SPATIAL UNFOLDING OF HOMOCLINIC BIFURCATIONS

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Abstract. We consider solutions which are homogeneous in space, periodic in time, and close to being homoclinic for a partial differential equation. We show that such solutions are generically unstable with respect to large wavelength perturbations, and that the instability can be of two different types : either the well-known Kuramoto phase instability, or a fundamentally different kind of instability, called *self-parametric*, displaying a period-doubling and an intrinsic wavelength. We also consider the case where the spatial parity symmetry breaks.

1. Introduction

Codimension one bifurcations of simple solutions (fixed points and limit cycles) of ordinary differential equation have been extensively studied, in particular by the Russian school [4]. They are frequently observed in Physical, Chemical and Biological systems. This is indeed one of the great merit of the Poincaré qualitative theory [10] and the Andronov [1] subsequent work to provide a language in order to describe the behavior of complex systems when some external parameters are varied, particularly in situations where the equations governing those systems are not exactly known (coarse systems). Unfolding a bifurcation among ODEs, and adding to the resulting family of ODEs a spatial dimension and spatial coupling terms (of low order in space derivatives), one obtains what we call a *spatial unfolding* of the bifurcation. Spatial unfoldings are well understood for local bifurcations ([6]), but, to our knowledge, they have not been much studied for global ones like Andronov homoclinic bifurcation. This study is the subject of this paper.

We consider PDEs of the form

$$\partial_t u = F(u, \partial_x), \tag{1}$$

i.e. invariant with respect to translations of time (autonomous) and space. Here u is in $\mathbf{R}^d$, $d \geq 1$, and for simplicity the space variable x is one-

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dimensional ($x \in \mathbf{R}$). Spatially homogeneous solutions of this PDE are solutions of the equation

$$\frac{du}{dt} = F(u,0) = f(u) \tag{2}$$

(we write $f(u)$ for $F(u,0)$), which is an autonomous ordinary differential equation in dimension d.

Among the solutions of equation (2), of prime interest are those which correspond to an asymptotic behavior, in particular attractive fixed points and attractive periodic orbits. Consider a solution $t \mapsto u_h(t)$ of equation (2) which is an attractive (linearly stable) periodic orbit. The corresponding homogeneous solution for the PDE (1) is thus stable with respect to homogeneous perturbations. We adress the question of the behavior of inhomogeneous perturbations. A small inhomogeneous perturbation $u(x,t)$ of $u_h(t)$ obeys the equation

$$\partial_t u = DF(u_h(t), \partial_x)u. \tag{3}$$

This equation being linear, it reduces in Fourier coordinates to

$$\partial_t \hat{u}(k) = DF(u_h(t), ik)\hat{u}(k) \tag{4}$$

which is just an ordinary differential equation parametrized by k, with periodic coefficients.

It turns out that the behavior of inhomogeneous perturbations of $u_h(t)$ is, without further hypotheses, by far a too general problem. An interesting way to make this problem more specific, altogether preserving some generality, is to look at it close to a bifurcation : normal forms or unfoldings of bifurcations are both "particular" and "universal" examples. Here we will suppose that the solution $t \mapsto u_h(t)$ is close to an Andronov homoclinic bifurcation, and we will denote by μ the bifurcation parameter (we suppose that $\mu < 0$ and μ is close to 0, the bifurcation occurring at $\mu = 0$).

At leading order in space derivatives, equation (4) reads

$$\frac{d}{dt}\hat{u}(k) = \Big(L(t) + ikC(t) - k^2 D(t) + \mathcal{O}(|k|^3)\Big)\hat{u}(k) \tag{5}$$

where $L(t)$, $C(t)$, $D(t)$ are $d \times d$ real matrices which are periodic) in time (remark that $L(t)$ is nothing else than $Df(u_h(t))$).

Differential equation (5) depends on two small parameters μ and k, and we know that, when $k = 0$, a homoclinic bifurcation occurs at $\mu = 0$. The main question is : are the spatial effects destabilizing or not ? in other words, is the homogeneous bifurcation anticipated (for $k \neq 0$, before $\mu = 0$) by another bifurcation due to the spatial effects ? We will see that the answer is always (generically) "yes", and study the nature of the instability, together with its weakly nonlinear development. This instability result was announced in [3] and proved in [11] and [5]. It holds in any dimension $d \geq 2$ (see [11]), but here, for simplicity, we shall restrict ourselves to the case $d = 2$.

