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A DIRECT COMPUTATION OF THE PHASE STABILITY CRITERION FOR SPATIALLY HOMOGENEOUS TIME PERIODIC SOLUTIONS CLOSE TO A HOPF BIFURCATION

EMMANUEL RISLER
Institut Non Linéaire de Nice, UMR CNRS-UNSA 6618,
1361 route des Lucioles, F-06560 Valbonne, France

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We provide an elementary direct computation of the classical criterion governing the (linear) phase stability under inhomogeneous perturbations of a spatially homogeneous periodic orbit close to a spatially homogeneous supercritical Hopf bifurcation.

1. Introduction

We consider PDEs of the form

$$\partial_t u = F_\mu(u, \partial_x)\,, \tag{1}$$

i.e. invariant with respect to translations of time (autonomous) and space, and depending on a real parameter μ. We suppose that u is in $\mathbf{R}^d$, $d \geq 1$, and that the space coordinate x belongs to $\mathbf{R}^n$, $n \geq 1$, or to a domain of $\mathbf{R}^n$ with boundary conditions of type Neumann or periodic. Spatially homogeneous solutions of this PDE are solutions of the equation

$$\frac{du}{dt} = F_\mu(u, 0) = f_\mu(u) \tag{2}$$

(we write $f_\mu(u)$ for $F_\mu(u, 0)$), which is an autonomous ordinary differential equation in dimension d.

Among the solutions of Eq. (2), of prime interest are those which correspond to an asymptotic behavior, in particular attractive fixed points and attractive periodic orbits. The corresponding homogeneous solutions for the PDE (1) are thus stable with respect to homogeneous perturbations but might be unstable with respect to inhomogeneous perturbations; this is at the origin of many phenomena displaying "patterns" or "spatiotemporal chaos" in nonlinear physics [Cross & Hohenberg, 1993].

In particular, an attractive periodic orbit might present the well-known Kuramoto phase instability [Benjamin & Feir, 1967; Newell, 1974; Yamada & Kuramoto, 1976]. Moreover, if this periodic orbit is born through a supercritical Hopf bifurcation, then, close to the threshold, the occurrence of the phase instability is governed by a criterion which can be explicitly computed, and which is currently known as the "Benjamin–Feir" criterion. Classically [Benjamin & Feir, 1967; Newell, 1974; Yamada & Kuramoto, 1976], this criterion is derived through an amplitude equation and expressed in terms of the coefficients of this equation.

Usually, such amplitude equations are derived using scaling and averaging arguments, but their rigorous justification is not a easy task (see [Collet, 1998] and references therein). In this paper, we show how to recover this criterion directly from the starting partial differential equation, by a perturbative computation of the critical Floquet exponent. Although elementary and to some extent classical, this computation is not easy to find out in the literature, buried in papers which are technically difficult and devoted to specific problems, often issued from hydrodynamics [Stuart & Di Prima, 1978; Bridges & Mielke, 1995]. We hope to provide here an easy access to a simple and rigorous justification of this celebrated criterion.

Notations: For $n \in \mathbf{N}^*$, we denote by $\mathcal{M}_n(\mathbf{R})$ the space of $n \times n$-real matrices and by $\mathrm{Id}_{\mathbf{R}^n}$ the identity map of $\mathbf{R}^n$. For α in $\mathbf{R}$ or in $\mathbf{R}/2\pi\mathbf{Z}$, we denote by Rot_α the rotation of angle α in $\mathbf{R}^2$.

2. Setup and Statement of the Result

2.1. *Hopf bifurcation*

We keep the previous notations. We suppose that the map: $\mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$, $(\mu, u) \mapsto f_\mu(u)$ is smooth, and we denote by 0 the origin $(0, \ldots, 0)$ of $\mathbf{R}^d$. We suppose that $d \geq 2$, that $f_0(0) = 0$, and that $Df_0(0)$ has two simple eigenvalues $\pm i\omega$, $\omega > 0$, all its other eigenvalues having strictly negative real part. Up to a linear change of variables, we can suppose that the matrix of $Df_0(0)$ reads

$$\begin{pmatrix} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} & 0 \\ 0 & L \end{pmatrix}$$

where $L \in \mathcal{M}_{d-2}(\mathbf{R})$ and its spectrum lies in the open left half-plane.

According to center manifold and normal form arguments (see e.g. [Iooss & Adelmeyer, 1992]), there exists a neighborhood I of 0 in $\mathbf{R}$, a neighborhood U of 0 in $\mathbf{R}^d$, and a smooth map $I \times U \to \mathbf{R}^d$, $(\mu, u) \mapsto h(\mu, u) = h_\mu(u)$, with the following properties (see Fig. 1):

