# Uniqueness of the pendent drop of infinite length

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July 16, 2016

We prove the uniqueness of the infinite length axisymmetric solution to the capillary equation. We observe that capillary equation can be viewed, at large depth, as a perturbation of an integrable two-dimensional differential system. Uniqueness is then proved by an elementary perturbation argument.

## 1 Introduction

An orientable hypersurface is said to satisfy a prescribed mean curvature equation if the mean curvature at each point of the hypersurface is prescribed by a scalar field. When this scalar field is proportional to the "vertical" coordinate, this equation is commonly referred as the *capillary equation*, since it is satisfied by a static interface between two liquid phases of different densities, where the capillary force (proportional to the mean curvature) has to balance the difference of pressures (which, up to a change of the origin, is proportional to this vertical coordinate).

Let $n$ be an integer not smaller than 2, and let $(x_1,\dots,x_n,z)$ denote the coordinates in $\mathbb{R}^{n+1}$ (the "vertical" coordinate will be $z$). A hypersurface of $\mathbb{R}^{n+1}$ is axisymmetric with respect to the $z$-axis if, at every point of the hypersurface, the distance $r=\sqrt{x_1^2+\cdots+x_n^2}$ to the $z$-axis is a function of $z$. Let us introduce the angle $\theta$ as displayed on figure 1 (namely, satisfying $dr/dz=\tan\theta$). Then, at every point of this axisymmetric hypersurface, the capillary equation reads:

$$(n-1)\frac{\cos\theta}{r}+\cos\theta\,\frac{d\theta}{dz}=\kappa z\,, \qquad (1)$$

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where $\kappa$ is a fixed real quantity. Let us assume without loss of generality that $\kappa$ is positive (the vertical coordinate $z$ increases "downwards"). When $n$ equals 2 equation (1) governs the shape of an axisymmetric liquid pendent drop in $\mathbb{R}^3$.

Up to a change of scale (in both $r$ and $z$), we can choose whatever positive value for $\kappa$. Let us make the convenient choice $\kappa = n-1$; indeed, with this choice, equation (1) can be rewritten as the following two-dimensional differential system:

$$(2) \qquad \begin{cases} dr/dz = -\tan\theta \\ d\theta/dz = (n-1)\Big(\dfrac{z}{\cos\theta} - \dfrac{1}{r}\Big) \end{cases} \qquad \text{where} \quad r > 0 \quad \text{and} \quad \theta \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big).$$

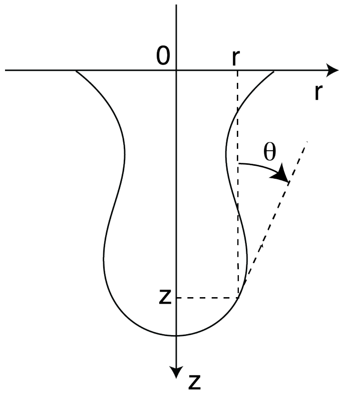


Figure 1: Definition of angle $\theta$.

Drops of finite length correspond to solutions of the axisymmetric capillary equation (1) — or equivalently system (2) — that do not exist for $z$ larger than a certain value $z_{\max}$ (the height at the tip of the drop) where $r(z)$ vanishes. In this paper we are interested in solutions that are defined up to $+\infty$ in $z$, namely drops of infinite length. The main result is the following.

**Theorem 1.** *There exists a unique solution $z \mapsto (r, \theta)$ of system (2) that is defined on $(0, +\infty)$; it satisfies:*

$$r(z) \sim 1/z \quad \textit{and} \quad \theta(z) \to 0 \quad \textit{when} \quad z \to +\infty\,.$$

Historically, the problem goes back to P.-S. Laplace [8], T. Young [12], and Lord Kelvin [11]. From a modern mathematical point of view, the main contributions are due to P. Concus and R. Finn. In [3, 4] (see also Finn's book [7]) they proved the existence of solutions corresponding to drops of every prescribed finite length, and studied their shapes. In [5], they proved the existence of a solution under the form of a function $z$ of the radius $r$, for which $z(r)$ approaches $+\infty$ when $r$ approaches 0, thus corresponding to a drop of infinite length, and computed its asymptotic expansion when $r$ approaches 0. In [6] they achieved a first step towards its uniqueness, by proving uniqueness among all singular solutions asymptotically sufficiently close to that asymptotic expansion. Further steps towards uniqueness were achieved by M.-F. Bidaut-Véron ([1, 2]), who proved uniqueness under a weaker explicit criterion on the closeness to the asymptotic expansion. Finally, the full proof of the uniqueness was achieved by R. Nickolov ([9]) who proved that the Bidaut-Véron uniqueness criterion is actually satisfied by every singular solution.