We shall moreover see that the instability can be of two types : either the well-known Kuramoto phase instability, of a fundamentally different kind of instability, which occurs with a period-doubling and at an intrinsic wavelength. To this respect this second instability shares features analogous to the classical parametric instability. We called it "self-parametric" since it can be viewed as the result of a self-forcing of the oscillations, when the cycle becomes highly anharmonic through the approach of a fixed point.

It is often the case that the system described by the PDE (1) admits an additional parity symmetry with respect to the space variable x. In this case, there exists a linear involution I of $\mathbf{R}^d$ such that the equation is invariant by the transformation $(x, u) \mapsto (-x, Iu)$. We will say in the following that the problem considered here (the local study around u_h) is *$(x \leftrightarrow -x)$-invariant* if the PDE admits a symmetry $(x, u) \mapsto (-x, Iu)$, and if moreover vector coordinates of $u_h(t)$ vanish identically (i.e. $Iu_h = u_h$). If this is the case, the linear PDE (3) around u_h involves only derivatives of even order, and in equation (5), the "convective" matrix $C(t)$ vanishes identically. We will distinguish along the paper between cases where $(x \leftrightarrow -x)$-invariance holds or not.

2. Preliminaries

We suppose that $t \mapsto u_h(t)$ is a periodic solution of the homogeneous equation (2) (denote by T its period). Along this paragraph, we forget that u_h depends on the parameter μ and is close to a homoclinic bifurcation, and we recall basic computations that will be used later.

Denote by $\Phi_k(t)$ the flow of the differential equation (5) over one period T. For $k = 0$, $\Phi_0(T)$ is a first return (monodromy) map for the differential equation (2) around $u_h(.)$, thus one of its eigenvalue is always equal to 1 (it corresponds to phase translation, in the direction of the flow). We suppose that u_h is linearly stable with respect to homogeneous perturbations, i.e. that all the other eigenvalues of $\Phi_0(T)$ are strictly inside the unit circle.

For k close to 0, the eigenvalues of $\Phi_k(T)$ are close to those of $\Phi_0(T)$, let us denote by $\lambda(k)$ the one which is close to 1. The stability with respect to the wavenumber k depends on the size of $|\lambda(k)|$ with respect to 1. Write

$$\lambda(k) = 1 + k\lambda_1 + k^2\lambda_2 + \mathcal{O}(|k|^3).$$

The coefficient λ_2 is real, while λ_1 has a vanishing real part.

(a) ($x \leftrightarrow -x$)-invariant case. In this case $C(.) \equiv 0$, λ_1 vanishes, and the stability with respect to large wavelength perturbations is given by the sign of λ_2. If $\lambda_2 < 0$, then $u_h(.)$ is phase stable, while it exhibits the well-known phase Kuramoto instability ([8]) if $\lambda_2 > 0$.

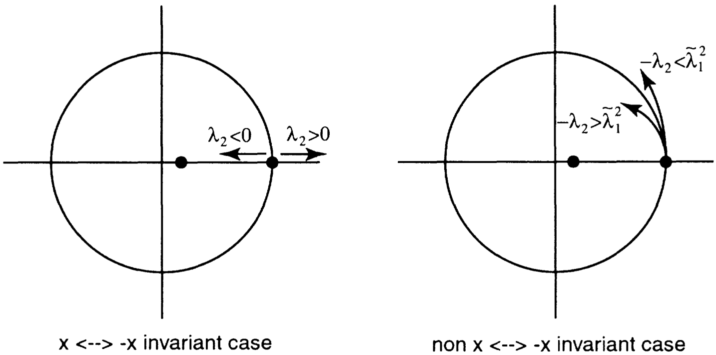

Figure 1. Behavior of $\lambda(k)$ for $|k|$ small.

(b) Non ($x \leftrightarrow -x$)-invariant case. In this case $C(.)$ is not identically vanishing, thus λ_1 is generically nonvanishing and purely imaginary. Write $\lambda_1 = i\tilde{\lambda}_1$. The stability with respect to small k now depends on both coefficients $\tilde{\lambda}_1$ and λ_2. More precisely, if $-\lambda_2 > \tilde{\lambda}_1^2$ (resp. $-\lambda_2 < \tilde{\lambda}_1^2$), then $u_h(.)$ is stable (resp. unstable) with respect to sufficiently small wavenumbers k.