1. $h_0(0) = 0$ and $Dh_0(0) = \mathrm{Id}_{\mathbf{R}^d}$;
2. for $\mu \in I$, the local diffeomorphism h_μ maps a center manifold of f_μ to the subspace $\mathbf{R}^2 \times \{(0, \ldots, 0)\}$. More precisely, for $v \in h_\mu(U)$, let $\tilde{f}_\mu(v) = Dh_\mu(h_\mu^{-1}(v)) f_\mu(h_\mu^{-1}(.))$; then, for any $v \in h_\mu(U) \cap (\mathbf{R}^2 \times \{(0, \ldots, 0)\})$, $\tilde{f}_\mu(v) \in \mathbf{R}^2 \times \{(0, \ldots, 0)\}$;
3. denote by $\tilde{g}_\mu(.)$ the smooth map defined on a neighborhood of 0 in $\mathbf{R}^2$, with values in $\mathbf{R}^2$, as the restriction of $\tilde{f}_\mu(.)$ to $\mathbf{R}^2 \times \{(0, \ldots, 0)\}$; then, in complex coordinates $Z = x + iy$ of $\mathbf{R}^2 \simeq \mathbf{C}$, $\tilde{g}_\mu$ reads:

$$\tilde{g}_\mu(Z) = i\omega Z + Q(|Z|^2, \mu)Z + R(Z, \overline{Z}, \mu), \qquad (3)$$

where $Q(|Z|^2, \mu)$ is a complex degree one polynomial in $|Z|^2$, depending smoothly on μ, $R(., ., .)$ is a smooth function, and $R(Z, \overline{Z}, \mu) = \mathcal{O}(|Z|^5)$.

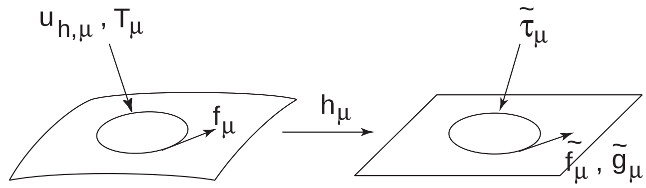

Fig. 1. Central manifold reduction.

As $Q(0, 0) = 0$, we can write: $Q(|Z|^2, \mu) = \gamma(\mu)\mu + \delta(\mu)|Z|^2$. Let us write $\gamma(\mu) = \gamma_r(\mu) + i\gamma_i(\mu)$ (real and imaginary parts of $\gamma(\mu)$) and $\delta(\mu) = \delta_r(\mu) + i\delta_i(\mu)$. In polar coordinates (r, θ), $Z = re^{i\theta}$, the differential equation $dZ/dt = \tilde{g}_\mu(Z)$ reads

$$\frac{dr}{dt} = \mu r \gamma_r(\mu) + r^3 \delta_r(\mu) + \mathcal{O}(r^5) \qquad (4)$$

$$\frac{d\theta}{dt} = \omega + \mu\gamma_i(\mu) + r^2\delta_i(\mu) + \mathcal{O}(r^4)\,. \qquad (5)$$

Generically, $\gamma_r(0) \neq 0$ and $\delta_r(0) \neq 0$; in this case, when the parameter μ is small and has the sign of $-\delta_r(0)/\gamma_r(0)$, this differential equation admits in a small uniform neighborhood of 0 in $\mathbf{R}^2$ a unique periodic orbit close to the circle of center 0 and radius $\sqrt{-\mu[\gamma_r(0)/\delta_r(0)]}$ (see [Iooss & Adelmeyer, 1992]). These periodic orbits are attractive if $\mu\gamma_r(0) > 0$, and repulsive otherwise. We suppose that they are attractive (the Hopf bifurcation is called *supercritical* in this case) and that they arise for $\mu > 0$. Finally, the hypotheses are: $\gamma_r(0) > 0$ and $\delta_r(0) < 0$.

For small $\mu > 0$, denote by $\tilde{\tau}_\mu$ the trajectory (in $\mathbf{R}^2$) of the above mentioned periodic orbit, and let $\mathcal{T}_\mu \subset \mathbf{R}^d$ denote the trajectory of the corresponding periodic orbit for f_μ (i.e. $\mathcal{T}_\mu = h_\mu^{-1}(\tilde{\tau}_\mu \times \{(0, \ldots, 0)\})$). Let $t \mapsto u_{h,\mu}(t)$ denote a solution of the differential equation (2) having the trajectory $\mathcal{T}_\mu$ (see Fig. 1).

2.2. *Reduction of the linear stability problem*

A small inhomogeneous perturbation $u(x, t)$ of $u_{h,\mu}(t)$ formally obeys at first order the linear equation

$$\partial_t u = DF_\mu(u_{h,\mu}(t), \partial_x)u\,,$$

which reduces in Fourier coordinates to

$$\partial_t \hat{u}(\mathbf{k}) = DF_\mu(u_{h,\mu}(t), i\mathbf{k})\hat{u}(\mathbf{k})$$

which is just an ordinary differential equation parametrized by $\mathbf{k}$. We make the supplementary hypothesis of space isotropy. The preceding equation thus only depends on $|\mathbf{k}|^2$, and can be rewritten

$$\partial_t \hat{u}(\mathbf{k}) = (Df_\mu(u_{h,\mu}(t)) + \mathcal{C}(u_{h,\mu}(t), -|\mathbf{k}|^2))\hat{u}(\mathbf{k}),$$

where $\mathcal{C} : \mathbf{R}^d \times \mathbf{R} \to \mathcal{L}(\mathbf{R}^d)$ satisfies $\mathcal{C}(., 0) \equiv 0$ (we denote by $\mathcal{L}(\mathbf{R}^d)$ the space of linear maps: $\mathbf{R}^d \to \mathbf{R}^d$). Thus we can write: $\mathcal{C}(u, \lambda) = \lambda C(u, \lambda)$, where the map $C : \mathbf{R}^d \times \mathbf{R} \to \mathcal{L}(\mathbf{R}^d)$ is regular.