In [10], C. Riera and the author proposed to view and study system (2) as a dynamical system. This point of view, which surprisingly enough does not seem to have been adopted before, turned out to be fruitful: classical dynamical systems techniques led us to describe the dynamics of all solutions, and to recover in a simple way many results on the shape of pendent drops of finite length, together with the existence of (at least) one singular solution corresponding to a drop of infinite length. The aim of the present paper is to provide, by the same approach, a short, elementary, and self-contained proof of its uniqueness.

## 2 Rescaling and statement of main result

Expression of $d\theta/dz$ in system (2) suggests the following change of variables:

$$(r,z) \mapsto (R,Z) = \Big(zr, \frac{z^2}{2}\Big),$$

which corresponds to a blow-up of factor $z$ at depth $z$, both in the directions of $r$ and $z$, see figures 1 and 3. Rewritten using $(R,Z)$ variables, system (2) becomes:

$$\text{(3)} \qquad \begin{cases} dR/dZ = -\tan\theta + \dfrac{R}{2Z} \\ d\theta/dZ = (n-1)\Big(\dfrac{1}{\cos\theta} - \dfrac{1}{R}\Big) \end{cases} \quad \text{where} \quad Z>0\,, \quad R>0\,, \quad \theta \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)\,,$$

which asymptotically reduces, when $Z$ approaches $+\infty$, to the two-dimensional autonomous differential system

$$\text{(4)} \qquad \begin{cases} dR/dZ = -\tan\theta \\ d\theta/dZ = (n-1)\Big(\dfrac{1}{\cos\theta} - \dfrac{1}{R}\Big) \end{cases}$$

for which the quantity

$$\mathcal{H}(R,\theta) = R^{n-1}\Big(\cos\theta - \frac{n-1}{n}R\Big)$$

is conserved (see figure 2). Indeed,

$$\text{(5)} \qquad \begin{cases} \partial_R\mathcal{H} = (n-1)R^{n-2}(\cos\theta - R) \\ \partial_\theta\mathcal{H} = -R^{n-1}\sin\theta \end{cases} \qquad \text{thus} \qquad \begin{cases} dR/dZ = \dfrac{1}{R^{n-1}\cos\theta}\partial_\theta\mathcal{H} \\ d\theta/dZ = -\dfrac{1}{R^{n-1}\cos\theta}\partial_R\mathcal{H} \end{cases}$$

which shows that, up to a scale change in $Z$, system (4) is nothing but the Hamiltonian system deriving from $\mathcal{H}(.,.)$ by the usual symplectic form on $\mathbb{R}^2$.

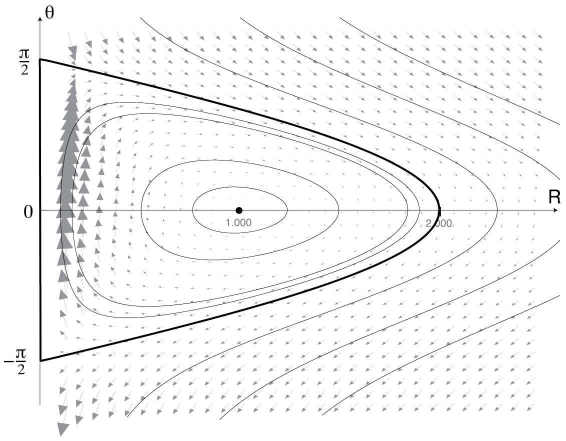


Figure 2: Phase portrait of system (4) (the time variable is $t$ with $dZ = -\cos\theta\, dt$, instead of $Z$) in the case $n = 2$.

