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Generic Instability of Spatial Unfoldings of Almost Homoclinic Periodic Orbits

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Abstract: We consider spatially homogeneous time periodic solutions of general partial differential equations. We prove that, when such a solution is close enough to a homoclinic orbit or a homoclinic bifurcation for the differential equation governing the spatially homogeneous solutions of the PDE, then it is generically unstable with respect to large wavelength perturbations. Moreover, the instability is of one of the two following types: either the well-known Kuramoto phase instability, corresponding to a Floquet multiplier becoming larger than 1, or a fundamentally different kind of instability, occurring with a period doubling at an intrinsic finite wavelength, and corresponding to a Floquet multiplier becoming smaller than -1.

1. Introduction

We consider PDEs of the form

$$\partial_t u = F(u, \partial_x), \tag{1}$$

i.e. invariant with respect to translations of time (autonomous) and space. We suppose that u is in $\mathbf{R}^d$, $d \geq 1$, and that the space coordinate x belongs to $\mathbf{R}^n$, $n \geq 1$, or to a domain of $\mathbf{R}^n$ with boundary conditions of type Neumann or periodic. Spatially homogeneous solutions of this PDE are solutions of the equation

$$\frac{du}{dt} = F(u, 0) = f(u) \tag{2}$$

(we write $f(u)$ for $F(u, 0)$), which is an autonomous ordinary differential equation in dimension d.

Among the solutions of Eq. (2), of prime interest are those which correspond to an asymptotic behavior, in particular attractive fixed points and attractive periodic orbits. Consider such a solution $t \mapsto u_h(t)$ of Eq. (2). The corresponding homogeneous solution for the PDE (1) is thus stable with respect to homogeneous perturbations. However, it might be unstable with respect to inhomogeneous perturbations, this is at the origin of many phenomena displaying "patterns" or "spatio-temporal chaos" in nonlinear Physics ([6]).

The question of the stability with respect to inhomogeneous perturbations turns out to be, without further hypotheses, by far a too general problem. It is thus necessary to specify, particularize this problem in order to be able to provide significant results. An interesting way to do so is to look close to a bifurcation. Indeed, bifurcation theory tells us that this greatly simplifies the problem, and at the same time preserves its generality: normal forms of unfoldings of bifurcations are both "particular" and "universal" examples.

Thus we will suppose that the solution $t \mapsto u_h(t)$ is close to a bifurcation as a solution of the differential Eq. (2). This is still not sufficient and we will moreover restrict ourselves to large wavelength (small wavenumber) perturbations. In [5], this approach, called *spatial unfolding of bifurcations*, is developed systematically, and all bifurcations occurring generically for fixed points and periodic orbits in dimension one and two are treated (results of the present paper are quoted, but only rough ideas of the proofs are given).

Here we will concentrate on almost homoclinic periodic orbits: we will suppose that $t \mapsto u_h(t)$ is periodic and close to a homoclinic orbit or to a homoclinic bifurcation, i.e. that it spends almost all its time close to a hyperbolic fixed point of Eq. (2). Moreover, we will assume that space is isotropic, and that the solution $t \mapsto u_h(t)$ itself does not break space isotropy. The aim of this paper is to show that such solutions are generically unstable with respect to inhomogeneous large wavelength perturbations.

A small inhomogeneous perturbation $u(x,t)$ of $u_h(t)$ formally obeys at first order the linear equation

$$\partial_t u = DF(u_h(t), \partial_x)u, \tag{3}$$

which reduces in Fourier coordinates to

$$\partial_t \hat{u}(\boldsymbol{k}) = DF(u_h(t), i\boldsymbol{k})\hat{u}(\boldsymbol{k}) \tag{4}$$

which is just an ordinary differential equation parametrized by $\boldsymbol{k}$. Because of the above hypotheses on space isotropy, the preceding equation only depends on $|\boldsymbol{k}|^2$ and can be rewritten

$$\partial_t \hat{u}(\boldsymbol{k}) = \big(Df(u_h(t)) + \mathcal{C}(u_h(t), -|\boldsymbol{k}|^2)\big)\hat{u}(\boldsymbol{k}), \tag{5}$$

where $\mathcal{C} : \mathbf{R}^d \times \mathbf{R} \to \mathcal{L}(\mathbf{R}^d)$ satisfies $\mathcal{C}(.,0) \equiv 0$ (we denote by $\mathcal{L}(\mathbf{R}^d)$ the space of linear maps: $\mathbf{R}^d \to \mathbf{R}^d$). Thus we can write: $\mathcal{C}(u,\lambda) = \lambda C(u,\lambda)$, where the map $C : \mathbf{R}^d \times \mathbf{R} \to \mathcal{L}(\mathbf{R}^d)$ is regular.

In the following, we will forget about the exact nature of the PDE (1), and just consider the ordinary differential equation

$$\frac{du}{dt} = \big(Df(u_h(t)) + \lambda C(u_h(t), \lambda)\big)u, \tag{6}$$

depending on the parameter λ (which corresponds to $-|\boldsymbol{k}|^2$, thus which will be supposed to be small negative).

For $\lambda \le 0$, denote by Φ_λ the (linear) flow over one period of u_h of this differential equation, and denote by $\rho(\Phi_\lambda)$ the spectral radius of Φ_λ. We know that 1 is always an eigenvalue of Φ_0 (the "neutral" Floquet multiplier in the direction of the flow). Thus, even for values of λ arbitrarily close to 0, the eigenvalue 1 of Φ_0 may become larger than 1. This is the well known Kuramoto phase instability ([4,9,11]).

We are going to show that, when the solution u_h passes close enough to a hyperbolic fixed point, then this solution is generically unstable with respect to inhomogeneous large wavelength perturbations, i.e. there are small negative values of λ for which $\rho(\Phi_\lambda) > 1$. Moreover, we shall see that the instability can be of two different types: either the Kuramoto phase instability, or a "period-doubling" instability, corresponding to a real eigenvalue of Φ_λ becoming smaller than -1.

This result was conjectured on the basis of numerical observations by Médéric Argentina and Pierre Coullet ([1]), and their observations, conjectures, and questions were the starting point of this work. The reader interested in these observations, in the physical interpretations and implications of these results, and in the nonlinear development of these instabilities is invited to consult the references [2] and [5]. Let us also mention that this generic instability result extends to the case where space isotropy is broken ([5]).

1.1. Statement of the results. We give ourselves and fix a C^1-vector field $f_0 : \mathbf{R}^d \to \mathbf{R}^d$, $d \geq 2$, and we make the following hypotheses (see Fig. 1):

- $f_0(0) = 0$ (0 denotes the origin $(0, \dots, 0)$ of $\mathbf{R}^d$);
- $Df_0(0)$ has a simple real eigenvalue $b_+ > 0$; if $d = 2$, then the second eigenvalue is not larger than $-b_+$; if $d \geq 3$, then the real part of any other eigenvalue is strictly smaller than $-b_+$;
- one of the following statements holds:
  - (a) the differential equation $\frac{du}{dt} = f_0(u)$ admits a solution $t \mapsto u_0(t)$ which is homoclinic to the fixed point 0 (i.e. $u_0(.) \not\equiv 0$ and $u_0(t) \to 0$ when $t \to \pm\infty$);
  - (b) the differential equation $\frac{du}{dt} = f_0(u)$ admits two solutions $t \mapsto u_0(t)$ and $t \mapsto \tilde{u}_0(t)$ (with distinct trajectories) which are homoclinic to the fixed point 0.

Let us consider any C^1-vector field $f_1 : \mathbf{R}^d \to \mathbf{R}^d$, with the following properties:

- $f_1(.)$ is close to $f_0(.)$ in the C^1-topology (this hypothesis will be formulated more precisely below);
- the differential equation $\frac{du}{dt} = f_1(u)$ admits a periodic solution $t \mapsto u_1(t)$ whose trajectory is, in case (a), close to the trajectory of $u_0(.)$, and, in case (b), close to the union of the trajectories of $u_0(.)$ and $\tilde{u}_0(.)$ (again, this hypothesis will be formulated more precisely below);
- if $d = 2$, then the periodic orbit $u_1(.)$ is not linearly unstable.

Here the hypothesis on the closeness of the trajectories holds in the sense of the Hausdorff distance between two sets (recall that this distance can be defined the following way: $\mathrm{dist}(A, B) = \inf\{\delta > 0 \mid A \subset \mathrm{Neighb}_\delta(B) \text{ and } B \subset \mathrm{Neighb}_\delta(A)\}$).

Remark. In the case $d \geq 3$, these hypotheses (in particular the ones on $Df_0(0)$) imply that the periodic orbit $u_1(.)$ is linearly stable; the same is true in the case $d = 2$ if the second eigenvalue of $Df_0(0)$ is strictly smaller than $-b_+$. On the other hand, the hypotheses on $Df_0(0)$ are almost necessary if we want $u_1(.)$ not to be linearly unstable. Indeed, in the case $d = 2$, if the second eigenvalue of $Df_0(0)$ was strictly larger than $-b_+$, then the hypotheses would imply that $u_1(.)$ is linearly unstable; the same would generically be true in the case $d \geq 3$ if $Df_0(0)$ had an eigenvalue different from b_+ with a real part strictly larger than $-b_+$.

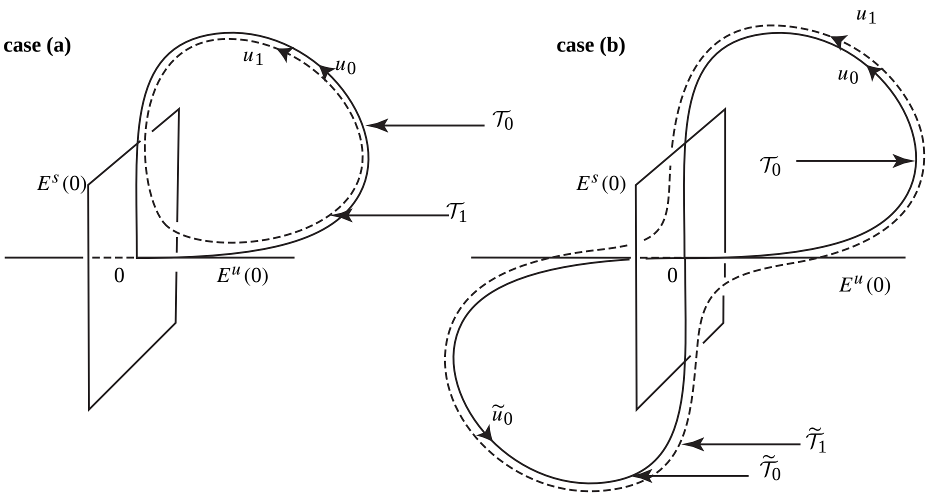


Fig. 1.

Now let us define the coupling terms to be added to the two previous vectors fields. We give ourselves and fix a C^0-map $C_0 : \mathbf{R}^d \times \mathbf{R} \to \mathcal{L}(\mathbf{R}^d)$ and we consider any C^0-map $C_1 : \mathbf{R}^d \times \mathbf{R} \to \mathcal{L}(\mathbf{R}^d)$ close to C_0 in the C^0-topology (this hypothesis will be formulated more precisely below).

Denote by $\mathcal{L}^+(\mathbf{R}^d)$ the subset of $\mathcal{L}(\mathbf{R}^d)$ consisting of linear maps having no eigenvalue with a strictly negative real part. We will suppose that the maps C_0 and C_1 take their values in $\mathcal{L}^+(\mathbf{R}^d)$. This hypothesis is natural, since, as $\lambda \leq 0$, it excludes the existence of instabilities uniquely due to the coupling. However, the results are to a large extent independent of this hypothesis (which will be necessary only in dimension $d = 2$, and mainly for the phase stability results in case 2 of Theorem 2 and case 2 of Theorem 3 below).

For $\lambda \leq 0$, denote by Φ_λ the (linear) flow over one period of u_1 of the differential equation

$$\frac{du}{dt} = \big(Df_1(u_1(t)) + \lambda C_1(u_1(t), \lambda)\big)u, \tag{7}$$

and denote by $\rho(\Phi_\lambda)$ the spectral radius of Φ_λ.

Let $||\dots||_{C^1}$ denote a uniform C^1-norm on $C^1(\mathbf{R}^d, \mathbf{R}^d)$ and let $||\dots||_{C^0}$ denote a uniform C^0-norm on $C^0(\mathbf{R}^d \times \mathbf{R}, \mathcal{L}(\mathbf{R}^d))$; let $\mathcal{T}_0$, $\mathcal{T}_1$, and, in case (b), $\tilde{\mathcal{T}}_0$ denote the respective trajectories of $u_0(.)$, $u_1(.)$, and $\tilde{u}_0(.)$.

Our result is the following.

Theorem 1. *Let $f_0(.)$ and $C_0(.,.)$ be as above. Then, if a generic condition (which will be detailed below) on $f_0(.)$ and $C_0(.,.)$ is satisfied, there exists $\varepsilon_0 > 0$ (small) such that, for any $f_1(.)$ and $C_1(.,.)$ as above, if $||f_1(.) - f_0(.)||_{C^1} \leq \varepsilon_0$ and $||C_1(.,.) - C_0(.,.)||_{C^0} \leq \varepsilon_0$ and if, in case (a), $\operatorname{dist}(\mathcal{T}_0, \mathcal{T}_1) < \varepsilon_0$, and in case (b), $\operatorname{dist}(\mathcal{T}_0 \cup \tilde{\mathcal{T}}_0, \mathcal{T}_1) < \varepsilon_0$, one can find $\lambda < 0$ (arbitrarily close to 0 if ε_0 is small enough) such that $\rho(\Phi_\lambda) > 1$.*

We are going to be more precise.

Let $f_0(.)$, $C_0(.,.)$, $f_1(.)$, and $C_1(.,.)$ be as above. Up to conjugating $f_1(.)$ by a (small) translation of $\mathbf{R}^d$, we will suppose that $f_1(0) = 0$. Fix $\delta_0 > 0$ small, let $B_0 = \{x \in \mathbf{R}^d \;\big|$

$||x|| \le \delta_0\}$, and let $W_1^{\mathrm{s,loc}}(0)$ denote the local stable manifold of 0 for $f_1(.)$, i.e. say the set of points of B_0 whose forward trajectory by $f_1(.)$ remains in B_0.

According to the hypotheses (for ε_0 sufficiently small), the set $\mathcal{T}_1 \cap \partial B_0$ contains, in case (a), exactly two points, and, in case (b), exactly four points; in dimension $d = 2$, this is due to an elementary plane topology argument, and in dimension $d \ge 3$, this is due to the hypotheses on $Df_0(0)$ (and related to the fact that $u_1(.)$ is linearly attractive). In case (a) (resp. in case (b)), denote by ζ_1 (resp. by ζ_1 and $\tilde{\zeta}_1$) the point(s) of $\mathcal{T}_1 \cap \partial B_0$ as shown on Fig. 2.

