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A VARIATIONAL PROOF OF GLOBAL STABILITY FOR BISTABLE TRAVELLING WAVES

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Abstract. We give a variational proof of global stability for bistable travelling waves of scalar reaction-diffusion equations on the real line. In particular, we recover some of the classical results by P. Fife and J.B. McLeod (1977) without any use of the maximum principle. The method that is illustrated here in the simplest possible setting has been successfully applied to more general parabolic or hyperbolic gradient-like systems.

1. Introduction

The purpose of this work is to revisit the stability theory for travelling waves of reaction-diffusion systems on the real line. We are mainly interested in *global* stability results, which assert that, for a wide class of initial data with a specified behavior at infinity, the solutions approach for large times a travelling wave with nonzero velocity. In the case of scalar reaction-diffusion equations, such properties have been established by Kolmogorov, Petrovski and Piskunov [11], by Kanel [9, 10], and by Fife & McLeod [4, 5] under various assumptions on the nonlinearity. The proofs of all these results use a priori estimates and comparison theorems based on the parabolic maximum principle. Therefore they cannot be extended to general reaction-diffusion systems nor to scalar equations of a different type, such as damped hyperbolic equations or higher-order parabolic equations, for which no maximum principle is available. However, these methods have been successfully

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applied to monotone reaction-diffusion systems [15, 18], as well as to scalar equations on infinite cylinders [14, 16].

Recently, a different approach to the global stability of bistable travelling waves has been developped by the second author [13]. The new method is of variational nature and is therefore restricted to systems which admit a gradient structure, but it does not make any use of the maximum principle and is therefore potentially applicable to a wide class of problems. The goal of this paper is to explain how this method works in the simplest possible case, namely the scalar parabolic equation

$$u_t = u_{xx} - F'(u) \ , \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$, and $t \geq 0$. We shall thus recover the main result of Fife and McLeod [4] under slightly different assumptions on the nonlinearity F, with a completely different proof. The present article can also serve as an introduction to the more elaborate work [13], where the method is developed in its full generality and applied to the important case of gradient reaction-diffusion systems of the form $u_t = u_{xx} - \nabla V(u)$, with $u \in \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$. A further application of our techniques is given in [7], where the global stability of travelling waves is established for the damped hyperbolic equation $\alpha u_{tt} + u_t = u_{xx} - F'(u)$, with $\alpha > 0$.

We thus consider the scalar parabolic equation (1.1), which models the propagation of fronts in chemical reactions [2], in combustion theory [9, 10], and in population dynamics [1, 6]. We suppose that the "potential" $F : \mathbb{R} \to \mathbb{R}$ is a smooth, coercive function with a unique global minimum and at least one additional local minimum. More precisely, we assume that $F \in \mathcal{C}^2(\mathbb{R})$ satisfies

$$\liminf_{|u|\to\infty} uF'(u) > 0. \tag{1.2}$$

In particular, $F(u) \to +\infty$ as $|u| \to \infty$. We also assume that F reaches its global minimum at $u = 1$:

$$F(1) = -A < 0 \ , \quad F'(1) = 0 \ , \quad F''(1) > 0 \ , \tag{1.3}$$

and has in addition a local minimum at $u = 0$:

$$F(0) = F'(0) = 0 \ , \quad F''(0) = \beta > 0. \tag{1.4}$$

Finally, we suppose that all the other critical values of F are positive, namely

$$\Big\{u \in \mathbb{R} : F'(u) = 0,\ F(u) \leq 0\Big\} = \{0; 1\}. \tag{1.5}$$

A typical potential satisfying the above requirements is represented in Figure 1.

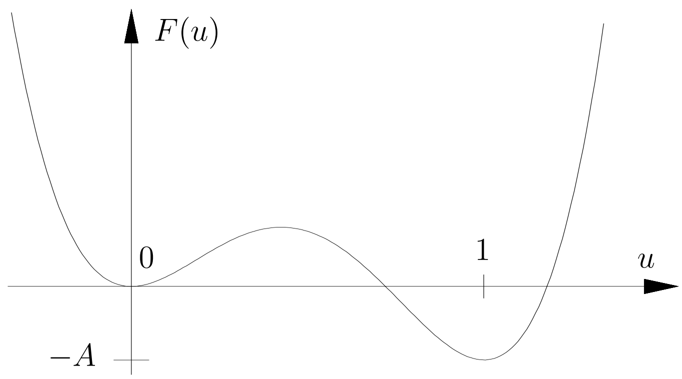

Fig. 1: The simplest example of a nonlinearity F satisfying assumptions (1.2)–(1.5).

Under assumptions (1.3)–(1.5), it is well-known that equation (1.1) has a family of travelling waves of the form $u(x,t) = h(x - c_* t)$ connecting the stable equilibria $u = 1$ and $u = 0$ [1]. More precisely, there exists a unique speed $c_* > 0$ such that the boundary-value problem

$$\begin{cases} h''(y) + c_* h'(y) - F'(h(y)) = 0 \ , \quad y \in \mathbb{R} \ , \\ h(-\infty) = 1 \ , \quad h(+\infty) = 0 \ , \end{cases} \tag{1.6}$$

has a solution $h : \mathbb{R} \to (0,1)$, in which case the profile h itself is unique up to a translation. Moreover, $h \in \mathcal{C}^3(\mathbb{R})$, $h'(y) < 0$ for all $y \in \mathbb{R}$, and $h(y)$ converges exponentially to its limits as $y \to \pm\infty$.

This family of travelling waves plays a major role in the dynamics of equation (1.1), as is shown by the following global convergence result:

Theorem 1.1. *Let $F \in \mathcal{C}^2(\mathbb{R})$ satisfy assumptions* (1.2)–(1.5)*. Then there exist $\delta > 0$ and $\nu > 0$ such that, for all initial data $u_0 \in \mathcal{C}^0(\mathbb{R})$ with*

$$\limsup_{x \to -\infty} |u_0(x) - 1| \le \delta \ , \quad \limsup_{x \to +\infty} |u_0(x)| \le \delta \ , \tag{1.7}$$

equation (1.1) *has a unique global bounded solution satisfying $u(x,0) = u_0(x)$ for all $x \in \mathbb{R}$. In addition, there exists $x_0 \in \mathbb{R}$ such that*

$$\sup_{x \in \mathbb{R}} \Big| u(x,t) - h(x - c_* t - x_0) \Big| = \mathcal{O}(\mathrm{e}^{-\nu t}) \ , \quad \textit{as } t \to +\infty. \tag{1.8}$$

Theorem 1.1 was first proved by Fife and McLeod [4, 5] under the additional assumption that $0 \le u_0(x) \le 1$ for all $x \in \mathbb{R}$. In that case

$u(x,t) \in [0,1]$ for all $x \in \mathbb{R}$ and all $t \geq 0$ by the maximum principle, so that the coercivity assumption (1.2) is not needed. As is mentioned in [3], the results of [4] can be extended to arbitrary initial data satisfying (1.7) provided that $uF'(u) > 0$ for all $u \notin [0,1]$, a condition that is more restrictive than (1.2) in the sense that F is not allowed to have critical points outside the interval $[0,1]$. The simplest case considered in [4] is when F has exactly one critical point in the open interval $(0,1)$, a situation in which condition (1.5) is clearly met. However, Fife and McLeod also study the case where F has three critical points in the open interval, including a local minimum at $u = u_* \in (0,1)$. In this situation there exists a travelling-wave solution of (1.1) with speed $c_1 > 0$ connecting $u = 1$ to $u = u_*$, and also a travelling wave with speed $c_2 \in \mathbb{R}$ connecting $u = u_*$ to $u = 0$. If $c_1 > c_2$, which is always the case if (1.5) holds, there exists $c_* \in (0,1)$ such that (1.6) has a solution $h : \mathbb{R} \to (0,1)$, and the conclusion of Theorem 1.1 is still valid. If $c_1 < c_2$, there exists no travelling wave connecting $u = 1$ to $u = 0$, and the solution of (1.1) with initial data satisfying (1.7) converges as $t \to \infty$ to a superposition of two travelling waves [4].

Theorem 1.1 is a particular case of the general results obtained in [13]; see Theorem 4 in Section 9.6 of that reference. Therefore, there is no need to give here a complete proof. Instead we shall prove the convergence result (1.8) under the additional assumption that the initial data $u_0(x)$ decay rapidly to zero as $x \to +\infty$. It is intuitively clear that the precise behavior of $u_0(x)$ near $x = +\infty$ should not play an important role, because the equilibrium $u = 0$ ahead of the front is stable (this is in sharp contrast with the case of a monostable front invading an unstable equilibrium, where the behavior ahead of the front is of crucial importance). However, this restriction allows us to shortcut many technicalities and to give a much simpler proof in which the essence of the argument can be easily understood.