In the following, we will forget about the exact nature of the PDE (1), and just consider the ordinary differential equation

$$\frac{du}{dt} = (Df_\mu(u_{h,\mu}(t)) + \lambda C(u_{h,\mu}(t), \lambda))u, \qquad (6)$$

depending on the parameter λ (which corresponds to $-|\mathbf{k}|^2$, and should therefore be thought of as negative).

For $\lambda \in \mathbf{R}$, denote by $\mathcal{F}_{\mu,\lambda}$ the (linear) flow over one period of $u_{h,\mu}$ of this differential equation, and denote by $\rho(\mathcal{F}_{\mu,\lambda})$ the spectral radius of $\mathcal{F}_{\mu,\lambda}$. We know that 1 is always an eigenvalue of $\mathcal{F}_{\mu,0}$ (the "neutral" Floquet multiplier in the direction of the flow). Thus, even for values of λ arbitrarily close to 0, the eigenvalue 1 of $\mathcal{F}_{\mu,0}$ may become larger than 1, which yields the phase instability.

2.3. *Statement of the result*

Write $\alpha = -2\gamma_r(0)[\delta_i(0)/\delta_r(0)]$ and $\beta = -2\gamma_r(0)$ (remark that $\beta < 0$).

Denote by $(C_{i,j})_{1\le i,j\le d}$ the coefficients of the matrix $C(0, 0)$ (where the first 0 denotes the origin in $\mathbf{R}^d$), and write $D = (C_{1,1} + C_{2,2})/2$, $P = (C_{2,1} - C_{1,2})/2$, and

$$\mathcal{B} = -\alpha P - \beta D.$$

To avoid complications, we make the hypothesis that $D \neq 0$.

The sign of $\mathcal{B}$ governs the occurrence of the phase instability:

Theorem 1. *If* $\mathcal{B} > 0$ (*resp.* $\mathcal{B} < 0$), *then, for* $\mu > 0$ *sufficiently small and* $\lambda < 0$ *sufficiently small* (*depending on* μ), *we have* $\rho(\mathcal{F}_{\mu,\lambda}) < 1$ (*resp.* $\rho(\mathcal{F}_{\mu,\lambda}) > 1$).

More precisely:

Theorem 2. *There exists* (*see Fig. 2*):

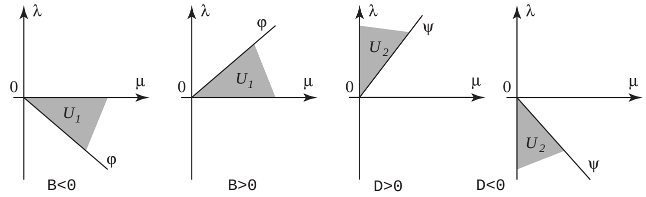

Fig. 2. Illustration of Theorem 2; areas in gray correspond to parameters for which instability occurs.

- $\varepsilon > 0$ (*write* $I =]-\varepsilon; \varepsilon[$ *and* $J =]0; \varepsilon[$);
- *a smooth function* $\varphi : I \to \mathbf{R}$ *satisfying*: $\varphi(\nu) = (\mathcal{B}/(D^2 + P^2))\nu^2 + \mathcal{O}(\nu^3)$;
- *a smooth function* $\psi : I \to \mathbf{R}$ *satisfying*: $\psi(\nu) = (-\beta/2D)\nu^2 + \mathcal{O}(\nu^3)$;

such that, with the following notations:

- $\mathcal{U}_1 = \{(\mu, \lambda) \in J \times I$ *s.t.* $\min(\varphi(\sqrt{\mu}), 0) < \lambda < \max(\varphi(\sqrt{\mu}), 0)\}$;
- *if* $D > 0$, $\mathcal{U}_2 = \{(\mu, \lambda) \in J \times I$ *s.t.* $\lambda > \psi(\sqrt{\mu})\}$ (*in this case we have* $\psi(\sqrt{\mu}) > 0$); *if* $D < 0$, $\mathcal{U}_2 = \{(\mu, \lambda) \in J \times I$ *s.t.* $\lambda < \psi(\sqrt{\mu})\}$ (*in this case we have* $\psi(\sqrt{\mu}) < 0$);
- $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, $\mathcal{S} = (J \times I)\backslash\overline{\mathcal{U}}$, *and* $\mathcal{N} = (J \times I)\backslash(\mathcal{S} \cup \mathcal{U})$,

for $(\mu, \lambda) \in \mathcal{S}$ (*resp.* $\mathcal{N}$, $\mathcal{U}$), *we have* $\rho(\mathcal{F}_{\mu,\lambda}) < 1$ (*resp.* $\rho(\mathcal{F}_{\mu,\lambda}) = 1$, $\rho(\mathcal{F}_{\mu,\lambda}) > 1$).