The phase portrait of the autonomous system (4) is shown (in the case $n$ equals 2) on figure 2. It admits an elliptic equilibrium at $(1, 0)$ and no other equilibrium in $(0, +\infty) \times (-\pi/2, \pi/2)$. The Hessian matrix of $\mathcal{H}$ at $(1, 0)$ reads:

$$\begin{pmatrix} -(n-1) & 0 \\ 0 & -1 \end{pmatrix}$$

This matrix being negative definite, the point $(1, 0)$ is a strict local maximum point of $\mathcal{H}$. You could actually check that this point is also the strict global maximum point of $\mathcal{H}$ on $[0, +\infty) \times (-\pi, \pi)$.

As was extensively studied by Riera and the author in [10], the effect of the additional "perturbation" term $R/(2Z)$ in the expression of $dR/dZ$ in the "full" (non-autonomous) system (3) is to take the solution away from the maximum point $(1, 0)$ of $\mathcal{H}(.,.)$ when $Z$ increases (in other words, to decrease the value of $\mathcal{H}(R, \theta)$ along the solution). See figure 3. Using this repulsive effect of the perturbation term, the following existence result was proved in [10] (once repulsion is quantified, it follows from an elementary argument, namely Cantor's intersection theorem). This result is almost identical to the existence result established by Concus and Finn in [5, 6].

**Theorem 2.** *There exists (at least) one solution $Z \mapsto \big(R(Z), \theta(Z)\big)$ of system (3) that is defined on $(0, +\infty)$, and every such solution satisfies:*

$$\big(R(Z), \theta(Z)\big) \to (1, 0) \quad \textit{when} \quad Z \to +\infty\,.$$

# 3 Sketch of the proof

As can be expected as this stage, the proof of uniqueness of the singular solution relies on the same "repulsion" effect displayed on figure 3. We shall assume that two distinct

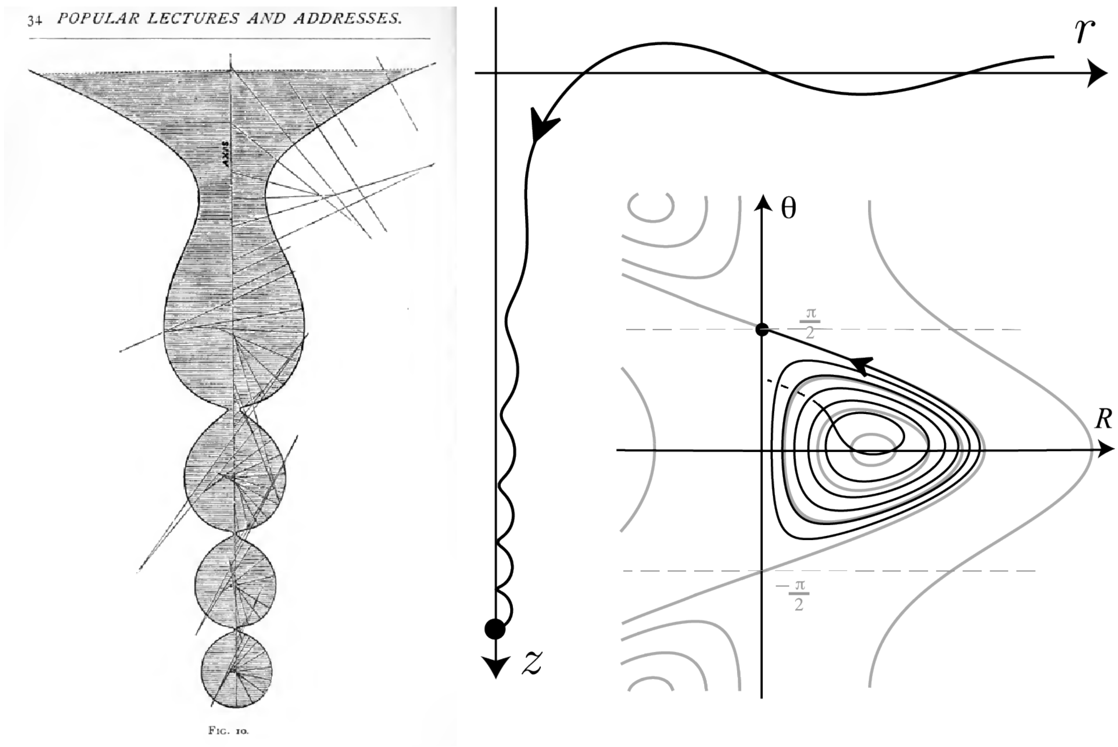


Figure 3: Left: pendent drop with four nodes, drawn by John Perry, a collaborator of Lord Kelvin, and reproduced from [11, p. 34]. Center and right: numerical computation (reproduced from [10, p. 1850]) of the profile of a (finite) pendent drop (with six nodes, in the case $n = 2$) and corresponding behaviour in the $(R, \theta)$-plane. The tip of the drop corresponds to the final point $(\pi/2, 0)$ of the trajectory in the $(R, \theta)$-plane.