FORMAL COMPUTATION OF λ_1 AND λ_2

Let $e_1(t) = f(u_h(t))$ and $e_2(t) = \text{Rot}_{\pi/2}(e_1(t))$, $t \in \mathbf{R}$. This defines a local frame $(e_1(.), e_2(.))$ along the periodic solution $u_h(.)$. Let us formulate the differential equation (5) using coordinates in this local frame. It takes the form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} + ik\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} - k^2\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} + \mathcal{O}(|k|^3)\right)\begin{pmatrix} x \\ y \end{pmatrix} \quad (6)$$

where a, b, c_j, and d_j are real and T-periodic. Write

$$B_s^t = e^{\int_s^t b(v)dv}.$$

The quantity B_0^T is equal to the second eigenvalue of $\Phi_0(T)$ (the first one being equal to 1). By hypothesis (linear stability), it belongs to $]0;1[$.

For k close to 0, denote by ϵ_k the (unique) vector belonging to the eigendirection of $\Phi_k(T)$ corresponding to the eigenvalue $\lambda(k)$ and having first coordinate equal to 1. Write $\epsilon_k = (1, y_k)$. For $k = 0$, we have $y_0 = 0$. For $t \in \mathbf{R}$, write $\epsilon_k(t) = \Phi_k(t)\epsilon_k$, write $\epsilon_k(t) = (x_k(t), y_k(t))$, and write

$$x_k(t) = 1 + kx_1(t) + k^2x_2(t) + \dots \quad \text{and} \quad y_k(t) = ky_1(t) + k^2y_2(t) + \dots$$

The relation $\epsilon_k(T) = \lambda(k)\epsilon_k(0)$ yields

$$\lambda_1 = x_1(T), \;\; y_1(t) = y_1, \;\; \lambda_2 = x_2(T), \;\; \text{and} \;\; y_2(T) = y_2(0) + \lambda_1 y_1(0),$$

and the differential equation (6) reads, at the first order in k,

$$\frac{dx_1}{dt} = ay_1 + ic_1$$

$$\frac{dy_1}{dt} = by_1 + ic_3,$$

and, at the second order in k,

$$\frac{dx_2}{dt} = ay_2 + i(c_1x_1 + c_2y_1) - d_1$$

$$\frac{dy_2}{dt} = by_2 + i(c_3x_1 + c_4y_1) - d_3.$$

(a) ($x \leftrightarrow -x$)-invariant case. In this case $C(.) \equiv 0$. Thus, $x_1 \equiv 0$, $y_1 \equiv 0$, $y_2(.)$ is the unique T-periodic solution of the differential equation $\frac{dY}{dt} = bY - d_3$, i.e. $y_2(t) = B_0^t y_2(0) - \int_0^t B_s^t d_3(s)ds$, where $y_2(0) = -(1 - B_0^T)^{-1} \int_0^T B_s^T d_3(s)ds$, and we obtain

$$\lambda_1 = 0 \quad \text{and } \lambda_2 = \int_0^T (ay_2 - d_1)(s)ds.$$

(b) Non ($x \leftrightarrow -x$)-invariant case. Here $C(.) \not\equiv 0$, $y_1(.)$ is the unique T-periodic solution of the differential equation $\frac{dY}{dt} = bY + ic_3$, $x_1(t) = \int_0^t (ay_1 + ic_1)(s)ds$, $y_2(.)$ is the unique solution of the differential equation $\frac{dY}{dt} = bY + i(c_3x_1 + c_4y_1) - d_3$ satisfying $y_2(T) = y_2(0) + \lambda_1 y_1(0)$, and λ_1 and λ_2 read

$$\lambda_1 = x_1(T), \quad \text{and} \quad \lambda_2 = \int_0^T (ay_2 + i(c_1x_1 + c_2y_1) - d_1)(s)ds.$$

3. Homoclinic bifurcation

Now we suppose again that the periodic solution u_h depends on the parameter $\mu \leq 0$, and that a homoclinic bifurcation occurs at $\mu = 0$ for this solution. We denote by $u \mapsto f_0(u)$ the function $u \mapsto f(u)$ when $\mu = 0$. We place ourselves close to the bifurcation, i.e. we suppose that μ is close to 0 but strictly negative ; thus, $f(.)$ is close to (but different from) $f_0(.)$.