Our approach is based on the fact that equation (1.1) possesses (at least formally) a *gradient structure*, not only in the laboratory frame but also in any frame moving to the right with a positive velocity. To see this, we introduce the following notation. If $u(x,t)$ is a solution of (1.1), we define for any $c > 0$

$$v(y,t) = u(y+ct,t)\ , \quad \text{or equivalently} \quad u(x,t) = v(x-ct,t). \tag{1.9}$$

Setting $y = x - ct$ we see that the new function $v(y,t)$ satisfies

$$v_t = v_{yy} + cv_y - F'(v). \tag{1.10}$$

We now introduce the *energy functional*

$$\mathcal{E}_c[v] = \int_{\mathbb{R}} e^{cy}\Big(\frac{1}{2}v_y^2 + F(v)\Big)\,\mathrm{d}y\ , \tag{1.11}$$

and the corresponding *energy dissipation functional*

$$\mathcal{D}_c[v] = \int_{\mathbb{R}} e^{cy}\Big(v_{yy} + cv_y - F'(v)\Big)^2\,\mathrm{d}y. \tag{1.12}$$

We also denote by $H_c^1(\mathbb{R})$ the Banach space

$$H_c^1(\mathbb{R}) = \Big\{v \in L^\infty(\mathbb{R}) : e^{cy/2}v \in H^1(\mathbb{R})\Big\}\ , \tag{1.13}$$

equipped with the norm $\|v\|_{H_c^1} = \|v\|_{L^\infty} + \|e^{cy/2}v\|_{H^1}$. Note that any $v \in H_c^1(\mathbb{R})$ decays to zero faster than $e^{-cy/2}$ as $y \to +\infty$. Since $F(v) \sim \beta v^2/2$ as $v \to 0$ by (1.4), it follows that $\mathcal{E}_c[v] < \infty$ for all $v \in H_c^1(\mathbb{R})$. Conversely, any $v \in L^\infty(\mathbb{R})$ such that $v(y) \to 0$ as $y \to +\infty$ belongs to $H_c^1(\mathbb{R})$ as soon as $\mathcal{E}_c[v] < \infty$.

If $v(y,t)$ is a solution of (1.10) with initial data $v_0 \in H_c^1(\mathbb{R})$, then $v(\cdot,t) \in H_c^1(\mathbb{R})$ for all $t \ge 0$ and a direct calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_c[v(\cdot,t)] = -\mathcal{D}_c[v(\cdot,t)] \le 0\ , \quad t > 0. \tag{1.14}$$

In other words, the energy $\mathcal{E}_c$ is a *Lyapunov function* of system (1.10) in $H_c^1(\mathbb{R})$. This observation is of course not new: in their original proof, Fife and McLeod [4] already used a suitable truncation of the functional $\mathcal{E}_c$ for the particular value $c = c_*$ to show that the solution $v(y,t)$ of (1.10) approaches a travelling wave for a sequence of times. However, the fact that equation (1.1) has a whole family of (nonequivalent) Lyapunov functions has not been fully exploited until recently. The only reference we know where the implications of this rich Lyapunov structure are really discussed is a recent paper by Muratov [12], which contains a lot of interesting observations and a few general results concerning a wider class of systems than equation (1.1), but fails to prove the convergence to travelling waves. The goal of the present article is to show that, in the simple case of equation (1.1), the gradient structure *alone* is sufficient to establish convergence, at least if we restrict ourselves to solutions which decay to zero rapidly enough as $x \to +\infty$ so that the energy functionals are properly defined.

The main difficulty of this purely variational approach is that we do not have good a priori estimates on the solution $v(y,t) = u(y+ct,t)$ in a moving frame with speed $c > 0$. First of all, it is not clear a priori that the energy $\mathcal{E}_c[v(\cdot,t)]$ is bounded from below (this will not be the case typically if c is too small), and without this information it is difficult to really exploit the dissipation relation (1.14). Next, if we have a lower bound on $\mathcal{E}_c[v(\cdot,t)]$, we can deduce from (1.14) that the solution $v(y,t)$ converges uniformly on compact sets, at least for a sequence of times, towards a stationary solution of (1.10), but we cannot exclude a priori that this limit is just the trivial equilibrium $v \equiv 0$ (this will be the case typically if c is too large). To overcome these difficulties, the main idea is to track the position of the front interface in the following way. We fix positive constants β_1 and β_2 such that $\beta_1 < F''(0) < \beta_2$, and we choose $\varepsilon > 0$ small enough so that

$$\beta_1 \le F''(u) \le \beta_2 \ , \quad \text{for all } u \in [-2\varepsilon, 2\varepsilon]. \tag{1.15}$$

Given a continuous solution of (1.1) satisfying the boundary conditions

$$\lim_{x\to-\infty} u(x,t) = 1 \ , \quad \lim_{x\to+\infty} u(x,t) = 0 \ , \quad t \ge 0 \ , \tag{1.16}$$

we define the *invasion point* $\bar{x}(t)$ as the first point starting from the right where the solution $u(x,t)$ leaves an ε-neighborhood of the equilibrium $u = 0$:

$$\bar{x}(t) = \max\Big\{x \in \mathbb{R} : |u(x,t)| \ge \varepsilon\Big\}. \tag{1.17}$$

In view of (1.16), it is clear that $-\infty < \bar{x}(t) < \infty$ for all $t \ge 0$, and that $|u(\bar{x}(t),t)| = \varepsilon$. A quantity similar to $\bar{x}(t)$ was also introduced in [12], where it is called the "leading edge."

The strategy of the proof is to show that the solution $u(x,t)$ converges uniformly on compact sets around the invasion point $\bar{x}(t)$ towards a suitable translate of the travelling wave (1.6). Using only the gradient structure, we can prove the following result:

Proposition 1.2. *Let $F \in \mathcal{C}^2(\mathbb{R})$ satisfy assumptions* (1.2)–(1.5). *If $u_0 \in H^1_c(\mathbb{R})$ for some sufficiently large $c > 0$ and $u_0 - 1 \in H^1(\mathbb{R}_-)$, then the solution $u(x,t)$ of equation* (1.1) *with initial data u_0 satisfies, for all $L > 0$,*

$$\sup_{z\in[-L,+\infty)} |u(\bar{x}(t)+z,t) - h_\varepsilon(z)| \xrightarrow[t\to\infty]{} 0 \ , \tag{1.18}$$

where $\bar{x}(t)$ is the invasion point (1.17) *and h_ε is the travelling wave* (1.6) *normalized so that $h_\varepsilon(0) = \varepsilon$. Moreover, the map $t \mapsto \bar{x}(t)$ is $\mathcal{C}^1$ for t sufficiently large and $\bar{x}'(t) \to c_*$ as $t \to \infty$.*

As is explained above, the assumption $u_0 \in H^1_c(\mathbb{R})$ is needed in order to use the energy functional $\mathcal{E}_c$ without truncating the unbounded exponential factor e^{cy}. The proof will show that it is sufficient to take here $c > \sqrt{2A}/\varepsilon$, where A is defined in (1.3) and ε in (1.15). On the other hand, the assumption $u_0 - 1 \in H^1(\mathbb{R}_-)$ is just a convenient way to guarantee that the first condition in (1.16) is satisfied, but with minor modifications we can treat the more general case where $|u_0(x) - 1|$ is assumed to be small for large $x < 0$, as in (1.7).

The local convergence established in Proposition 1.2 is the key step in proof of Theorem 1.1. Once (1.18) is known, it remains to show that the solution $u(x,t)$ converges uniformly to 1 in the region far behind the invasion point $\bar{x}(t)$. Such a "repair" is certainly expected because $u = 1$ is the point where the potential F reaches its global minimum. A convenient way to prove this is to use a truncated version of the functional

$$\mathcal{E}[u] = \int_{\mathbb{R}} \Big(\frac{1}{2}u_x^2 + \overline{F}(u)\Big)\,\mathrm{d}x \ , \tag{1.19}$$

where $\overline{F}(u) = F(u) - F(1) \geq 0$. In this way, we can show that the solution $u(x,t)$ approaches uniformly on $\mathbb{R}$ a travelling wave (at least for a sequence of times), and using in addition the local stability results established in [17] we obtain (1.8). We thus have

Corollary 1.3. *Under the assumptions of Proposition* 1.2, *there exist* $x_0 \in \mathbb{R}$ *and* $\nu > 0$ *such that* (1.8) *holds.*

We conclude this introduction with a few comments on the scope of our method. First, it is clear that the assumptions (1.2)–(1.5) are not the weakest ones under which Proposition 1.2 holds. A careful examination of the proof reveals that the only hypotheses that we really use are the following:

- **H1:** *For all bounded initial data* u_0*, equation* (1.1) *has a (unique) global bounded solution.* This is certainly true if (1.2) holds, but it is sufficient to assume, for instance, that $F(u) \to +\infty$ as $|u| \to \infty$, or that $uF'(u) > 0$ whenever $|u|$ is sufficiently large.
- **H2:** $F(0) = F'(0) = 0$*, and there exists* $\varepsilon > 0$ *such that* $F''(u) \geq 0$ *for all* $u \in [-\varepsilon, \varepsilon]$. This is automatically true if (1.4) holds, but $u = 0$ need not be a strict local minimum of F. In particular, Proposition 1.2 holds for the nonlinearities of combustion type considered in [9, 10].
- **H3:** *There exists a unique* $c > 0$ *such that the differential equation* $v_{yy} + cv_y - F'(v) = 0$ *has a bounded solution satisfying* $|v(0)| = \varepsilon$, $|v(y)| \leq \varepsilon$ *for all* $y \geq 0$, *and* $v(y) \to 0$ *as* $y \to +\infty$*; furthermore, this solution is unique.* Under assumptions (1.3)–(1.5), we have $c = c_*$ and $v = h_\varepsilon$. In general, we can assume without loss of generality that v is positive and converges to 1 as $y \to -\infty$, so that $F(1) < 0$ and $F'(1) = 0$. It

also follows that $F(u) \geq 0$ for all $u \leq 0$ and that F has no critical point $u_* < 1$ with $F(u_*) < 0$.

On the other hand, to prove that the solution of (1.1) given by Proposition 1.2 converges uniformly on $\mathbb{R}$ to a travelling wave we need the additional assumption:

H4: *There exists $\varepsilon' > 0$ such that the only bounded solution of the differential equation $u_{xx} - F'(u) = 0$ with $|u(0) - 1| \leq \varepsilon'$ is $u \equiv 1$.* This requires that F attains its global minimum at $u = 1$, and nowhere else.