singular solutions exist, then we shall prove by an elementary perturbation argument that the resulting effect of the "additional" term $R/(2Z)$ in the "full" system (3) is to increase the distance between those two solutions, at least for $Z$ sufficiently large, a contradiction with the fact that both must converge to $(1,0)$ when $Z$ approaches $+\infty$.

The sole obstacle to overcome is that the perturbation argument requires an a priori estimate on singular solutions slightly more precise than: "$\big(R(Z),\theta(Z)\big)$ approaches $(1,0)$ when $Z$ approaches $+\infty$". What will be required is the following fist order approximation:

$$\begin{cases} R(Z) = 1 + o_{Z\to+\infty}\Big(\dfrac{1}{Z}\Big) \\ \theta(Z) = \dfrac{Z}{2} + o_{Z\to+\infty}\Big(\dfrac{1}{Z}\Big) \end{cases} \tag{6}$$

The remaining of the paper is organized as follows. In section 4 we shall describe the (unique) formal singular solution (as an asymptotic expansion in powers of $1/Z$), and observe that it matches first order expansion (6). In section 5 we shall carry on the main perturbation argument and, assuming that (6) holds, we shall prove uniqueness of the singular solution. In section 6, the very same argument will be used to prove (6) — by the way we shall actually prove that the singular solution matches the asymptotic expansion of section 4 up to every order.

## 4 Formal singular solution (asymptotic expansion)

In the following proposition, a *formal solution* means an expansion of the form (7) below such that, when it is injected in system (3) where the terms $\tan\theta$ and $\cos\theta$ are replaced by their expansions in power series of $\theta$, then the two equations of system (3) hold at every order in $Z^{-1}$.

**Proposition 1.** *System (3) admits a unique formal solution of the form*

$$\begin{cases} R(Z) = \displaystyle\sum_{k\in\mathbb{N}} \frac{R_k}{Z^k} \\ \theta(Z) = \displaystyle\sum_{k\in\mathbb{N}} \frac{\theta_k}{Z^k} \end{cases} \quad \textit{with} \quad (R_k,\theta_k)\in\mathbb{R}^2 \quad \textit{and} \quad R_0>0 \quad \textit{and} \quad -\frac{\pi}{2}<\theta_0<\frac{\pi}{2}. \tag{7}$$

*Moreover,*

- *expansion $\sum_{k\in\mathbb{N}} R_k/Z^k$ is even with respect to $Z$ (that is $R_k=0$ if $k$ is odd);*
- *expansion $\sum_{k\in\mathbb{N}} \theta_k/Z^k$ is odd with respect to $Z$ (that is $\theta_k=0$ if $k$ is even);*

*Proof.* Replacing in system (3) the quantities $R(Z)$ and $\theta(Z)$ by their formal expansions (7) gives:

$$\sum_{k\geq 2} -\frac{(k-1)R_{k-1}}{Z^k} = -\tan\Big(\sum_{k\geq 0}\frac{\theta_k}{Z^k}\Big) + \sum_{k\geq 1}\frac{R_{k-1}}{2Z^k}, \tag{8}$$

$$\sum_{k\geq 2} -\frac{(k-1)\theta_{k-1}}{Z^k} = (n-1)\bigg(\frac{1}{\cos\big(\sum_{k\geq 0}\theta_k/Z^k\big)} - \frac{1}{\sum_{k\geq 0} R_k/Z^k}\bigg). \tag{9}$$

Thus,

- at order $k = 0$, equality (8) yields $\theta_0 = 0$ and equality (9) yields $R_0 = 1$;
- at order $k = 1$, equality (8) yields $\theta_1 = 1/2$ and equality (9) yields $R_0 = 0$.