We suppose that $f_0(0) = 0 = f(0)$, that $Df_0(0)$ reads $\begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix}$, where $0 < b_+ < -b_-$, and that the differential equation $\frac{du}{dt} = f_0(u)$ admits a solution $u_{h,0}(.)$ homoclinic to the fixed point 0, and we suppose that the trajectories of $u_h(.)$ and $u_{h,0}(.)$ are close.

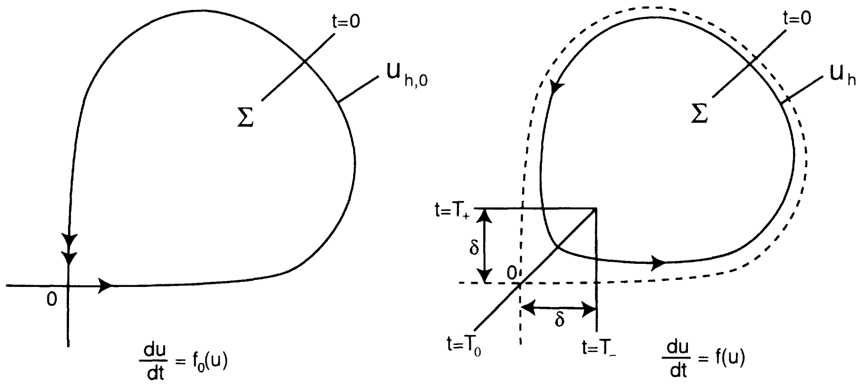

Figure 2. Homoclinic and almost homoclinic orbits.

We are going to show that, for $\mu < 0$ close enough to 0, the periodic solution $u_h(.)$ is always (generically) linearly unstable with respect to inhomogeneous perturbations. In the $(x \leftrightarrow -x)$-invariant case, this result was conjectured in [2], announced in [3], and rigorously proved in [11]. The case where the $(x \leftrightarrow -x)$-symmetry breaks was considered in [5].

Consider the differential equation (6) (in the local frame along $u_h(.)$) where the origin of times is fixed on a section Σ transverse to some point of the trajectory of $u_{h,0}(.)$ (see the figure). We can write down a similar differential equation in the local frame of the homoclinic solution $u_{h,0}(.)$; let us write it

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} 0 & a_0 \\ 0 & b_0 \end{pmatrix} + ik \begin{pmatrix} c_{1,0} & c_{2,0} \\ c_{3,0} & c_{4,0} \end{pmatrix} - k^2 \begin{pmatrix} d_{1,0} & d_{2,0} \\ d_{3,0} & d_{4,0} \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix}.$$

This differential equation is not periodic any more, its coefficients are limits (on any bounded time interval) of the coefficients of (6) when $\mu \to 0$. We have $b_0(t) \to b_+ - b_-$ when $t \to +\infty$, $b_0(t) \to b_- - b_+$ when $t \to -\infty$, $a_0(t) \to 0$ when $t \to \pm\infty$, and each coefficient $c_{j,0}(t)$ (resp. $d_{j,0}(t)$) admits limits when $t \to +\infty$ and $t \to -\infty$, say $c_{j,0,+}$ and $c_{j,0,-}$ (resp. $d_{j,0,+}$ and $d_{j,0,-}$).

Following Andronov's classical idea, we are going to decompose the periodic solution $u_h(.)$ into two parts, one close to and the other one far from the fixed point 0. Let δ be a small positive parameter. Denote by T_+ (resp. T_-) the first positive time when $u_h(.)$ enters (resp. escapes) the box of size 2δ centered in 0. Denote by T_0 the time between T_+ and T_- where $u_h(.)$ belongs to the diagonal $x = y$ (see the figure). The quantities $T_- - T_0$ and $T_0 - T_+$ are both large, but $T_- - T_0$ is larger than $T_0 - T_+$ (more precisely, the ratio $\frac{T_- - T_0}{T_0 - T_+}$ is close to $\frac{|b_-|}{b_+} > 1$).