Finally, if we want the convergence to be exponential in time as in (1.8), we need to assume that $F''(1) > 0$.

Another comment concerns the variational structure of equation (1.1). Due to the exponential weight e^{cy}, it is clear that the energy functional $\mathcal{E}_c$ is not translation invariant. In fact, for any $v \in H^1_c(\mathbb{R})$ and any $\ell \in \mathbb{R}$, we have the relation $\mathcal{E}_c[v(\cdot - \ell)] = e^{c\ell}\mathcal{E}_c[v]$. This implies that the infimum of $\mathcal{E}_c[v]$ is either 0 or $-\infty$. Under our assumptions on F, the transition between both regimes occurs precisely at the critical speed c_* for which travelling waves exist:

$$\inf_{v \in H^1_c(\mathbb{R})} \mathcal{E}_c[v] = \begin{cases} 0 & \text{if} \quad c \geq c_* \ , \\ -\infty & \text{if} \quad c < c_*. \end{cases}$$

Indeed, as was observed by Muratov [12], for any $c < c_* + \sqrt{c_*^2 + 4F''(0)}$ we have the identity

$$c\mathcal{E}_c[h] = (c - c_*) \int_{\mathbb{R}} e^{cy} h'(y)^2 \, \mathrm{d}y \ ,$$

where h is the solution of (1.6). This shows in particular that $\mathcal{E}_c[h] < 0$ when $c < c_*$; hence, $\inf \mathcal{E}_c = -\infty$ in that case. The fact that $\mathcal{E}_c \geq 0$ when $c \geq c_*$ is not obvious a priori, and will be established in the course of the proof of Proposition 1.2; see Corollary 4.3. Note also that $\mathcal{E}_{c_*}[h] = 0$, so that $\inf \mathcal{E}_{c_*} = \min \mathcal{E}_{c_*} = 0$.

The rest of the paper is organized as follows. In Section 2, we establish the basic inequalities relating the energy $\mathcal{E}_c$, the dissipation $\mathcal{D}_c$, and the invasion point. Using these relations, we prove in Section 3 that the average speed of the invasion point $\bar{x}(t)$ has a limit $c_\infty > 0$ as $t \to \infty$. The core of the paper is Section 4, where we show that $c_\infty = c_*$ and prove Proposition 1.2. The proof of Corollary 1.3 is then performed in the final Section 5.

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2. Preliminary estimates

As the potential F is smooth and coercive, it is well-known that the Cauchy problem for the semilinear equation (1.1) is globally well-posed in the space of bounded functions; see e.g. [8]. Due to parabolic regularization, the solutions are smooth for $t > 0$ and satisfy (1.1) in the classical sense. Under assumption (1.2), one can also show that our system has a *bounded absorbing set* in the following sense:

Lemma 2.1. *There exists a constant $B > 0$ depending only on F such that, for all initial data $u_0 \in L^\infty(\mathbb{R})$, the (unique) solution $u(x,t)$ of* (1.1) *satisfies, for all sufficiently large $t \ge 0$,*

$$\sup_{x\in\mathbb{R}}\Big(|u(x,t)| + |u_x(x,t)| + |u_{xx}(x,t)|\Big) \le B. \tag{2.1}$$

Moreover, $u(\cdot,t)$ is bounded in $H^s_{\mathrm{loc}}(\mathbb{R})$ for some $s > 5/2$ and all $t \ge 1$.

The uniform bound on $|u(x,t)|$ follows easily from the maximum principle, but it can also be established using localized energy estimates; see [13, Section 9.1]. The bounds on the derivatives are then obtained in a standard way using parabolic regularization.

From now on, we suppose that $u_0 \in H^1_{c_0}(\mathbb{R})$ for some $c_0 > 0$ (which will be specified later) and that $u_0 - 1 \in H^1(\mathbb{R}_-)$. Then the solution of (1.1) with initial data u_0 satisfies $u(\cdot,t) \in H^1_{c_0}(\mathbb{R})$ and $u(\cdot,t) - 1 \in H^1(\mathbb{R}_-)$ for all $t \ge 0$, because $u = 0$ and $u = 1$ are (stable) equilibria of (1.1). In particular, the boundary conditions (1.16) hold for all times, so that one can define the invasion point $\bar{x}(t)$ by (1.17). Also, since we are interested in the long-time behavior of $u(x,t)$, we can assume without loss of generality that estimate (2.1) is valid for all $t \ge 0$.

As is explained in the introduction, we shall use the energy functionals $\mathcal{E}_c$ (for various values of $c > 0$) to prove that the solution $u(x,t)$ converges to a travelling wave h locally around the invasion point $\bar{x}(t)$. A technical problem we shall encounter is that the invasion point, as defined in (1.17), need not be a continuous function of time and can therefore jump back and forth in an uncontrolled way. It is possible to avoid this difficulty using a more clever definition than (1.17) (see [13]), but we follow here another approach and just introduce a second invasion point defined by

$$\bar{X}(t) = \max\Big\{x \in \mathbb{R} : |u(x,t)| \ge 2\varepsilon\Big\}. \tag{2.2}$$

Clearly, $-\infty < \bar{X}(t) < \bar{x}(t) < +\infty$ for all $t \geq 0$. The important point is that information on $\bar{x}$ at a given time provides an upper bound on $\bar{X}$ at later times:

Lemma 2.2. *There exists $T_0 > 0$ and $C_0 > 0$ such that, for all $t_0 \geq 0$, one has*

$$\bar{X}(t) \leq \bar{x}(t_0) + C_0 \quad \textit{for all } t \in [t_0, t_0 + T_0]. \tag{2.3}$$

Proof. Fix $t_0 \geq 0$. The solution of (1.1) satisfies

$$u(t) = S(t - t_0)u(t_0) - \int_{t_0}^{t} S(t - s)F'(u(s))\,\mathrm{d}s \equiv u_1(t) + u_2(t), \quad t \geq t_0,$$

where $S(t) = e^{t\partial_x^2}$ is the heat semigroup. Take $K > 0$ such that $|F'(u)| \leq K$ whenever $|u| \leq B$, where B is as in (2.1). Then $\|u_2(t)\|_{L^\infty} \leq K(t - t_0)$. On the other hand, by definition of $\bar{x}$, we have $|u(x, t_0)| \leq \varepsilon$ if $x \geq \bar{x}(t_0)$ and $|u(x, t_0)| \leq B$ if $x \leq \bar{x}(t_0)$. Using the explicit form of the heat kernel, we deduce that

$$|u_1(x, t)| \leq \frac{1}{\sqrt{4\pi(t - t_0)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-t_0)}} |u(y, t_0)|\,\mathrm{d}y \leq \varepsilon + \frac{B}{2}\mathrm{erfc}\Big(\frac{x - \bar{x}(t_0)}{\sqrt{4(t - t_0)}}\Big),$$

where $\mathrm{erfc}(x) = (2/\sqrt{\pi}) \int_x^\infty e^{-z^2}\,\mathrm{d}z$. We first choose $T_0 > 0$ such that $KT_0 < \varepsilon/2$, and then $C_0 > 0$ such that $B\mathrm{erfc}(C_0/\sqrt{4T_0}) < \varepsilon$. Then, for all $t \in [t_0, t_0 + T_0]$ and all $x \geq \bar{x}(t_0) + C_0$ we have $|u(x, t)| < 2\varepsilon$, which implies that $\bar{X}(t) \leq \bar{x}(t_0) + C_0$. $\square$

We now derive the basic estimates on the energy (1.11) and the energy dissipation (1.12) which will be used throughout the proof. Given $c \in (0, c_0)$, we define $v(y, t) = u(y + ct, t)$ as in (1.9), and we set

$$E_c(t) = \mathcal{E}_c[v(\cdot, t)]\ , \quad D_c(t) = \mathcal{D}_c[v(\cdot, t)]\ , \quad t \geq 0. \tag{2.4}$$

Of course, $v(y, t)$ depends also on the speed parameter c, but to simplify the notation this dependence will not be indicated explicitly. We also denote by $\bar{y}_c(t)$ and $\bar{Y}_c(t)$ the invasion points in the moving frame:

$$\bar{y}_c(t) = \bar{x}(t) - ct\ , \qquad \bar{Y}_c(t) = \bar{X}(t) - ct\ . \tag{2.5}$$

By construction, $|v(\bar{y}_c(t), t)| = \varepsilon$ and $|v(y, t)| \leq \varepsilon$ for $y \geq \bar{y}_c(t)$, and $|v(y, t)| \leq 2\varepsilon$ for $y \geq \bar{Y}_c(t)$. Remark that, by (1.15), the following inequalities hold whenever $|v| \leq 2\varepsilon$:

$$\frac{\beta_1}{2}v^2 \leq F(v) \leq \frac{\beta_2}{2}v^2\ , \quad \beta_1 v^2 \leq vF'(v) \leq \beta_2 v^2\ , \quad \beta_1 \leq F''(v) \leq \beta_2. \tag{2.6}$$

Lower bound on E_c: Using (1.11), (2.6), and the fact that $F(u) \geq -A$ for all $u \in \mathbb{R}$, we find

$$
\begin{aligned}
E_c(t) = & \int_{-\infty}^{\bar{y}_c(t)} e^{cy}\Big(\frac{1}{2}v_y^2(y,t) + F(v(y,t))\Big)\,\mathrm{d}y \\
& + \int_{\bar{y}_c(t)}^{\infty} e^{cy}\Big(\frac{1}{2}v_y^2(y,t) + F(v(y,t))\Big)\,\mathrm{d}y \qquad (2.7)\\
\geq & \int_{-\infty}^{\bar{y}_c(t)} e^{cy}(-A)\,\mathrm{d}y + \int_{\bar{y}_c(t)}^{\infty} e^{cy}\Big(\frac{1}{2}v_y^2(y,t) + \frac{\beta_1}{2}v^2(y,t)\Big)\,\mathrm{d}y.
\end{aligned}
$$