More generally, at every order $k$ in $\mathbb{N}$,

- equality (8) provides an expression of $\theta_k$ depending on $R_0, \dots, R_{k-1}$ and $\theta_0, \dots, \theta_{k-1}$;
- equality (9) provides an expression of $R_k$ depending on $R_0, \dots, R_{k-1}$ and $\theta_0, \dots, \theta_{k-1}$.

This proves the uniqueness of the expansion (7) satisfying system (3).

Now assume that this expansion of $R(Z)$ is even up to order $k-1$ and this expansion of $\theta(Z)$ is odd up to some order $k-1$. Then,

- if $k$ is even then the expression of $\theta_k$ provided by equality (8) equals zero; this follows from the facts that $R_{k-1} = 0$ and that tangent function is odd;
- if $k$ is odd then the expression of $R_k$ provided by equality (9) equals zero; this follows from the facts that $\theta_{k-1} = 0$ and that cosine function is even.

Thus expansion of $R(Z)$ ($\theta(Z)$) remains even (respectively, odd) up to order $k$. This completes the proof. $\square$

## 5 Perturbation argument

Let

$$Z \mapsto \big(R(Z), \theta(Z)\big) \quad \text{and} \quad Z \mapsto \big(\tilde{R}(Z), \tilde{\theta}(Z)\big)$$

denote two singular solutions of system (3) (that is, defined up to $+\infty$ in $Z$); according to Theorem 2, both must approach $(1, 0)$ when $Z$ approaches $+\infty$). Let us write:

$$\begin{aligned} \rho(Z) &= \tilde{R}(Z) - R(Z)\,, \\ \varphi(Z) &= \tilde{\theta}(Z) - \theta(Z)\,. \end{aligned}$$

Since the two solutions share the same limit when $Z$ approaches $+\infty$, it follows that:

$$\big(\rho(Z), \varphi(Z)\big) \to (0, 0) \quad \text{when} \quad Z \to +\infty\,.$$

To enforce the contradiction argument sketched in section 3, we would like to show that, if these two solutions differ, then the distance between them — that is, the size of the pair $\big(\rho(Z), \varphi(Z)\big)$ — increases with $Z$. Since $(1, 0)$ is a strict local maximum point of $\mathcal{H}$, a possible approach for that is to show that the function

$$h(Z) = \mathcal{H}\big(1 + \rho(Z), \varphi(Z)\big)$$

decreases with $Z$. Basically,

$$\frac{dh}{dZ} = \partial_R \mathcal{H}\big(1+\rho(Z), \varphi(Z)\big) \cdot \frac{d\rho}{dZ} + \partial_\theta \mathcal{H}\big(1+\rho(Z), \varphi(Z)\big) \cdot \frac{d\varphi}{dZ}\,. \tag{10}$$

From expressions (5) of $\partial_R \mathcal{H}$ and $\partial_\theta \mathcal{H}$, it follows that:

$$\begin{aligned} \partial_R \mathcal{H}\big(1+\rho(Z), \varphi(Z)\big) &= (n-1)(1+\rho)^{n-2}(\cos\varphi - 1 - \rho) \\ &= -(n-1)\Big(\rho + o_{Z\to+\infty}\big(|(\rho,\varphi)|\big)\Big)\,, \qquad (11)\\ \partial_\theta \mathcal{H}\big(1+\rho(Z), \varphi(Z)\big) &= -(1+\rho)^{n-1}\sin\varphi \\ &= -\Big(\varphi + o_{Z\to+\infty}\big(|(\rho,\varphi)|\big)\Big) \qquad (12)\end{aligned}$$

with the notation

$$|(\rho,\varphi)| = \sqrt{\rho^2+\varphi^2}\,.$$

Let us define functions $\varepsilon_\rho(Z)$ and $\varepsilon_\varphi(Z)$ by:

$$\begin{cases} d\rho/dZ = -\tan\varphi + \varepsilon_\rho \\ d\varphi/dZ = (n-1)\Big(\dfrac{1}{\cos\varphi} - \dfrac{1}{1+\rho}\Big) + \varepsilon_\varphi \end{cases}$$