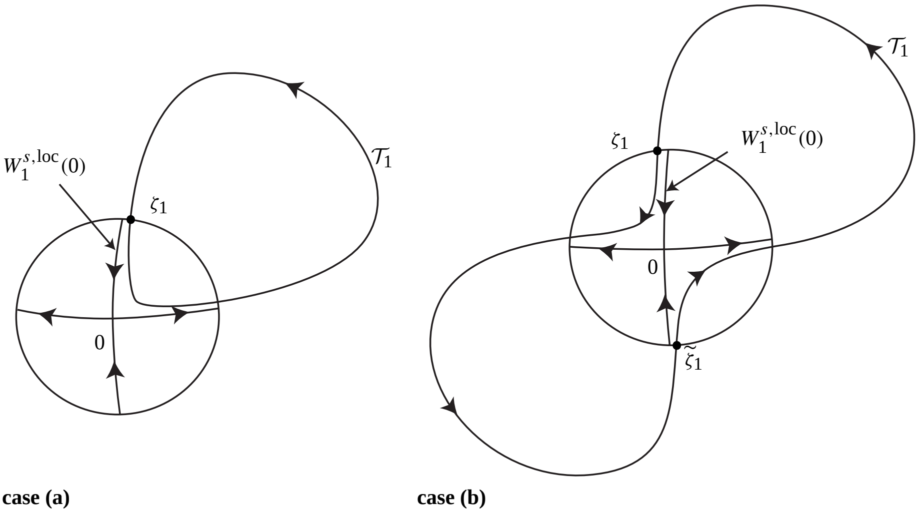


Fig. 2. Definition of ζ_1 and $\tilde{\zeta}_1$

Let

$$\mu = \mathrm{dist}(\zeta_1, W_1^{\mathrm{s,loc}}(0)) \quad \text{and, in case (b),} \quad \tilde{\mu} = \mathrm{dist}(\tilde{\zeta}_1, W_1^{\mathrm{s,loc}}(0))$$

(these quantities can be considered as bifurcation parameters: they measure the proximity to the homoclinic orbit or to the homoclinic bifurcation).

In the following (Sect. 2), we will show how to associate to each triplet (f_0, u_0, C_0) as above an index $\sigma(f_0, u_0, C_0)$ in $\{-1, 0, 1\}$, which vanishes if, for each (t, λ), the map $C_0(u_h(t), \lambda)$ is positively proportional to $\mathrm{Id}_{\mathbf{R}^d}$, but which is generically different from 0 for a general $C_0(.,.)$, and whose sign governs the nature of the instability. With this index, we can formulate the following more precise results (for sake of clarity, we distinguish cases (a) and (b)).

Theorem 2. *Let $f_0(.)$, $u_0(.)$, and C_0 be as above, in case (a). Then, if $\sigma(f_0, u_0, C_0) \neq 0$, there exists $\varepsilon_0 > 0$ (small) such that, for any $f_1(.)$ as above, if $||f_1(.) - f_0(.)||_{C^1} \le \varepsilon_0$, $\mathrm{dist}(\mathcal{T}_0, \mathcal{T}_1) < \varepsilon_0$, and $||C_1(.) - C_0(.)||_{C^0} \le \varepsilon_0$, then,*

1. *if $\sigma(f_0, u_0, C_0) = 1$, then for any $\lambda \in]-\varepsilon_0; 0[$, Φ_λ has an eigenvalue which is real and strictly larger than 1 (phase instability);*

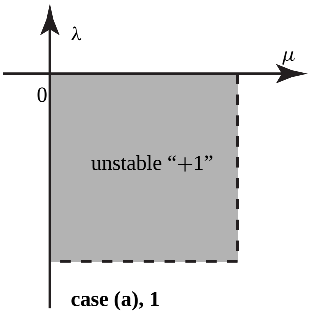


Fig. 3. Illustration of Theorem 2 (case (a))

2. *if* $\sigma(f_0, u_0, C_0) = -1$*, then there are constants* $K' > K > 0$*, depending only on* $f_0(.)$ *and* C_0*, such that, for any* $\lambda \in]-K\mu; 0[$*,* $\rho(\Phi_\lambda) \leq 1$ *(no phase instability), and for any* $\lambda \in]-\varepsilon_0; -K'\mu[$*,* Φ_λ *has an eigenvalue which is real and strictly smaller than* -1 *("period-doubling" instability).*

Theorem 3. *Let* $f_0(.)$*,* $u_0(.)$*,* $\tilde{u}_0(.)$*, and* C_0 *be as above, in case (b). Then, if* $\sigma(f_0, u_0, C_0) \neq 0$ *and* $\sigma(f_0, \tilde{u}_0, C_0) \neq 0$*, there exists* $\varepsilon_0 > 0$ *(small) such that, for any* $f_1(.)$ *as above, if* $||f_1(.) - f_0(.)||_{C^1} \leq \varepsilon_0$*,* $\mathrm{dist}(\mathcal{T}_0 \cup \tilde{\mathcal{T}}_0, \mathcal{T}_1) < \varepsilon_0$*, and* $||C_1(.) - C_0(.)||_{C^0} \leq \varepsilon_0$*, then,*

1. *if* $\sigma(f_0, u_0, C_0) = -1$ *and* $\sigma(f_0, \tilde{u}_0, C_0) = -1$*, then for any* $\lambda \in]-\varepsilon_0; 0[$*,* Φ_λ *has an eigenvalue which is real and strictly larger than* 1 *(combination of two phase instabilities);*
2. *if* $\sigma(f_0, u_0, C_0) = +1$ *and* $\sigma(f_0, \tilde{u}_0, C_0) = +1$*, then there are constants* $K' > K > 0$*, depending only on* $f_0(.)$ *and* C_0*, such that, for any* $\lambda \in]-K\min(\mu, \tilde{\mu}); 0[$*,* $\rho(\Phi_\lambda) \leq 1$ *(no phase instability), and for any* $\lambda \in]-\varepsilon_0; -K'\max(\mu, \tilde{\mu})[$*,* Φ_λ *has an eigenvalue which is real and strictly larger than* 1 *(combination of two "period-doubling" instabilities);*
3. *if* $\sigma(f_0, u_0, C_0)$ *and* $\sigma(f_0, \tilde{u}_0, C_0)$ *have opposite signs, then there is a constant* $K'' > 0$ *such that, for any* $\lambda \in]-\varepsilon_0; -K''\max(\mu, \tilde{\mu})[$*,* Φ_λ *has an eigenvalue which is real and strictly smaller than* -1 *(combination of a phase and a "period-doubling" instability).*

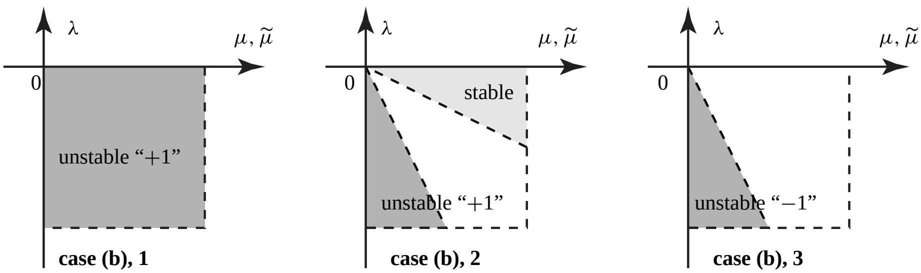


Fig. 4. Illustration of Theorem 3 (case (b))

In case 2 of this last result, the instability is of the same nature as the period-doubling instability (it can be viewed as the composition of two period-doubling instabilities). Case 3 is a bit more involved, but occurs less frequently than cases 1 and 2. For instance, it never occurs when the two homoclinic orbits $u_0(.)$ and $\tilde{u}_0(.)$ are symmetric.

1.2. Examples. The hypotheses of Theorems 1, 2, and 3 cover essentially two kinds of situations: homoclinic bifurcations of attractive periodic orbits in one-parameter families of ordinary differential equations on one hand, and families of periodic orbits bounded by homoclinic orbits in two-dimensional conservative ordinary differential equations on the other hand (this second case corresponds to $f_1 = f_0$). Moreover, these hypotheses take into account cases where, because of the presence of a symmetry or of a conserved quantity, the limit of the periodic orbits consists of two (instead of one) homoclinic orbits. We now give some examples (for other examples and references, see [2]).

1. Consider the following nonlinear wave equation:

$$u_{tt} + (\nu + u)u_t + u - u^2 = \Delta_x u$$

parametrized by $\nu \in \mathbf{R}$. This is the equation governing a chain of coupled second order oscillators in the potential $V(u) = \frac{1}{2}u^2 - \frac{1}{3}u^3$, submitted to the nonlinear damping $-(\nu + u)u_t$. This equation can be rewritten

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} v \\ -(\nu + u)v - u + u^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta_x u \\ \Delta_x v \end{pmatrix},$$

and thus can be viewed as a spatial extension of the ordinary differential equation

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} v \\ -(\nu + u)v - u + u^2 \end{pmatrix}$$

with respect to the "coupling" matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ (here the map $C(.\,,.)$ is constant and equal to this matrix).

This family of differential equations appears in the universal unfolding of the Bogdanov–Takens bifurcation ([7]). Its dynamics displays the following features. For $\nu > 0$, the fixed point $(0, 0)$ is linearly stable. At $\nu = 0$, it undergoes a supercritical Hopf bifurcation and becomes unstable for $\nu < 0$. The bifurcation gives rise to an attractive periodic orbit around $(0, 0)$ for $\nu < 0$ close to 0. At a certain value $\nu = \nu_c < 0$ of the parameter, this attractive periodic orbit disappears through homoclinic bifurcation (see Fig. 5), the limiting orbit being homoclinic to the hyperbolic fixed point $(1, 0)$. For $\nu < \nu_c$, forward orbits generically go to infinity.

Theorem 1 claims that, for $\nu > \nu_c$, ν close to ν_c, the attractive periodic orbit is unstable with respect to inhomogeneous perturbations. More generally, a possible physical interpretation of our results is the following: for a spatially extended dynamical system, it is impossible to cross a potential barrier in a synchronous way.

According to Theorem 2, it is possible to predict the nature of the instability. We use the definitions and notations of Subsect. 2.2. On one hand, the homoclinic orbit is backward oriented, thus $\sigma_{\rm or} = -1$. On the other hand, we can see from the expression of $C(., .)$ that $c_{3,0}(t)$ will be negative for all times, which shows that $Y_-(.) > 0$, that $Y_+(.) < 0$, and thus that $Y_-(.) - Y_+(.) > 0$. Thus, $\sigma_Y = +1$, and, according to

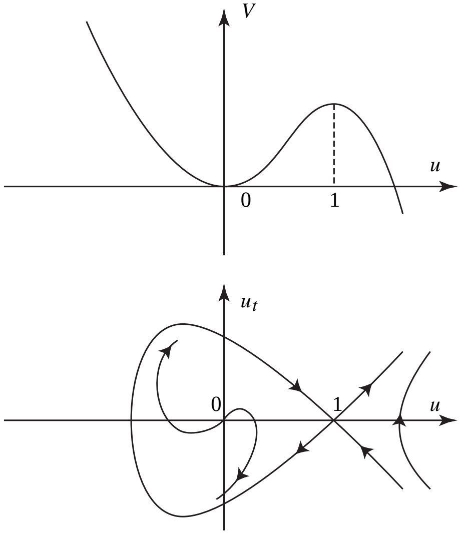


Fig. 5. Phase portrait when $\nu = \nu_c$

Theorem 2, the instability is a phase instability (for more details on the links between the expression of $C(.,.)$ and the nature of the instability, see [10]).

2. Consider the following partial differential equation:

$$u_{tt} + V'(u) = u_{xx},$$

where $V(u) = -\frac{1}{2}u^2 + \frac{1}{3}u^3 + \frac{1}{4}u^4$. It represents a chain of coupled conservative oscillators in the bistable potential $V(.)$. It can be viewed as a spatial extension of an ordinary differential equation with respect to the same coupling matrix as above. The phase space of the differential equation is as follows. It is foliated by periodic orbits, bounded by the fixed points and by two orbits homoclinic to $(0, 0)$ and having an energy $\frac{1}{2}u_t^2 + V(u)$ equal to 0. According to Theorem 1, any periodic orbit having an energy E close enough to 0 is unstable with respect to inhomogeneous perturbations; moreover, it is phase unstable (case 1 of Theorem 2) if $E < 0$, and not phase unstable (but "period-doubling-like" unstable, case 2 of Theorem 3) if $E > 0$.

3. We end up our series of examples with the celebrated sine-Gordon equation

$$u_{tt} + \sin u = u_{xx}.$$

The phase space of the corresponding ordinary differential equation on $(\mathbf{R}/2\pi\mathbf{Z}) \times \mathbf{R}$ is foliated by periodic orbits, bounded by the fixed points and by two orbits homoclinic to $(\pi, 0)$ and having an energy $\frac{1}{2}u_t^2 - \cos u$ equal to 1. We can easily deal with the fact that the phase space is 2π-periodic on the horizontal variable. According to Theorem 1, any periodic orbit having an energy E close enough to 1 is unstable with respect to inhomogeneous perturbations; moreover, it is phase unstable (case 1 of Theorem 3) if $E < 1$, and period-doubling unstable (case 2 of Theorem 2) if $E > 1$.

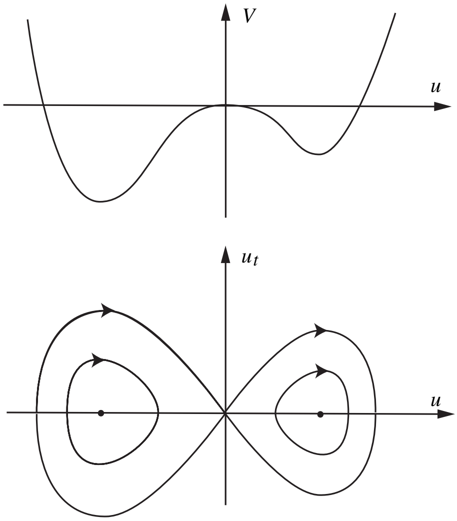


Fig. 6.

1.3. Sketch of the proof and organization of the paper. Let us describe rapidly how the proof goes. To simplify, we suppose that we are in case (a) and that the dimension d equals 2.

We shall take a small parameter $\delta > 0$ and cut the trajectory of $u_1(.)$ into two parts, as shown on the Fig 7. Consider the local frame $(e_1(t), e_2(t)) = (f_1(u_1(t)), \mathrm{Rot}_{\frac{\pi}{2}} f_1(u_1(t)))$ along this trajectory. Denote by ψ_λ (resp. by ϕ_λ) the flow of the differ-

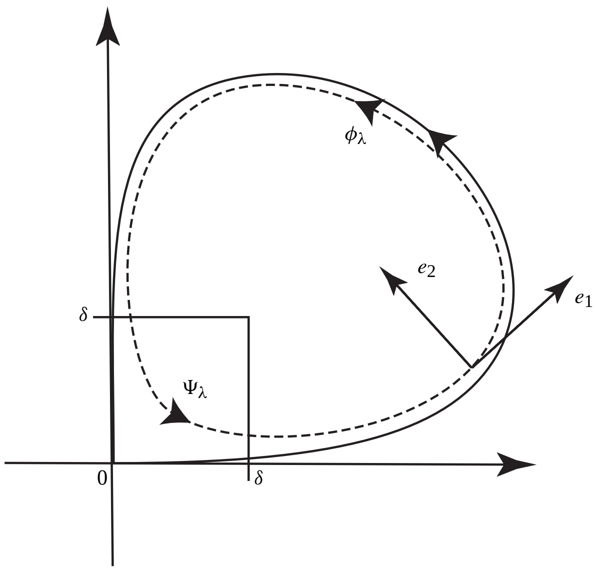


Fig. 7.

ential Eq. (6), expressed in this local frame, along the part of the trajectory which lies inside (resp. outside) the box of size δ around 0. The flow $\psi_\lambda \circ \phi_\lambda$ is conjugated to Φ_λ, and we want to study its spectral radius.