To estimate the last integral in (2.7) we recall that $v(\bar{y}_c(t),t)^2 = \varepsilon^2$, so that

$$
\begin{aligned}
e^{c\bar{y}_c(t)}\varepsilon^2 &= -\int_{\bar{y}_c(t)}^{\infty} \partial_y\Big(e^{cy}v^2(y,t)\Big)\,\mathrm{d}y \\
&= -\int_{\bar{y}_c(t)}^{\infty} e^{cy}\Big(2v(y,t)v_y(y,t) + cv^2(y,t)\Big)\,\mathrm{d}y.
\end{aligned}
$$

Given $d > -c$, we have $|2vv_y| \leq (c+d)v^2 + (c+d)^{-1}v_y^2$; hence,

$$
e^{c\bar{y}_c(t)}\varepsilon^2 \leq \int_{\bar{y}_c(t)}^{\infty} e^{cy}\Big(\frac{1}{c+d}v_y^2(y,t) + dv^2(y,t)\Big)\,\mathrm{d}y. \qquad (2.8)
$$

If we choose d such that $d(c+d) = \beta_1$ and insert the resulting inequality into (2.7), we obtain

$$
E_c(t) \geq e^{c\bar{y}_c(t)}\Big(-\frac{A}{c} + \kappa\varepsilon^2\Big), \text{ where } 0 \leq \kappa \leq \frac{1}{4}(c + \sqrt{c^2 + 4\beta_1}). \qquad (2.9)
$$

This estimate shows in particular that the energy $E_c(t)$ is bounded from below as long as the invasion point $\bar{y}_c(t)$ is bounded from above. Moreover, the lower bound is close to zero if $\bar{y}_c(t)$ is large and negative.

Variation of D_c: It follows from (1.10) and (1.12) that

$$
D_c(t) = \int_{\mathbb{R}} e^{cy}v_t^2(y,t)\,\mathrm{d}y.
$$

Differentiating this relation with respect to t and integrating by parts, we find

$$
\begin{aligned}
\frac{1}{2}D_c'(t) &= \int_{\mathbb{R}} e^{cy}(v_t v_{tt})(y,t)\,\mathrm{d}y = \int_{\mathbb{R}} e^{cy}v_t(v_{tyy} + cv_{ty} - F''(v)v_t)\,\mathrm{d}y \\
&= -\int_{\mathbb{R}} e^{cy}v_{ty}^2(y,t)\,\mathrm{d}y - \int_{\mathbb{R}} e^{cy}F''(v(y,t))v_t^2(y,t)\,\mathrm{d}y.
\end{aligned}
$$

Take $C_1 > 0$ such that $F''(u) \ge -C_1/2$ whenever $|u| \le B$, where B is as in (2.1). Then the above relation shows that

$$D_c'(t) \le C_1 D_c(t) \ , \quad t \ge 0. \tag{2.10}$$

This differential inequality implies that, if $D_c \in L^1(\mathbb{R}_+)$, then $D_c(t) \to 0$ as $t \to \infty$. Since $E_c'(t) = -D_c(t)$ by (1.14), this will be the case as soon as E_c is bounded from below.

Lower bound on D_c: Using (1.12) again and integrating by parts, we find

$$\begin{aligned} D_c(t) &= \int_{\mathbb{R}} e^{cy}(v_{yy} + cv_y - F'(v))^2(y,t)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} e^{cy}\Big(v_{yy}^2 + 2F''(v)v_y^2 + F'(v)^2\Big)(y,t)\,\mathrm{d}y. \end{aligned}$$

We split the integration domain into $(-\infty, \bar{Y}_c(t))$ and $(\bar{Y}_c(t), +\infty)$. Using (2.1), (2.6) and the lower bound $F''(v) \ge -C_1/2$, we obtain

$$D_c(t) \ge -\frac{C_1 B^2}{c} e^{c\bar{Y}_c(t)} + \int_{\bar{Y}_c(t)}^{\infty} e^{cy}\Big(v_{yy}^2(y,t) + 2\beta_1 v_y^2(y,t) + \beta_1^2 v^2(y,t)\Big)\,\mathrm{d}y.$$

Observe that, for any $y_0 \in \mathbb{R}$,

$$\int_{y_0}^{\infty} e^{cy} v_y^2\,\mathrm{d}y \ge \frac{c^2}{4}\int_{y_0}^{\infty} e^{cy} v^2\,\mathrm{d}y \quad \text{and} \quad \int_{y_0}^{\infty} e^{cy} v_{yy}^2\,\mathrm{d}y \ge \frac{c^2}{4}\int_{y_0}^{\infty} e^{cy} v_y^2\,\mathrm{d}y. \tag{2.11}$$

Indeed the first inequality is just (2.8) with $\varepsilon = 0$, $d = -c/2$, and $\bar{y}_c(t)$ replaced by y_0, and the second inequality is similar. Thus, for any $d \ge 0$ we have

$$\begin{aligned} D_c(t) \ge\ & -\frac{C_1 B^2}{c} e^{c\bar{Y}_c(t)} \\ & + \int_{\bar{Y}_c(t)}^{\infty} e^{cy}\left\{\Big(2\beta_1 + \frac{c^2}{4} - d\Big)v_y^2 + \Big(\beta_1^2 + \frac{dc^2}{4}\Big)v^2\right\}(y,t)\,\mathrm{d}y. \end{aligned} \tag{2.12}$$

In an analogous way we find

$$\begin{aligned} E_c(t) &= \int_{-\infty}^{\bar{Y}_c(t)} e^{cy}\Big(\frac{1}{2}v_y^2(y,t) + F(v(y,t))\Big)\,\mathrm{d}y \\ &\quad + \int_{\bar{Y}_c(t)}^{\infty} e^{cy}\Big(\frac{1}{2}v_y^2(y,t) + F(v(y,t))\Big)\,\mathrm{d}y \\ &\le \frac{K}{c} e^{c\bar{Y}_c(t)} + \int_{\bar{Y}_c(t)}^{\infty} e^{cy}\Big(\frac{1}{2}v_y^2(y,t) + \frac{\beta_2}{2}v^2(y,t)\Big)\,\mathrm{d}y\ , \end{aligned} \tag{2.13}$$

where $K = (B^2/2) + K'$ and $K' = \sup\{F(u) | |u| \le B\}$. If we now combine (2.12) and (2.13), and choose the particular value $d = \beta_2 - (\beta_2 - \beta_1)^2/(\beta_2 + c^2/4) \ge 0$, we arrive at

$$D_c(t) \ge \gamma E_c(t) - \frac{C_2}{c} e^{c\bar{Y}_c(t)} , \quad t \ge 0 , \tag{2.14}$$

where

$$0 < \gamma \le \frac{1}{2}\frac{(c^2 + 4\beta_1)^2}{c^2 + 4\beta_2} , \quad \text{and} \quad C_2 = C_1 B^2 + \gamma K.$$

Inequality (2.14) means that, if the invasion point $\bar{Y}_c(t)$ is large and negative, the energy dissipation $D_c = -E_c'$ is essentially proportional to the energy itself. This gives a differential inequality for $E_c(t)$ which, in view of Lemma 2.2, can be integrated as follows:

$$E_c(t) \le e^{-\gamma(t-t_0)} E_c(t_0) + \frac{C_2 T_0}{c} e^{c(\bar{y}_c(t_0)+C_0)} , \quad t \in [t_0, t_0 + T_0]. \tag{2.15}$$

Remark 2.3. The constants C_0, C_1 and T_0 introduced in this section depend only on the potential F. In particular, they are independent of the solution $u(x,t)$ and of the speed parameter c. Similarly, if we choose $\kappa = \sqrt{\beta_1}/2$ and $\gamma = 2\beta_1^2/\beta_2$, then the constants κ, γ and C_2 depend only on F.

3. Existence of the invasion speed

As in the previous section, we suppose that $u(x,t)$ is a solution of (1.1) with initial data u_0 satisfying the assumptions of Proposition 1.2. We also assume that the bounds (2.1) hold for all $t \ge 0$. If $\bar{x}(t)$ denotes the invasion point (1.17), we define

$$c_- = \liminf_{t\to\infty} \frac{\bar{x}(t)}{t} , \qquad c_+ = \limsup_{t\to\infty} \frac{\bar{x}(t)}{t}. \tag{3.1}$$

Our first result shows that the solution $u(x,t)$ invades the stable equilibrium $u = 0$ at a positive, but finite, speed.