3. Proofs

In what follows, we denote by K an interval $]-\varepsilon; \varepsilon[$, where $\varepsilon > 0$ is as small as necessary. Let $\tilde{\tau}_0 = \{(0, 0)\}$ and $\mathcal{T}_0 = \{(0, \ldots, 0)\}$ (in $\mathbf{R}^d$).

First we have to reparametrize by $\pm\sqrt{\mu}$ to eliminate the singularity at $\mu = 0$. It is known by an implicit function argument (see e.g. [Iooss & Adelmeyer, 1992]) that there exists a smooth function $m : K \to \mathbf{R}$, $m(\rho) = -(\delta_r(0)/\gamma_r(0))\rho^2 + \mathcal{O}(\rho^3) \geq 0$, such that, for any $\rho \in K$, the point $(\rho, 0)$ belongs to $\tilde{\tau}_{m(\rho)}$ (see Fig. 3).

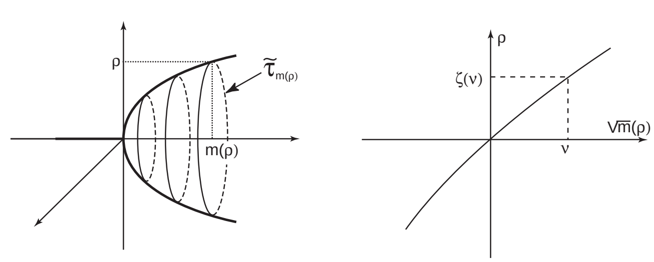

Fig. 3. Hopf bifurcation and reparametrization by ν instead of μ.

Write $m(\rho) = -(\delta_r(0)/\gamma_r(0))\rho^2(1+\Delta m(\rho))$, and denote by $\sqrt{m}$ the map $\rho \mapsto \sqrt{-(\delta_r(0)/\gamma_r(0))}\rho\sqrt{1+\Delta m(\rho)}$; this map is invertible in a neighborhood of 0; denote by ζ its inverse (thus we have $m(\rho) = (\zeta^{-1}(\rho))^2$). Then, for any $\nu \in K$, the point $(\zeta(\nu), 0)$ (which depends smoothly on ν) belongs to $\tilde{\tau}_{\nu^2}$; in the following, the bifurcation parameter μ will correspond to ν^2. For $\mu > 0$ small, denote by $T(\mu)$ the period of the periodic orbit of trajectory $\tilde{\tau}_\mu$ (or $\mathcal{T}_\mu$), and write: $T(0) = 2\pi/\omega$. According to the differential equation (5), the map $\nu \mapsto T(\nu^2)$ is smooth on K.

For $\nu \in K$, write $z(\nu) = h_{\nu^2}^{-1}(\zeta(\nu), 0, \ldots, 0)$; this point belongs to $\mathcal{T}_{\nu^2}$. Let $t \mapsto v_{h,\nu}(t)$ denote the solution of the differential equation $du/dt = f_{\nu^2}(u)$ with initial condition $z(\nu)$ at $t = 0$; this solution is periodic (or fixed if $\nu = 0$), its trajectory is $\mathcal{T}_{\nu^2}$ and its period is $T(\nu^2)$ (if $\nu \neq 0$), and $v_{h,\nu}(t)$ depends smoothly on ν and t. Denote by $\Phi_{\nu,\lambda}$ the flow of the differential equation

$$\frac{du}{dt} = (Df_{\nu^2}(v_{h,\nu}(t)) + \lambda C(v_{h,\nu}(t), \lambda))u$$

between $t = 0$ and $t = T(\nu^2)$ (up to a linear conjugacy, $\Phi_{\nu,\lambda}$ equals $\mathcal{F}_{\nu^2,\lambda}$). According to the previous smoothness assertions, $\Phi_{\nu,\lambda}$ depends smoothly on (ν, λ) on $K \times \mathbf{R}$.

Write $M_0 = Df_0(0)$. We have

$$\Phi_{0,0} = \exp\left(\frac{2\pi}{\omega}M_0\right) = \begin{pmatrix} \mathrm{Id}_{\mathbf{R}^2} & 0 \\ 0 & e^{\frac{2\pi}{\omega}L} \end{pmatrix}.$$