The differential equation $\frac{du}{dt} = Df_1(u_1(t))u$, expressed in the local frame, takes the form $\frac{du}{dt} = \hat{M}_1(t)u$, where the first column of $\hat{M}_1(t)$ vanishes; this shows that ψ_0 and ϕ_0 are of the form $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$. When the distances between f_0 and f_1 and between $\mathcal{T}_0$ and $\mathcal{T}_1$ go to 0, the flow ϕ_0 converges to a limit, while ψ_0 becomes singular. Indeed, writing $\psi_0 = \begin{pmatrix} 1 & \eta \\ 0 & \zeta \end{pmatrix}$, we will see that η goes to $+\infty$ (or to $-\infty$ if the orbits have the converse orientation) while ζ remains bounded (if $b_- = -b_+$) or goes to 0 (if $b_- < -b_+$). More precisely, we will see that η is of the order of μ^{-1}.

The flow ϕ_λ is a non-singular perturbation of ϕ_0. Writing $\phi_\lambda = \phi_0 + \lambda \begin{pmatrix} w & x \\ y & z \end{pmatrix}$, we can see that the trace of $\psi_\lambda \circ \phi_\lambda$ reads

$$\mathrm{tr}\ \psi_\lambda \circ \phi_\lambda = \mathrm{tr}\ \psi_0 \circ \phi_0 + \lambda \eta y + \dots .$$

We will show that, when δ is small, y is large and has a definite sign (actually, when $u_1(.)$ is oriented as on the figure above, the index $\sigma(f_0, u_0, C_0)$ will be equal to ± 1 according to this sign). Thus, we can already see on this expression of the trace what we will actually prove: for $|\lambda\eta| \geq 1$ (which corresponds to $|\lambda|$ being at least of the order of μ), this trace is large and its sign is governed by the sign of y (i.e. by the sign of $\sigma(f_0, u_0, C_0)$). This already proves the instability.

For the case $|\lambda\eta| < 1$, we will need slightly more precise estimates, either to prove the phase instability (if y is positive) or to prove some stability (for $|\lambda\eta|$ small, if y is negative). The proofs in dimension $d = 2$ and in dimension $d \geq 3$ differ noticeably at this point: in case $d = 2$, we will simply estimate the determinant of $\psi_\lambda \circ \phi_\lambda$, while in case $d \geq 3$, we will have to construct an invariant cone for this map (none of these two strategies seems to be convenient for the other case: in dimension $d \geq 3$, estimates on the trace and the determinant are not sufficient to control the eigenvalues, while the construction of an invariant cone seems to be delicate in dimension 2 in case $b_- = -b_+$).

The paper is organized as follows. Section 2 is devoted to some notations and to the definition of the index $\sigma(.,.,.)$. This definition is very simple when $d = 2$, and slightly more involved when $d \geq 3$, thus we distinguish theses two cases (Sects. 2.2 and 2.3). The proof of the results in case (a) is given in Sect. 3. After a preliminary setup (Sect. 3.1), we again distinguish the cases $d = 2$ (Sect. 3.2) and $d \geq 3$ (Sect. 3.3). Finally, we explain in Sect. 4 how to adapt the previous arguments in order to prove the results in case (b).

Notations. For $n \in \mathbf{N}$, we will denote by $\mathcal{B}_{\mathrm{can}}(\mathbf{R}^n)$ the canonical basis of $\mathbf{R}^n$ and by $\epsilon_1, \dots, \varepsilon_n$ the vectors forming this canonical basis. We will denote by $||\dots||$ the usual euclidean norm on $\mathbf{R}^n$, by $\mathcal{M}_n(\mathbf{R})$ the space of $n \times n$ real matrices, and by $|||\dots|||$ the usual norm on $\mathcal{M}_n(\mathbf{R})$.

2. Definition of the Index σ

2.1. Notations related to the local frames. Throughout the proofs, we will have to work in local frames along the solutions $u_0(.)$ (or $\tilde{u}_0(.)$) and $u_1(.)$. Here we introduce some notations related to these local frames.

For $k \in \{0, 1\}$ and $t \in \mathbf{R}$, write

$$M_k(t) = Df_k(u_k(t))$$

and

$$e_{1,k}(t) = f_k(u_k(t)).$$

In dimension $d = 2$, write

$$e_{2,k}(t) = \mathrm{Rot}_{\frac{\pi}{2}} e_{1,k}(t).$$

In dimension $d \geq 3$, the local frame is not canonical, but we will define vectors $e_{2,k}(t), \ldots, e_{d,k}(t)$, C^1 and periodic (of the same period as $u_1(.)$) with respect to t, such that the family $(e_{1,k}(t), \ldots, e_{d,k}(t))$ defines for each t a basis of $\mathbf{R}^d$.

Then,

- let $P_k(t)$ denote the matrix whose columns are the coordinates of $e_{1,k}(t)$ and $e_{2,k}(t)$,
- let $\hat{M}_k(t) = -P_k(t)^{-1}\frac{dP_k}{dt}(t) + P_k(t)^{-1}M_k(t)P_k(t)$ and $\hat{C}_k(t,\lambda) = P_k(t)^{-1} C(u_k(t),\lambda)P_k(t)$.

The change of variables $u = P_k(t)v$ transforms the differential equation

$$\frac{du}{dt} = (M_k(t) + \lambda C(u_k(t), \lambda))u \tag{8}$$

into

$$\frac{dv}{dt} = (\hat{M}_k(t) + \lambda \hat{C}_k(t, \lambda))v. \tag{9}$$

The definition of $e_{1,k}(t)$ ensures that the first column of $\hat{M}_k(t)$ vanishes. Let us write

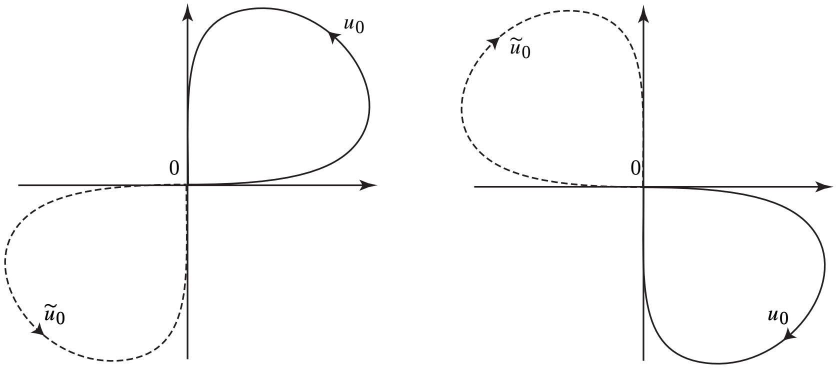
$$\hat{M}_k(t) = \begin{pmatrix} 0 & a_k(t) \\ 0 & b_k(t) \end{pmatrix} \quad \text{and} \quad \hat{C}_k(t, 0) = \begin{pmatrix} c_{1,k}(t) & c_{2,k}(t) \\ c_{3,k}(t) & c_{4,k}(t) \end{pmatrix},$$

where $c_{1,k}(t)$ is a number, $a_k(t)$ and $c_{2,k}(t)$ $1 \times (d-1)$-matrices, $c_{3,k}(t)$ is a $(d-1) \times 1$-matrix, and $b_k(t)$ and $c_{4,k}(t)$ are $(d-1) \times (d-1)$-matrices.

2.2. Definition of σ in dimension two. We suppose that the dimension d equals 2, and we give ourselves a vector field $f_0(.)$ and a map $C_0(.,.)$ as in Subsect. 1.1.

Up to a linear change of coordinates preserving the orientation, we can suppose that $E^{\mathrm{u}}(0)$ and $E^{\mathrm{s}}(0)$ (the unstable and stable spaces of $Df_0(0)$) are respectively equal to $\mathbf{R} \times \{0\}$ and $\{0\} \times \mathbf{R}$. We will say that $u_0(.)$ (or $\tilde{u}_0(.)$) is *forward oriented* or *backward oriented* according to the orientation of its trajectory in $\mathbf{R}^2$ (see Fig. 8). Remark that, in case (b), $u_0(.)$ and $\tilde{u}_0(.)$ necessarily have the same orientation. We are going to define the index $\sigma(f_0, u_0, C_0)$ (in case (b), $\sigma(f_0, \tilde{u}_0, C_0)$ would be defined similarly). Write $Df_0(0) = \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}$. With the notations of the preceding paragraph, we have

$$b_0(t) \to b_- - b_+ < 0 \text{ when } t \to -\infty \quad \text{and} \quad b_0(t) \to b_+ - b_- > 0 \text{ when } t \to +\infty \tag{10}$$



"forward oriented" **"backward oriented"**

Fig. 8.

(see assertion (11) below). Thus, the differential equation

$$\frac{dY}{dt} = b_0(t)Y + c_{3,0}(t), \quad t \in \mathbf{R},$$

has a unique solution $Y_+(.)$ (resp. $Y_-(.)$) which is bounded when $t \to +\infty$ (resp. when $t \to -\infty$). The difference $Y_-(.) - Y_+(.)$ is either identically 0, or does not vanish, and in this case its sign is constant.

Let $\sigma_{\text{or}} = +1$ (resp. $\sigma_{\text{or}} = -1$) if $u_0(.)$ is forward (resp. backward) oriented. Let $\sigma_Y = +1$ (resp. $\sigma_Y = 0$, $\sigma_Y = -1$) if $Y_-(.) - Y_+(.) > 0$ (resp. $Y_-(.) - Y_+(.) \equiv 0$, $Y_-(.) - Y_+(.) < 0$). Finally, let us define our index $\sigma(f_0, u_0, C_0)$ by

$$\sigma(f_0, u_0, C_0) = -\sigma_{\text{or}}\sigma_Y.$$

The condition $Y_-(.) - Y_+(.) \neq 0$ is generic, except if the map $C_0(.,.)$ is identically proportional to the identity (in this case, we have $c_{3,0}(.) \equiv 0$, and thus $Y_-(.) \equiv Y_+(.) \equiv 0$), and the condition $\sigma(f_0, u_0, C_0) \neq 0$ is thus also generic.

If $C_0(.,.)$ is constant and not proportional to the identity and if its two eigenvalues are either complex conjugated or equal, then one can check that $c_{3,0}(.) \not\equiv 0$ and that the sign of $c_{3,0}(.)$ is constant, given by the "sense of rotation" of the flow $t \mapsto \exp(-tC_0)$ (for more precisions on these "monotonic" matrices, see [10]); in this case, $Y_-(.)$ has the sign of $c_{3,0}(.)$, and $Y_+(.)$ has the opposite sign, and the condition $Y_-(.) - Y_+(.) \neq 0$ (and $\sigma(f_0, u_0, C_0) \neq 0$) is thus always fulfilled. Moreover, in this last case, the sign of $\sigma(f_0, u_0, C_0)$, and thus the nature of the instability, can be predicted geometrically, from the orientation (forward or backward) of the homoclinic orbit and the "sense of rotation" of C_0 ([10]).

We finish with a rapid computation which will justify the limits (10), and which will be used later. For $k = 0$ or 1, denote by $\theta_k(t)$ the angle between the vectors $(1, 0)$ and $e_{1,k}(t)$, and write $M_k(t) = \begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix}$.

Claim. We have

$$\begin{pmatrix} a_k(t) \\ b_k(t) \end{pmatrix} = \text{Rot}_{-2\theta_k(t)} \begin{pmatrix} M_{1,2} + M_{2,1} \\ M_{2,2} - M_{1,1} \end{pmatrix}. \tag{11}$$

Indeed, we have (forgetting the indices k and the dependence with respect to t),

$$\begin{pmatrix} a \\ b \end{pmatrix} = \hat{M} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = P^{-1}\Big(-\frac{dP}{dt} + MP\Big) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Besides,

$$P \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathrm{Rot}_{\frac{\pi}{2}} e_1 \quad \text{and thus} \quad \frac{dP}{dt} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathrm{Rot}_{\frac{\pi}{2}} M e_1.$$

Thus,

$$\begin{pmatrix} a \\ b \end{pmatrix} = P^{-1}[M, \mathrm{Rot}_{\frac{\pi}{2}}] P \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and we have

$$\begin{aligned} P^{-1}[M, \mathrm{Rot}_{\frac{\pi}{2}}]P &= \mathrm{Rot}_{-\theta} \begin{pmatrix} M_{1,2} + M_{2,1} & M_{2,2} - M_{1,1} \\ M_{2,2} - M_{1,1} & -M_{1,2} - M_{2,1} \end{pmatrix} \mathrm{Rot}_{\theta} \\ &= \mathrm{Rot}_{-2\theta} \begin{pmatrix} M_{1,2} + M_{2,1} & M_{2,2} - M_{1,1} \\ M_{2,2} - M_{1,1} & -M_{1,2} - M_{2,1} \end{pmatrix}, \end{aligned}$$

which proves the claim.

2.3. Definition of σ in dimension higher than two. We suppose that $d \geq 3$, and we give ourselves a vector field $f_0(.)$ and a map $C_0(.,.)$ as in Subsect. 1.1.

We are going to define the index $\sigma(f_0, u_0, C_0)$ (in case (b), $\sigma(f_0, \tilde{u}_0, C_0)$ would be defined similarly). Up to a linear change of coordinates, we can suppose that $E^{\mathrm{u}}(0)$ and $E^{\mathrm{s}}(0)$ (the unstable and stable spaces of $Df_0(0)$) are respectively equal to $\mathrm{Vect}(\epsilon_1)$ and $\{0\} \times \mathbf{R}^{d-1}$, and that the first coordinate of $u_0(t)$ is positive when t is large negative. Write $Df_0(0) = \begin{pmatrix} b_+ & 0 \\ 0 & B_- \end{pmatrix}$, $B_- \in \mathcal{M}_{d-1}(\mathbf{R})$. We can suppose that B_- is diagonal by blocks, i.e. that it reads

$$\begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_s \end{pmatrix},$$

each block B_j corresponding to an eigenvalue b_j. We can suppose that the non-real eigenvalues of B_- are $b_{s'+1}, \ldots, b_s$, where $0 \leq s' \leq s$. For $j \geq s' + 1$, denote by ρ_j (resp. by θ_j) the real part (resp. the imaginary part) of b_j. We can suppose that, for $j \geq s' + 1$, B_j takes the form

$$\begin{pmatrix} \begin{pmatrix} \rho_j & -\theta_j \\ \theta_j & \rho_j \end{pmatrix} & & * \\ & \ddots & \\ 0 & & \begin{pmatrix} \rho_j & -\theta_j \\ \theta_j & \rho_j \end{pmatrix} \end{pmatrix}.$$

For $t \in \mathbf{R}$ denote by R_t the linear map of $\mathbf{R}^d$ whose restriction to the characteristic spaces corresponding to the eigenvalues b_+ and b_j, $j \leq s'$, is the identity, and whose restriction to the characteristic space corresponding to any eigenvalue b_j, $j \geq s' + 1$, reads

$$\begin{pmatrix} \mathrm{Rot}_{t\theta_j} & & 0 \\ & \ddots & \\ 0 & & \mathrm{Rot}_{t\theta_j} \end{pmatrix} \text{ where } \mathrm{Rot}_{t\theta_j} = \begin{pmatrix} \cos t\theta_j & -\sin t\theta_j \\ \sin t\theta_j & \cos t\theta_j \end{pmatrix}.$$

The change of variables $v = R_{-t}u$ transforms the differential equation $\frac{du}{dt} = f_0(u)$ into $\frac{dv}{dt} = g_0(v, t)$, where

$$g_0(v, t) = R_{-t} f_0(R_t v) + \frac{dR_{-t}}{dt} R_t v.$$

Write $R = \frac{dR_{-t}}{dt} R_t$; this matrix does not depend on t and we have $D_v g_0(0, t) = Df_0(0) + R$. Thus, $D_v g_0(0, t)$ does not depend on t, and we can see that its eigenvalues are real (these eigenvalues are $b_+, b_1, \dots, b_{s'}, \rho_{s'+1}, \dots, \rho_s$).