The quantities $\varepsilon_\rho(Z)$ and $\varepsilon_\varphi(Z)$ thus measure the discards in $d\rho/dZ$ and $d\varphi/dZ$ with respect to what these two derivatives would be if the pair $\big(1+\rho(Z),\varphi(Z)\big)$ was (exactly) governed by the asymptotic differential system (4). Since $\mathcal{H}(.,.)$ is a conserved quantity for this system, expression (10) of $dh/dZ$ then reduces to

$$\frac{dh}{dZ} = \partial_R\mathcal{H}\cdot\varepsilon_\rho + \partial_\theta\mathcal{H}\cdot\varepsilon_\varphi$$

and, substituting expansions (11) and (12) of $\partial_R\mathcal{H}$ and $\partial_\theta\mathcal{H}$ in this expression, it follows that

$$\frac{dh}{dZ} = -(n-1)\Big(\rho + o_{Z\to+\infty}\big(|(\rho,\varphi)|\big)\Big)\cdot\varepsilon_\rho - \Big(\varphi + o_{Z\to+\infty}\big(|(\rho,\varphi)|\big)\Big)\cdot\varepsilon_\varphi,. \tag{13}$$

This leads us to compute the quantities $\varepsilon_\rho(Z)$ and $\varepsilon_\varphi(Z)$. It follows from system (3) that

$$\begin{aligned} \varepsilon_\rho &= -(\tan\tilde\theta - \tan\theta) + \frac{\rho}{2Z} + \tan\varphi \\ &= \frac{\rho}{2Z} - \tan\varphi\tan\theta\tan(\theta+\varphi)\,, \\ \varepsilon_\varphi &= (n-1)\bigg(\Big(\frac{1}{\cos(\theta+\varphi)} - \frac{1}{\cos\theta}\Big) - \Big(\frac{1}{\cos\varphi} - 1\Big) + \Big(-\frac{1}{R+\rho} + \frac{1}{R}\Big) + \Big(\frac{1}{1+\rho} - 1\Big)\bigg) \\ &= (n-1)\bigg(\frac{\sin\varphi\sin\theta}{\cos\theta\cos(\theta+\varphi)} \\ &\qquad + \frac{1-\cos\varphi}{\cos(\theta+\varphi)}\Big((1-\cos\theta) + \frac{\sin\varphi\sin\theta}{\cos\varphi}\Big) + \rho(1-R)\frac{1+R+\rho}{R(R+\rho)(1+\rho)}\bigg)\,. \end{aligned}$$

At first glance these expressions look a bit intricate, but if we make the additional hypothesis (6), that is if we assume that

$$R(Z) = 1 + o_{Z\to+\infty}\Big(\frac{1}{Z}\Big)\,,$$
$$\theta(Z) = \frac{1}{2Z} + o_{Z\to+\infty}\Big(\frac{1}{Z}\Big)\,,$$

then expressions of $\varepsilon_\rho$ and $\varepsilon_\varphi$ above yield:

$$\varepsilon_\rho(Z) = \frac{\rho}{2Z} + \frac{o_{Z\to+\infty}\big(|(\rho,\varphi)|\big)}{Z}\,,$$
$$\varepsilon_\varphi(Z) = (n-1)\frac{\varphi}{2Z} + \frac{o_{Z\to+\infty}\big(|(\rho,\varphi)|\big)}{Z}\,.$$

Expansion (13) of $dh/dZ$ thus becomes

$$\frac{dh}{dZ} = -(n-1)\frac{\rho^2+\varphi^2}{2Z}\big(1 + o_{Z\to+\infty}(1)\big)\,,$$

and as a consequence, for all $Z$ sufficiently large,

$$\frac{dh}{dZ} \le 0. \tag{14}$$

On the other hand, since $\big(\rho(Z),\varphi(Z)\big)$ approaches $(0,0)$ when $Z$ approaches $+\infty$, it follows that:

$$h(Z) \to \mathcal{H}(1,0) \quad \text{when} \quad Z\to+\infty\,.$$

Since $(1,0)$ is a strict local maximum point of $\mathcal{H}$, this shows, together with inequality (14), that, for all $Z$ sufficiently large, $h(Z) = \mathcal{H}(1,0)$, in other words:

$$\big(\rho(Z),\varphi(Z)\big) = (0,0)\,.$$

The two singular solutions $(R,\theta)$ and $(\tilde{R},\tilde{\theta})$ are thus equal. This proves the desired uniqueness under hypothesis (6).