Thus, for $(\nu, \lambda) \simeq (0, 0)$, $\Phi_{\nu,\lambda}$ has exactly two eigenvalues (counted with multiplicity) close to 1; let $E_{\nu,\lambda}$ denote the two-dimensional subspace of $\mathbf{R}^d$, invariant by $\Phi_{\nu,\lambda}$, corresponding to these two eigenvalues, and $P_{\nu,\lambda}$ denote the projector on $E_{\nu,\lambda}$ along $\{(0, 0)\} \times \mathbf{R}^{d-2}$. We have: $P_{0,0} = \begin{pmatrix} \mathrm{Id}_{\mathbf{R}^2} & 0 \\ 0 & 0 \end{pmatrix}$ and $P_{\nu,\lambda} = \begin{pmatrix} \mathrm{Id}_{\mathbf{R}^2} & 0 \\ * & 0 \end{pmatrix}$, and we know (see [Kato, 1980]) that $P_{\nu,\lambda}$ depends smoothly on (ν, λ). Write

$$P_{0,0}\Phi_{\nu,\lambda}P_{\nu,\lambda} = \begin{pmatrix} \phi_{\nu,\lambda} & 0 \\ 0 & 0 \end{pmatrix}.$$

The matrix $\phi_{\nu,\lambda} \in \mathcal{M}_2(\mathbf{R})$ defined this way is conjugated to the restriction of $\Phi_{\nu,\lambda}$ to $E_{\nu,\lambda}$. The stability of $\Phi_{\nu,\lambda}$ thus reduces, for (ν, λ) close to $(0,0)$, to the stability of $\phi_{\nu,\lambda}$.

We have $\phi_{0,0} = \mathrm{Id}_{\mathbf{R}^2}$. Write $\phi_{\nu,\lambda} = \mathrm{Id}_{\mathbf{R}^2} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (a, b, c and d are smooth functions of (ν, λ)) and write $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (2\pi/\omega)\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$. We now have to estimate these functions.

Lemma 1. *We have*

$$\tilde{a} = \beta\nu^2 + D\lambda + \cdots \qquad \tilde{b} = -P\lambda + \cdots$$
$$\tilde{c} = \alpha\nu^2 + P\lambda + \cdots \qquad \tilde{d} = D\lambda + \cdots$$

where the "$\cdots$" mean higher order terms, i.e. $\mathcal{O}(\nu^3) + \mathcal{O}(\nu\lambda) + \mathcal{O}(\lambda^2)$.

Proof. Let us first estimate $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ when $\nu = 0$ and $\lambda \neq 0$. Denote by $\Psi_{0,\lambda}(t)$ the flow between the times 0 and t of the differential equation

$$\frac{du}{dt} = (M_0 + \lambda C(0, \lambda))u\,.$$

We have $\Psi_{0,\lambda}(T(0)) = \Phi_{0,\lambda}$. Write $\Psi_{0,\lambda}(t) = e^{tM_0}(\mathrm{Id}_{\mathbf{R}^d} + \lambda\psi_\lambda(t))$; then we have $\psi_\lambda(0) = 0$ and

$$\frac{d\psi_\lambda}{dt} = e^{-tM_0}C(0, \lambda)e^{tM_0}(\mathrm{Id}_{\mathbf{R}^d} + \lambda\psi_\lambda(t))\,,$$

which shows that

$$\Phi_{0,\lambda} = \Phi_{0,0}\left(\mathrm{Id}_{\mathbf{R}^d} + \lambda\int_0^{T(0)} e^{-tM_0}C(0, 0)e^{tM_0}dt + \mathcal{O}(\lambda^2)\right).$$

Moreover, as $e^{tM_0} = \begin{pmatrix} \mathrm{Rot}_{\omega t} & 0 \\ 0 & e^{tL} \end{pmatrix}$, we have

$$\int_0^{T(0)} e^{-tM_0}C(0, 0)e^{tM_0}dt = \begin{pmatrix} \begin{pmatrix} D & -P \\ P & D \end{pmatrix} & * \\ * & * \end{pmatrix}.$$

Thus,

$$P_{0,0}\Phi_{0,\lambda}P_{0,\lambda} = P_{0,0}\Phi_{0,0}P_{0,\lambda} + \lambda P_{0,0}\Phi_{0,0}\begin{pmatrix} \begin{pmatrix} D & -P \\ P & D \end{pmatrix} & * \\ * & * \end{pmatrix}P_{0,\lambda} + \mathcal{O}(\lambda^2)\,,$$

and finally, according to the expressions of $P_{0,0}$ and $P_{0,\lambda}$,

$$\phi_{0,\lambda} = \mathrm{Id}_{\mathbf{R}^2} + \lambda\begin{pmatrix} D & -P \\ P & D \end{pmatrix} + \mathcal{O}(\lambda^2)\,. \qquad (7)$$

We now have to estimate the functions $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ when $\lambda = 0$ and $\nu \neq 0$. For $\nu \in K$ and

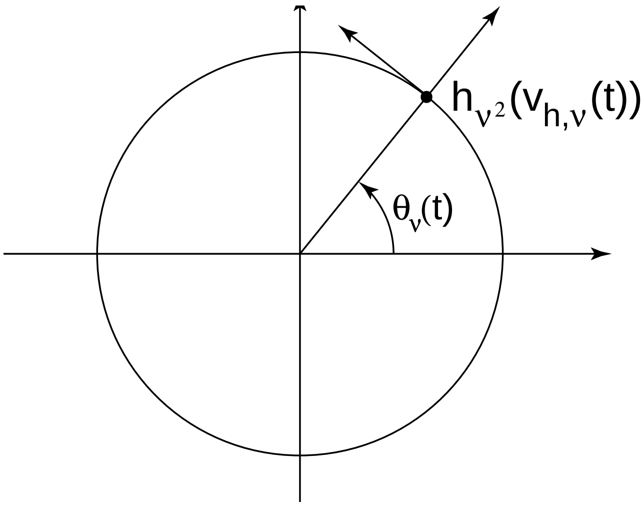

Fig. 4. Definition of the angle $\theta_\nu(t)$.