For $t \in [T_-; T_+]$, i.e. when $u_h(t)$ lies inside the small box, we have

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \mathrm{Rot}_{-2\theta(t)} \begin{pmatrix} 0 \\ b_- - b_+ \end{pmatrix} + \ldots$$

where $\theta(t)$ is the angle $(\epsilon_1, e_1(t))$ (here $\epsilon_1 = (1,0)$ and $e_1(t) = f(u_h(t))$ is the speed vector), and the remaining terms "..." are small if δ and μ are close to 0. The angle $\theta(t)$ is close to $-\pi/2$ for $t - T_0 \ll 0$, close to 0 for $t - T_0 \gg 0$, and jumps between these two values during a bounded time interval around T_0. Thus the qualitative behaviors of $a(t)$ and $b(t)$ for $t \in [T_+; T_-]$ are as follows (see figure 3): the coefficient $b(.)$ is close to $b_+ - b_-$ for $t - T_0 \ll 0$, close to $b_- - b_+$ for $t - T_0 \gg 0$, and jumps between these two values during a bounded time interval around T_0 ; the coefficient $a(.)$ is close to 0 except during this bounded time interval, where it takes finite positive values between 0 and $b_+ - b_-$.

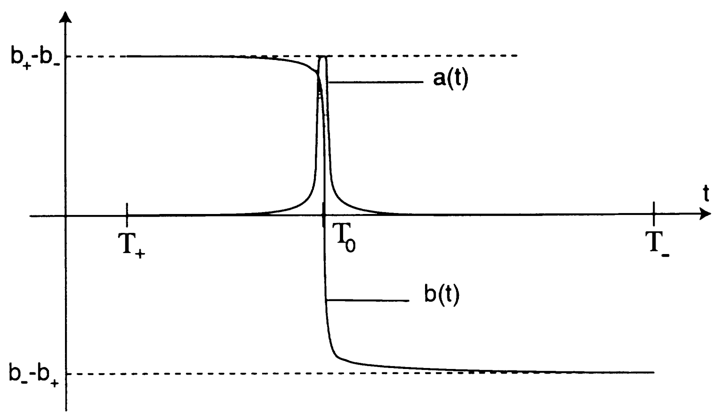

Figure 3. Behavior of $a(t)$ and $b(t)$ for $t \in [T_+; T_-]$.

3.1. $(X \leftrightarrow -X)$-INVARIANT CASE

(a) Phase instability. We are first going to estimate λ_2. We know that y_2 is the unique T-periodic solution of the differential equation $\frac{dy}{dt} = by - d_3$. Let us consider the limit when $\mu \to 0$ of this equation, i.e.

$$\frac{dy}{dt} = b_0 y - d_{3,0}.$$

The asymptotic behavior of b_0 and $d_{3,0}$ show that this equation admits a unique solution $y_-(.)$ (resp. $y_+(.)$) which is bounded when $t \to -\infty$ (resp. when $t \to +\infty$). Generically, these two solutions are different and the sign of $y_-(.) - y_+(.)$ is constant (remark that the only case where this genericity result does not hold is when the matrix $D(t)$ is proportional to the identity : in this case the coefficient $d_3(.)$ vanishes identically, and so do y_-, and y_+ ; the periodic solution $u_h(.)$ is then stable, because the coupling has only a trivial stabilizing effect).

The sign of $y_-(.) - y_+(.)$ governs the nature of the instability. Indeed, when $t \to +\infty$, $|y_-(.)| \to +\infty$ and has the sign of $y_-(.) - y_+(.)$. The fact that $T_- - T_0 > T_0 - T_+$ shows that the behavior of $y_2(.)$ is the following : on $[T_- - T; T_+]$ (i.e. when u_h is "far" from 0), it is very close to $y_-(.)$; on $[T_+; T_0]$, $|y_2(.)|$ grows exponentially and y_2 has the same sign as $y_- - y_+$; $|y_2(.)|$ takes a maximal value around $t = T_0$, and decreases exponentially afterwards. Thus, the main contribution in the expression $\lambda_2 = \int_0^T (ay_2 -$

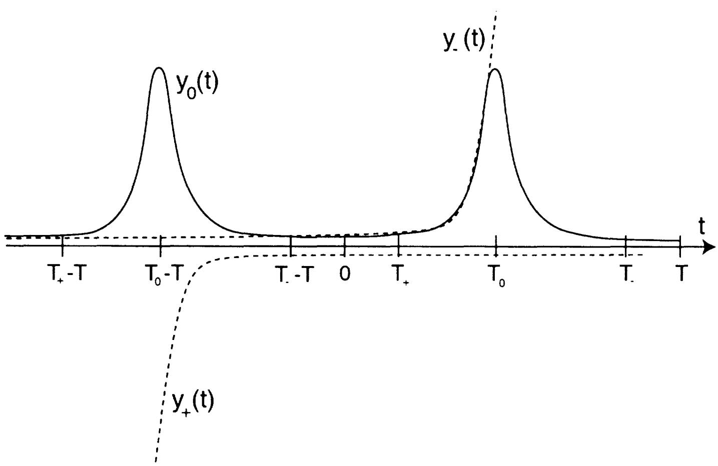

Figure 4. Behaviors of $y_0(.)$, $y_+(.)$, and $y_-(.)$ on **R**.