Proposition 3.1. *One has $c_- > 0$ and $c_+ < \infty$.*

Proof. The proof relies on the lower bound (2.9). Assume that the initial data u_0 belong to $H^1_{c_0}(\mathbb{R})$ for some $c_0 > \sqrt{2A}/\varepsilon$, where $A = -F(1)$ and ε is as in (1.15). Using (2.9) with $c = c_0$ and $\kappa = c/2$, we find that $E_c(t) \ge \alpha e^{c\bar{y}_c(t)}$ for some $\alpha > 0$. Since $E_c(t) \le E_c(0)$ for all $t \ge 0$, it follows that $\bar{y}_c(t) = \bar{x}(t) - ct$ is bounded from above; hence,

$$c_+ = c + \limsup_{t\to\infty} \frac{\bar{y}_c(t)}{t} \le c < \infty.$$

On the other hand, since $u_0 - 1 \in H^1(\mathbb{R}_-)$ and $F(1) = -A < 0$, it is easy to verify that $E_c(0) = \mathcal{E}_c[u_0] \sim -A/c$ as $c \to 0$. Thus if we take $c > 0$ sufficiently small so that $E_c(0) < 0$, it follows from (2.9) that $0 > E_c(0) \geq E_c(t) \geq (-A/c)e^{c\bar{y}_c(t)}$ for all $t \geq 0$. This implies that $\bar{y}_c(t) = \bar{x}(t) - ct$ is bounded from below; hence,

$$c_- = c + \liminf_{t\to\infty} \frac{\bar{y}_c(t)}{t} \geq c > 0.$$

This concludes the proof. □

We next prove that the average invasion speed $\bar{x}(t)/t$ converges to a limit as $t \to \infty$.

Proposition 3.2. *One has $c_- = c_+$.*

Proof. We argue by contradiction. Assume that $c_- < c_+$, and choose time sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{t'_n\}_{n\in\mathbb{N}}$ such that $t_n \to \infty$, $t'_n \to \infty$ and

$$\frac{\bar{x}(t'_n)}{t'_n} \xrightarrow[n\to\infty]{} c_- \;, \qquad \frac{\bar{x}(t_n)}{t_n} \xrightarrow[n\to\infty]{} c_+ .$$

Due to Lemma 2.1, upon extracting a subsequence we can assume that $u(\bar{x}(t_n) + z, t_n)$ converges in $H^2_{\mathrm{loc}}(\mathbb{R})$ to some limit $w_\infty(z)$. More precisely, for any $L > 0$,

$$\begin{aligned} u(\bar{x}(t_n) + z, t_n) &\xrightarrow[n\to\infty]{} w_\infty(z) \quad \text{in} \quad H^2([-L, L]) \;, \\ u_t(\bar{x}(t_n) + z, t_n) &\xrightarrow[n\to\infty]{} \hat{w}_\infty(z) \quad \text{in} \quad L^2([-L, L]) \;, \end{aligned}$$

where $w_\infty \in H^2_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\hat{w}_\infty \in L^\infty(\mathbb{R})$ satisfy $\hat{w}_\infty = w''_\infty - F'(w_\infty)$. Moreover, by definition of the invasion point, we have $|w_\infty(0)| = \varepsilon$.

Now, we fix any $c \in (c_-, c_+)$, and we observe that the invasion point $\bar{y}_c(t) = \bar{x}(t) - ct$ satisfies $\bar{y}_c(t'_n) \to -\infty$ and $\bar{y}_c(t_n) \to +\infty$ as $n \to \infty$. Using first the lower bound (2.9), we find

$$E_c(t'_n) \geq -\frac{A}{c} e^{c\bar{y}_c(t'_n)} \xrightarrow[n\to\infty]{} 0 \;; \tag{3.2}$$

hence (since E_c is non-increasing), $E_c(t) \geq 0$ for all $t \geq 0$. As $E'_c(t) = -D_c(t)$, we deduce that

$$\int_0^\infty D_c(t)\,\mathrm{d}t \leq E_c(0) < \infty \;, \tag{3.3}$$

and using in addition (2.10) we conclude that $D_c(t) \to 0$ as $t \to \infty$.

Next, we observe that, for all $n \in \mathbb{N}$,

$$\begin{aligned} D_c(t_n) &= \int_{\mathbb{R}} e^{cy} v_t^2(y,t_n)\,\mathrm{d}y = e^{c\bar{y}_c(t_n)} \int_{\mathbb{R}} e^{cz} v_t^2(\bar{y}_c(t_n)+z,t_n)\,\mathrm{d}z \\ &= e^{c\bar{y}_c(t_n)} \int_{\mathbb{R}} e^{cz} (u_t + cu_x)^2(\bar{x}(t_n)+z,t_n)\,\mathrm{d}z. \end{aligned} \tag{3.4}$$

Since $D_c(t_n) \to 0$ as $n \to \infty$, the last integral in (3.4) converges to zero as $n \to \infty$; hence, the limits w_∞ and $\hat{w}_\infty$ satisfy $\hat{w}_\infty + cw'_\infty = 0$. Incidentally, this means that $w''_\infty + cw'_\infty - F'(w_\infty) = 0$; i.e., w_∞ is a travelling-wave solution of (1.1) with speed c. Now the crucial point is that $c \in (c_-, c_+)$ is *arbitrary*. Obviously, the relation $\hat{w}_\infty + cw'_\infty = 0$ can be satisfied for two different values of c only if $w'_\infty \equiv 0$, i.e., if w_∞ is identically constant. But then we must have $F'(w_\infty) = 0$, which is impossible in view of (2.6) since $|w_\infty| = \varepsilon$. This contradicts the assumption $c_- < c_+$ and concludes the proof. □

Remark 3.3. Another way to obtain a contradiction in the proof of Proposition 3.2, which works even if $u = 0$ is not a strict local minimum of F (see hypothesis **H2** in the introduction), is to observe that the limiting function $w_\infty(z)$ converges to zero as $z \to +\infty$. Indeed, proceeding as in (2.7) and (3.4), and using (2.11), we find for all $n \in \mathbb{N}$

$$\begin{aligned} E_c(t_n) &= e^{c\bar{y}_c(t_n)} \int_{\mathbb{R}} e^{cz} \Big(\frac{1}{2}u_x^2 + F(u)\Big)(\bar{x}(t_n)+z,t_n)\,\mathrm{d}z \\ &\geq e^{c\bar{y}_c(t_n)} \int_0^\infty e^{cz} \Big(\frac{1}{2}u_x^2 + \frac{\beta_1}{2}u^2\Big)(\bar{x}(t_n)+z,t_n)\,\mathrm{d}z - \frac{A}{c} e^{c\bar{y}_c(t_n)} \\ &\geq e^{c\bar{y}_c(t_n)} \frac{\kappa'}{2} \int_0^\infty e^{cz} (u_x^2 + u^2)(\bar{x}(t_n)+z,t_n)\,\mathrm{d}z - \frac{A}{c} e^{c\bar{y}_c(t_n)}, \end{aligned} \tag{3.5}$$

where $\kappa' = \min\{1, (c^2+4\beta_1)(c^2+4)^{-1}\}$. As $E_c(t_n) \leq E_c(0)$ and $\bar{y}_c(t_n) \to +\infty$ as $n \to \infty$, we have by Fatou's lemma

$$\begin{aligned} &\int_0^\infty e^{cz} (w'_\infty(z)^2 + w_\infty(z)^2)\,\mathrm{d}z \\ &\qquad \leq \liminf_{n\to\infty} \int_0^\infty e^{cz} (u_x^2 + u^2)(\bar{x}(t_n)+z,t_n)\,\mathrm{d}z \leq \frac{2A}{c\kappa'} < \infty. \end{aligned}$$

Thus $w_\infty \in H_c^1(\mathbb{R})$, and in particular $w_\infty(z) \to 0$ as $z \to +\infty$. This is clearly impossible if $w'_\infty \equiv 0$ and $|w_\infty(0)| = \varepsilon$.

4. Local convergence to a travelling wave

This section is devoted to the proof of Proposition 1.2. Using the same notation as in the previous sections, we first prove that the solution $u(x,t)$ of (1.1) converges for a *sequence of times* towards a travelling wave, locally in space around the invasion point. On this occasion we identify the invasion speed given by Proposition 3.2 with the unique speed c_* for which travelling waves exist.

Proposition 4.1. *Let $c_\infty = c_- = c_+$. There exists a sequence $t_n \to \infty$ such that, for all $L > 0$,*

$$\int_{-L}^{L} e^{c_\infty z}(u_t + c_\infty u_x)^2(\bar{x}(t_n) + z, t_n)\,\mathrm{d}z \xrightarrow[n\to\infty]{} 0. \tag{4.1}$$

Proof. Since the left-hand side of (4.1) is a nondecreasing function of L, it is sufficient to prove that, for any $L > 0$, there exists a sequence $t_n \to \infty$ such that (4.1) holds. We argue by contradiction and assume that there exist $L > 0$ and $\delta > 0$ such that

$$\int_{-L}^{L} e^{c_\infty z}(u_t + c_\infty u_x)^2(\bar{x}(t) + z, t)\,\mathrm{d}z \ge \delta\ , \tag{4.2}$$

for all sufficiently large t. In fact, upon changing the origin of time, we can assume that (4.2) holds for all $t \ge 0$. In analogy with (2.5), we denote $\bar{y}(t) = \bar{x}(t) - c_\infty t$. Two situations may occur:

Case 1: There exists a time sequence $t_n \to \infty$ such that $\{\bar{y}(t_n)\}_{n\in\mathbb{N}}$ is bounded from below. Without loss of generality we can assume that $t_{n+1} \ge t_n + 1$ and $\bar{y}(t_n) \ge 1$ for all $n \in \mathbb{N}$ (the second condition is easily achieved by translating the origin).