$t \in \mathbf{R}$, let $\tilde{M}_\nu(t) = D\tilde{f}_{\nu^2}(h_{\nu^2}(v_{h,\nu}(t)))$ and let $H_\nu(t) = Dh_{\nu^2}(v_{h,\nu}(t))$. Under the change of variables $v = H_\nu(t)u$, the differential equation $du/dt = Df_{\nu^2}(v_{h,\nu}(t))u$ becomes

$$\frac{dv}{dt} = \tilde{M}_\nu(t)v\,. \tag{8}$$

Let $\tilde{\Phi}_\nu$ denote the flow between $t = 0$ and $t = T(\nu^2)$ of (8).

For $t \in \mathbf{R}$ and $\nu \neq 0$, denote by $\theta_\nu(t)$ the polar angle defined by the point $h_{\nu^2}(v_{h,\nu}(t))$ (see Fig. 4), and write $\theta_0(t) = \omega t$. Write

$$P_\nu(t) = \begin{pmatrix} \mathrm{Rot}_{\theta_\nu(t)} & 0 \\ 0 & \mathrm{Id}_{\mathbf{R}^{d-2}} \end{pmatrix}.$$

Under the change of variables $v = P_\nu(t)w$, the differential equation (8) becomes

$$\frac{dw}{dt} = \hat{M}_\nu(t)w$$

(this defines $\hat{M}_\nu(t)$). As $\theta_\nu(0) = 0$ or π, the flow of this differential equation between $t = 0$ and $t = T(\nu^2)$ is equal to $\tilde{\Phi}_\nu$.

Differentiating Eqs. (4) and (5) at a point of $\tilde{\tau}_\nu$ yields (writing $r = r_0 + r'$ and $\theta = \theta_0 + \theta'$)

$$\frac{dr'}{dt} = (-2\gamma_r(0)\mu + \mathcal{O}(\mu^2))r' + \mathcal{O}(\mu^2)\theta'$$

$$\frac{d\theta'}{dt} = \sqrt{\mu\frac{\gamma_r(0)}{-\delta_r(0)}}(2\delta_i(0) + \mathcal{O}(\mu))r' + \mathcal{O}(\mu^2)\theta'\,.$$

This shows that

$$\hat{M}_\nu(t) = \begin{pmatrix} \begin{pmatrix} -2\gamma_r(0)\nu^2 + \mathcal{O}(\nu^4) & \mathcal{O}(\nu^3) \\ -2\gamma_r(0)\dfrac{\delta_i(0)}{\delta_r(0)}\nu^2 + \mathcal{O}(\nu^4) & \mathcal{O}(\nu^4) \end{pmatrix} & * \\ 0 & * \end{pmatrix}$$

and thus that

$$\tilde{\Phi}_\nu = \begin{pmatrix} \mathrm{Id}_{\mathbf{R}^2} + \dfrac{2\pi}{\omega}\begin{pmatrix} \beta\nu^2 + \mathcal{O}(\nu^4) & \mathcal{O}(\nu^3) \\ \alpha\nu^2 + \mathcal{O}(\nu^4) & \mathcal{O}(\nu^4) \end{pmatrix} & * \\ 0 & * \end{pmatrix}.$$

We have $\Phi_{\nu,0} = H_\nu(0)^{-1}\tilde{\Phi}_\nu H_\nu(0)$ and we see that $E_{\nu,0} = H_\nu(0)^{-1}(\mathbf{R}^2 \times \{(0,\ldots,0)\})$. As $H_\nu(0) = \mathrm{Id}_{\mathbf{R}^d} + \mathcal{O}(\nu)$, this shows that the restriction of $\Phi_{\nu,0}$ to $E_{\nu,0}$ is conjugated to a matrix of the form: $\mathrm{Id}_{\mathbf{R}^2} + (2\pi/\omega)\left(\begin{smallmatrix} \beta\nu^2+\mathcal{O}(\nu^4) & \mathcal{O}(\nu^3) \\ \alpha\nu^2+\mathcal{O}(\nu^4) & \mathcal{O}(\nu^4) \end{smallmatrix}\right)$ by a conjugacy of the form: $\mathrm{Id}_{\mathbf{R}^2} + \mathcal{O}(\nu)$. Thus, the same is true for $\phi_{\nu,0}$ and we obtain

$$\phi_{\nu,0} = \mathrm{Id}_{\mathbf{R}^2} + \frac{2\pi}{\omega}\begin{pmatrix} \beta\nu^2 + \mathcal{O}(\nu^3) & \mathcal{O}(\nu^3) \\ \alpha\nu^2 + \mathcal{O}(\nu^3) & \mathcal{O}(\nu^3) \end{pmatrix}. \tag{9}$$

The lemma follows from (7) and (9). ■

We continue the proof of Theorems 1 and 2. Let $\rho(\phi_{\nu,\lambda})$ denote the spectral radius of $\phi_{\nu,\lambda}$.