To complete the proof of Theorem 1, the sole remaining thing to prove is that hypothesis (6) holds for every singular solution.

# 6 Matching with asymptotic expansion

The following proposition states that every singular solution coincides, up to every order, with the formal singular solution defined in Proposition 1.

**Proposition 2.** *For every solution $Z\mapsto\big(\tilde{R}(Z),\tilde{\theta}(Z)\big)$ of system (3), defined on $(0,+\infty)$, and for all $k$ in $\mathbb{N}$, the following expansions hold:*

$$\tilde{R}(Z) = \sum_{j=0}^{k}\frac{R_j}{Z^j} + o_{Z\to+\infty}\Big(\frac{1}{Z^k}\Big)\,,$$
$$\tilde{\theta}(Z) = \sum_{j=0}^{k}\frac{\theta_j}{Z^j} + o_{Z\to+\infty}\Big(\frac{1}{Z^k}\Big)\,.$$

*Proof.* Let $Z \mapsto \big(\tilde{R}(Z), \tilde{\theta}(Z)\big)$ denote a solution of of system (3) defined on $(0, +\infty)$. According to Theorem 2, the pair $\big(\tilde{R}(Z), \tilde{\theta}(Z)\big)$ approaches $(1, 0)$ when $Z$ approaches $+\infty$. Since $R_0 = 1$ and $\theta_0 = 0$, this proves the desired result for $k = 0$.

Now take $k$ in $\mathbb{N}^*$ and let us consider the following two functions of $Z$:

$$R(Z) = \sum_{j=0}^{k} \frac{R_j}{Z^j} \quad \text{and} \quad \theta(Z) = \sum_{j=0}^{k} \frac{\theta_j}{Z^j}$$

(note that these two functions depend on the choice of $k$). Then, the definitions of the coefficients $R_j$ and $\theta_j$ in the proof of Proposition 1 show that the following estimates hold:

$$\begin{cases} dR/dZ = -\tan\theta + \dfrac{R}{2Z} + \mathcal{O}_{Z\to+\infty}\Big(\dfrac{1}{Z^{k+1}}\Big) \\ d\theta/dZ = (n-1)\Big(\dfrac{1}{\cos\theta} - \dfrac{1}{R}\Big) + \mathcal{O}_{Z\to+\infty}\Big(\dfrac{1}{Z^{k+1}}\Big) \end{cases} \tag{15}$$

Moreover, since $k \geq 1$, hypothesis (6) holds for $(R, \theta)$:

$$\begin{aligned} R(Z) &= 1 + o_{Z\to+\infty}\Big(\frac{1}{Z}\Big), \\ \theta(Z) &= \frac{1}{2Z} + o_{Z\to+\infty}\Big(\frac{1}{Z}\Big). \end{aligned}$$

To complete the proof, we will apply exactly the same approach (and adopt the same notation) as in section 5, the only difference being the remaining terms $\mathcal{O}(1/Z^{k+1})$ on the right-hand side of system (15) above. Let us write:

$$\begin{aligned} \rho(Z) &= \tilde{R}(Z) - R(Z), \\ \varphi(Z) &= \tilde{\theta}(Z) - \theta(Z), \\ h(Z) &= \mathcal{H}\big(1 + \rho(Z), \varphi(Z)\big). \end{aligned}$$

Expansions (11) and (12) of $\partial_R\mathcal{H}$ and $\partial_\theta\mathcal{H}$ hold unchanged. Let use define functions $\varepsilon_\rho(Z)$ and $\varepsilon_\varphi(Z)$ by:

$$\begin{cases} d\rho/dZ = -\tan\varphi + \varepsilon_\rho \\ d\varphi/dZ = (n-1)\Big(\dfrac{1}{\cos\varphi} - \dfrac{1}{1+\rho}\Big) + \varepsilon_\varphi \end{cases}$$