Let $K > 0$ be such that $E_c(0) \le K$ for all $c \in [c_\infty, c_0]$, where $c_0 > c_\infty$ is as in the proof of Proposition 3.1. Take $c \in (c_\infty, c_0)$ sufficiently close to c_∞ so that

$$e^{(c-c_\infty)L} \le 2\ , \quad \text{and} \quad (c-c_\infty)^2 B^2 \int_{-L}^{L} e^{c_\infty z}\,\mathrm{d}z \le \frac{\delta}{4}\ , \tag{4.3}$$

where B is as in (2.1). Let $\bar{y}_c(t) = \bar{x}(t) - ct = \bar{y}(t) - (c - c_\infty)t$. Since $\bar{y}(t_n) \ge 1$ for all $n \in \mathbb{N}$, it is clear that the cardinality of the set

$$S_c = \{n \in \mathbb{N} : \bar{y}_c(t_n) \ge 0\} = \{n \in \mathbb{N} : \bar{y}(t_n) \ge (c - c_\infty)t_n\}$$

becomes arbitrarily large as $c \to c_\infty$; see Figure 2. On the other hand, $\bar{y}_c(t_n) \to -\infty$ as $n \to \infty$, and this implies (as in the proof of Proposition 3.2)

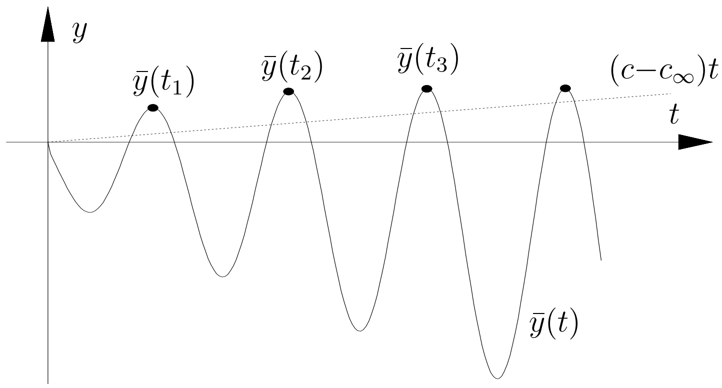

Fig. 2: If there exists a sequence $t_n \to \infty$ such that $\bar{y}(t_n)$ is bounded from below, a contradiction is obtained by considering the dissipation of the energy E_c in a moving frame with speed $c > c_\infty$ (c close to c_∞). If $\bar{y}(t_n) \geq 1$ for all $n \in \mathbb{N}$, the set S_c consisting of all n such that $\bar{y}(t_n) \geq (c - c_\infty)t_n$ increases as $c \to c_\infty$, and $\mathrm{card}(S_c) \to \infty$.

that $E_c(t) \geq 0$ for all $t \geq 0$. But for all $n \in S_c$, we have by (3.4), (4.2) and (4.3)

$$\begin{aligned} D_c(t_n) &= e^{c\bar{y}_c(t_n)} \int_{\mathbb{R}} e^{cz}(u_t + cu_x)^2(\bar{x}(t_n) + z, t_n)\,\mathrm{d}z \\ &\geq \int_{-L}^{L} e^{cz}(u_t + cu_x)^2(\bar{x}(t_n) + z, t_n)\,\mathrm{d}z \geq \frac{\delta}{8}\ , \end{aligned}$$

because $e^{cz} \geq \frac{1}{2}e^{c_\infty z}$ for $|z| \leq L$ and $(u_t + cu_x)^2 \geq \frac{1}{2}(u_t + c_\infty u_x)^2 - (c - c_\infty)^2B^2$. Moreover, it follows from (2.10) that $D_c(t) \geq D_c(t_n)e^{-C_1}$ for all $t \in [t_n - 1, t_n]$; hence,

$$E_c(t_n - 1) - E_c(t_n) = \int_{t_n-1}^{t_n} D_c(t)\,\mathrm{d}t \geq \frac{\delta}{8}e^{-C_1}\ , \quad n \in S_c.$$

If we choose c close enough to c_∞ so that $\mathrm{card}(S_c) > 8Ke^{C_1}/\delta$, we obtain a contradiction with the fact that $E_c(t)$ is positive, nonincreasing, and $E_c(0) \leq K$.

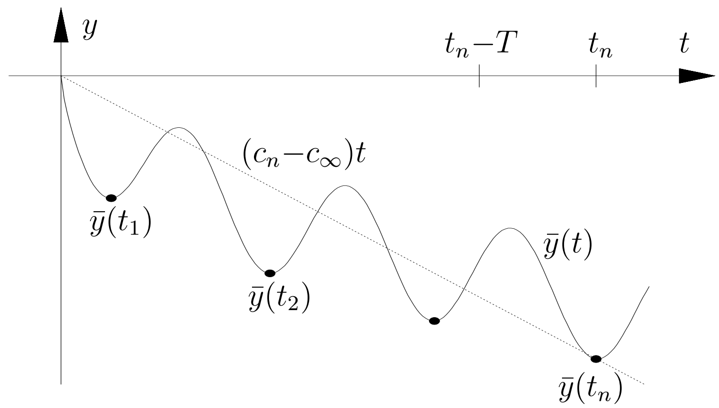

Fig. 3: If $\bar{y}(t) \to -\infty$ a contradiction is obtained by considering the dissipation of the energy E_{c_n} in a moving frame with speed $c_n < c_\infty$ on the time interval $[0, t_n]$, where $\bar{y}(t_n) = (c_n - c_\infty)t_n$. We choose $T \gg 1$ and then n large enough so that $(c_\infty - c_n)T \le 1$.

Case 2: $\bar{y}(t) \to -\infty$ as $t \to \infty$. In this case, there exists a sequence $t_n \to \infty$ such that

$$\bar{y}(t_n) \le \inf_{0 \le s \le t_n} \bar{y}(s) + 1 \ , \quad \text{for all } n \in \mathbb{N} \ ; \tag{4.4}$$

see Figure 3. Indeed, the function $\mu(t) = \inf\{\bar{y}(s) | 0 \le s \le t\}$ is nonincreasing and $\mu(t) \to -\infty$ as $t \to \infty$. For each $n \in \mathbb{N}$, we choose $t_n \in [0, n]$ such that $\bar{y}(t_n) \le \mu(n) + 1$. Then $\mu(n) \le \mu(t_n)$; hence, (4.4) holds.

Given some (large) $n \in \mathbb{N}$, we take $c_n < c_\infty$ such that $(c_n - c_\infty)t_n = \bar{y}(t_n)$, or equivalently $\bar{y}_{c_n}(t_n) = 0$. Since $c_n \to c_\infty$ as $n \to \infty$, we can assume that $c_n \ge c_\infty/2$ and

$$e^{(c_\infty - c_n)L} \le 2 \ , \quad (c_\infty - c_n)^2 B^2 \int_{-L}^{L} e^{c_\infty z} \,\mathrm{d}z \le \frac{\delta}{4}. \tag{4.5}$$

If $t \in [0, t_n]$, we have by (4.4)

$$\bar{y}_{c_n}(t) = \bar{y}(t) + (c_\infty - c_n)t \ge \bar{y}(t_n) - 1 + (c_\infty - c_n)t = (c_n - c_\infty)(t_n - t) - 1.$$

Using (4.2) and (4.5), and proceeding as in the previous case, we obtain

$$D_{c_n}(t) = e^{c_n \bar{y}_{c_n}(t)} \int_{\mathbb{R}} e^{c_n z} (u_t + c_n u_x)^2 (\bar{x}(t) + z, t) \,\mathrm{d}z$$

$$\geq e^{c_n((c_n-c_\infty)(t_n-t)-1)} \int_{-L}^{L} e^{c_n z}(u_t + c_n u_x)^2(\bar{x}(t_n)+z, t_n)\,\mathrm{d}z$$
$$\geq e^{c_\infty((c_n-c_\infty)(t_n-t)-1)} \frac{\delta}{8};$$

hence, for all $T \leq t_n$,

$$\int_{t_n-T}^{t_n} D_{c_n}(t)\,\mathrm{d}t \geq T e^{c_\infty((c_n-c_\infty)T-1)} \frac{\delta}{8}. \tag{4.6}$$

On the other hand, there exists $K > 0$ such that $E_{c_n}(0) \leq K$ for all n, and since $\bar{y}_{c_n}(t_n) = 0$ we know from (2.9) that $E_{c_n}(t_n) \geq -A/c_n$. Thus

$$\int_0^{t_n} D_{c_n}(t)\,\mathrm{d}t = E_{c_n}(0) - E_{c_n}(t_n) \leq K + \frac{A}{c_n} \leq K + \frac{2A}{c_\infty}. \tag{4.7}$$

If we now choose $T > 0$ large enough so that $Te^{-2c_\infty}\delta > 8K + 16A/c_\infty$, and then $n \in \mathbb{N}$ large enough so that $t_n \geq T$ and $(c_\infty - c_n)T \leq 1$, we obtain the desired contradiction by comparing (4.6) and (4.7). $\square$

Corollary 4.2. *One has $c_- = c_+ = c_*$, and there exists a sequence $t_n \to \infty$ such that, for all $L > 0$,*

$$\sup_{z\in[-L,L]} |u(\bar{x}(t_n)+z, t_n) - h_\varepsilon(z)| \xrightarrow[n\to\infty]{} 0.$$

Proof. We argue as in the proof of Proposition 3.2. If $\{t_n\}_{n\in\mathbb{N}}$ is the sequence given by Proposition 4.1, we know that (upon extracting a subsequence) $u(\bar{x}(t_n)+z, t_n)$ converges in $H^2_{\mathrm{loc}}(\mathbb{R})$ to a limit $w_\infty(z)$ which satisfies

$$w_\infty''(z) + c_\infty w_\infty'(z) - F'(w_\infty(z)) = 0\ , \quad z \in \mathbb{R}.$$

Moreover, $|w_\infty(z)| \leq \varepsilon$ for all $z \geq 0$, $|w_\infty(z)| \leq B$ for all $z \leq 0$, and $|w_\infty(0)| = \varepsilon$. Arguing as in Remark 3.3, one can also show that $w_\infty(z) \to 0$ as $z \to +\infty$. These properties together imply that $c_\infty = c_*$ and that $w_\infty = h_\varepsilon$; see hypothesis **H3** in the introduction. $\square$

Corollary 4.3. *For all $c \geq c_*$ and all $w \in H^1_c(\mathbb{R})$, one has*

$$\mathcal{E}_c[w] = \int_{\mathbb{R}} e^{cx}\Big(\frac{1}{2}w'(x)^2 + F(w(x))\Big)\,\mathrm{d}x \geq 0.$$

Proof. Assume first that $w \in H^1_{c'}(\mathbb{R})$ for all $c' > 0$, and that $w - 1 \in H^1(\mathbb{R}_-)$. If $u(x,t)$ is the solution of (1.1) with initial data $u(x,0) = w(x)$, we know from Proposition 3.2 and Corollary 4.2 that the invasion point $\bar{x}(t)$ defined by (1.17) satisfies $\bar{x}(t)/t \to c_*$ as $t \to \infty$. Thus, for any $c > c_*$, the quantity

$\bar{y}_c(t) = \bar{x}(t) - ct$ converges to $-\infty$ as $t \to \infty$, so that $E_c(t) \geq 0$ for all $t \geq 0$. In particular, $E_c(0) = \mathcal{E}_c[w] \geq 0$. Letting $c \to c_*$, we also obtain $\mathcal{E}_{c_*}[w] \geq 0$.