Write $v_0(t) = R_{-t}u_0(t)$, $t \in \mathbf{R}$. The following lemma is classical (see for instance [3]), and we shall omit its proof.

Lemma 1. *The quantity* $\frac{v_0(t)}{||v_0(t)||}$ *has a limit when* $t \to +\infty$*, and this limit is an eigenvector of* $D_v g_0(0, .)$*.*

Denote by w this eigenvector. It belongs to one of the characteristic spaces of $Df_0(0)$, corresponding to an eigenvalue b_{j_0} of $Df_0(0)$. We know that $\rho_{j_0} < -b_+$.

Remark. Generically, we have $\rho_{j_0} \geq \mathrm{Re}\, b_j$, $1 \leq j \leq s$, but we shall not need this in the following.

Denote by $\epsilon_1, \dots, \epsilon_d$ the canonical basis of $\mathbf{R}^d$. Up to another change of coordinates, we can suppose that $w = \epsilon_2$, and that, if b_{j_0} is real, then $Df_0(0)$ reads

$$\begin{pmatrix} b_+ & 0 & 0 \\ 0 & b_{j_0} & * \\ 0 & 0 & \tilde{B}_- \end{pmatrix}$$

with $\tilde{B}_- \in \mathcal{M}_{d-2}(\mathbf{R})$ (in this case, write $E = \mathrm{Vect}(\epsilon_2)$), and, if b_{j_0} is non-real, then $Df_0(0)$ reads

$$\begin{pmatrix} b_+ & 0 & 0 \\ 0 & \begin{pmatrix} \rho_{j_0} & -\theta_{j_0} \\ \theta_{j_0} & \rho_{j_0} \end{pmatrix} & * \\ 0 & 0 & \tilde{B}_- \end{pmatrix}$$

with $\tilde{B}_- \in \mathcal{M}_{d-3}(\mathbf{R})$ (in this case, write $E = \mathrm{Vect}(\epsilon_2, \epsilon_3)$).

We can now define the moving frame $(e_{1,0}(t) \dots, e_{d,0}(t))$, $t \in \mathbf{R}$. Let $e_{1,0}(t) = f_0(u_0(t))$, $t \in \mathbf{R}$. This vector $e_{1,0}(t)$ is almost parallel to ϵ_1 when t is large negative, and almost parallel to E when t is large positive. Denote by Π_1 (resp. by Π_E) the orthogonal projection onto $\mathrm{Vect}(\epsilon_1)$ (resp. onto E) in $\mathbf{R}^d$. There exists $T > 0$ (large) such that, for $t < -T$, $\Pi_1(e_{1,0}(t)) \neq 0$, and, for $t > T$, $\Pi_E(e_{1,0}(t)) \neq 0$.

For $t < -T$, let $e_{j,0}(t) = ||e_{1,0}(t)||\epsilon_j$, $2 \leq j \leq d$.

For $t > T$, let $e_{2,0}(t) = ||e_{1,0}(t)||\epsilon_1$, and,

- if b_{j_0} is real, then let $e_{j,0}(t) = ||e_{1,0}(t)||\epsilon_j$, $3 \leq j \leq d$;

- if b_{j_0} is non-real, then let $e_{j,0}(t) = ||e_{1,0}(t)||\epsilon_j$, $4 \leq j \leq d$, and let

$$e_{3,0}(t) = \frac{||e_{1,0}(t)||}{||\Pi_E e_{1,0}(t)||}\mathrm{Rot}_{-\frac{\pi}{2}}\Pi_E e_{1,0}(t)$$

(here $\mathrm{Rot}_{-\frac{\pi}{2}}$ denotes the rotation of angle $-\frac{\pi}{2}$ in the subspace E equipped with the orientation of the basis (ϵ_2, ϵ_3)).

We can see that, for any t with $|t| > T$, the family $(e_{1,0}(t)\dots, e_{d,0}(t))$ defines a basis of $\mathbf{R}^d$; it depends smoothly on t, it is almost orthogonal for large $|t|$, it satisfies $||e_{j,0}(t)|| = ||e_{1,0}(t)||$, $2 \leq j \leq d$, and it has the direct orientation. It is thus possible to extend smoothly each map $t \mapsto e_{j,0}(t)$ to the whole real line, in such a way that, for any $t \in \mathbf{R}$, $(e_{1,0}(t), \dots, e_{d,0}(t))$ defines a (positively oriented) basis of $\mathbf{R}^d$.

We use the notations of Subsect. 2.1. We have, when $t \to -\infty$, $||e_{1,0}(t)||^{-1}P_0(t) \to \mathrm{Id}_{\mathbf{R}^d}$, and, by calculus, $P_0(t)^{-1}\frac{dP_0}{dt}(t) \to b_+\,\mathrm{Id}_{\mathbf{R}^d}$. Thus,

$$b_0(t) \to B_- - b_+\,\mathrm{Id}_{\mathbf{R}^{d-1}} \text{ when } t \to -\infty. \tag{12}$$

Suppose that b_{j_0} is real. Then, when $t \to +\infty$, $||e_{1,0}(t)||^{-1}P_0(t) \to \begin{pmatrix} \varsigma & 0 \\ 0 & \mathrm{Id}_{\mathbf{R}^{d-2}} \end{pmatrix}$, where $\varsigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and, by calculus, $P_0(t)^{-1}\frac{dP_0}{dt}(t) \to b_{j_0}\,\mathrm{Id}_{\mathbf{R}^d}$. We thus have in this case

$$b_0(t) \to \begin{pmatrix} b_+ - b_{j_0} & 0 \\ 0 & \tilde{B}_- - b_{j_0}\,\mathrm{Id}_{\mathbf{R}^{d-2}} \end{pmatrix} \text{ when } t \to +\infty. \tag{13}$$

Now suppose that b_{j_0} is non-real. Then, when $t \to +\infty$, $||e_{1,0}(t)||^{-1}P_0(t)$ is close to be of the form

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ * & 0 & * \\ * & 0 & * \end{pmatrix} & 0 \\ 0 & \mathrm{Id}_{\mathbf{R}^{d-3}} \end{pmatrix},$$

and, by calculus,

$$P_0(t)^{-1}\frac{dP_0}{dt}(t) \to \rho_{j_0}\,\mathrm{Id}_{\mathbf{R}^d} + \begin{pmatrix} \begin{pmatrix} 0 & 0 & -\theta_{j_0} \\ 0 & 0 & 0 \\ \theta_{j_0} & 0 & 0 \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}.$$

We thus have in this case

$$b_0(t) \to \begin{pmatrix} b_+ - \rho_{j_0} & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & \tilde{B}_- - \rho_{j_0}\,\mathrm{Id}_{\mathbf{R}^{d-3}} \end{pmatrix} \text{ when } t \to +\infty \tag{14}$$

(where the terms $*$ may depend on time).

Consider the differential equation

$$\frac{dY}{dt} = b_0(t)Y + c_{3,0}(t), \quad Y \in \mathbf{R}^{d-1}, \quad t \in \mathbf{R}.$$

According to (12), this equation has a unique solution $t \mapsto Y_-(t)$ which is bounded when $t \to -\infty$. On the other hand, according to (13) and (14), this equation admits a (unique) affine hyperplane of solutions $\mathcal{Y}$ (of dimension $d-2$) such that, for any $Y(.) \in \mathcal{Y}$, the vector $e^{b_{j_0}t}Y(t)$ is bounded when $t \to +\infty$. Let $\mathcal{S}$ denote the set of all solutions of the preceding differential equation, and denote by $\mathcal{S}_+$ (resp. by $\mathcal{S}_-$) the set of solutions $Y(.)$ such that the first coordinate of the vector $e^{b_{j_0}t}Y(t)$ goes to $+\infty$ (resp. to $-\infty$) when $t \to +\infty$. We have

$$\mathcal{S} \setminus \mathcal{Y} = \mathcal{S}_+ \sqcup \mathcal{S}_- .$$

Let us define our index $\sigma(f_0, u_0, C_0)$ by

$$\begin{gathered}\sigma(f_0, u_0, C_0) = +1 \text{ (resp. } \sigma(f_0, u_0, C_0) = 0,\ \sigma(f_0, u_0, C_0) = -1)\\ \text{if } Y_- \in \mathcal{S}_+ \text{ (resp. } Y_- \in \mathcal{Y},\ Y_- \in \mathcal{S}_-).\end{gathered}$$

The condition $Y_- \in \mathcal{S}_+ \sqcup \mathcal{S}_-$ (and thus $\sigma(f_0, u_0, C_0) \neq 0$) is again generic, except if the map $C_0(.,.)$ is identically proportional to the identity (in this case, we have $c_{3,0}(.) \equiv 0$, and thus $Y_-(.) \equiv 0$ and $\mathcal{Y} \equiv \{0\} \times \mathbf{R}^{d-2}$).

3. Proof in Case (a)

3.1. Setup for the proof. We give ourselves and fix a vector field $f_0(.)$ and a map $C_0(.,.)$ as in Subsect. 1.1, in case (a). We adopt the conventions (choice of a convenient basis) and notations of Sect. 2 and we suppose that $\sigma(f_0, u_0, C_0) \neq 0$.

Let $\delta > 0$ and $\varepsilon_0 > 0$ be two constants to be chosen later. Throughout the proof, we will often have to make the hypotheses that δ or ε_0 are small. The hypotheses on δ will always depend only on C_0 and $f_0(.)$ (although this will not be stated explicitly), and the ones on ε_0 only on C_0, $f_0(.)$, and δ. Thus the final convenient choices of δ and ε_0 will only depend on C_0 and $f_0(.)$.

Consider any vector field $f_1(.)$ and any map $C_1(.,.)$ with the same hypotheses as in Theorems 1 and 2, in particular

$$||f_1(.) - f_0(.)||_{C^1} < \varepsilon_0, \quad ||C_1(.) - C_0(.)||_{C^0} < \varepsilon_0, \text{ and } \quad \text{dist}(\mathcal{T}_0, \mathcal{T}_1) < \varepsilon_0,$$

and let

$$\nu = ||f_1(.) - f_0(.)||_{C^1}.$$

Because of the continuous dependence of a local stable manifold on the vector field, we have

$$\mu \to 0 \quad \text{when} \quad \varepsilon_0 \to 0$$

(recall, see Subsect. 1.1, that $\mu = \text{dist}(\zeta_1, W_1^{\text{s,loc}}(0))$).

Let Σ_0 be a small hypersurface crossing transversally $\mathcal{T}_0$ at $u_0(0)$ (see Fig. 9). For ε_0 sufficiently small, $\Sigma_0 \cap \mathcal{T}_1 \neq \emptyset$, and, up to reparametrizing $t \mapsto u_1(t)$ we will suppose that $u_1(0) \in \Sigma_0$. Let

$$\begin{aligned}\Sigma &= \{(x, y) \mid x \in [-\delta; \delta] \text{ and } y \in \mathbf{R}^{d-1}, ||y|| = \delta\},\\ \Sigma' &= \{(x, y) \mid x = \pm\delta \text{ and } y \in \mathbf{R}^{d-1}, ||y|| \leq \delta\}.\end{aligned}$$

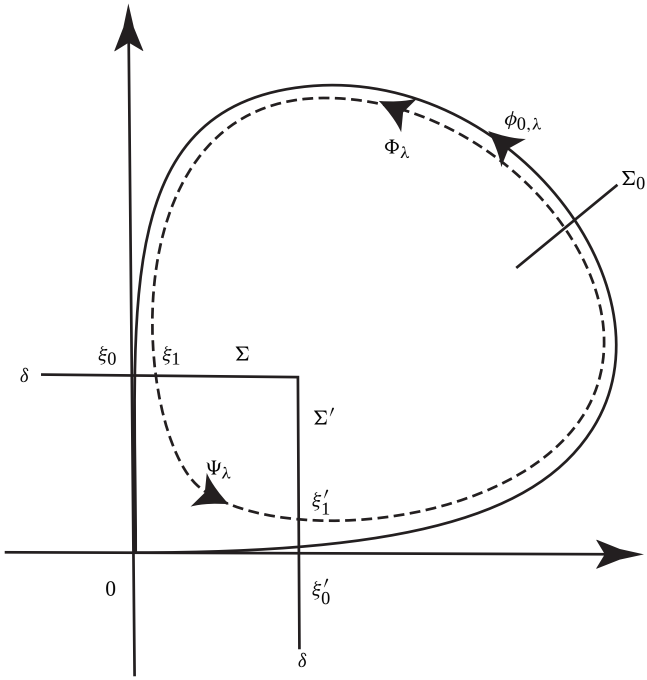


Fig. 9.

For δ and ε_0 sufficiently small, the intersection $\mathcal{T}_0 \cap \Sigma$ (resp. $\mathcal{T}_0 \cap \Sigma'$, $\mathcal{T}_1 \cap \Sigma$, $\mathcal{T}_1 \cap \Sigma'$) contains exactly one point (same reason as in Subsect. 1.1); denote it by ξ_0 (resp. ξ_0', ξ_1, ξ_1'). Denote by T the period of $u_1(.)$, define t_0, t_0', t_1, and t_1' by:

$$u_0(t_0) = \xi_0, \quad u_0(t_0') = \xi_0', \quad u_1(t_1) = \xi_1, \quad u_1(t_1') = \xi_1', \quad t_1' < 0 < t_1 < t_1' + T,$$

and write $t_1'' = t_1' + T$.

Let

$$\mu' = \mathrm{dist}(u_1(t_1), W_1^{\mathrm{s,loc}}(0)) \quad \text{and} \quad \varepsilon = \frac{\mu'}{\delta}$$

($W_1^{\mathrm{s,loc}}(0)$ was defined in Subsect. 1.1, we suppose that $\delta < \delta_0$).

We remark that μ, μ', and ε are of the same order (they are equal up to multiplicative constants depending on the choice of δ).