Claim. *For (ν, λ) sufficiently close to $(0, 0)$, we have*

- $\rho(\phi_{\nu,\lambda}) > 1 \Leftrightarrow ad - bc < 0$ *or* $ad - bc > -(a + d)$;
- $\rho(\phi_{\nu,\lambda}) < 1 \Leftrightarrow 0 < ad - bc < -(a + d)$;
- $\rho(\phi_{\nu,\lambda}) = 1 \Leftrightarrow 0 = ad - bc \leq -(a + d)$ *or* $0 \leq ad - bc = -(a + d)$.

Indeed, the characteristic polynomial of $\phi_{\nu,\lambda}$ reads

$$\chi(x) = x^2 - (2 + a + d)x + (1 + (a + d) + (ad - bc))\,,$$

and its discriminant reads $\Delta = (a + d)^2 - 4(ad - bc)$.

- if $ad - bc > -(a + d)$, then $\det\phi_{\nu,\lambda} > 1$ and thus $\rho(\phi_{\nu,\lambda}) > 1$;
- if $ad - bc < 0$, then $\Delta > 0$ and the eigenvalues of $\phi_{\nu,\lambda}$ are real and equal to $1 + ((a + d)/2) \pm (\sqrt{\Delta}/2)$; in particular, $\rho(\phi_{\nu,\lambda}) \geq 1 + ((a+d)/2) + (\sqrt{\Delta}/2) > 1$;
- if $0 \leq ad - bc \leq -(a + d)$, then we see that in both cases $\Delta \geq 0$ and $\Delta < 0$, we have $\rho(\phi_{\nu,\lambda}) \leq 1$ (for (ν, λ) sufficiently close to $(0, 0)$).

This proves the first assertion of the claim, and the other ones follow by similar arguments.

We know that $\phi_{\nu,0}$ always has an eigenvalue which is equal to 1 (it is the neutral Floquet multiplier in the direction of the flow), the second one being smaller than 1 (strictly smaller if $\nu \neq 0$); thus, we always have det $\phi_{\nu,0} < 1$ and $\rho(\phi_{\nu,0}) = 1$, which yields $ad - bc = 0$. We can thus write $ad - bc = \lambda F(\nu, \lambda)$, where F depends smoothly on ν and λ; according to Lemma 1, we have

$$\left(\frac{\omega}{2\pi}\right)^2 F(\nu, \lambda) = (\alpha P + \beta D)\nu^2 + (D^2 + P^2)\lambda + \mathcal{O}(\nu^3) + \mathcal{O}(\nu^2\lambda) + \mathcal{O}(\lambda^2)$$

As $D^2 + P^2 > 0$, the equation $F(\nu, \lambda) = 0$ defines a smooth implicit function $\varphi : K \to \mathbf{R}$, $\nu \mapsto \varphi(\nu)$, satisfying: $\varphi(\nu) = [(-\alpha P - \beta D)/(D^2 + P^2)]\nu^2 + \mathcal{O}(\nu^3)$. For any $(\nu, \lambda) \in K \times K$, we have:

- if $\lambda < \min(0, \varphi(\nu))$ or $\lambda > \max(0, \varphi(\nu))$, then $ad - bc > 0$;
- if $\min(0, \varphi(\nu)) < \lambda < \max(0, \varphi(\nu))$, then $ad - bc < 0$;
- if $\lambda = 0$ or $\lambda = \varphi(\nu)$, then $ad - bc = 0$.

According to Lemma 1, we have

$$\frac{\omega}{2\pi}(a + d + ad - bc) = \beta\nu^2 + 2D\lambda + \mathcal{O}(\nu^3) + \mathcal{O}(\nu\lambda) + \mathcal{O}(\lambda^2)\,.$$

As $D \neq 0$, the equation $a + d + ad - bc = 0$ defines a smooth implicit function $\psi : K \to \mathbf{R}$, $\nu \mapsto \psi(\nu)$, satisfying: $\psi(\nu) = (-\beta/2D)\nu^2 + \mathcal{O}(\nu^3)$. Remark that, for $\nu \neq 0$ sufficiently small, $\psi(\nu)$ has the sign of D. For any $(\nu, \lambda) \in K \times K$, we have

- if $\lambda < \psi(\nu)$ and $D > 0$, or if $\lambda > \psi(\nu)$ and $D < 0$, then $a + d + ad - bc < 0$;
- if $\lambda > \psi(\nu)$ and $D > 0$, or if $\lambda < \psi(\nu)$ and $D < 0$, then $a + d + ad - bc > 0$;
- if $\lambda = \psi(\nu)$, then $a + d + ad - bc = 0$.