$d_1)(s)ds$ is the integral of ay_2 on a bounded interval around $t = T_0$. As $a > 0$ on this interval, we see that λ_2 is large and has the sign of $y_-(.) - y_+(.)$.

The conclusion is that, if $y_-(.) - y_+(.) > 0$, then the periodic solution $u_h(.)$ is phase unstable close to the homoclinic bifurcation.

(b) Self-parametric instability. If $y_-(.) - y_+(.) < 0$, then $u_h(.)$ is phase stable, i.e. stable with respect to values of k (very) close to 0. We are going to see, however, that in this case another instability holds, with respect to small but finite values of k.

Denote by Φ_k^R (resp. Φ_k^S) the flow of the differential equation (6) between the times $T_- - T$ and T_+, i.e. outside of the small box of size 2δ (resp. between the times T_+ and T_-, i.e. inside the small box of size 2δ). The composition $\Phi_k^S \circ \Phi_k^R$ represents the flow over one period.

Let us write $\Phi_0^R = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}$. The flow Φ_k^R is a perturbation of Φ_0^R which remains non-singular when one approaches the homoclinic bifurcation ; we can thus write

$$\Phi_k^R = \Phi_0^R + k^2 \begin{pmatrix} w & x \\ y & z \end{pmatrix} + \mathcal{O}(k^2).$$

Remark that the quantity y in this expression of Φ_k^R is the value at time T_+ of the solution of the differential equation $\frac{dy}{dt} = by - d_3$, with initial condition 0 at time $T_- - T$. If δ is sufficiently small, the behavior of this solution is once again governed by the behavior of y_- ; it grows exponentially, and has the sign of $y_- - y_+$ (thus negative in the case considered here) when t approaches T_+. Thus, the value y at time T_+ is negative and arbitrarily large if δ is sufficiently small.

Let us write $\Phi_0^S = \begin{pmatrix} 1 & \eta \\ 0 & \zeta \end{pmatrix}$. If $|\mu|$ equals the distance of $u_h(T_+)$ (i.e. at the entrance of the small box) to the stable manifold of 0, and denoting $\delta^{-1}|\mu|$ by ε and $|b_-|/b_+$ by γ, we find that

$$\eta = \gamma(1 + \mathcal{O}(\delta))\varepsilon^{-1+\mathcal{O}(\delta)} \text{ and } \zeta = \gamma^2(1 + \mathcal{O}(\delta))\varepsilon^{\gamma-1+\mathcal{O}(\delta)}.$$

The very singular form of Φ_0^S deserves comments. Although this linear map has distinct eigenvalues (namely 1 and η), it can hardly be put in a diagonal form because its two eigenvectors are almost parallel. An amplitude perturbations of the limit cycle when it enters the small box transforms into a strong phase perturbation. There is thus a strong coupling between amplitude and phase perturbations, and even if one of the Floquet multipliers (ζ) tends to zero while the other remains finite, no dimensional reduction is possible.

The flow Φ_k^S is a perturbation of Φ_0^S which becomes singular when one approaches the homoclinic bifurcation ; nevertheless, we have the following estimate :

$$\Phi_k^S = q_k(\Phi_0^S + \eta k^2\, \mathcal{O}(1)),$$

where $q_k = \varepsilon^{\mathcal{O}(k^2)}$. Denote by T_k the trace of $\Phi_k^S \circ \Phi_k^R$. The previous expressions of Φ_k^R and Φ_k^S show that

$$T_k = q_k(T_0 + \eta k^2(y + \mathcal{O}(1))).$$

If y is sufficiently large (i.e. if δ is sufficiently small), and if ηk^2 is large (i.e. if $k^2 \gg \varepsilon$), we see that the dominant term in this expression of T_k is the term $q_k \eta k^2 y$; it is large and negative. As on the other hand the determinant of $\Phi_k^S \circ \Phi_k^R$ is small, we see finally that, when k^2 is small but $k^2 \gg \varepsilon$, this first return map has two real eigenvalues, one close to 0, and the other one large negative. This proves the instability in this case.