For $\lambda \in \mathbf{R}$, denote by ϕ_λ (resp. ψ_λ) the flow of the differential Eq. (9) with $k = 1$, between the times $t = t_1'$ and $t = t_1$ (resp. between the times $t = t_1$ and $t = t_1''$) (in the case $d \geq 3$, the local frames will be defined in Subsect. 3.3). Denote by $\phi_{0,\lambda}$ the flow of the differential Eq. (9)) with $k = 0$, between the times $t = t_0'$ and $t = t_0$.

Write

$$\phi_0 = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}, \quad \psi_0 = \begin{pmatrix} 1 & \eta \\ 0 & \zeta \end{pmatrix}, \quad \text{and} \quad \phi_{0,0} = \begin{pmatrix} 1 & \alpha_0 \\ 0 & \beta_0 \end{pmatrix}$$

(where α, η, and α_0 are $1 \times (d-1)$-matrices, and β, ζ, and β_0 are $(d-1) \times (d-1)$-matrices) and write

$$\phi_\lambda = \phi_0 + \lambda \begin{pmatrix} w_\lambda & x_\lambda \\ y_\lambda & z_\lambda \end{pmatrix} \quad \text{and} \quad \phi_{0,\lambda} = \phi_{0,0} + \lambda \begin{pmatrix} w_{0,\lambda} & x_{0,\lambda} \\ y_{0,\lambda} & z_{0,\lambda} \end{pmatrix}$$

(with similar conventions).

The quantities $w_{0,\lambda}$, $x_{0,\lambda}$, $y_{0,\lambda}$, and $z_{0,\lambda}$ have limits $w_{0,0}$, $x_{0,0}$, $y_{0,0}$, and $z_{0,0}$ when $\lambda \to 0$; these limits can be obtained as values at time t_0 of solutions of explicit differential equations involving $a_0(.)$, $b_0(.)$, and $c_{j,0}(.)$, $1 \le j \le 4$); the differential equation for $y_{0,0}$ reads

$$\frac{dy}{dt} = b_0(t)y + c_{3,0}(t) \tag{15}$$

(it is the differential equation used in paragraphs 2.2 and 2.3 for the definition of σ_Y). According to classical results on continuous dependence with respect to parameters for solutions of ordinary differential equations, the quantities w_λ, x_λ, y_λ, and z_λ are arbitrarily close to $w_{0,0}$, $x_{0,0}$, $y_{0,0}$, and $z_{0,0}$ if $|\lambda|$ and ε_0 are sufficiently small (depending on δ).

For the remainder of the proof, we impose $\lambda \in]-\varepsilon_0; 0[$; moreover, we will suppose that ε_0 is small enough (depending on δ) in order to have $\delta > \nu$, $\delta > |\lambda|$, and $\delta > \varepsilon$. Thus, in all the following estimates, the terms of the order of $\mathcal{O}(\nu)$, $\mathcal{O}(\lambda)$ or $\mathcal{O}(\varepsilon)$ will be absorbed in the terms $\mathcal{O}(\delta)$.

3.2. Estimates in dimension two.

Estimates on ψ_λ. Denote by Ψ_λ the flow of the differential equation (8) with $k = 1$ between the times t_1 and t_1''. Write $Q = P_1(t_1)$ and $Q' = P_1(t_1'')$. We have

$$\psi_\lambda = Q'^{-1} \Psi_\lambda Q.$$

A cone-invariance argument on the flow of (8) shows that Ψ_λ has two eigenvectors i_λ and j_λ of the form

$$i_\lambda = \begin{pmatrix} 1 \\ \mathcal{O}(\delta) \end{pmatrix} \quad \text{and} \quad j_\lambda = \begin{pmatrix} \mathcal{O}(\delta) \\ 1 \end{pmatrix}$$

(the terms $\mathcal{O}(\nu)$ and $\mathcal{O}(\lambda)$ are absorbed in $\mathcal{O}(\delta)$). Denote by R_λ the matrix of $\mathcal{M}_2(\mathbf{R})$ whose columns are the coordinates of i_λ and j_λ (we have $R_\lambda = \mathrm{Id}_{\mathbf{R}^2} + \mathcal{O}(\delta)$). The matrix $R_\lambda^{-1} \Psi_\lambda R_\lambda$ is diagonal; denote it by L_λ and write

$$L_\lambda = \begin{pmatrix} A_\lambda & 0 \\ 0 & a_\lambda \end{pmatrix}.$$

Let us estimate ψ_0. Write $\gamma = \frac{|b_-|}{b_+} \ge 1$. As $\varepsilon = \mu'/\delta$, we have

$$t_1'' - t_1 = \frac{1}{b_+ + \mathcal{O}(\delta)} \log \varepsilon^{-1}$$

and thus

$$A_0 = \varepsilon^{-1+\mathcal{O}(\delta)} \gg 1 \quad \text{and} \quad a_0 = \varepsilon^{\gamma + \mathcal{O}(\delta)} \ll 1$$

(these last estimates are not optimal but are sufficient for the moment; we will prove a more precise estimate on A_0 in the following).

We have $u_1(t_1) = \begin{pmatrix} \mathcal{O}(\delta^2) + \mu' \\ \sigma_{\rm or}\delta \end{pmatrix} = \begin{pmatrix} \mathcal{O}(\delta^2) \\ \sigma_{\rm or}\delta \end{pmatrix}$, and thus

$$Q = \sigma_{\rm or}\delta|b_-|\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \mathcal{O}(\delta)\right).$$

We have $u_1(t_1'') = \begin{pmatrix} \delta \\ \mathcal{O}(\delta^2) + \sigma_{\rm or}\delta\varepsilon^{\gamma+\mathcal{O}(\delta)} \end{pmatrix} = \begin{pmatrix} \delta \\ \mathcal{O}(\delta^2) \end{pmatrix}$, thus $Q' = \delta b_+(\mathrm{Id}_{\mathbf{R}^2} + \mathcal{O}(\delta))$ and

$$Q'^{-1} = \frac{1}{\delta b_+}(\mathrm{Id}_{\mathbf{R}^2} + \mathcal{O}(\delta)).$$

Finally, as

$$\psi_0 = Q'^{-1}R_0L_0R_0^{-1}Q, \tag{16}$$

we get

$$\zeta = \det\psi_0 = \gamma^2(1+\mathcal{O}(\delta))A_0a_0 = \gamma^2(1+\mathcal{O}(\delta))\varepsilon^{\gamma-1+\mathcal{O}(\delta)} \tag{17}$$

and, identifying in the expression (16) of ψ_0, we find

$$\eta = \sigma_{\rm or}\gamma(1+\mathcal{O}(\delta))A_0 = \sigma_{\rm or}\gamma(1+\mathcal{O}(\delta))\varepsilon^{-1+\mathcal{O}(\delta)}. \tag{18}$$

Now we estimate ψ_λ. Write $q_\lambda = \frac{A_\lambda}{A_0}$ and $d_\lambda = \frac{A_0}{A_\lambda}a_\lambda - a_0$; then we have

$$L_\lambda = q_\lambda\begin{pmatrix} A_0 & 0 \\ 0 & a_0 + d_\lambda \end{pmatrix} = q_\lambda\left(L_0 + \begin{pmatrix} 0 & 0 \\ 0 & d_\lambda \end{pmatrix}\right).$$

A cone-invariance argument shows that

$$R_\lambda = R_0 + \mathcal{O}(\lambda),$$

and we have

$$A_\lambda = A_0e^{(t_1''-t_1)\,\mathcal{O}(\lambda)} = A_0\varepsilon^{\mathcal{O}(\lambda)};$$

$$a_\lambda = a_0e^{(t_1''-t_1)\,\mathcal{O}(\lambda)} = a_0\varepsilon^{\mathcal{O}(\lambda)};$$

thus $q_\lambda = \varepsilon^{\mathcal{O}(\lambda)}$ and $d_\lambda = a_0(\varepsilon^{\mathcal{O}(\lambda)} - 1)$.

Now we have

$$\begin{aligned}
\psi_\lambda &= Q'^{-1}R_\lambda L_\lambda R_\lambda^{-1}Q \\
&= q_\lambda\left((\mathrm{Id}+\mathcal{O}(\lambda))(Q'^{-1}R_0)\left(L_0 + \begin{pmatrix} 0 & 0 \\ 0 & d_\lambda \end{pmatrix}\right)(R_0^{-1}Q)(\mathrm{Id}+\mathcal{O}(\lambda))\right) \qquad (19) \\
&= q_\lambda\Big((\mathrm{Id}+\mathcal{O}(\lambda))\psi_0(\mathrm{Id}+\mathcal{O}(\lambda)) + \mathcal{O}(d_\lambda)\Big)
\end{aligned}$$

and we obtain

$$\psi_\lambda = q_\lambda(\psi_0 + S),$$

where $S = (S_{i,j})_{1\le i,j\le 2}$ satisfies

$$S_{i,j} = \eta\,\mathcal{O}(\lambda) + \mathcal{O}(d_\lambda) \text{ if } (i,j) \neq (2,1);$$
$$S_{2,1} = (1+\zeta)\,\mathcal{O}(\lambda) + \eta\,\mathcal{O}(\lambda^2) + \mathcal{O}(d_\lambda).$$

We remark that

$$d_\lambda = a_0(e^{\mathcal{O}(\lambda)\log\varepsilon} - 1) = \varepsilon^{\gamma+\mathcal{O}(\delta)}(\log\varepsilon)\,\mathcal{O}(\lambda)$$

which shows that

$$S_{i,j} = \eta\,\mathcal{O}(\lambda) \quad \text{if} \quad (i,j) \neq (2,1) \quad \text{and} \quad S_{2,1} = (1+\zeta)\,\mathcal{O}(\lambda) + \eta\,\mathcal{O}(\lambda^2). \tag{20}$$

Estimates on the trace of $\psi_\lambda \circ \phi_\lambda$. Denote by T_λ the trace of $\psi_\lambda \circ \phi_\lambda$. We have

$$T_0 = 1 + \zeta\beta$$

and calculus yields

$$T_\lambda = q_\lambda\Big(T_0 + \lambda\eta(y_\lambda + r(\lambda))\Big), \tag{21}$$

where (forgetting the indices λ)

$$r(\lambda) = \eta^{-1}(w+\zeta z) + (\lambda\eta)^{-1}\Big(S_{1,1}(1+\lambda w) + S_{1,2}\lambda y + S_{2,1}(\alpha+\lambda x) + S_{2,2}(\beta+\lambda z)\Big).$$

Lemma 2. *The quantity β_0 is bounded by a constant which does not depend on δ.*

Proof. We have

$$\beta_0 = \exp\int_{t_0'}^{t_0} b_0(s)ds.$$

Write $u_0(t) = (x_0(t), y_0(t))$. We have $\log|y_0(t)|^{-1} \sim t|b_-|$ when $t \to +\infty$ and $\log|x_0(t)|^{-1} \sim |t|b_+$ when $t \to -\infty$. In particular, we have

$$t_0 \sim |b_-|^{-1}\log\delta^{-1} \quad \text{and} \quad t_0' \sim -b_+^{-1}\log\delta^{-1} \quad \text{when} \quad \delta \to 0.$$

As $b_0(t) \to \pm(b_+ - b_-)$ when $t \to \pm\infty$, this shows that, if $|b_-| > b_+$, then $\beta_0 \to 0$ when $\delta \to 0$, and this proves the lemma in this case.

In the remaining case $|b_-| = b_+$, we have to be slightly more precise. When $t \to +\infty$, we have $x_0(t) = \mathcal{O}(y_0(t)^2)$ and thus, according to claim (11), $b_0(t) = b_- - b_+ + \mathcal{O}(y_0(t))$. Similarly, when $t \to -\infty$, we have $b_0(t) = b_+ - b_- + \mathcal{O}(x_0(t))$. Thus, β_0 is equal, up to a multiplicative constant independent of δ, to the quantity $e^{(b_+-b_-)(t_0-|t_0'|)}$.

On the other hand, we have

$$\frac{dy_0}{dt} = b_- y_0 + \mathcal{O}(y_0^2) \quad \text{when} \quad t \to +\infty,$$

which shows that $\log|y_0(t)|^{-1} - t|b_-|$ is bounded when $t \to +\infty$. Similarly, $\log|x_0(t)|^{-1} - |t|b_+$ is bounded when $t \to -\infty$, which shows that $t_0 - |t_0'|$ is bounded independently of δ, and the lemma follows. □

According to estimates (17) on ζ, (18) on η, (20) on $S_{i,j}$, and to the lemma above, $r(\lambda)$ is bounded, for ε_0 sufficiently small (depending on δ) by a constant which does not depend on δ.

The quantity $y_{0,0}$ is the value at time t_0 of the solution of the differential Eq. (15), namely

$$\frac{dY}{dt} = b_0(t)Y + c_{3,0}(t)$$

with initial condition $Y = 0$ at time $t = t_0'$.

This differential equation is precisely the one governing the functions $Y_-(.)$ and $Y_+(.)$ of Subsect. 2.2. We know that $\sigma_Y = \pm 1$, and, as $b_0(t) \to b_+ - b_-$ when $t \to +\infty$, we see that $\sigma_Y Y_-(t) \to +\infty$ when $t \to +\infty$. We thus have

$$\sigma_Y y_{0,0} \to +\infty \quad \text{when} \quad \delta \to 0.$$

Before we can conclude, we need a more precise estimate on η.

Lemma 3. *We have*

$$A_0 = (1 + \mathcal{O}(\delta))\varepsilon^{-1}.$$

Proof. We could use Hartman's C^1 linearization theorem ([8]) but we will give a more elementary proof.

There is a smooth map g_1, defined on a neighborhood of 0 in $\mathbf{R}^2$, with values in $\mathbf{R}^2$, satisfying $g_1(0) = 0$, and mapping $W_1^{\mathrm{u,loc}}(0)$ (resp. $W_1^{\mathrm{s,loc}}(0)$) to the x-axis (resp. to the y-axis). We have $Dg_1(0) = \mathrm{Id}_{\mathbf{R}^2} + \mathcal{O}(\nu)$.

Denote by $\hat{f}_1$ the vector field obtained by conjugating f_1 by g_1 (i.e. $\hat{f}_1(.) = Dg_1(g_1^{-1}(.)) f_1(g_1^{-1}(.))$), denote by $\hat{f}_{1,1}$ the first component of $\hat{f}_1$, and let $b_{+,1} = b_+ + \mathcal{O}(\nu)$ and $b_{-,1} = b_- + \mathcal{O}(\nu)$ denote the two eigenvalues of $Df_1(0)$. Then we have

$$\hat{f}_{1,1}(x, y) = x\big(b_{+,1} + \mathcal{O}(||(x, y)||)\big). \tag{22}$$

Write $\hat{u}_1(t) = g_1(u_1(t))$, $t \in \mathbf{R}$, and denote by $\hat{x}_1(t)$ the first coordinate of $\hat{u}_1(t)$. We have

$$\hat{x}_1(t_1) = \mu'(1 + \mathcal{O}(\delta)), \quad \hat{x}_1(t_1'') = \delta(1 + \mathcal{O}(\delta)), \tag{23}$$

and, according to (22),

$$\frac{d\hat{x}_1}{dt} = \hat{x}_1(t)(b_{+,1} + \mathcal{O}(||\hat{u}_1(t)||)), \quad t \in [t_1; t_1'']. \tag{24}$$

On the other hand, the dynamics close to 0 shows that

$$\int_{t_1}^{t_1''} \mathcal{O}(||\hat{u}_1(t)||)dt = \mathcal{O}(\delta). \tag{25}$$

Thus, we deduce from (23) and (24) that

$$e^{b_{+,1}(t_1''-t_1)} = (1 + \mathcal{O}(\delta))\varepsilon^{-1}. \tag{26}$$

Denote by $\hat{A}_0$ the largest eigenvalue of the flow of the differential equation

$$\frac{du}{dt} = D\hat{f}_1(\hat{u}_1(t))u \tag{27}$$

between the times $t = t_1$ and $t = t_1''$, and denote by v the corresponding eigenvector (with the normalization constraint that the first coordinate of v is equal to 1). We have $\hat{A}_0 = (1 + \mathcal{O}(\delta))A_0$.