The proof of Theorems 1 and 2 is complete.

Remark. For $\nu \neq 0$ small, and for λ very small, the smallness of λ depending on ν, we have (with the notations of the preceding proof), $\Delta > 0$ and

$$\rho(\phi_{\nu,\lambda}) = 1 + \frac{a + d}{2} + \frac{\sqrt{\Delta}}{2}$$

and

$$\frac{\omega}{2\pi}(a + d) = \beta\nu^2 + \mathcal{O}(\lambda) + \cdots < 0\,.$$

Thus,

$$\rho(\phi_{\nu,\lambda}) = 1 + \frac{a + d}{2}\left(1 - \sqrt{1 - 4\frac{ad - bc}{(a + d)^2}}\right)\,.$$

As $\lambda = 0$ forces $ad - bc = 0$, this yields

$$\frac{\partial}{\partial\lambda}(\rho(\phi_{\nu,\lambda}))(\nu, 0) = \left(\frac{1}{a + d}\frac{\partial}{\partial\lambda}(ad - bc)\right)(\nu, 0)$$

and, according to the above estimates on $ad - bc$,

$$\frac{\partial}{\partial\lambda}(\rho(\phi_{\nu,\lambda}))(\nu, 0) = \frac{2\pi}{\omega}\frac{\alpha P + \beta D}{\beta} + \mathcal{O}(\nu) = -\frac{2\pi}{\omega\beta}\mathcal{B} + \mathcal{O}(\nu)\,.$$

4. Approach via an Amplitude Equation

Here we briefly recall the classical approach via an amplitude equation, which enables to recover the same criterion involving the quantity $\mathcal{B}$. The rigorous justification of the derivation of the amplitude equation is a difficult problem (see [Collet, 1998] and references therein), which will not be considered here. We will simply invoke the usual (and nonrigorous) scaling and averaging arguments.

For simplification, we just consider the spatially extended differential equation

$$\partial_t u = f_\mu(u) + C\Delta u\,,$$

where Δ is the Laplace operator, the dimension d of the variable u equals 2, C is a constant 2×2 matrix, and the vector field f_μ is already in the normal form (3). Using complex coordinates $Z = x + iy$, the equation becomes

$$\partial_t Z = i\omega Z + (\mu\gamma(\mu) + \delta(\mu)|Z|^2)Z + R(Z, \overline{Z}, \mu) + C\Delta Z\,.$$

Write $Z_0(t) = \sqrt{\mu(\gamma_r(0)/ - \delta_r(0))}e^{i(\omega + \mu\gamma_i(0))t}$ and $Z(t) = Z_0(t)A(t)$ (here $A(.)$ is a function depending on the space variable, and represents the "amplitude"). Then, neglecting higher order terms, the previous equation becomes

$$\partial_t A = \mu\gamma_r(0)\left(A - \left(1 + i\frac{\delta_i(0)}{\delta_r(0)}\right)|A|^2 A\right) + (Z_0^{-1}CZ_0)\Delta A$$

(multiplication by $Z_0(t)$ commutes with Δ, but not necessarily with C). Writing $\tau = \mu\gamma_r(0)t$, this gives

$$\partial_\tau A = A - \left(1 + i\frac{\delta_i(0)}{\delta_r(0)}\right)|A|^2 A + \frac{1}{\mu\gamma_r(0)}(Z_0^{-1}CZ_0)\Delta A\,.$$

As $Z_0^{-1}CZ_0$ oscillates at a frequency $\omega/2\pi\mu\gamma_r(0)$ large with respect to the time scale of the variable τ, it is legitimate to replace it by its mean over one period, i.e. in complex notations by the complex number $D + iP$. Finally, if we suppose $D > 0$, we obtain

$$\partial_\tau A = A - (1 + i\alpha_0)|A|^2 A + (1 + i\beta_0)\left(\frac{D}{\mu\gamma_r(0)}\Delta\right)A$$

where $\alpha_0 = \delta_i(0)/\delta_r(0)$ and $\beta_0 = P/D$.

This last equation is nothing else than the classical complex Ginzburg–Landau equation. This equation admits a spatially homogeneous periodic solution $A_0(t) = e^{-i\alpha_0 t}$, whose stability is governed, as an easy calculus shows, by the celebrated Benjamin–Feir criterion, i.e. by the sign of the number $\mathcal{B}_0 = 1 + \alpha_0\beta_0$ (it is stable if this number is positive, unstable if it is negative). We can see that

$$\mathcal{B} = 2\gamma_r(0)D\,\mathcal{B}_0\,,$$

and thus the stability criterion $\mathcal{B}_0 > 0$ is, as expected, equivalent to the phase stability criterion $\mathcal{B} > 0$ established before.

5. Conclusion

By an elementary perturbation calculus, we have provided a rigorous justification of the criterion (often called "Benjamin–Feir criterion") governing the phase stability of spatially homogeneous oscillations close to a Hopf bifurcation; moreover we have determined precisely the values of the bifucation parameter and of the wavenumber where stability and instability occur.

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