Assume now that $c \geq c_*$ and that $w \in H^1_c(\mathbb{R})$. For any $n \geq 1$ we define

$$w_n(x) = w(x)\chi(x-n) + (1 - w(x))\chi(x+n+1) \ , \quad x \in \mathbb{R} \ ,$$

where $\chi \in \mathcal{C}^\infty(\mathbb{R})$, $\chi(x) = 1$ for $x \leq 0$ and $\chi(x) = 0$ for $x \geq 1$. Clearly $w_n(x) = w(x)$ for $x \in [-n, n]$, whereas $w_n(x) = 0$ for $x \geq n+1$ and $w_n(x) = 1$ for $x \leq -n-1$. Thus $w_n \in H^1_{c'}(\mathbb{R})$ for all $c' > 0$ and $w_n - 1 \in H^1(\mathbb{R}_-)$, so that $\mathcal{E}_c[w_n] \geq 0$ for all $n \in \mathbb{N}$ by the preceding argument. Moreover, it is straightforward to verify that $\mathcal{E}_c[w_n] \to \mathcal{E}_c[w]$ as $n \to \infty$, hence $\mathcal{E}_c[w] \geq 0$. $\square$

Equipped with these results, we are now able to prove that the solution $u(x,t)$ converges for *all times* towards a travelling wave, locally in space around the invasion point.

Proposition 4.4. *For all $L > 0$ we have*

$$\int_{-L}^{L} e^{c_* z}(u_t + c_* u_x)^2(\bar{x}(t) + z, t)\,\mathrm{d}z \xrightarrow[t\to\infty]{} 0. \tag{4.8}$$

Proof. We argue by contradiction and assume that there exist $L > 0$, $\delta > 0$, and a sequence $t_n \to \infty$ such that

$$\int_{-L}^{L} e^{c_* z}(u_t + c_* u_x)^2(\bar{x}(t_n) + z, t_n)\,\mathrm{d}z \geq \delta \ , \tag{4.9}$$

for all $n \in \mathbb{N}$. Let $\bar{y}(t) = \bar{x}(t) - c_* t$. If the sequence $\{\bar{y}(t_n)\}_{n\in\mathbb{N}}$ has a subsequence that is bounded from below, then we easily get a contradiction as in the proof of Proposition 4.1 (case 1). So it remains to consider the case where $\bar{y}(t_n) \to -\infty$, which requires a new argument. Without loss of generality, we can suppose that $t_{n+1} \geq t_n + T_0$ for all $\in \mathbb{N}$, where $T_0 > 0$ is as in Lemma 2.2, and that $\bar{y}(t_n) \leq -n-1$. Upon extracting a subsequence, we can also assume that $u(\bar{x}(t_n) + z, t_n)$ converges in $H^2_{\mathrm{loc}}(\mathbb{R})$ towards a limit $w_\infty(z)$.

Given some (large) $n \in \mathbb{N}$, we take $c_n < c_*$ such that $\bar{y}(t_n) = (c_n - c_*)t_n$; see Figure 4. Since $c_n \to c_*$ as $n \to \infty$, we can assume that $c_n \geq c_*/2$. Let $\bar{y}_{c_n}(t) = \bar{y}(t) + (c_* - c_n)t$, so that $\bar{y}_{c_n}(t_n) = 0$. For each $k = 0, 1, \ldots, n$ we have by (2.15)

$$E_{c_n}(t_{k+1}) \leq E_{c_n}(t_k + T_0) \leq e^{-\gamma T_0} E_{c_n}(t_k) + \frac{C_2 T_0}{c_n} e^{c_n(\bar{y}_{c_n}(t_k) + C_0)} \ ;$$

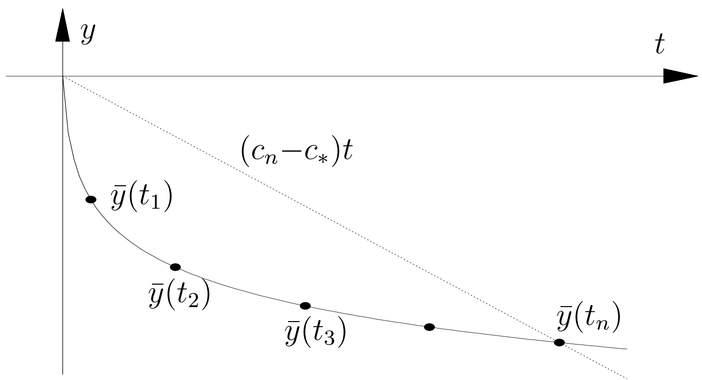

Fig. 4: If $\bar{y}(t_n) \to -\infty$ a contradiction is obtained by considering the dissipation of the energy E_{c_n} in a moving frame with speed $c_n < c_*$ on the time interval $[0, t_n]$, where $\bar{y}(t_n) = (c_n - c_*)t_n$.

hence,

$$E_{c_n}(t_k) \le e^{-k\gamma T_0} E_{c_n}(t_0) + \frac{C_2 T_0 e^{c_n C_0}}{c_n} \sum_{j=1}^{k} e^{-(j-1)\gamma T_0} e^{c_n \bar{y}_{c_n}(t_{k-j})}. \tag{4.10}$$

We now define $k(n)$ as the largest integer $k \in \mathbb{N}$ such that

$$(c_* - c_n)t_j \le 1 + \frac{j}{2}\,, \quad \text{for all} \quad j = 0, 1, \ldots, k.$$

Since $c_n \to c_*$, it is clear that $k(n) \to \infty$ as $n \to \infty$. Moreover, $k(n) < n$ as $(c_* - c_n)t_n = -\bar{y}(t_n) \ge n + 1$ by assumption. For $k = k(n)$ and $j \le k$, we have

$$\bar{y}_{c_n}(t_{k-j}) = \bar{y}(t_{k-j}) + (c_* - c_n)t_{k-j} \le -(k-j)/2\ ;$$

hence, it follows from (4.10) that

$$E_{c_n}(t_n) \le E_{c_n}(t_{k(n)}) \le e^{-k(n)\gamma T_0} E_{c_n}(0) + \frac{C_2 T_0 e^{c_n C_0}}{c_n} k(n) e^{-\gamma^*(k(n)-1)}\ ,$$

where $\gamma^* = \min(\gamma T_0, c_*/4)$. Taking the limit $n \to \infty$ and using the fact that $E_{c_n}(0)$ is uniformly bounded, we conclude that

$$\limsup_{n\to\infty} E_{c_n}(t_n) \le 0.$$

Now, since $\bar{y}_{c_n}(t_n) = 0$ by our choice of c_n, we have

$$E_{c_n}(t_n) = \int_{\mathbb{R}} e^{c_n z}\Big(\frac{1}{2}u_x^2 + F(u)\Big)(\bar{x}(t_n)+z, t_n)\,\mathrm{d}z\ ;$$

hence, taking the limit $n \to \infty$ and using Fatou's lemma we obtain

$$\mathcal{E}_{c_*}[w_\infty] = \int_{\mathbb{R}} e^{c_* z}\Big(\frac{1}{2}w_\infty'(z)^2 + F(w_\infty(z))\Big)\,\mathrm{d}z \le \liminf_{n\to\infty} E_{c_n}(t_n) \le 0.$$

In particular, $w_\infty \in H^1_{c_*}(\mathbb{R})$; hence, it follows from Corollary 4.3 that $\mathcal{E}_{c_*}[w_\infty] = 0$. On the other hand, in view of (4.9), we have

$$\begin{aligned}
\mathcal{D}_{c_*}[w_\infty] &= \int_{\mathbb{R}} e^{c_* z}\Big(w_\infty''(z) + c_* w_\infty'(z) - F'(w_\infty(z))\Big)^2\,\mathrm{d}z \\
&\ge \int_{-\infty}^{L} e^{c_* z}\Big(w_\infty''(z) + c_* w_\infty'(z) - F'(w_\infty(z))\Big)^2\,\mathrm{d}z \\
&= \lim_{n\to\infty}\int_{-\infty}^{L} e^{c_* z}(u_t + c_* u_x)^2(\bar{x}(t_n)+z, t_n)\,\mathrm{d}z \ge \delta.
\end{aligned}$$

Thus, if $u(x,t)$ is the solution of (1.1) with initial data $u(x,0) = w_\infty(x)$, then $E_{c_*}(0) = \mathcal{E}_{c_*}[w_\infty] = 0$ and $E_{c_*}'(0) = -\mathcal{D}_{c_*}[w_\infty] \le -\delta$; hence, $E_{c_*}(t) < 0$ for all $t > 0$. This contradicts the conclusion of Corollary 4.3. □

It is now a straightforward task to conclude the proof of Proposition 1.2. Using Proposition 4.4 and proceeding as in Corollary 4.2, we see that $u(\bar{x}(t)+z, t)$ converges to $w_\infty(z) \equiv h_\varepsilon(z)$ in $H^2([-L, L])$ for any $L > 0$. On the other hand, arguing as in (3.5), we find for any $c \in (0, c_*)$,

$$\limsup_{t\to\infty}\int_0^\infty e^{cz}(u_x^2 + u^2)(\bar{x}(t)+z, t)\,\mathrm{d}z \le \frac{2A}{c\kappa'} < \infty.$$

This implies in particular that $u(\bar{x}(t)+z, t)$ converges to zero as $z \to +\infty$ uniformly in $t \ge 0$; hence, $u(\bar{x}(t)+z, t)$ converges as $t \to \infty$ to $h_\varepsilon(z)$ uniformly for all $z \in [-L, +\infty)$. This proves (1.18).