Let $v(t)$ denote the solution of the differential equation (27) with initial condition v at time $t = t_1$. Write $v(t) = (v_1(t), v_2(t))$. Then $v_1(t_1'') = \hat{A}_0$. A cone-invariance argument shows that, for any $t \in [t_1; t_1'']$, we have $v_2(t)/v_1(t) = \mathcal{O}(\delta)$. Thus, according to (27), we have

$$\frac{dv_1}{dt} = v_1(t)\big(b_{+,1}) + \mathcal{O}(||\hat{u}_1(t)||)\big).$$

The lemma thus follows from (25) and (26). $\square$

According to this lemma and to estimate (18) on η, we have

$$\eta = \sigma_{\mathrm{or}}\gamma(1 + \mathcal{O}(\delta))\varepsilon^{-1}. \tag{28}$$

End of the proof. To conclude, we will distinguish two cases.

(i) $|\lambda| \geq \varepsilon$. In this case, write $\varepsilon = s|\lambda|$, $0 < s \leq 1$. We have $1 < T_0 \leq 2$. Thus, according to (28), the formula (21) for T_λ yields

$$T_\lambda = q_\lambda \lambda \eta y_\lambda (1 + \dots) = (-\sigma_Y \sigma_{\mathrm{or}})\gamma s^{-1+\mathcal{O}(\lambda)}|y_\lambda|(1 + \cdots),$$

where the " ... " denote terms which are arbitrarily small if δ is sufficiently small and ε_0 is sufficiently small (depending on δ). Thus, for δ sufficiently small and ε_0 sufficiently small (depending on δ), the quantity $(-\sigma_Y\sigma_{\mathrm{or}})T_\lambda$ is arbitrarily large, in particular larger than 2. On the other hand, we know, as the trace of $C_1(.,.)$ is nonnegative (according to the hypothesis that $C_1(.,.) \in \mathcal{L}^+(\mathbf{R}^d)$, see §1.1), that the determinant of $\psi_\lambda \circ \phi_\lambda$ is not larger than 1. Thus, $(-\sigma_Y\sigma_{\mathrm{or}})T_\lambda > 2$ implies that $\psi_\lambda \circ \phi_\lambda$ has an eigenvalue which is real and strictly larger than one in modulus, its sign being the sign of $-\sigma_Y\sigma_{\mathrm{or}}$. This proves the instability in case $|\lambda| \geq \varepsilon$; in particular, this proves the instability in case 2 of Theorem 2 (i.e. when $-\sigma_Y\sigma_{\mathrm{or}} = -1$); indeed, as we already mentioned, the quantity $\frac{\varepsilon}{\mu}$ is bounded from above by a constant (which depends on the choice of δ) which is convenient for the choice of the constant K' appearing in the theorem.

(ii) $|\lambda| < \varepsilon$. In this case, write $|\lambda| = t\varepsilon$, $0 < t < 1$. Write $T_\lambda = T_0 + tT'_\lambda$. According to (28), we have

$$T'_\lambda = (-\sigma_Y\sigma_{\mathrm{or}})\gamma|y_\lambda|(1 + \dots).$$

In particular, T'_λ is arbitrarily large, and has the sign of $(-\sigma_Y\sigma_{\mathrm{or}})$, if δ is sufficiently small and ε_0 is sufficiently small (depending on δ).

Denote by D_λ the determinant of $\psi_\lambda \circ \phi_\lambda$. We have

$$\det \psi_\lambda = \det Q'^{-1}(A_\lambda a_\lambda) \det Q = \varepsilon^{\mathcal{O}(\lambda)} \det \psi_0$$

and

$$\det \phi_\lambda = \det \phi_0 + \mathcal{O}(\lambda) = (1 + \mathcal{O}(\lambda)) \det \phi_0$$

(be careful that in this last expression, the term $\mathcal{O}(\lambda)$ depends on δ !); thus

$$D_\lambda = \varepsilon^{\mathcal{O}(\lambda)}(1+\mathcal{O}(\lambda))D_0.$$

Write $D_\lambda = D_0 + tD'_\lambda$. As $D_0 \le 1$, we see that D'_λ is arbitrarily small if ε_0 is sufficiently small (depending on δ). Write $\Delta_\lambda = T_\lambda^2 - 4D_\lambda$ and $\Delta_\lambda = \Delta_0 + t\Delta'_\lambda$. We have

$$\Delta'_\lambda = 2T_0T'{}_\lambda + tT'{}^2_\lambda - 4D'_\lambda.$$

If $\Delta_\lambda \ge 0$, denote by m_λ the largest eigenvalue of $\psi_\lambda \circ \phi_\lambda$. We have $m_0 = 1$ and

$$m_\lambda = 1 + \frac{1}{2}\left(tT'{}_\lambda + \sqrt{\Delta_0 + t\Delta'_\lambda} - \sqrt{\Delta_0}\right).$$

Now we can conclude. We know that $\Delta_0 \ge 0$. If $-\sigma_{\mathrm{or}}\sigma_Y = 1$, we see that $\Delta'_\lambda > 0$ (thus $\Delta_\lambda > 0$) and $m_\lambda > 1$. This proves the instability result in case 1 of Theorem 2.

If on the other hand $-\sigma_{\mathrm{or}}\sigma_Y = -1$, then we see that, for t sufficiently small (depending on δ), $\Delta'_\lambda < 0$, and, if $\Delta_\lambda \ge 0$, then the two eigenvalues of $\psi_\lambda \circ \phi_\lambda$ are strictly between 0 and 1. Finally, if $\Delta_\lambda < 0$, then we know that $D_\lambda \le 1$ (according to the hypothesis that $C_1(.,.) \in \mathcal{L}^+(\mathbf{R}^d)$, the trace of $C_1(.,.)$ is nonnegative) and the spectral radius of $\psi_\lambda \circ \phi_\lambda$ is thus not larger than 1. This proves the stability result in case 2 of Theorem 2 (the value of t "sufficiently small" provides a convenient choice for the constant K).

The proof in dimension 2 of Theorem 2 (and thus of Theorem 1 in case (a)) is now complete. □

3.3. Estimates in dimension higher than two. For $t \in [t'_1; t_1]$, let $e_{j,1}(t) = e_{j,0}(t)$, $j = 2, \dots, d$ (the vectors $e_{j,0}(t)$ were defined in Subsect. 2.3). If ε_0 is sufficiently small, then, for any $t \in [t'_1; t_1]$, the family $(e_{1,1}(t), \dots, e_{d,1}(t))$ defines a basis of $\mathbf{R}^d$. This enables to define $P_1(t)$, $\hat{M}_1(t)$, and $\hat{C}_{01}(t, \lambda)$ for $t \in [t'_1; t_1]$ as in Subsect. 2.1. We can thus define $\phi_{0,\lambda}$ and ϕ_λ as in Subsect. 3.1. To define ψ_λ, we do not have to define explicitly the local frame between $t = t_1$ and $t = t''_1$; indeed, ψ_λ actually depends only on the local frame at $t = t_1$ and $t = t'_1$. Write $Q = P_1(t_1)$ and $Q' = P_1(t'_1)$, and denote by Ψ_λ the flow of the differential Eq. (8)) between the times t_1 and t''_1. We can define ψ_λ by:

$$\psi_\lambda = Q'^{-1}\Psi_\lambda Q.$$

Estimates on ψ_λ. We suppose, as in the case $d = 2$, that ε_0 is sufficiently small (depending on δ) to have $\delta > \nu$, $\delta > |\lambda|$, and $\delta > \varepsilon$, so that the terms $\mathcal{O}(\mu)$, $\mathcal{O}(\lambda)$, and $\mathcal{O}(\varepsilon)$ are absorbed by terms $\mathcal{O}(\delta)$.

A cone-invariance argument shows that Ψ_λ has two invariant subspaces I_λ and J_λ, with $\dim I_\lambda = 1$ and $\dim J_\lambda = d-1$. The subspace I_λ (resp. J_λ) is almost parallel to ϵ_1 (resp. to $\{0\} \times \mathbf{R}^{d-1}$). Denote by Π_I (resp. by Π_J) the projector on I_λ along $\{0\} \times \mathbf{R}^{d-1}$ (resp. the projector on J_λ along $\mathrm{Vect}(\epsilon_1)$), and write $\epsilon_{1,\lambda} = \Pi_I\epsilon_1$ and $\epsilon_{j,\lambda} = \Pi_J\epsilon_j$, $j = 2, \dots, d$. These vectors define a basis of $\mathbf{R}^d$, and we have

$$\epsilon_{j,\lambda} = \epsilon_j + \mathcal{O}(\delta), \quad j = 1, \dots, d.$$

Denote by R_λ the matrix of $\mathcal{M}_d(\mathbf{R})$ whose columns are the coordinates of the vectors $\epsilon_{j,\lambda}$, $j = 1, \dots, d$, and write $L_\lambda = R_\lambda^{-1}\Psi_\lambda R_\lambda$. The matrix L_λ reads

$$\begin{pmatrix} A_\lambda & 0 \\ 0 & a_\lambda \end{pmatrix}$$

with $a_\lambda \in \mathcal{M}_{d-1}(\mathbf{R})$.

Let us estimate ψ_0. Fix a real number $b_- < 0$ satisfying $\max_{j=1,\dots,s} \operatorname{Re} b_j < b_- < -b_+$ and let $\gamma = \frac{|b_-|}{b_+} > 1$. We have (as in the case $d = 2$)

$$t_1'' - t_1 = \frac{1}{b_+ + \mathcal{O}(\delta)} \log \varepsilon^{-1},$$

and thus

$$A_0 = \varepsilon^{-1+\mathcal{O}(\delta)} \gg 1,$$

and, for δ sufficiently small (according to the margin between $\max_{j=1,\dots,s} \operatorname{Re} b_j$ and b_-),

$$|||a_0||| < \varepsilon^\gamma \ll 1.$$

Write $\eta = (\eta_1, \dots, \eta_{d-1})$. According to the estimates of Subsect. 2.3 on $P_0(t)$, $|t| > T$, and computing $\psi_0 = Q'^{-1} R_0 L_0 R_0^{-1} Q$, we get

$$\eta_1 = \frac{|b_{j_0}|}{b_+} A_0(1 + \mathcal{O}(\delta)) \quad \text{and} \quad \eta_j = A_0\, \mathcal{O}(\delta), \quad j = 2, \dots, d-1. \tag{29}$$

Lemma 4. *For δ sufficiently small and ε_0 sufficiently small (depending on δ), we have*

$$|||\zeta||| < \varepsilon^{\gamma-1}.$$

Proof. For $t \in [t_1; t_1'']$, write $u_1(t) = (x_1(t), y_1(t))$, $x_1(t) \in \mathbf{R}$, $y_1(t) \in \mathbf{R}^{d-1}$. We have

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \left(\begin{pmatrix} b_+ & 0 \\ 0 & B_- \end{pmatrix} + \mathcal{O}(\delta) \right) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

which shows that, for δ sufficiently small, there exists a unique time $t_1''' \in]t_1; t_1''[$ such that $x_1(t_1''') = ||y_1(t_1''')||$.

For $t \in [t_1'''; t_1'']$, write $e_{1,1}(t) = f_1(u_1(t))$ and $e_{j,1}(t) = \epsilon_j$, $2 \le j \le d$. For δ sufficiently small, these vectors define, for any $t \in [t_1'''; t_1'']$, a basis of $\mathbf{R}^d$. Let us define the matrices $P_1(t)$ and $\hat{M}_1(t)$ as in Subsect. 2.1.

Let $Q'' = P_1(t_1''')$, and let $\Psi_{(1)}$ denote the flow of the differential Eq. (8)) between the times t_1 and t_1'''. Write $\psi_{(1)} = Q''^{-1}\Psi_{(1)} Q$ and let $\psi_{(2)}$ denote the flow of the differential equation $\frac{du}{dt} = \hat{M}_1(t)u$ between the times t_1''' and t_1''. We have $\psi_0 = \psi_{(2)} \circ \psi_{(1)}$ and we can write

$$\psi_{(1)} = \begin{pmatrix} 1 & \eta_{(1)} \\ 0 & \zeta_{(1)} \end{pmatrix} \text{ and } \psi_{(2)} = \begin{pmatrix} 1 & \eta_{(2)} \\ 0 & \zeta_{(2)} \end{pmatrix},$$

where $\zeta_{(1)}$ and $\zeta_{(2)}$ belong to $\mathcal{M}_{d-1}(\mathbf{R})$. Then we have

$$\zeta = \zeta_{(2)} \circ \zeta_{(1)}.$$

We have $|||Q||| = \mathcal{O}(\delta)$ and $|||Q''^{-1}||| = ||e_{1,1}(t_1''')||^{-1}\,\mathcal{O}(1)$. We have

$$t_1''' - t_1 < \frac{1}{|b_-|} \log \frac{\delta}{||e_{1,1}(t_1''')||} \tag{30}$$

and we get

$$|||\Psi_{(1)}||| < \left(\frac{\delta}{||e_{1,1}(t_1''')||}\right)^{\frac{b_+}{|b_-|}} \text{ and thus } |||\psi_{(1)}||| < \left(\frac{\delta}{||e_{1,1}(t_1''')||}\right)^{1+\frac{1}{\gamma}}$$

(the margin between $\max_{j=1,\ldots,s} \operatorname{Re} b_j$ and b_- enables to absorb the terms $\mathcal{O}(\delta)$).