Expressions of $\varepsilon_\rho$ and $\varepsilon_\varphi$ are identical to those of section 5, except the fact that they comprise additional terms $\mathcal{O}(1/Z^{k+1})$. Thus,

$$\begin{aligned} \varepsilon_\rho &= \frac{\rho}{2Z} + \frac{o_{Z\to+\infty}\big(|(\rho,\varphi)|\big)}{Z} + \mathcal{O}_{Z\to+\infty}\Big(\frac{1}{Z^{k+1}}\Big), \\ \varepsilon_\varphi &= (n-1)\frac{\varphi}{2Z} + \frac{o_{Z\to+\infty}\big(|(\rho,\varphi)|\big)}{Z} + \mathcal{O}_{Z\to+\infty}\Big(\frac{1}{Z^{k+1}}\Big), \end{aligned}$$

and it follows from unchanged expansion (13) that:

$$\begin{aligned}\frac{dh}{dZ} &= -(n-1)\frac{\rho^2+\varphi^2}{2Z}\bigl(1+o_{Z\to+\infty}(1)\bigr)+\mathcal{O}_{Z\to+\infty}\Bigl(\frac{|(\rho,\varphi)|}{Z^{k+1}}\Bigr)\\ &= -(n-1)\frac{|(\rho,\varphi)|}{2Z}\biggl(|(\rho,\varphi)|\bigl(1+o_{Z\to+\infty}(1)\bigr)+\mathcal{O}_{Z\to+\infty}\Bigl(\frac{1}{Z^k}\Bigr)\biggr).\end{aligned}\tag{16}$$

We are going to deduce from this last estimate that

$$|(\rho,\varphi)| = \mathcal{O}_{Z\to+\infty}\Bigl(\frac{1}{Z^k}\Bigr).$$

Indeed, according to expression (16) of $dh/dZ$, there exist positive quantities $C_1$ and $Z_0$ such that, for all $Z$ superior or equal to $Z_0$,

$$\bigl|\bigl(\rho(Z),\varphi(Z)\bigr)\bigr| \geq \frac{C_1}{Z^k} \Longrightarrow \frac{dh}{dZ}\leq 0.$$

Besides, since $(1,0)$ is a strict local maximum point of $\mathcal{H}$, since the Hessian matrix of $\mathcal{H}$ at $(0,1)$ is negative definite, and since $\mathcal{H}(1,0)$ equals $1/n$, the quantities $|(\rho,\varphi)|$ and $\sqrt{1/n-h(Z)}$ do not differ, if $Z$ is large, by more than a fixed multiplicative factor. As a consequence, there exists a positive quantity $C_2$ such that, for all $Z$ greater than or equal to $Z_0$,

$$\sqrt{\frac{1}{n}-h(Z)} \geq \frac{C_2}{Z^k} \Longrightarrow \frac{dh}{dZ}(Z)\leq 0\,,$$

in other words:

$$h(Z) \leq \frac{1}{n}-\Bigl(\frac{C_2}{Z^k}\Bigr)^2 \Longrightarrow \frac{dh}{dZ}(Z)\leq 0\,.\tag{17}$$

But since $h(Z)$ approaches $1/n$ when $Z$ approaches $+\infty$, the left-hand inequality of implication (17) can actually not occur if $Z\geq Z_0$. Indeed, by contradiction, if there existed a quantity $Z_1$ superior or equal to $Z_0$ such that

$$h(Z_1)\leq \frac{1}{n}-\Bigl(\frac{C_2}{Z1^k}\Bigr)^2,$$

then according to implication (17), it would follow that, for all $Z$ superior or equal to $Z_1$,

$$h(Z)\leq \frac{1}{n}-\Bigl(\frac{C_2}{Z_1^k}\Bigr)^2 \leq \frac{1}{n}-\Bigl(\frac{C_2}{Z^k}\Bigr)^2,$$

a contradiction with the fact that $h(Z)$ approaches $1/n$ when $Z$ approaches $+\infty$.

We thus proved that, for all $Z$ sufficiently large,

$$h(Z)\geq 1/n - C_2^2/Z_1^{2k} \qquad \text{or equivalently} \qquad \sqrt{1/n-h(Z)}\leq C_2/Z^k\,,$$

and this finally yields the desired estimate:

$$|(\rho,\varphi)| = \mathcal{O}_{Z\to+\infty}\Bigl(\frac{1}{Z^k}\Bigr).$$

Since the positive integer $k$ was arbitrary, this finishes the proof of Proposition 2. □

In view of the comments in section 5, proof of Theorem 1 is complete.

### Acknowledgements

I thank Pierre Coullet who introduced me to the capillary equation and this perturbation approach, and Christophe Riera who carried out the numerical simulations and with whom I had many fruitful discussions.


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