat C(t)}(0)=(c_1(t)/c_3(t))\le\cot\varphi_0$, and if $c_3(t)=0$, then $c_1(t)\ge 0$. In both cases, we have $c_1(t)\ge c_3(t)\cot\varphi_0$. On the other hand, if $\varphi_{\hat M(t)}(0^+)$ is defined, then it belongs to $]\varphi_0;\varphi_0+\pi[$. Thus, if $b(t)<0$, then $\cot\varphi_{\hat M(t)}(0^+)>\cot\varphi_0$; if $b(t)>0$, then $\cot\varphi_{\hat M(t)}(0^+)<\cot\varphi_0$; and if $b(t)=0$, then $a(t)\le 0$. In particular, we always have $a(t)\le b(t)\cot\varphi_0$.

Now, as $Y(\cdot)>0$, we deduce from these estimates that

$$\mathcal{I}>\int_0^T(\cot\varphi_0(b(s)Y(s))+c_1(s))\,\mathrm{d}s=\cot\varphi_0\int_0^T\frac{\mathrm{d}Y}{\mathrm{d}s}(s)\,\mathrm{d}s+\int_0^T(c_1(s)-c_3(s)\cot\varphi_0)\,\mathrm{d}s.$$

As $Y(T)=Y(0)$, the first term in the last expression vanishes, and we see that the second term is positive. Thus, $\mathcal{I}>0$ and we have phase stability.

If (1b) holds, the symmetric argument again shows that $\mathcal{I}>0$. Now, suppose that (2a) holds. We have again $c_3(\cdot)\le 0$, $\varphi_0<0$, and $Y(\cdot)>0$. If $\varphi_{-\hat C(t)}(0)$ is defined, then it belongs to $]-\pi;\varphi_0]$. Thus, if $c_3(t)<0$, then $\cot\varphi_{-\hat C(t)}(0)=(c_1(t)/c_3(t))\ge\cot\varphi_0$, and if $c_3(t)=0$, then necessarily $c_1(t)=0$. In both cases, we have $c_1(t)\le c_3(t)\cot\varphi_0$.

On the other hand, if $\varphi_{\hat M(t)}(0^+)$ is defined, then it belongs to $]-\pi;\varphi_0[\cup]\varphi_0+\pi;\pi]$. Thus, if $b(t)<0$, then $\cot\varphi_{\hat M(t)}(0^+)<\cot\varphi_0$; if $b(t)>0$, then $\cot\varphi_{\hat M(t)}(0^+)>\cot\varphi_0$; and if $b(t)=0$, then $a(t)\ge 0$. In particular, we always have $a(t)\ge b(t)\cot\varphi_0$.

As $Y(\cdot)>0$, we deduce from these estimates that

$$\mathcal{I}<\int_0^T(\cot\varphi_0(b(s)Y(s))+c_1(s))\,\mathrm{d}s=\cot\varphi_0\int_0^T\frac{\mathrm{d}Y}{\mathrm{d}s}(s)\,\mathrm{d}s+\int_0^T(c_1(s)-c_3(s)\cot\varphi_0)\,\mathrm{d}s.$$

The first term in the last expression vanishes, and we see that the second term is negative. Thus $\mathcal{I}<0$ and we have phase instability.

If (2b) holds, the symmetric argument again shows that $\mathcal{I}<0$. Finally, suppose that (3) holds. Necessarily, we have $\varphi_0'-\varphi_0<\pi$, and $\cot\varphi_0<\cot\varphi_0'$.

If $c_3(t)<0$, then $\varphi_{-\hat C(t)}(0)\in[\varphi_0;0[$ and thus $\cot\varphi_{-\hat C(t)}(0)\le\cot\varphi_0$, which yields $c_1(t)\ge c_3(t)\cot\varphi_0>c_3(t)\cot\varphi_0'$. If $c_3(t)<0$, then $\varphi_{-\hat C(t)}(0)\in]0;\varphi_0']$ and thus $\cot\varphi_{-\hat C(t)}(0)\ge\cot\varphi_0'$, which yields $c_1(t)\ge c_3(t)\cot\varphi_0'>c_3(t)\cot\varphi_0$. Finally, if $c_3(t)=0$, then $c_1(t)\ge 0$. In all cases, we have

$$c_1(t)\ge\max(c_3(t)\cot\varphi_0,c_3(t)\cot\varphi_0')$$

and there are values of t for which this inequality is strict.