On the other hand, we have

$$t_1'' - t_1''' = \frac{1}{b_+ + \mathcal{O}(\delta)} \log \frac{\delta}{||e_{1,1}(t_1''')||}$$

and the expression of $\hat{M}_1(t)$ shows that

$$|||\zeta_{(2)}||| < \left(\frac{\delta}{||e_{1,1}(t_1''')||}\right)^{1+\gamma}.$$

Finally we get

$$|||\zeta||| < \left(\frac{\delta}{||e_{1,1}(t_1''')||}\right)^{\gamma-\frac{1}{\gamma}}.$$

Besides, we have

$$t_1''' - t_1 = \frac{1}{b_+ + \mathcal{O}(\delta)} \log \frac{\delta}{||e_{1,1}(t_1''')||}$$

which yields, according to (30) (and absorbing the term $\mathcal{O}(\delta)$ by the margin between $\max_{j=1,\ldots,s} \operatorname{Re} b_j$ and b_-),

$$\frac{\delta}{||e_{1,1}(t_1''')||} < \varepsilon^{\frac{\gamma}{1+\gamma}},$$

and the result follows. □

Now we estimate ψ_λ. Write $q_\lambda = \frac{A_\lambda}{A_0}$ and $d_\lambda = \frac{A_0}{A_\lambda} a_\lambda - a_0$. We have

$$L_\lambda = q_\lambda \left(L_0 + \begin{pmatrix} 0 & 0 \\ 0 & d_\lambda \end{pmatrix} \right).$$

A cone-invariance criterion shows that

$$R_\lambda = R_0 + \mathcal{O}(\lambda)$$

and we have

$$A_\lambda = A_0 \varepsilon^{\mathcal{O}(\lambda)}.$$

Moreover, comparing the differential equations the flows of which give rise to a_0 and a_λ, we get, for ε_0 sufficiently small, and using the margin between $\max_{j=1,\ldots,s} \operatorname{Re} b_j$ and b_-,

$$|||a_\lambda - a_0||| < \mathcal{O}(\lambda)\varepsilon^\gamma$$

which yields

$$|||d_\lambda||| < \varepsilon^\gamma(\varepsilon^{\mathcal{O}(\lambda)} - 1).$$

Proceeding as in the case $d = 2$, we obtain

$$\psi_\lambda = q_\lambda(\psi_0 + S),$$

where, if $(s_{i,j})_{1\leq i,j\leq d}$ are the coefficients of the matrix S, and writing $S = \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2}\end{pmatrix}$, we have

$$\begin{aligned} s_{i,j} &= \eta_1\, \mathcal{O}(\lambda) \quad \text{if } s_{i,j} \text{ does not belong to } S_{2,1}, \\ s_{i,j} &= \mathcal{O}(\lambda) + \eta_1\, \mathcal{O}(\lambda^2) \quad \text{if } s_{i,j} \text{ belongs to } S_{2,1}. \end{aligned} \tag{31}$$

Looking for an unstable eigenvector for $\psi_\lambda \circ \phi_\lambda$. The matrix $q_\lambda^{-1}\psi_\lambda \circ \phi_\lambda$ reads (forgetting the indices λ)

$$\begin{pmatrix} 1 + \lambda(w + \eta y) + S_{1,1}(1 + \lambda w) + \lambda S_{1,2} y & \alpha + \eta\beta + \lambda(x + \eta z) + S_{1,1}(\alpha + \lambda x) + S_{1,2}(\beta + \lambda z) \\ \lambda\zeta y + S_{2,1}(1 + \lambda w) + \lambda S_{2,2} y & \zeta\beta + \lambda\zeta z + S_{2,1}(\alpha + \lambda x) + S_{2,2}(\beta + \lambda z) \end{pmatrix}.$$

Let c be a large constant to be chosen later. We are looking for an unstable eigenvector for $\psi_\lambda \circ \phi_\lambda$, in the cone $\mathcal{C} = \{(x, y) \in \mathbf{R} \times \mathbf{R}^{d-1} \mid ||y|| < c|\lambda||x|\}$. Let φ be any vector of $\mathbf{R}^{d-1}$ satisfying $||\varphi|| = c$, and write

$$\begin{pmatrix} \chi \\ \xi \end{pmatrix} = \psi_\lambda \circ \phi_\lambda \begin{pmatrix} 1 \\ \lambda\varphi \end{pmatrix}.$$

The existence of an unstable eigenvector for $\psi_\lambda \circ \phi_\lambda$ will be proved if we get the following estimates:

$$|\lambda|^{-1}||\xi|| < c|\chi| \quad \text{and} \quad |\chi| > 1.$$

Let us first estimate χ. Write $y_{0,0} = (y_{0,0,1}, \ldots, y_{0,0,d-1})$, $y_\lambda = (y_{\lambda,1}, \ldots, y_{\lambda,d-1})$ and

$$\chi = q_\lambda\big(1 + \lambda\eta_1(y_{\lambda,1} + r(\lambda))\big). \tag{32}$$

We can write

$$r(\lambda) = \eta_1^{-1}(y_{\lambda,2}\,\eta_2 + \cdots + y_{\lambda,d-1}\,\eta_{d-1}) + (\lambda\eta_1)^{-1} S_{1,1} + \eta_1^{-1}\eta\beta\varphi + \ldots,$$

where "…" denotes terms which are arbitrarily small if ε_0 is sufficiently small (depending on δ and c). Let us consider the remaining terms.

According to (31), the term $(\lambda\eta_1)^{-1}S_{1,1}$ is bounded (independently of δ and c), and, according to (29) and to the following lemma, the term $\eta_1^{-1}\eta\beta\varphi$ goes to 0 when $\delta \to 0$ and c is fixed.

Lemma 5. *We have $|||\beta_0||| \to 0$ when $\delta \to 0$.*

The proof of this lemma is actually simpler than that of Lemma 2 (since we have here $|b_-| > b_+$) and we leave it to the reader.

The quantity $y_{0,0}$ is the value at time t_1 of the solution of the differential Eq. (15), namely

$$\frac{dY}{dt} = b_0(t)Y + c_{3,0}(t)$$

with initial condition $Y = 0$ at time $t = t_1'$. Thus, we see from the definition of $\sigma(f_0, u_0, C_0)$ that $\sigma(f_0, u_0, C_0)y_{0,0,1} \to +\infty$ when $\delta \to 0$, and from this differential equation that the ratio $y_{0,0,j}/y_{0,0,1}$ goes to 0 when $\delta \to 0$.

Thus, for δ sufficiently small (depending on c) and for ε_0 sufficiently small (depending on δ), we have $\sigma(f_0, u_0, C_0)y_{\lambda,1} > 0$ and

$$\chi = q_\lambda(1 + \lambda\eta_1 y_{\lambda,1}(1 + \dots)) \tag{33}$$

where " ... " is small.

In the following, $\sigma(f_0, u_0, C_0)$ will simply be denoted by σ. Let us consider ξ. We have

$$\lambda^{-1}q_\lambda^{-1}\xi = \lambda^{-1}S_{2,1}(1 + \lambda w_\lambda) + S_{2,2}y_\lambda + \big(S_{2,1}(\alpha + \lambda x_\lambda) + S_{2,2}(\beta + \lambda z_\lambda)\big)\varphi + \dots,$$

where "... " denotes terms which are arbitrarily small if ε_0 is sufficiently small (depending on δ and c). We thus have, according to (31),

$$|\lambda|^{-1}q_\lambda^{-1}||\xi|| = \mathcal{O}(1) + y_{\lambda,1}\eta_1\,\mathcal{O}(\lambda) + c\eta_1\,\mathcal{O}(\lambda). \tag{34}$$

As in the case $d = 2$, we have the following more precise estimate on A_0:

$$A_0 = (1 + \mathcal{O}(\delta))\varepsilon^{-1}$$

(the proof of this estimate is similar to that of Lemma 3). According to (29), this yields

$$\eta_1 = \frac{|b_{j_0}|}{b_+}(1 + \mathcal{O}(\delta))\varepsilon^{-1}. \tag{35}$$

To conclude, we have, as in the case $d = 2$, to distinguish two cases.

(i) $|\lambda| \geq \varepsilon$. In this case, write $\varepsilon = s|\lambda|$, $0 < s \leq 1$. We deduce from (33) and (35) that

$$q_\lambda^{-1}\chi = -\sigma\frac{|b_{j_0}|}{b_+}s^{-1}|y_{\lambda,1}|(1 + \dots) \tag{36}$$

and thus

$$\chi = -\sigma\frac{|b_{j_0}|}{b_+}s^{-1+\mathcal{O}(\lambda)}|y_{\lambda,1}|(1 + \dots), \tag{37}$$

where the terms " ... " are small.

Thus, if δ is sufficiently small and c is sufficiently large, we see from (34) and (36) that $|\lambda|^{-1}||\xi|| < c|\chi|$. This shows the existence of an eigendirection in the cone $\mathcal{C}$ for $\psi_\lambda \circ \phi_\lambda$, the corresponding eigenvalue being real. According to (37), the modulus of this eigenvalue is strictly larger than 1, and its sign is the sign of $y_{\lambda,1}$, i.e. the sign of $-\sigma$. In particular, this proves the instability in case 2 of Theorem 2.

(ii) $|\lambda| < \varepsilon$. In this case, write $|\lambda| = t\varepsilon$, $0 < t < 1$. We see from the expression (32) that

$$\chi = 1 - t\sigma\frac{|b_{j_0}|}{b_+}|y_{\lambda,1}|(1 + \dots), \tag{38}$$

where " ... " are small, and from (34) that

$$|\lambda|^{-1}||\xi|| = \mathcal{O}(1) + t(y_{\lambda,1}\,\mathcal{O}(1) + c\,\mathcal{O}(1)). \tag{39}$$

Let us suppose that $-\sigma = 1$ (we are proving the instability in case 1 of Theorem 2). Then, we see that $\chi > 1$ and that, for δ sufficiently small and c sufficiently large, $|\lambda|^{-1}||\xi|| < c|\chi|$. This shows the existence of an eigendirection in the cone $\mathcal{C}$ for $\psi_\lambda \circ \phi_\lambda$, the corresponding eigenvalue being real and strictly larger than 1. This finishes the proof of the instability results when $d \geq 3$.

A stability result. Here we suppose that $-\sigma = 1$ and that $|\lambda| < \varepsilon$, and we still write $|\lambda| = t\varepsilon$, $0 < t < 1$. It remains to prove that, for t sufficiently small, the eigenvalues of $\psi_\lambda \circ \phi_\lambda$ are not larger than 1 in modulus.

We see from (38) and (39) that, for t sufficiently small (depending on δ) and c sufficiently large, the cone $\mathcal{C}$ is still invariant by $\psi_\lambda \circ \phi_\lambda$. Thus, this linear map admits two invariant subspaces E_1 and E_2, with $\mathbf{R}^d = E_1 \oplus E_2$, $\dim E_1 = 1$, $\dim E_2 = d-1$, $E_1 \subset \mathcal{C}$, and $E_2 \subset \mathbf{R}^d \setminus \mathcal{C}$. The eigenvalue corresponding to the eigenspace E_1 is, according to (38) (and for t sufficiently small) between 0 and 1.

Let v be any vector of E_2. Write $v = \begin{pmatrix} \lambda^{-1}\hat{x} \\ \hat{\varphi} \end{pmatrix}$, $\hat{x} \in \mathbf{R}$, $\hat{\varphi} \in \mathbf{R}^{d-1}$, and suppose that $||\hat{\varphi}|| = 1$. Then, as $v \notin \mathcal{C}$, we have $|\hat{x}| \le c^{-1}$. Write

$$\begin{pmatrix} \hat{\chi} \\ \hat{\xi} \end{pmatrix} = \psi_\lambda \circ \phi_\lambda \begin{pmatrix} \lambda^{-1}\hat{x} \\ \hat{\varphi} \end{pmatrix}.$$

We have

$$\hat{\xi} = \Big(\zeta y_\lambda + \lambda^{-1} S_{2,1}(1+\lambda w_\lambda) + S_{2,2} y_\lambda\Big)\hat{x} + (S_{2,2}(\beta + \lambda z_\lambda))\hat{\varphi} + \dots,$$

where "…" denotes terms which are arbitrarily small if ε_0 is sufficiently small. We thus have, for δ sufficiently small (depending on c),

$$||\hat{\xi}|| \le (1 + t y_{\lambda,1}) c^{-1}\, \mathcal{O}(1).$$

In particular, for t sufficiently small (depending on δ) and c sufficiently large, we have $||\hat{\xi}|| < 1$, which shows that all the eigenvalues of $(\psi_\lambda \circ \phi_\lambda)_{|E_2}$ are strictly smaller than 1 in modulus. This proves the desired stability result, and finishes the proof of Theorems 1 and 2 in dimension $d \ge 3$. □

Remark. This method of construction of an invariant cone works all the same in dimension $d = 2$, providing that $b_- < -b_+$. Thus, under the hypotheses of Subsect. 3.2, if $b_- < -b_+$, then we can say that the unstable eigendirection of $\psi_\lambda \circ \phi_\lambda$ (which was proved to exist via estimation of the trace and determinant) is actually close to the horizontal direction (it belongs to a cone $\mathcal{C} = \{(x, y) \in \mathbf{R}^2 \;\big|\; |y| < c|\lambda|\,|x|\}$, for c sufficiently large).

4. Proof in Case (b)

4.1. Setup for the proof. We give ourselves and fix $f_0(.)$, $C_0(.,.)$ as in Subsect. 1.1, in case (b), and we suppose that $\sigma(f_0, u_0, C_0) \neq 0$ and $\sigma(f_0, \tilde{u}_0, C_0) \neq 0$.

We introduce δ, ε_0, $f_1(.)$, $C_1(.,.)$, and ν as in Subsect. 3.1 (see Fig. 10).

Up to a linear change of coordinates, we suppose that $Df_0(0)$ reads $\begin{pmatrix} b_+ & 0 \\ 0 & B_- \end{pmatrix}$. Let Σ_0, the parametrization of $u_0(.)$, Σ, and Σ' be as in Subsect. 3.1. For δ and ε_0 sufficiently small, the intersection $\mathcal{T}_0 \cap \Sigma$ (resp. $\mathcal{T}_0 \cap \Sigma'$, $\tilde{\mathcal{T}}_0 \cap \Sigma$, $\tilde{\mathcal{T}}_0 \cap \Sigma'$) contains exactly one point; denote it by ξ_0 (resp. ξ_0', $\tilde{\xi}_0$, $\tilde{\xi}_0'$); moreover, the intersection $\mathcal{T}_1 \cap \Sigma$ (resp. $\mathcal{T}_1 \cap \Sigma'$) contains exactly two points (see Subsect. 1.1); denote them by ξ_1, $\tilde{\xi}_1$ (resp. by ξ_1', $\tilde{\xi}_1'$), in such a way that $\xi_1 \simeq \xi_0$, $\xi_1' \simeq \xi_0'$, $\tilde{\xi}_1 \simeq \tilde{\xi}_0$, $\tilde{\xi}_1' \simeq \tilde{\xi}_0'$.

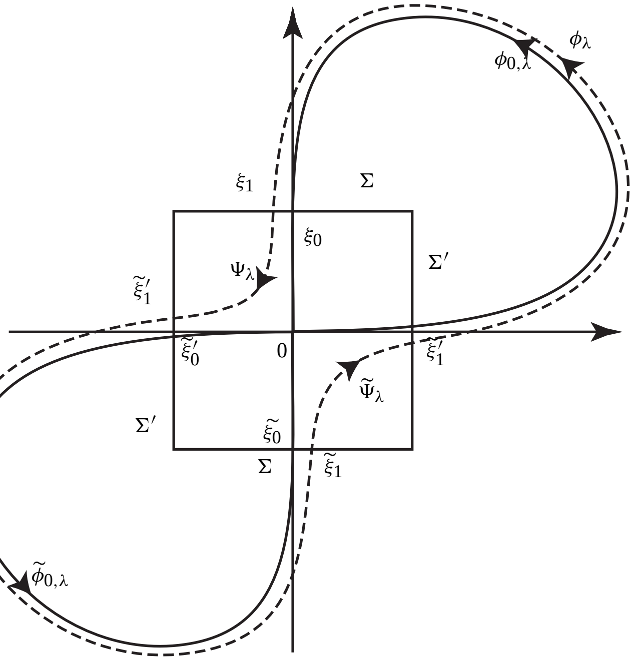


Fig. 10.