3.2. NON $(X \leftrightarrow -X)$-INVARIANT CASE

In this case, the same kind of computation as in the previous paragraph (period doubling instability) can be achieved. It shows that, generically, for $|k|$ small but $|k| \gg \varepsilon$, the trace of the monodromy map $\Phi_k^S \circ \Phi_k^R$ has a large modulus (and an argument close to $\pm\pi/2$), which proves already the instability. Nevertheless, we want to be more precise and show that the instability occurs for arbitrarily small values of k, i.e. that the phase instability criterion $-\lambda_2 < \tilde{\lambda}_1^2$ holds.

Write $y_1 = i\tilde{y}_1$ and $x_1 = i\tilde{x}_1$. Then $\tilde{y}_1$ is defined as the unique T-periodic solution of the differential equation $\frac{dy}{dt} = by + c_3$. Consider the corresponding limit differential equation :

$$\frac{dy}{dt} = b_0 y + c_{3,0}.$$

Again, this differential equation admits a unique solution $y_-(.)$ (resp. $y_+(.)$) which is bounded when $t \to -\infty$ (resp. when $t \to +\infty$). Generically, these two solutions are different, and the sign of $y_-(.) - y_+(.)$ is constant.

As in the previous paragraph, the behaviour of $|\tilde{y}_1(.)|$ is the following : it grows exponentially on $[T_+; T_0]$, and decreases exponentially on $[T_0; T_-]$ (on these two intervals $\tilde{y}_1(.)$ has the sign of $y_-(.) - y_+(.)$). The quantity $|\tilde{y}_1(t)|$ thus takes a maximal value when t is around T_0. We deduce from this the behavior of $|\tilde{x}_1(t)| = |\int_0^t (a\tilde{y}_1 + c_1)(s)ds|$: it grows very fast when t belongs to a bounded interval around T_0, and varies much slower outside of this interval. We have

$$\frac{dy_2}{dt} = by_2 - c_3\tilde{x}_1 - c_4\tilde{y}_1 - d_3.$$

The constraint $y_2(T) = y_2(0) - \tilde{x}_1(T)\tilde{y}_1(0)$ yields

$$y_2(0)(1 - B_0^T) = -\tilde{x}_1(T)\tilde{y}_1(0) - \int_0^T B_s^T\Big(-c_3\tilde{x}_1 - c_4\tilde{y}_1 - d_3\Big)(s)ds.$$

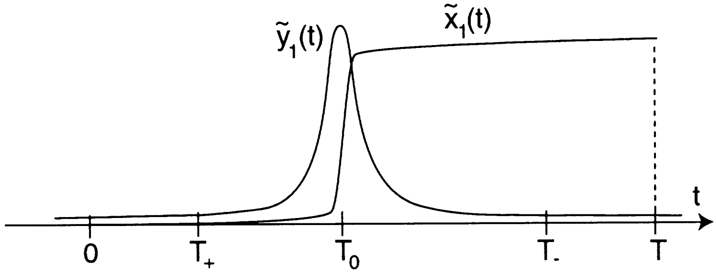

Figure 5. Behaviors of $\tilde{x}_1(t)$ and $\tilde{y}_1(t)$ for $t \in [0;T]$.

Now, according to the behaviors of b and a, we have, for δ sufficiently small, on one hand

$$\tilde{y}_1(0) = (1 - B_0^T)^{-1} \int_0^T B_s^T c_3(s)ds \simeq \int_{T_-}^T B_s^T c_3(s)ds,$$

and on the other hand

$$\int_0^T B_s^T\Big(-c_3\tilde{x}_1 - c_4\tilde{y}_1 - d_3\Big)(s)ds \simeq \tilde{x}_1(T) \int_{T_-}^T B_s^T c_3(s)ds.$$

Finally, we obtain $|y_2(0)| \ll |\tilde{x}_1(T)|$, which shows that $\max_{t\in[0;T]} |y_2(t)| \ll \tilde{x}_1(T)^2$, and finally that

$$|\lambda_2| \ll \tilde{\lambda}_1^2.$$

This proves the instability.

3.3. NONLINEAR BEHAVIOR

In summary, close to an Andronov bifurcation, a limit cycle is always unstable with respect to spatially inhomogeneous perturbations. Depending on the form of the coupling, this instability is either the phase instability or the self-parametric instability.