It remains to verify that the map $t \mapsto \bar{x}(t)$ is $\mathcal{C}^1$ for large t and satisfies $\bar{x}'(t) \to c_*$ as $t \to \infty$. Using (1.18), (2.1), and an interpolation argument, we find for any $L > 0$,

$$\sup_{z\in[-L,L]} |u_x(\bar{x}(t)+z, t) - h_\varepsilon'(z)| \xrightarrow[t\to\infty]{} 0.$$

As $h_\varepsilon'(0) < 0$, this implies in particular that $u_x(\bar{x}(t), t)$ is bounded away from zero for t sufficiently large. Since $u(\bar{x}(t), t) = \varepsilon$ for t large, the implicit function theorem then asserts that $\bar{x}(t)$ is differentiable with

$$\bar{x}'(t) = -\frac{u_t(\bar{x}(t), t)}{u_x(\bar{x}(t), t)} \,, \quad \text{for all sufficiently large } t > 0.$$

On the other hand, $\sup_{|z|\le L} |u_t(\bar{x}(t){+}z, t) + c_* u_x(\bar{x}(t){+}z, t)| \to 0$ as $t \to \infty$ by (2.1) and (4.8); hence, $\bar{x}'(t) \to c_*$ as $t \to \infty$. □

5. Repair behind the front

This final section is devoted to the proof of Corollary 1.3. We follow closely the arguments given in [13, Section 9.6], with a few simplifications.

Let $u(x,t)$ be a solution of (1.1) with initial data u_0 satisfying the assumptions of Proposition 1.2. According to (1.18), we can find a time sequence $t_n \to \infty$ such that $t_{n+1} \ge t_n + n + 1$ for all $n \in \mathbb{N}$, and

$$\sup_{z\in[-2n,+\infty)} |u(\bar{x}(t) + z, t) - h_\varepsilon(z)| \le \frac{1}{n+1} \,, \quad \text{for all } t \ge t_n. \tag{5.1}$$

Let $\theta : \mathbb{R} \to [0,1]$ be a smooth, nondecreasing function satisfying $\theta(x) = 0$ for $x \le 0$ and $\theta(x) = 1$ for $x \ge 1$. We define a map $\hat{x} : [0, +\infty) \to \mathbb{R}$ by imposing, for all $n \in \mathbb{N}$,

$$\hat{x}(t) = \bar{x}(t) - n - \theta\Big(\frac{t - t_n}{t_{n+1} - t_n}\Big) \,, \quad \text{for all } t \in [t_n, t_{n+1}].$$

It is clear that $\bar{x}(t) - n - 1 \le \hat{x}(t) \le \bar{x}(t) - n$ for all $t \in [t_n, t_{n+1}]$. Moreover, there exists $T > 0$ such that $\hat{x}(t)$ is differentiable for $t \ge T$, with $\hat{x}'(t) \le \bar{x}'(t) \le c_* + 1$ for all $t \ge T$. For later use we observe that, for any $L > 0$,

$$\sup_{z\in[-L,L]} \Big(|u(\hat{x}(t) + z, t) - 1| + |u_x(\hat{x}(t) + z, t)|\Big) \xrightarrow[t\to\infty]{} 0. \tag{5.2}$$

Indeed, since $\hat{x}(t) \approx \bar{x}(t) - n$ for $t \in [t_n, t_{n+1}]$, the estimate on $|u - 1|$ is a consequence of (5.1) and of the fact that $h_\varepsilon(z) \to 1$ as $z \to -\infty$. The result for $|u_x|$ then follows from the a priori bound (2.1) by interpolation.

We next consider the truncated energy function

$$E(t) = \int_{\mathbb{R}} \phi(x,t) \Big(\frac{1}{2} u_x^2(x,t) + \overline{F}(u(x,t))\Big)\, \mathrm{d}x \,,$$

where $\overline{F}(u) = F(u) - F(1) \ge 0$ and

$$\phi(x,t) = \begin{cases} 1 & \text{if} \quad x \le \hat{x}(t) \,, \\ e^{\hat{x}(t) - x} & \text{if} \quad x \ge \hat{x}(t). \end{cases}$$

Since $u(\cdot,t)-1 \in H^1(\mathbb{R}_-)$ and $u(\cdot,t) \in H^1(\mathbb{R}_+)$, it is clear that $E(t)$ is well-defined and finite for all $t \geq 0$. Moreover, $E(t)$ is differentiable for $t \geq T$, and a direct calculation shows that

$$\begin{aligned} E'(t) &= -\int_{\mathbb{R}} \phi(x,t) u_t^2(x,t)\,\mathrm{d}x + \int_{\hat{x}(t)}^{\infty} \phi(x,t)\Big\{\hat{x}'(t)\Big(\frac{1}{2}u_x^2 + \overline{F}(u)\Big) + u_x u_t\Big\}\,\mathrm{d}x \\ &\leq -\frac{1}{2}\int_{\mathbb{R}} \phi(x,t) u_t^2(x,t)\,\mathrm{d}x + (c_*{+}1)\int_0^{\infty} e^{-z}(u_x^2 + \overline{F}(u))(\hat{x}(t)+z,t)\,\mathrm{d}z. \end{aligned}$$

In view of (2.1) and (5.2), the last integral in the right-hand side converges to zero as $t \to \infty$. Since $E(t) \geq 0$ for all $t \geq 0$, it follows that there exists a time sequence $t'_n \to \infty$ such that

$$\int_{\mathbb{R}} \phi(x,t'_n) u_t^2(x,t'_n)\,\mathrm{d}x \xrightarrow[n\to\infty]{} 0. \tag{5.3}$$

Now, we claim that

$$\sup_{x\in(-\infty,\hat{x}(t'_n)]} |u(x,t'_n) - 1| \xrightarrow[n\to\infty]{} 0. \tag{5.4}$$

Indeed, if this is not the case, there exist a positive constant ε', a subsequence $\{t''_n\}_{n\in\mathbb{N}}$ of $\{t'_n\}_{n\in\mathbb{N}}$, and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $x_n \leq \hat{x}(t''_n)$ and $|u(x_n,t''_n) - 1| = \varepsilon'$ for all $n \in \mathbb{N}$. Without loss of generality, we can assume that $\varepsilon' > 0$ is sufficiently small so that the only bounded solution of the differential equation $w_{xx} - F'(w) = 0$ with $|w(0)-1| \leq \varepsilon'$ is $w \equiv 1$; see hypothesis **H4** in the introduction. In view of (5.2), it is clear that $x_n - \hat{x}(t''_n) \to -\infty$ as $n \to \infty$. On the other hand, upon extracting a subsequence, we can assume that, for all $L > 0$,

$$\begin{aligned} u(x_n + z, t''_n) &\xrightarrow[n\to\infty]{} w_\infty(z) \quad \text{in} \quad H^2([-L,L])\ , \\ u_t(x_n + z, t''_n) &\xrightarrow[n\to\infty]{} \hat{w}_\infty(z) \quad \text{in} \quad L^2([-L,L])\ , \end{aligned}$$

where $w_\infty \in H^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\hat{w}_\infty \in L^\infty(\mathbb{R})$ satisfy $\hat{w}_\infty(z) = w''_\infty(z) - F'(w_\infty(z))$. However, it follows from (5.3) that $\hat{w}_\infty = 0$; hence, w_∞ is a bounded solution of the differential equation $w''_\infty - F'(w_\infty) = 0$ that satisfies $|w_\infty(0) - 1| = \varepsilon'$. This contradicts the assumption above on ε'; hence, (5.4) must hold.

Finally, if we combine (5.1) and (5.4), we obtain

$$\sup_{z\in\mathbb{R}} |u(\bar{x}(t'_n) + z, t'_n) - h_\varepsilon(z)| \xrightarrow[n\to\infty]{} 0.$$

In other words, the solution $u(x,t)$ approaches uniformly on $\mathbb{R}$ a translate of the travelling wave h_ε for a sequence of times $t'_n \to \infty$. On the other hand, the classical results of Sattinger [17] show that, if assumptions (1.3) and (1.4) are satisfied, the travelling wave h is *asymptotically stable with shift* in the space $L^\infty(\mathbb{R})$. In other words, equation (1.8) holds for any solution of (1.1) which is sufficiently close (uniformly on $\mathbb{R}$) to a translate of h. This is the case for $u(\cdot, t'_n)$ if n is sufficiently large; hence, Corollary 1.3 is proved. □

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