Define t_0, t_0', $\tilde{t}_0$, and $\tilde{t}_0'$ by:

$$u_0(t_0) = \xi_0, \quad u_0(t_0') = \xi_0', \quad \tilde{u}_0(\tilde{t}_0) = \tilde{\xi}_0, \quad \text{and} \quad \tilde{u}_0(\tilde{t}_0') = \tilde{\xi}_0'.$$

Denote by T the period of $u_1(.)$, define t_1, t_1', $\tilde{t}_1$, and $\tilde{t}_1'$ by:

$$u_1(t_1) = \xi_1, \quad u_1(t_1') = \xi_1', \quad \tilde{u}_1(\tilde{t}_1) = \tilde{\xi}_1,$$
$$\tilde{u}_1(\tilde{t}_1') = \tilde{\xi}_1', \quad t_1' < 0 < t_1 < \tilde{t}_1' < \tilde{t}_1 < t_1' + T,$$

and write $t_1'' = t_1' + T$. Define μ', ε as in Subsect. 3.1, and define $\tilde{\mu}'$, $\tilde{\varepsilon}$ similarly. Define ϕ_λ and $\phi_{0,\lambda}$ as in Subsect. 3.1, and define $\tilde{\phi}_\lambda$ and $\tilde{\phi}_{0,\lambda}$ similarly.

Let ψ_λ (resp. $\tilde{\psi}_\lambda$) denote the flow of the differential Eq. (9) with $k = 1$, between the times $t = t_1$ and $t = \tilde{t}_1'$ (resp. between the times $t = \tilde{t}_1$ and $t = t_1''$). We adopt the same notations as in in Subsect. 3.1 for ϕ_λ, ψ_λ, $\phi_{0,\lambda}$, and similar notations (with a tilde) for $\tilde{\phi}_\lambda$, $\tilde{\psi}_\lambda$, $\tilde{\phi}_{0,\lambda}$.

The map $\tilde{\psi}_\lambda \circ \tilde{\phi}_\lambda \circ \psi_\lambda \circ \phi_\lambda$ is conjugated to Φ_λ, and our aim is to study its spectral radius.

4.2. Estimates in dimension two. Estimates on ψ_λ are the same as in Subsect. 3.2 (in particular estimates (17) on ζ and (20) on $S_{i,j}$), and similar estimates hold for $\tilde{\psi}_\lambda$. Estimates on β_0 and A_0 are the same as in Subsect. 3.2 (Lemmas 2 and 3) and similar estimates hold for $\tilde{\beta}_0$ and $\tilde{A}_0$. We deduce from the estimates on A_0 and $\tilde{A}_0$ that

$$\eta = -\sigma_{\mathrm{or}}\gamma(1 + \mathcal{O}(\delta))\varepsilon^{-1} \quad \text{and} \quad \tilde{\eta} = -\sigma_{\mathrm{or}}\gamma(1 + \mathcal{O}(\delta))\tilde{\varepsilon}^{-1} \tag{40}$$

(these estimates are similar to estimate (28) of Subsect. 3.2 on η, except that σ_{or} is replaced by $-\sigma_{\text{or}}$).

Denote by T_λ the trace of $\tilde{\psi}_\lambda \circ \tilde{\phi}_\lambda \circ \psi_\lambda \circ \phi_\lambda$. We have

$$T_0 = 1 + \tilde{\zeta}\tilde{\beta}\zeta\beta.$$

According to the expression of $\psi_\lambda \circ \phi_\lambda$ (see Subsect. 3.3) and to estimates on η, ζ, and $S_{i,j}$ of Subsect. 3.2, we have

$$q_\lambda^{-1}\psi_\lambda \circ \phi_\lambda = \begin{pmatrix} 1 + \lambda\eta y_\lambda(1 + \dots) & \eta(\beta + \dots) \\ \lambda y_\lambda\big(\zeta(1 + \dots) + \lambda\eta\,\mathcal{O}(1)\big) + \lambda\,\mathcal{O}(1) & \zeta\beta + \lambda\eta\,\mathcal{O}(1) \end{pmatrix},$$

where the "$\mathcal{O}(1)$" denote quantities which, for ε_0 sufficiently small (depending on δ), are bounded independently of δ, and the "..." denote quantities which are arbitrarily small if δ is sufficiently small and ε_0 is sufficiently small (depending on δ).

A similar expression holds for $\tilde{q}_\lambda^{-1}\tilde{\psi}_\lambda \circ \tilde{\phi}_\lambda$. We thus have

$$\begin{aligned}(q_\lambda\tilde{q}_\lambda)^{-1}T_\lambda &= T_0 + \lambda\eta y_\lambda(1 + \dots) + \lambda\tilde{\eta}\tilde{y}_\lambda(1 + \dots) + \lambda\eta\lambda\tilde{\eta}y_\lambda\tilde{y}_\lambda(1 + \dots) \\ &\quad + \eta(\beta + \dots)\lambda\tilde{y}_\lambda\tilde{\zeta} + \tilde{\eta}(\tilde{\beta} + \dots)\lambda y_\lambda\zeta + \lambda\eta\tilde{\zeta}\tilde{\beta}\,\mathcal{O}(1) + \lambda\tilde{\eta}\zeta\beta\,\mathcal{O}(1),\end{aligned}$$

where the "..." denote quantities which are arbitrarily small if δ is sufficiently small and ε_0 is sufficiently small (depending on δ).

Lemma 6. *We have*

$$a_0 = (1 + \mathcal{O}(\delta))\varepsilon^\gamma.$$

We omit the proof which is very similar to that of Lemma 3.

According to this lemma and to estimate (17) on ζ, we have

$$\zeta = \gamma^2(1 + \mathcal{O}(\delta))\varepsilon^{\gamma-1} = \mathcal{O}(1).$$

As a consequence, in the above expression of $(q_\lambda\tilde{q}_\lambda)^{-1}T_\lambda$, the last two terms can be removed. Now, we once again distinguish several cases.

(i) $\max(\varepsilon, \tilde{\varepsilon}) \le |\lambda|$. In this case, write $\varepsilon = s|\lambda|$, $0 < s \le 1$, and $\tilde{\varepsilon} = \tilde{s}|\lambda|$, $0 < \tilde{s} \le 1$. For δ sufficiently small, the dominant term in T_λ reads, according to (40),

$$(q_\lambda\tilde{q}_\lambda)\lambda\eta\lambda\tilde{\eta}y_\lambda\tilde{y}_\lambda = \gamma^2(1 + \mathcal{O}(\delta))|\lambda|^{\mathcal{O}(\lambda)}s^{-1+\mathcal{O}(\lambda)}\tilde{s}^{-1+\mathcal{O}(\lambda)}y_\lambda\tilde{y}_\lambda.$$

If δ is small, this term is large, thus T_λ is large and has the sign of $y_\lambda\tilde{y}_\lambda$; this proves the desired instability (in particular, this proves the instability in cases 2 and 3 of Theorem 3).

(ii) $\min(\varepsilon, \tilde{\varepsilon}) \le |\lambda| < \max(\varepsilon, \tilde{\varepsilon})$. This situation has to be considered only in case 1 of Theorem 3, called "case (b),1", namely when $\sigma(f_0, u_0, C_0) = \sigma(f_0, \tilde{u}_0, C_0) = -1$. In this case, $\sigma_{\text{or}}y_\lambda < 0$ and $\sigma_{\text{or}}\tilde{y}_\lambda < 0$, and we can see that all the terms in the above expression of $(q_\lambda\tilde{q}_\lambda)^{-1}T_\lambda$ are positive.

Suppose for instance that $\varepsilon \le |\lambda| < \tilde{\varepsilon}$ and write $\varepsilon = s|\lambda|$, $0 < s \le 1$, and $\tilde{\varepsilon} = \tilde{s}|\lambda|$, $1 < \tilde{s}$. Then the term $\lambda\eta y_\lambda(1 + \dots)$ is large, and, as the other terms are positive, we find, according to (40),

$$T_\lambda \ge \tilde{q}_\lambda(q_\lambda\lambda\eta y_\lambda)(1 + \dots) = \tilde{\varepsilon}^{\mathcal{O}(\lambda)}\Big(\gamma(1 + \mathcal{O}(\delta))|\lambda|^{\mathcal{O}(\lambda)}s^{-1+\mathcal{O}(\lambda)}|y_\lambda|\Big)(1 + \dots).$$

As $|\lambda| < \tilde{\varepsilon}$, we have $\tilde{\varepsilon}^{\mathcal{O}(\lambda)} = \tilde{\varepsilon}^{\mathcal{O}(\tilde{\varepsilon})} \simeq 1$, thus T_λ is large positive, which proves the desired instability.

(iii) $|\lambda| < \min(\varepsilon, \tilde{\varepsilon})$. In this case, write $|\lambda| = t\varepsilon$, $0 < t < 1$, and $|\lambda| = \tilde{t}\tilde{\varepsilon}$, $0 < \tilde{t} < 1$. It remains to prove the instability in case 1 of Theorem 3 and the stability result in case 2 of Theorem 3. In these two cases, $\sigma(f_0, u_0, C_0)$ and $\sigma(f_0, \tilde{u}_0, C_0)$ have the same sign, and, equivalently, for δ sufficiently small, y_λ and $\tilde{y}_\lambda$ have the same sign.

Write $T_\lambda = T_0 + tT'_\lambda + \tilde{t}\tilde{T}'_\lambda + t\tilde{t}T''_\lambda$. We have $q_\lambda \tilde{q}_\lambda = 1 + t\,\mathcal{O}(\varepsilon)\log\varepsilon + \tilde{t}\,\mathcal{O}(\tilde{\varepsilon})\log\tilde{\varepsilon}$, and thus

$$\begin{aligned}
T'_\lambda &= \sigma_{\mathrm{or}}\gamma\Big(y_\lambda(1+\dots) + \tilde{\zeta}\,\tilde{y}_\lambda(1+\dots)\Big),\\
\tilde{T}'_\lambda &= \sigma_{\mathrm{or}}\gamma\Big(\tilde{y}_\lambda(1+\dots) + \zeta\, y_\lambda(1+\dots)\Big),\\
T''_\lambda &= \gamma^2 y_\lambda \tilde{y}_\lambda(1+\dots),
\end{aligned}$$

where the terms " ... " are small. As y_λ and $\tilde{y}_\lambda$ have the same sign, we see that T'_λ, $\tilde{T}'_\lambda$, and T''_λ are arbitrarily large in modulus if δ is sufficiently small.

Denote by D_λ the determinant of $\tilde{\psi}_\lambda \circ \tilde{\phi}_\lambda \circ \psi_\lambda \circ \phi_\lambda$, and write $D_\lambda = D_0 + tD'_\lambda + \tilde{t}\tilde{D}'_\lambda$. Proceeding as in Subsect. 3.2, we see that D'_λ and $\tilde{D}'_\lambda$ are arbitrarily small if ε_0 is sufficiently small (depending on δ).

Write $\Delta_\lambda = T_\lambda^2 - 4D_\lambda$, and, as in Subsect. 3.2, if $\Delta_\lambda \geq 0$, let

$$m_\lambda = 1 + \frac{1}{2}\big(T_\lambda - T_0 + \sqrt{\Delta_\lambda} - \sqrt{\Delta_0}\big).$$

Let us conclude. If $\sigma_{\mathrm{or}} y_\lambda > 0$ and $\sigma_{\mathrm{or}}\tilde{y}_\lambda > 0$, then we see that T'_λ, $\tilde{T}'_\lambda$, and T''_λ are all large positive, thus $T_\lambda > T_0$ and $\Delta_\lambda > \Delta_0 \geq 0$, and finally $m_\lambda > 1$. This finishes the proof of the instability result in case 1 of Theorem 3.

If on the other hand $\sigma_{\mathrm{or}} y_\lambda < 0$ and $\sigma_{\mathrm{or}}\tilde{y}_\lambda < 0$, then we see that, for t and $\tilde{t}$ sufficiently small (depending on δ), the term $t\tilde{t}T''_\lambda$ is dominated by $tT'_\lambda + \tilde{t}\tilde{T}'_\lambda$, which is negative. Thus we see $T_\lambda < T_0$ and $\Delta_\lambda < \Delta_0$, and thus that, if $\Delta_\lambda \geq 0$, then $m_\lambda < 1$. This proves the stability result in case 2 of Theorem 3.

The proof in dimension 2 of Theorem 3 (and thus of Theorem 1) is complete. □

4.3. Estimates in dimension higher than two. Estimates on ψ_λ (in particular on ζ, $s_{i,j}$, A_0) are the same as in Subsect. 3.3, and similar estimates hold for $\tilde{\psi}_\lambda$. We deduce from the estimates on A_0 and $\tilde{A}_0$ that

$$\eta_1 = -\frac{|b_{j_0}|}{b_+}(1+\mathcal{O}(\delta))\varepsilon^{-1} \quad \text{and} \quad \tilde{\eta}_1 = -\frac{|b_{j_0}|}{b_+}(1+\mathcal{O}(\delta))\tilde{\varepsilon}^{-1}.$$

As in Subsect. 3.3, let $\mathcal{C} = \{(x, y) \in \mathbf{R} \times \mathbf{R}^{d-1} \mid ||y|| < c|\lambda|\,|x|\}$, where c is a large constant to be chosen. Let us denote $\sigma(f_0, u_0, C_0)$ by σ and $\sigma(f_0, \tilde{u}_0, C_0)$ by $\tilde{\sigma}$. Then, proceeding as in Subsect. 3.3, we obtain that, if c is sufficiently large, δ sufficiently small (depending on c), and ε_0 sufficiently small (depending on δ), in the three following cases:

(i) $|\lambda| \geq \max(\varepsilon, \tilde{\varepsilon})$,
(ii) $|\lambda| < \max(\varepsilon, \tilde{\varepsilon})$ and $\sigma = 1$ and $\tilde{\sigma} = 1$,
(iii) $|\lambda| < \tau\min(\varepsilon, \tilde{\varepsilon})$, where τ is a small constant (depending on δ), and $\sigma = -1$ and $\tilde{\sigma} = -1$,

the cone $\mathcal{C}$ is invariant by $\psi_\lambda \circ \phi_\lambda$ and by $\tilde{\psi}_\lambda \circ \tilde{\phi}_\lambda$. Thus, it is also invariant by the composition $\tilde{\psi}_\lambda \circ \tilde{\phi}_\lambda \circ \psi_\lambda \circ \phi_\lambda$, which shows the existence of an eigendirection in the cone $\mathcal{C}$ for this map, the corresponding eigenvalue being real. Proceeding as in Subsect. 3.3, we obtain that, in cases (i) and (ii) above, this eigenvalue is strictly larger than 1 in modulus, and has the sign of $\sigma\tilde{\sigma}$ (this proves the instability results); in case (iii), if τ is small enough, this eigenvalue belongs to]0; 1[, and we can show as in Subsect. 3.3 that the other eigenvalues are smaller than 1 in modulus (this proves the stability result).

This finishes the proof of Theorem 3. $\square$

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