On the other hand, if $\varphi_{\hat{M}(t)}(0^+)$ is defined, then it belongs to $[\varphi_0'; \varphi_0 + \pi]$. Thus, if $(a(t), b(t)) \neq (0{,}0)$ then $b(t) < 0$, and we have $\cot\varphi_0 \leq \cot\varphi_{\hat{M}(t)}(0^+) \leq \cot\varphi_0'$. We thus always have

$$b(t)\cot\varphi_0' \leq a(t) \leq b(t)\cot\varphi_0.$$

Now, the sign of $Y(\cdot)$ may change on $[0; T]$; nevertheless, let us suppose first that this sign is constant, for instance $Y(\cdot) \geq 0$. Then,

$$\mathcal{I} \geq \int_0^T (b(s)Y(s)\cot\varphi_0' + c_1(s))\,\mathrm{d}s = \cot\varphi_0' \int_0^T \frac{\mathrm{d}Y}{\mathrm{d}s}(s)\,\mathrm{d}s + \int_0^T (c_1(s) - c_3(s)\cot\varphi_0')\,\mathrm{d}s.$$

The first term vanishes, and the second term is strictly positive, thus $\mathcal{I} > 0$.

If $Y(\cdot) \leq 0$, the symmetric argument yields again $\mathcal{I} > 0$. Finally, if the sign of $Y(\cdot)$ changes, we can apply the preceding estimates on each maximal interval where this sign remains constant, and once again we obtain $\mathcal{I} > 0$. This finishes the proof. □

Definition. We will say that the matrix $-C$ is *uniformly forward* (resp. *backward*) *monotonic* if there exists $\varepsilon > 0$ such that, for any $\psi \in \mathbf{R}/2\pi\mathbf{Z}$ such that $u_\psi \notin \ker C$, $\varphi_{-C}(\psi) \leq -\varepsilon$ (resp. $\varphi_{-C}(\psi) \geq \varepsilon$).

We will say that the periodic orbit $u_0(\cdot)$ is uniformly forward (resp. *backward*) *monotonic* if, for any $t \in \mathbf{R}$, $a(t) < 0$ (resp. $a(t) > 0$).

Remark. *The interest of this last definition is limited, because it is not invariant under conjugacy of the vector field $f(\cdot)$ by a diffeomorphism of $\mathbf{R}^2$, nor under a change in the choice of the local frame $(e_1(\cdot), e_2(\cdot))$. Nevertheless, it enables to derive the following corollary as an immediate consequence of Proposition* 4. *This corollary can be viewed as a weak form of Proposition* 3.

Corollary 6.

1. *Suppose that $u_0(\cdot)$ is uniformly forward* (*resp. backward*) *monotonic, and that $-C$ is forward* (*resp. backward*) *monotonic. If moreover $\Phi_{-C} \in [-\pi/2; 0]$* (*resp.* $\Phi_{-C} \in [0; \pi/2]$), *then the spatial extension of $u_0(\cdot)$ with respect to C is phase stable*.
2. *Suppose that $u_0(\cdot)$ is uniformly forward* (*resp. backward*) *monotonic. Then there exists $\varepsilon > 0$ such that, if $-C$ is backward* (*resp. forward*) *monotonic and moreover $\Phi_{-C} \in [\pi/2 - \varepsilon; \pi[$* (*resp.* $\Phi_{-C} \in]-\pi; -\pi/2 + \varepsilon]$), *then the spatial extension of $u_0(\cdot)$ with respect to C is phase unstable*.
3. *Suppose that $-C$ is uniformly forward* (*resp. backward*) *monotonic. Then there exists $\varepsilon > 0$ such that, if $u_0(\cdot)$ is uniformly backward* (*resp. forward*) *monotonic and moreover $\Phi_{\hat{M}}(0^+) \subset [-\pi; -\varepsilon[\cup]\pi-\varepsilon; \pi]$* (resp. $\Phi_{\hat{M}}(0^-) \subset [-\pi; -\pi+\varepsilon[\cup]\varepsilon; \pi]$), *then the spatial extension of $u_0(\cdot)$ with respect to C is phase unstable*.

Acknowledgements

I am grateful to Médéric Argentina, Pierre Coullet, and Nicolas Vandenberghe for their support through numerous discussions. The first two introduced me to spatially extended dynamical systems, and the content of this paper owes much to their ideas.

References

[1] T.B. Benjamin, J.E. Feir, J. Fluid Mech. 27 (1967) 417.
[2] M.C. Cross, P.C. Hohenberg, Pattern formation outside of equilibrium, Rev. Mod. Phys. 65 (1993) 851.

[3] G. Iooss, M. Adelmeyer, Topics in Bifurcation Theory and Applications, Advanced Series in Nonlinear Dynamics, Vol. 3, World Scientific, Singapore, 1992.
[4] A.C. Newell, Envelop equation, Lectures in Appl. Math. 15 (1974) 157.
[5] E. Risler, A general class of spatially extended differential equations: definition, existence of solutions, and some formal properties, Preprint INLN, 1999.
[6] E. Risler, Phase and Turing instabilities for a spatially extended Hopf bifurcation, Preprint INLN, 1999.
[7] A.M. Turing, The chemical basis of morphogenesis, Philos. Trans. Roy. Soc. B 237 (1953) 37–72. Bull. Math. Biol. 52 (1/2), (1990) 153–197 (reprint).
[8] T. Yamada, Y. Kuramoto, Progr. Theoret. Phys. 56 (1976) 681.