For the phase instability, amplitude equation with $u(t,x) = u_h(t-\phi)$ is the well known Kuramoto-Sivashinsky equation ([8]) :

$$\partial_\tau \phi = \mu\phi_{XX} + \alpha\phi_X^2 - \phi_{XXXX} \tag{7}$$

For the self-parametric instability in the $(x \leftrightarrow -x)$-invariant case, the non linear amplitude equation with $u(t,x) = u_h(t-\phi) + Ae^{ik_0x}\zeta(t-\phi) +$

c.c. + ... reads :

$$\partial_\tau A = \mu A \pm \alpha |A|^2 A + \alpha \phi_{XX} A + \beta \phi_X^2 + A_{XX} \quad (8)$$
$$\partial_\tau \phi = \delta \phi_{XX} + \phi_X^2 + \eta |A|^2 \quad (9)$$

where $\zeta(t)$ is the Floquet eigenvector corresponding to the period doubling.

We show numerical computation of the following equations :

$$u_t = v + \gamma u_x + u_{xx} - \beta v_{xx} \quad (10)$$
$$v_t = (\mu - u)v - u + u^2 + \beta u_{xx} + v_{xx} \quad (11)$$

The homogeneous part of this equation admits a stable periodic solution for $0 < \mu < 0.135$ which disappears via an Andronov bifurcation.

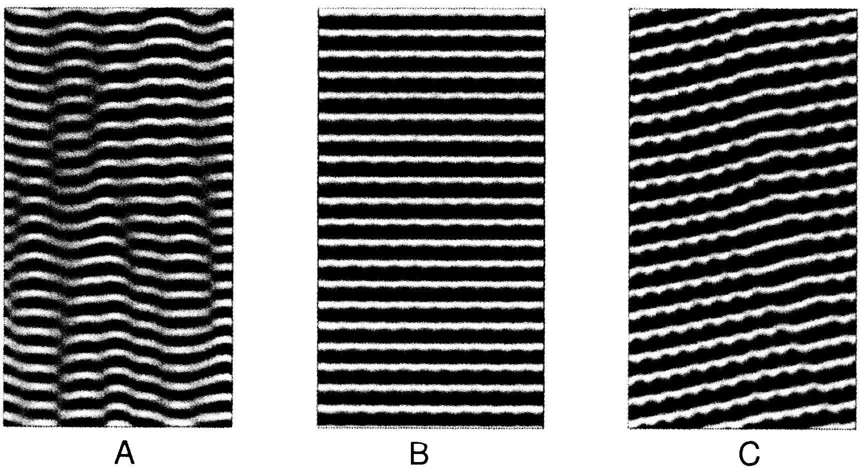

Figure 6. Numerical simulation of equations (10) and (11).

Results appear on figure 6 where abcissa is for the spatial coordinate and ordinate is for time. The intensity of gray corresponds to the value of u.

1. For A the parameters are $\mu = 0.075$, $\beta = 1$, $\gamma = 0$. For these values of parameter the system exhibits a phase instability. There is no wave number selected.
2. For B the parameters are $\mu = 0.075$, $\beta = -1$, $\gamma = 0$. The system exhibits this stable pattern after a bifurcation from the homogeneous state. It clearly shows period doubling, and a wave length is selected.
3. For C the parameters are $\mu = 0.075$, $\beta = -1$, $\gamma = 0.35$. The case is more involved and the limit cycle is unstable for small wavenumber

(phase instability) and is also unstable for a finite wavenumber with a negative real part of the Floquet multiplier. On this diagram there are traces of phase instability and of self-parametric instability.

4. Conclusion

We have shown that spatially homogeneous time periodic solutions of partial differential equations generically become unstable with respect to inhomogeneous perturbations when they approach a saddle fixed point (i.e. a homoclinic bifurcation). Although we restricted ourselves to dimension $d = 2$ for the field variable u, this result still holds in any higher dimension ([11]). Moreover we have seen that the instability can be of two different types : either the classical Kuramoto phase instability of a "self-parametric" instability, occurring with a period-doubling at a finite wavelength. We have related this second instability to the self-parametric forcing of the homogeneous periodic solution. Although this generic instability was observed in physical ([9]) and chemical systems ([7]) and model equations ([12]), no clear explanation of the mechanism was proposed up to now.

When the parity symmetry is broken, we have proved that a phase instability (instability with respect to arbitrarily large wavelength) always occurs close to the bifurcation. If this breaking is small and occurs in a system which previously displayed a self-parametric instability, we have co-existence of both instabilities.

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